

# On the rate of growth of Lévy processes with no positive jumps conditioned to stay positive .

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*Abstract:* In this article, we study the asymptotic behaviour of Lévy processes with no positive jumps conditioned to stay positive. We establish integral tests for the lower envelope at 0 and at  $+\infty$  and an analogue of Khintchin's law of the iterated logarithm at 0 and  $+\infty$ , for the upper envelope.

*Key words:* Lévy processes conditioned to stay positive, Future infimum process, First and last passage times, Rate of growth, integral tests.

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## 1 Introduction and main results.

Lévy processes conditioned to stay positive have been introduced by Bertoin at the beginning of the nineties (see [1], [2], [3]). These first studies are devoted to the special case where the Lévy processes are spectrally one-sided. In a recent work Chaumont and Doney [9] studied more general cases.

In this article, we are interested in the case when Lévy processes have no positive jumps (or spectrally negative Lévy processes). This case has been deeply studied by Bertoin in Chapter VII of [5]. This will be our basic reference.

The aim of this note is to describe the lower and upper envelope at 0 and at  $+\infty$  of Lévy processes with no positive jumps conditioned to stay positive throughout integral tests and laws of the iterated logarithm.

For our purpose, we will first introduce some important properties of Lévy processes with no positive jumps and then we will define the “conditioned” version.

Let  $\mathcal{D}$  denote the Skorokhod's space of càdlàg paths with real values and defined on

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the positive real half-line  $[0, \infty)$  and  $\mathbb{P}$  a probability measure defined on  $\mathcal{D}$  under which  $\xi$  will be a real-valued Lévy process with no positive jumps, that is its Lévy measure has support in the negative real-half line.

From the general theory of Lévy processes (see Bertoin [5] or Sato [19] for background), we know that  $\xi$  has finite exponential moments of arbitrary positive order. In particular  $\xi$  satisfies

$$\mathbb{E}\left(\exp\{\lambda\xi_t\}\right) = \exp\{t\psi(\lambda)\}, \quad \lambda, t \geq 0,$$

where

$$\psi(\lambda) = a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty, 0)} (e^{\lambda x} - 1 - \lambda x \mathbb{1}_{\{x > -1\}}) \Pi(dx), \quad \lambda \geq 0,$$

$a \in \mathbb{R}, \sigma \geq 0$  and  $\Pi$  is a measure that satisfies

$$\int_{(-\infty, 0)} (1 \wedge x^2) \Pi(dx).$$

The measure  $\Pi$  is well-known as the Lévy measure of the process  $\xi$ .

According to Bertoin [5], the mapping  $\psi : [0, \infty) \rightarrow (-\infty, \infty)$  is convex and ultimately increasing. We denote its right-inverse on  $[0, \infty)$  by  $\Phi$ . Let us introduce the first passage time of  $\xi$  by

$$T_x = \inf\{s : \xi_s \geq x\} \quad \text{for } x \geq 0.$$

From Theorem VII.1 in [5], we know that the process  $T = (T_x, x \geq 0)$  is a subordinator, killed at an independent exponential time if  $\xi$  drifts towards  $-\infty$ . The Laplace exponent of  $T$  is given by  $\Phi$ ,

$$\mathbb{E}\left(\exp\{-\lambda T_x\}\right) = \exp\{-x\Phi(\lambda)\}, \quad \lambda, t \geq 0.$$

According to Bertoin [5] Chapter III, we know that

$$\Phi(\lambda) = k + d\lambda + \int_{(0, \infty)} (1 - e^{-\lambda x}) \nu(dx), \quad \lambda \geq 0,$$

where  $k$  is the killing rate,  $d$  is the drift coefficient and  $\nu$  is the Lévy measure of the subordinator  $T$  which fulfils the following condition,

$$\int_{(0, \infty)} (1 \wedge x) \nu(dx) < \infty.$$

It is sometimes convenient to perform an integration by parts and rewrite  $\Phi$  as

$$\frac{\Phi(\lambda)}{\lambda} = d + \int_0^\infty e^{-\lambda x} \bar{\nu}(x) dx, \quad \text{with } \bar{\nu}(x) = k + \nu((x, \infty)).$$

Note that the killing rate and the drift coefficient are given by

$$k = \Phi(0) \quad \text{and} \quad d = \lim_{\lambda \rightarrow +\infty} \frac{\Phi(\lambda)}{\lambda}.$$

In particular, the life time  $\zeta$  has an exponential distribution with parameter  $k \geq 0$  ( $\zeta = +\infty$  for  $k = 0$ ). Hence, it is not difficult to deduce that  $\xi$  drifts towards  $-\infty$  if and only if  $\Phi(0) > 0$ .

In order to study the case when  $\xi$  drifts towards  $-\infty$ , we will define the following probability measure,

$$\mathbb{P}^{\natural}(A) = \mathbb{E}\left(\exp\{\Phi(0)\xi_t\}\mathbb{1}_A\right), \quad A \in \mathcal{F}_t,$$

where  $\mathcal{F}_t$  is the  $\mathbb{P}$ -complete sigma-field generated by  $(\xi_s, s \leq t)$ . Note that under  $\mathbb{P}^{\natural}$ , the process  $\xi$  is still a Lévy process with no positive jumps which drifts towards  $+\infty$  and its Laplace exponent is defined by  $\psi^{\natural}(\lambda) = \psi(\Phi(0) + \lambda)$ ,  $\lambda \geq 0$ . Moreover the first passage process  $T$  is still a subordinator with Laplace exponent  $\Phi^{\natural}(\lambda) = \Phi(\lambda) - \Phi(0)$ . Since  $\xi$  has no positive jumps, we can solve the so-called two-sided-exit problem in explicit form. This problem consists in determining the probability that  $\xi$  makes its first exit from an interval  $[-x, y]$  ( $x, y > 0$ ) at the upper boundary point. More precisely,

$$\mathbb{P}\left(\inf_{0 \leq t \leq T_y} \xi_t \geq -x\right) = \frac{W(x)}{W(x+y)},$$

where  $W : [0, \infty) \rightarrow [0, \infty)$  is the unique absolutely continuous increasing function with Laplace transform

$$\int_0^{\infty} e^{-\lambda x} W(x) dx = \frac{1}{\psi(\lambda)}, \quad \text{for } \lambda > \Phi(0). \quad (1.1)$$

The function  $W$  is well-known as the scale function and is necessary for the definition of Lévy processes conditioned to stay positive.

Using the Doob's theory of  $h$ -transforms, we construct a new Markov process by an  $h$ -transform of the law of the Lévy process killed at time  $R = \inf\{t \geq 0 : \xi_t < 0\}$  with the harmonic function  $W$  (see for instance Chapter VII in Bertoin [5], Chaumont [8] or Chaumont and Doney [9]), and its semigroup is given by

$$\mathbb{P}_x^{\uparrow}(\xi_t \in dy) = \frac{W(y)}{W(x)} \mathbb{P}_x(\xi_t \in dy, t < R) \quad \text{for } x > 0,$$

where  $\mathbb{P}_x$  denotes the law of  $\xi$  starting from  $x > 0$ . Under  $\mathbb{P}_x^{\uparrow}$ ,  $\xi$  is a process taking values in  $[0, \infty)$ . It will be referred to as the Lévy process started at  $x$  and conditioned to stay positive.

It is important to note that when  $\xi$  drifts towards  $-\infty$ , we have that  $\mathbb{P}_x^{\uparrow} = \mathbb{P}_x^{\natural\uparrow}$ , for all  $x \geq 0$ . Hence the study of this case is reduced to the study of the processes which drift towards  $+\infty$ .

Bertoin proved in [5] the existence of a measure  $\mathbb{P}_0^{\uparrow}$  under which the process starts at 0 and stays positive. In fact, the author in [5] proved that the probability measures  $\mathbb{P}_x^{\uparrow}$  converge as  $x$  goes to  $0+$  in the sense of finite-dimensional distributions to  $\mathbb{P}_0^{\uparrow} := \mathbb{P}^{\uparrow}$  and noted that this convergence also holds in the sense of Skorokhod. Several authors have studied this convergence, the most recent work is due to Chaumont and Doney [9]. In [9], the authors proved that this convergence holds in the sense of Skorokhod

under general hypothesis.

Chaumont and Doney [9] also give a path decomposition of the process  $(\xi, \mathbb{P}_x^\uparrow)$  at the time of its minimum. In particular, if  $m$  is the time at which  $\xi$ , under  $\mathbb{P}_x^\uparrow$ , attains its minimum, we have that under  $\mathbb{P}_x^\uparrow$  the pre-infimum process,  $(\xi_t, 0 \leq t < m)$ , and the post-infimum process,  $(\xi_{t+m} - \xi_m, t \geq 0)$ , are independent and the later has by law  $\mathbb{P}^\uparrow$ . Moreover, the process  $(\xi, \mathbb{P}_x^\uparrow)$  reaches its absolute minimum once only and its law is given by

$$\mathbb{P}_x^\uparrow(\xi_m \geq y) = \frac{W(x-y)}{W(x)} \mathbb{1}_{\{y \leq x\}}. \quad (1.2)$$

Recently, Chaumont and Pardo [10] studied the lower envelope of positive self-similar Markov processes (or pssMp for short). In particular, the authors obtained integral tests at 0 and at  $+\infty$ , for the lower envelope of stable Lévy processes with no positive jumps conditioned to stay positive and with index  $\alpha \in (1, 2]$ ; such processes are well-known examples of pssMp. More precisely, when  $X^{(x)}$  is the stable Lévy process with no positive jumps conditioned to stay positive and with starting point  $x \geq 0$ , we have the following integral test for the lower envelope at 0 and at  $+\infty$ : let  $f$  be an increasing function, then

$$\mathbb{P}(X_t^{(0)} < f(t), \text{i.o.}, \text{ as } t \rightarrow 0) = \begin{cases} 0 \\ 1 \end{cases}, \text{ according as } \int_{0+} \left(\frac{f(t)}{t}\right)^{1/\alpha} \frac{dt}{t} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Let  $f$  be an increasing function, then for all  $x \geq 0$ ,

$$\mathbb{P}(X_t^{(x)} < f(t), \text{i.o.}, \text{ as } t \rightarrow \infty) = \begin{cases} 0 \\ 1 \end{cases}, \text{ according as } \int^{+\infty} \left(\frac{f(t)}{t}\right)^{1/\alpha} \frac{dt}{t} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Later, Pardo [17] studied the upper envelope of pssMp at 0 and at  $+\infty$ . In particular, the author noted that  $X^{(x)}$ , the stable Lévy process with no positive jumps conditioned to stay positive, satisfies the following law of the iterated logarithm

$$\limsup_{t \rightarrow 0} \frac{X_t^{(0)}}{t^{1/\alpha} (\log |\log t|)^{1-1/\alpha}} = c(\alpha), \quad \mathbb{P}^\uparrow - a.s.,$$

where  $c(\alpha)$  is a positive constant which depends on  $\alpha$ .

Moreover, Pardo [17] also established that

$$\limsup_{t \rightarrow 0} \frac{J_t^{(0)}}{t^{1/\alpha} (\log |\log t|)^{1-1/\alpha}} = c(\alpha), \quad \mathbb{P}^\uparrow - a.s.,$$

$$\limsup_{t \rightarrow 0} \frac{X_t^{(0)} - J_t^{(0)}}{t^{1/\alpha} (\log |\log t|)^{1-1/\alpha}} = c(\alpha), \quad \mathbb{P}^\uparrow - a.s.,$$

where  $J^{(x)}$  is the future infimum process of  $X^{(x)}$ , i.e.  $J_t^{(x)} = \inf_{s \geq t} X_s^{(x)}$ . It is important to note that the above laws of the iterated logarithm have been also obtained at  $+\infty$ , for any starting point  $x \geq 0$ .

Bertoin [4] studied the local rate of growth of Lévy processes with no positive jumps. In

[4], the author noted that the sample path behaviour of a Lévy process with no positive jumps  $\xi$ , immediately after a local minimum is the same as that of its conditioned version  $(\xi, \mathbb{P}^\uparrow)$  at the origin. The main result in [4] gives us a remarkable law of the iterated logarithm at an instant when the path attains a local minimum on the interval  $[0, 1]$ . We will recall this result for Lévy processes conditioned to stay positive. With a misuse of notation, we will denote by  $\Phi$  and  $\psi$  for the functions  $\Phi^\natural$  and  $\psi^\natural$ , respectively; when we are in the case where the process  $\xi$  drifts towards  $-\infty$ .

**Theorem 1 (Bertoin, [4])** *There is a finite positive constant  $k$  such that*

$$\limsup_{t \rightarrow 0} \frac{\xi_t \Phi(t^{-1} \log |\log t|)}{\log |\log t|} = k, \quad \mathbb{P}^\uparrow - a.s.$$

It is important to note that the constant found by Bertoin satisfies  $k \in [c, c + \gamma]$ , where  $c$  is the same constant found below in Theorem 3 and  $\gamma \geq 6$ .

Bertoin presented in [4] a “geometric” proof using some path transformations that connect  $\xi$  and its conditioned version  $(\xi, \mathbb{P}^\uparrow)$ . Here, we will present standard arguments involving probability tail estimates.

Our main results requires the following hypothesis, for all  $\beta > 1$

$$(H1) \quad \liminf_{x \rightarrow 0} \frac{\psi(x)}{\psi(\beta x)} > 0 \quad \text{and} \quad (H2) \quad \liminf_{x \rightarrow +\infty} \frac{\psi(x)}{\psi(\beta x)} > 0.$$

**Theorem 2** *Let us suppose that (H2) is satisfied, then there is a positive constant  $k$  such that,*

$$\limsup_{t \rightarrow 0} \frac{\xi_t \Phi(t^{-1} \log |\log t|)}{\log |\log t|} = k, \quad \mathbb{P}^\uparrow - a.s.$$

Moreover, if condition (H1) is satisfied, then

$$\limsup_{t \rightarrow +\infty} \frac{\xi_t \Phi(t^{-1} \log \log t)}{\log |\log t|} = k, \quad \mathbb{P}^\uparrow - a.s.$$

Note that with our arguments, we found that  $k \in [c, c\eta]$  where of course  $\eta \geq 1$  and  $c\eta > 3$ . We also remark that in particular (H1) and (H2) are satisfied under the assumption that  $\psi$  is regularly varying at 0 and at  $\infty$  with index  $\alpha > 1$ . Under this assumption

$$k = c = (1/\alpha)^{-1/\alpha} (1 - 1/\alpha)^{\frac{1-\alpha}{\alpha}}.$$

The next result gives us a law of the iterated logarithm at 0 and at  $+\infty$  of the future infimum of  $(\xi, \mathbb{P}^\uparrow)$ . This result extends the result of Pardo [17] for the stable case.

**Theorem 3** *Let  $J$  denote the future infimum process of  $\xi$ , defined by  $J_t = \inf_{s \geq t} \xi_s$ , then there is a positive constant  $c$  such that,*

$$\limsup_{t \rightarrow 0} \frac{J_t \Phi(t^{-1} \log |\log t|)}{\log |\log t|} = c, \quad \mathbb{P}^\uparrow - a.s.,$$

and

$$\limsup_{t \rightarrow +\infty} \frac{J_t \Phi(t^{-1} \log \log t)}{\log |\log t|} = c, \quad \mathbb{P}^\uparrow - a.s.$$

If we assume that (H2) is satisfied, then

$$\limsup_{t \rightarrow 0} \frac{J_t \Phi(t^{-1} \log |\log t|)}{\log |\log t|} = k, \quad \mathbb{P}^\dagger - a.s.,$$

i.e. that  $c = k$ .

Moreover if (H1) is satisfied, then

$$\limsup_{t \rightarrow +\infty} \frac{J_t \Phi(t^{-1} \log \log t)}{\log |\log t|} = k, \quad \mathbb{P}^\dagger - a.s.,$$

i.e. that  $c = k$ .

We now turn our attention to the Lévy process conditioned to stay positive reflected at its future infimum. The following theorem extends the law of the iterated logarithm of Pardo [17] for the stable case.

Let us suppose that for all  $\beta < 1$

$$(H3) \quad \limsup_{x \rightarrow 0} \frac{W(\beta x)}{W(x)} < 1 \quad \text{and} \quad (H4) \quad \limsup_{x \rightarrow +\infty} \frac{W(\beta x)}{W(x)} < 1.$$

**Theorem 4** Under the hypothesis (H2) and (H3), we have that

$$\limsup_{t \rightarrow 0} \frac{(\xi_t - J_t) \Phi(t^{-1} \log |\log t|)}{\log |\log t|} = k, \quad \mathbb{P}^\dagger - a.s.$$

Moreover if (H1) and (H4) are satisfied, then

$$\limsup_{t \rightarrow +\infty} \frac{(\xi_t - J_t) \Phi(t^{-1} \log \log t)}{\log |\log t|} = k, \quad \mathbb{P}^\dagger - a.s.$$

Again conditions (H3) and (H4) are satisfied when  $\psi$  is regularly varying at 0 and at  $\infty$  with index  $\alpha > 1$ .

The lower envelope of  $(\xi, \mathbb{P}^\dagger)$  at 0 and at  $\infty$  is determined as follows,

**Theorem 5** If  $\xi$  has unbounded variation and  $f : [0, \infty) \rightarrow [0, \infty)$  is an increasing function such that  $t \rightarrow f(t)/t$  decreases, one has

$$\liminf_{t \rightarrow 0} \frac{\xi_t}{f(t)} = 0 \quad \mathbb{P}^\dagger - a.s. \quad \text{if and only if} \quad \int_0^\infty f(x) \nu(dx) = \infty.$$

Moreover,

$$\text{if} \quad \int_0^\infty f(x) \nu(dx) < \infty \quad \text{then} \quad \lim_{t \rightarrow 0} \frac{\xi_t}{f(t)} = \infty \quad \mathbb{P}^\dagger - a.s.$$

The lower envelope at  $+\infty$  is determined as follows: if  $\xi$  oscillates or drifts towards  $-\infty$  and the function  $f : [0, \infty) \rightarrow [0, \infty)$  is increasing such that  $t \rightarrow f(t)/t$  decreases, one has

$$\liminf_{t \rightarrow +\infty} \frac{\xi_t}{f(t)} = 0 \quad \mathbb{P}^\dagger - a.s. \quad \text{if and only if} \quad \int^{+\infty} f(x) \nu(dx) = \infty.$$

Moreover,

$$\text{if } \int^{+\infty} f(x)\nu(dx) < \infty \quad \text{then} \quad \lim_{t \rightarrow +\infty} \frac{\xi_t}{f(t)} = \infty \quad \mathbb{P}^\dagger\text{-a.s.}$$

The following result describes the lower envelope of the future infimum of  $(\xi, \mathbb{P}^\dagger)$ . In fact, we will deduce that  $(\xi, \mathbb{P}^\dagger)$  and its future infimum have the same lower functions.

**Theorem 6** (i) *If  $\xi$  has bounded variation, one has*

$$\lim_{t \rightarrow 0} \frac{J_t}{t} = \frac{1}{d} \quad \mathbb{P}^\dagger\text{-a.s.}$$

(ii) *If  $\xi$  has unbounded variation and  $f : [0, \infty) \rightarrow [0, \infty)$  is an increasing function such that  $t \rightarrow f(t)/t$  decreases, one has*

$$\liminf_{t \rightarrow 0} \frac{J_t}{f(t)} = 0 \quad \mathbb{P}^\dagger\text{-a.s.} \quad \text{if and only if} \quad \int_0^\infty f(x)\nu(dx) = \infty.$$

Moreover,

$$\text{if } \int_0^\infty f(x)\nu(dx) < \infty \quad \text{then} \quad \lim_{t \rightarrow 0} \frac{J_t}{f(t)} = \infty \quad \mathbb{P}^\dagger\text{-a.s.}$$

(iii) *If  $\xi$  drifts to  $+\infty$  one has*

$$\lim_{t \rightarrow +\infty} \frac{J_t}{t} = \frac{1}{\mathbb{E}(T_1)} \quad \mathbb{P}^\dagger\text{-a.s.}$$

(iv) *If  $\xi$  oscillates or drifts to  $-\infty$  and  $f : [0, \infty) \rightarrow [0, \infty)$  is an increasing function such that the mapping  $t \rightarrow f(t)/t$  decreases, one has*

$$\liminf_{t \rightarrow +\infty} \frac{J_t}{f(t)} = 0 \quad \mathbb{P}^\dagger\text{-a.s.} \quad \text{if and only if} \quad \int^{+\infty} f(x)\nu(dx) = \infty.$$

Moreover,

$$\text{if } \int^{+\infty} f(x)\nu(dx) < \infty \quad \text{then} \quad \lim_{t \rightarrow +\infty} \frac{J_t}{f(t)} = \infty \quad \mathbb{P}^\dagger\text{-a.s.}$$

Moreover, we have the following integral test for the lower functions in terms of  $\Phi$ .

**Proposition 1** (i) *Let  $f : [0, \infty) \rightarrow [0, \infty)$  be an increasing function.*

$$\text{If } \int_0^\infty x^{-1}f(x)\Phi(1/x)dx < \infty \quad \text{then} \quad \lim_{t \rightarrow 0} \frac{J_t}{f(t)} = \lim_{t \rightarrow 0} \frac{\xi_t}{f(t)} = \infty \quad \mathbb{P}^\dagger\text{-a.s.}$$

(ii) *Let  $f : [0, \infty) \rightarrow [0, \infty)$  be an increasing function.*

$$\text{If } \int^{+\infty} x^{-1}f(x)\Phi(1/x)dx < \infty \quad \text{then} \quad \lim_{t \rightarrow +\infty} \frac{J_t}{f(t)} = \lim_{t \rightarrow +\infty} \frac{\xi_t}{f(t)} = \infty \quad \mathbb{P}^\dagger\text{-a.s.}$$

The rest of this note consists of two sections, which are devoted to the following topics: Section 2 provides asymptotic results for the first and the last passage times of the process  $(\xi, \mathbb{P}^\dagger)$ . In Section 3, we will prove the results presented above.

## 2 First and last passage times.

Let us recall the definition of the first and last passage time of  $(\xi, \mathbb{P}^\dagger)$  or  $\xi^\dagger$  to simplify our notation,

$$T_x^\dagger = \inf \left\{ t \geq 0 : \xi_t^\dagger \geq x \right\} \quad \text{and} \quad U_x^\dagger = \sup \left\{ t \geq 0 : \xi_t^\dagger \leq x \right\} \quad \text{for } x \geq 0.$$

From Theorem VII.18 in [5], we know that  $(\xi_t, 0 \leq t \leq T_x)$ , the Lévy process killed at its first passage time above  $x$ , under  $\mathbb{P}^\dagger$ , has the same law as the Lévy process conditioned to stay positive time-reversed at its last passage time below  $x$ ,  $(x - \xi_{(U_x^\dagger - t)^-}^\dagger, 0 \leq t \leq U_x^\dagger)$ .

In particular, we deduce that  $U_x^\dagger$  has the same law as  $T_x$  and that  $U^\dagger = (U_x^\dagger, x \geq 0)$  is a subordinator with Laplace exponent  $\Phi(\lambda)$  and therefore we obtain that the process  $\xi^\dagger$  drifts towards  $+\infty$ .

There exist a huge variety of results on the upper envelope of subordinators. Fristedt and Pruitt [12] proved a general law of the iterated logarithm which is valid for a wide class of subordinators. The sharper result on the lower envelope for subordinators is due to Pruitt [18]. In his main result, he gave an important integral test.

Bertoin [5] presents a more precise law of the iterated logarithm of subordinators than the result obtained by Fristedt and Pruitt but for a more restrictive class of subordinators. In his result, Bertoin supposes that  $\psi$  is regularly varying at  $+\infty$  with index  $\alpha > 1$  (see Theorems III.11 and III.14 in [5]). In particular, we have the following lemma.

**Lemma 1** *The last passage time process  $U^\dagger$  under the assumption that  $\psi$  is regularly varying at  $+\infty$  with index  $\alpha > 1$ , satisfies*

$$\liminf_{x \rightarrow 0} \frac{U_x^\dagger \psi(x^{-1} \log |\log x|)}{\log |\log x|} = \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right)^{\alpha-1}, \quad \text{almost surely,}$$

and for large times, if we suppose that  $\psi$  is regularly varying at 0 with index  $\alpha > 1$ , then

$$\liminf_{x \rightarrow +\infty} \frac{U_x^\dagger \psi(x^{-1} \log \log x)}{\log \log x} = \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right)^{\alpha-1}, \quad \text{almost surely.}$$

Now, we turn our attention to the first passage time process. Note that due to the absence of positive jumps, for all  $x \geq 0$ ,  $\xi_{T_x^\dagger}^\dagger = x$ , a.s. Hence from the strong Markov property, we have that  $T^\dagger = (T_x^\dagger, x \geq 0)$  is an increasing process with independent increments but not stationary.

Here we will use the results presented in Bertoin [5] and Lemma 1 to obtain the following law of the iterated logarithm for the first and last passage time of  $\xi^\dagger$ .

**Proposition 2** *Suppose that  $\psi$  is regularly varying at  $+\infty$  with index  $\alpha > 1$ . Then the first passage time process satisfies the following law of the iterated logarithm,*

$$\liminf_{x \rightarrow 0} \frac{T_x^\dagger \psi(x^{-1} \log |\log x|)}{\log |\log x|} = \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right)^{\alpha-1}, \quad \text{almost surely,}$$

and for large times, if we suppose that  $\psi$  is regularly varying at 0 with index  $\alpha > 1$ , we have

$$\liminf_{x \rightarrow +\infty} \frac{T_x^\dagger \psi(x^{-1} \log \log x)}{\log \log x} = \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right)^{\alpha-1}, \quad \text{almost surely.}$$

*Proof of Proposition 2:* We will only prove the result for small times since the proof for large times is very similar. For all  $x \geq 0$ , we see that  $T_x^\dagger \leq U_x^\dagger$ , then from Lemma 1 we obtain the upper bound

$$\liminf_{x \rightarrow 0} \frac{T_x^\dagger \psi(x^{-1} \log |\log x|)}{\log |\log x|} \leq \liminf_{x \rightarrow 0} \frac{U_x^\dagger \psi(x^{-1} \log |\log x|)}{\log |\log x|} = \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right)^{\alpha-1}.$$

Next, we prove the lower bound. With this purpose we establish the following lemma.

**Lemma 2** *Assume that  $\psi$  is regularly varying at  $+\infty$  with index  $\alpha > 1$ . Then for every constant  $c_1 > 0$ , we have*

$$-\log \mathbb{P}\left(T_x^\dagger \leq c_1 g(x)\right) \sim \left(1 - \frac{1}{\alpha}\right) \left(\frac{1}{c_1 \alpha}\right)^{1/(\alpha-1)} \log |\log x|, \quad \text{as } x \rightarrow 0,$$

where

$$g(t) = \frac{\log |\log x|}{\psi(x^{-1} \log |\log x|)}.$$

*Proof of Lemma 2:* We know that  $U^\dagger$  is a subordinator and that  $\psi$  is the inverse of the function  $\Phi$ , then from Lemma III.12 in Bertoin [5], we see that

$$-\log \mathbb{P}\left(U_x^\dagger \leq c_1 g(x)\right) \sim \left(1 - \frac{1}{\alpha}\right) \left(\frac{1}{c_1 \alpha}\right)^{1/(\alpha-1)} \log |\log x|, \quad \text{as } x \rightarrow 0.$$

Then the upper bound is clear since for all  $x > 0$  we have that  $T_x^\dagger \leq U_x^\dagger$ .

For the lower bound, let us first define the supremum process  $S = (S_t, t \geq 0)$  by  $S_t = \sup_{0 \leq s \leq t} \xi_s$ . Next, we fix  $\epsilon > 0$ , then by the Markov property

$$\begin{aligned} \mathbb{P}^\dagger(J_{c_1 g(x)} > (1 - \epsilon)x) &\geq \mathbb{P}^\dagger(S_{c_1 g(x)} > x, J_{c_1 g(x)} > (1 - \epsilon)x) \\ &= \int_0^{c_1 g(x)} \mathbb{P}\left(T_x^\dagger \in dt\right) \mathbb{P}_x^\dagger(J_{c_1 g(x)-t} > (1 - \epsilon)x) \\ &\geq \mathbb{P}\left(T_x^\dagger < c_1 g(x)\right) \mathbb{P}_x^\dagger(J_0 > (1 - \epsilon)x). \end{aligned} \quad (2.3)$$

From the definition of the future infimum process, it is clear that  $J_0$  is the absolute minimum of  $(\xi, \mathbb{P}_x^\dagger)$  then by (1.2)

$$\mathbb{P}_x^\dagger(J_0 > (1 - \epsilon)x) = \frac{W(\epsilon x)}{W(x)}.$$

On the other hand, from (1.1) and applying the Tauberian and Monotone density theorems (see for instance Bertoin [5] or Bingham et al [7]) we deduce that

$$W(x) \sim \frac{\alpha}{\Gamma(1 + \alpha)} \frac{1}{x \psi(1/x)} \quad \text{as } x \rightarrow 0,$$

hence,

$$\mathbb{P}_x^\uparrow(J_0 > (1 - \epsilon)x) \rightarrow \epsilon^{(\alpha-1)} \quad \text{as } x \rightarrow 0. \quad (2.4)$$

Now, since the last passage time process is the right inverse of the future infimum process, we have that

$$\mathbb{P}^\uparrow(J_{c_1g(x)} > (1 - \epsilon)x) = \mathbb{P}\left(U_{(1-\epsilon)x}^\uparrow < c_1g(x)\right),$$

and applying Chebyshev's inequality, we have that for every  $\lambda > 0$

$$\mathbb{P}\left(U_{(1-\epsilon)x}^\uparrow < c_1g(x)\right) \leq \exp\left\{\lambda c_1g(x) - (1 - \epsilon)x\Phi(\lambda)\right\},$$

and thus

$$-\log \mathbb{P}\left(U_{(1-\epsilon)x}^\uparrow < c_1g(x)\right) \geq -\lambda c_1g(x) + (1 - \epsilon)x\Phi(\lambda).$$

Next, we choose  $\lambda = \lambda(x)$  such that  $(1 - \epsilon)x\Phi(\lambda) = K \log |\log x|$  for some positive constant  $K$ , that will be specified later on, then  $\lambda = \psi(K(1 - \epsilon)^{-1}x^{-1} \log |\log x|)$ . Since  $\psi$  is regularly varying at  $\infty$  with index  $\alpha$ , we see that

$$\lambda = \lambda(x) \sim K^\alpha(1 - \epsilon)^{-\alpha}\psi(x^{-1} \log |\log x|).$$

This implies

$$-\lambda c_1g(x) + (1 - \epsilon)x\Phi(\lambda) \sim (K - c_1K^\alpha(1 - \epsilon)^{-\alpha}) \log |\log x| \quad (x \rightarrow 0).$$

We now choose  $K$  in such way that  $K - c_1K^\alpha(1 - \epsilon)^{-\alpha}$  is maximal, that is

$$K = (1 - \epsilon)^{\alpha/(\alpha-1)} \left(\frac{1}{c_1\alpha}\right)^{1/(\alpha-1)},$$

and

$$K - c_1K^\alpha(1 - \epsilon)^{-\alpha} = (1 - \epsilon)^{\alpha/(\alpha-1)} \left(\frac{1}{c_1\alpha}\right)^{1/(\alpha-1)} \left(1 - \frac{1}{\alpha}\right).$$

In conclusion, we have established that

$$(1 - \epsilon)^{\alpha/(\alpha-1)} \left(1 - \frac{1}{\alpha}\right) \left(\frac{1}{c_1\alpha}\right)^{1/(\alpha-1)} \leq \liminf_{x \rightarrow 0} \frac{-\log \mathbb{P}\left(U_{(1-\epsilon)x}^\uparrow \leq c_1g(x)\right)}{\log |\log x|} \quad (2.5)$$

Hence from the inequality (2.3) and (2.4) and (2.5), we deduce

$$(1 - \epsilon)^{\alpha/(\alpha-1)} \left(1 - \frac{1}{\alpha}\right) \left(\frac{1}{c_1\alpha}\right)^{1/(\alpha-1)} \leq \liminf_{x \rightarrow 0} \frac{-\log \mathbb{P}\left(T_x^\uparrow \leq c_1g(x)\right)}{\log |\log x|},$$

and since  $\epsilon$  can be chosen arbitrarily small, the lemma is proved. ■

Now we can prove the lower bound of the law of the iterated logarithm for  $T^\uparrow$ . Let  $(x_n)$  be a decreasing sequence of positive real numbers which converges to 0 and let us

define the event  $A_n = \{T_{x_{n+1}}^\uparrow < c_1 g(x_n)\}$ . Now, we choose  $x_n = r^n$ , for  $r < 1$ . From the first Borel-Cantelli's Lemma, if  $\sum_n \mathbb{P}(A_n) < \infty$ , it follows

$$T_{r^{n+1}}^\uparrow \geq c_1 g(r^n) \quad \text{almost surely,}$$

for all large  $n$ . Since the function  $g$  and the process  $T^\uparrow$  are increasing, we have

$$T_x^\uparrow \geq c g(x) \quad \text{for } r^{n+1} \leq x \leq r^n.$$

Then, it is enough to prove that  $\sum_n \mathbb{P}(A_n) < \infty$ . In this direction, we take

$$0 < c_1 < c' < \left(\frac{1}{\alpha}\right)^\alpha (\alpha - 1)^{\alpha-1}.$$

Since  $\psi$  is regularly varying and we can chose  $r$  close enough to 1, we see that for  $n_0$  sufficiently large

$$\sum_{n \geq n_0} \mathbb{P}(A_n) \leq \sum_{n \geq n_0} \mathbb{P}\left(T_{r^{n+1}}^\uparrow < c' g(r^{n+2})\right) \leq \int_0^{r^{n_0+2}} \mathbb{P}\left(T_x^\uparrow \leq c' g(x)\right) \frac{dx}{x},$$

and from Lemma 2 this last integral is finite since

$$\left(1 - \frac{1}{\alpha}\right) \left(\frac{1}{c'\alpha}\right)^{1/(\alpha-1)} > 1,$$

with this we finish the proof. ■

There also exist a huge variety of results for the upper envelope of subordinators, see for instance Chapter III of Bertoin [5]. Here, we will state with out proofs the main results for the upper envelope of  $U^\uparrow$ . The proofs of the following results can be found in Chapter III of Bertoin [5].

**Proposition 3** (i) *If  $d > 0$  one has*

$$\lim_{x \rightarrow 0} \frac{U_x^\uparrow}{x} = d \quad \text{almost surely.}$$

(ii) *If  $d = 0$  and  $f : [0, \infty) \rightarrow [0, \infty)$  is an increasing function such that  $t \rightarrow f(t)/t$  increases, one has*

$$\limsup_{x \rightarrow 0} \frac{U_x^\uparrow}{f(x)} = \infty \quad \text{a.s.} \quad \text{if and only if} \quad \int_0 \bar{\nu}(f(t)) dt = \infty,$$

where  $\bar{\nu}(t) = \nu((t, \infty))$ .

Moreover,

$$\text{if } \int_0 \bar{\nu}(f(t)) dt < \infty \quad \text{then} \quad \lim_{x \rightarrow 0} \frac{T_x^\uparrow}{f(x)} = \lim_{x \rightarrow 0} \frac{U_x^\uparrow}{f(x)} = 0 \quad \text{almost surely.}$$

(iii) If  $\mathbb{E}(T_1) < \infty$  one has

$$\lim_{x \rightarrow +\infty} \frac{U_x^\uparrow}{x} = \mathbb{E}(T_1) \quad \text{almost surely.}$$

(iv) If  $\mathbb{E}(T_1)$  is infinite and  $f : [0, \infty) \rightarrow [0, \infty)$  is an increasing function such that the mapping  $t \rightarrow f(t)/t$  increases, one has

$$\limsup_{x \rightarrow +\infty} \frac{U_x^\uparrow}{f(x)} = \infty \quad \text{a.s.} \quad \text{if and only if} \quad \int^{+\infty} \bar{\nu}(f(t)) dt = \infty.$$

Moreover,

$$\text{if} \quad \int^{+\infty} \bar{\nu}(f(t)) dt < \infty \quad \text{then} \quad \lim_{x \rightarrow +\infty} \frac{T_x^\uparrow}{f(x)} = \lim_{x \rightarrow +\infty} \frac{U_x^\uparrow}{f(x)} = 0 \quad \text{almost surely.}$$

### 3 Proofs of the main results.

For simplicity, we introduce the notation

$$h(t) = \frac{\log |\log t|}{\Phi(t^{-1} \log |\log t|)}.$$

We start with the proof of the first part of Theorem 3, since a key result on subordinators due to Fristed and Pruitt [12] easily yields the result. The second part will be proved after the proof of Theorem 2, since the latter is necessary for its proof.

*Proof of Theorem 3 (first part):* First we will observe that  $\psi(\lambda) = O(\lambda^2)$ , as  $\lambda$  goes to  $+\infty$ , then  $\lambda^{1/2} = O(\Phi(\lambda))$ . Since the last passage time process  $U^\uparrow$  is a subordinator with Laplace exponent  $\Phi$  and the future infimum process is the right-inverse of the last passage times  $U^\uparrow$ , then according to Theorem 2 and Remark on p. 176 in Fristed and Pruitt [12], there exists a positive constant  $c$  such that

$$\limsup_{t \rightarrow 0} \frac{J_t \Phi(t^{-1} \log |\log t|)}{\log |\log t|} = c, \quad \mathbb{P}^\uparrow - \text{a.s.},$$

and

$$\limsup_{t \rightarrow +\infty} \frac{J_t \Phi(t^{-1} \log \log t)}{\log \log t} = c, \quad \mathbb{P}^\uparrow - \text{a.s.},$$

then the first part of Theorem 3 is proved. ■

*Proof of Theorem 2:* We only prove the result for small times since the proof for large times is very similar. The lower bound is easy to deduce from Theorem 3 and since  $J_t^\uparrow \leq \xi_t^\uparrow$ , where  $J_t^\uparrow$  denotes the future infimum of  $\xi^\uparrow$ . Hence

$$c = \limsup_{t \rightarrow 0} \frac{J_t \Phi(t^{-1} \log |\log t|)}{\log |\log t|} \leq \limsup_{t \rightarrow 0} \frac{\xi_t \Phi(t^{-1} \log |\log t|)}{\log |\log t|} \quad \mathbb{P}^\uparrow - \text{a.s.}$$

Now, we prove the upper bound. Let  $(x_n)$  be a decreasing sequence of positive real numbers which converges to 0, in particular we choose  $x_n = r^n$ , for  $r < 1$ .

Recall that  $S$  is the supremum process of  $\xi$ , i.e.  $S_t = \sup_{0 \leq u \leq t} \xi_u$ . We define the events  $A_n = \{S_{x_n} > \eta ch(x_{n+1})\}$ , where  $\eta \geq c^{-1}(2 + r^{-1})$  and  $S$  is the supremum process. From the first Borel-Cantelli's Lemma, if  $\sum_n \mathbb{P}^\dagger(A_n) < \infty$ , it follows

$$S_{r^n} \leq \eta ch(r^{n+1}) \quad \mathbb{P}^\dagger\text{-a.s.},$$

for all large  $n$ . Since the function  $h$  and the process  $S$  are increasing in a neighbourhood of 0, we have

$$S_t \leq \eta ch(t) \quad \text{for } r^{n+1} \leq t \leq r^n, \quad \text{under } \mathbb{P}^\dagger.$$

Then, it is enough to prove that  $\sum_n \mathbb{P}^\dagger(A_n) < \infty$ . In this direction, we will prove the following lemma,

**Lemma 3** *Let  $0 < \epsilon < 1$  and  $r < 1$ . If we assume that condition (H2) is satisfied then there exists a positive constant  $C(\epsilon)$  such that*

$$\mathbb{P}^\dagger(J_{r^n} > (1 - \epsilon)\eta ch(r^{n+1})) \geq C(\epsilon)\mathbb{P}^\dagger(A_n) \quad \text{as } n \rightarrow +\infty. \quad (3.6)$$

*Proof of Lemma 3:* From the inequality (2.3), we have that

$$\mathbb{P}^\dagger(J_{r^n} > (1 - \epsilon)\eta ch(r^{n+1})) \geq \mathbb{P}_{\eta ch(r^{n+1})}^\dagger(J_0 > (1 - \epsilon)\eta ch(r^{n+1}))\mathbb{P}^\dagger(S_{r^n} > \eta ch(r^{n+1})),$$

and since  $J_0$  is the absolute minimum of  $(\xi, \mathbb{P}_{\eta ch(r^{n+1})}^\dagger)$  then by (1.2)

$$\mathbb{P}_{\eta ch(r^{n+1})}^\dagger(J_0 > (1 - \epsilon)\eta ch(r^{n+1})) = \frac{W(\epsilon\eta ch(r^{n+1}))}{W(\eta ch(r^{n+1}))}.$$

On the other hand, an application of Proposition III.1 in Bertoin [5] gives that there exist a positive real number  $K_1$  such that

$$K_1 \frac{1}{x\psi(1/x)} \leq W(x) \leq K_1^{-1} \frac{1}{x\psi(1/x)}, \quad \text{for all } x > 0, \quad (3.7)$$

then it is clear that

$$\frac{W(\epsilon\eta ch(r^{n+1}))}{W(\eta ch(r^{n+1}))} \geq K_1^2 \epsilon^{-1} \frac{\psi(1/\eta ch(r^{n+1}))}{\psi(\epsilon^{-1}/\eta ch(r^{n+1}))}.$$

From this inequality and condition (H2), there exist a positive constant  $C(\epsilon)$  such that for  $n$  sufficiently large

$$\mathbb{P}^\dagger(J_{r^n} > (1 - \epsilon)\eta ch(r^{n+1})) \geq C(\epsilon)\mathbb{P}^\dagger(S_{r^n} > \eta ch(r^{n+1})),$$

which proves our result. ■

Now, we prove the upper bound for the law of the iterated logarithm of  $(\xi, \mathbb{P}^\dagger)$ . Fix  $0 < \epsilon < 1/(2 + r^{-1})$ . Since  $J$  can be seen as the right inverse of  $U$ , it is straightforward that

$$\mathbb{P}^\dagger(J_{r^n} > (1 - \epsilon)\eta ch(r^{n+1})) = \mathbb{P}^\dagger(U_{(1-\epsilon)\eta ch(r^{n+1})} < r^n),$$

and this probability is bounded from above by

$$\exp\{\lambda r^n\} \mathbb{E}^\dagger \left( \exp \left\{ -\lambda U_{(1-\epsilon)\eta c h(r^{n+1})} \right\} \right) = \exp \left\{ \lambda r^n - (1-\epsilon)\eta c h(r^{n+1}) \Phi(\lambda) \right\},$$

for every  $\lambda \geq 0$ . We choose  $\lambda = r^{-(n+1)} \log |\log r^{n+1}|$ , then

$$\mathbb{P}^\dagger (J_{r^n} > (1-\epsilon)\eta c h(r^{n+1})) \leq \exp \left\{ -((1-\epsilon)\eta c - r^{-1}) \log |\log r^{n+1}| \right\},$$

hence from the above inequality and Lemma 3, we have that

$$C(\epsilon) \sum_n \mathbb{P}^\dagger (A_n) \leq C_1 \sum_n (\log(n+1))^{(1-\epsilon)\eta c - r^{-1}} < +\infty,$$

since  $(1-\epsilon)\eta c - r^{-1} > 1$ .

Hence, we have

$$\limsup_{t \rightarrow 0} \frac{S_t}{h(t)} \leq \eta c, \quad \mathbb{P}^\dagger\text{-a.s.},$$

for  $\eta c > 3$ , since we can choose  $r$  close enough to 1.

The two preceding parts show that

$$\limsup_{t \rightarrow 0} \frac{\xi_t}{h(t)} \in [c, \eta c], \quad \mathbb{P}^\dagger\text{-a.s.}$$

By the Blumenthal zero-one law, it must be a constant number  $k$ ,  $\mathbb{P}^\dagger$ -a.s. ■

*Proof of Theorem 3 (second part):* First we prove the result for large times. Assume that the additional hypothesis (H2) is satisfied. Since  $J_t \leq \xi_t$  for every  $t \geq 0$  and Theorem 2, it is clear that

$$\limsup_{t \rightarrow +\infty} \frac{J_t}{h(t)} \leq \limsup_{t \rightarrow +\infty} \frac{\xi_t}{h(t)} = k \quad \mathbb{P}^\dagger\text{-a.s.},$$

then the upper bound is proved.

Now, fix  $\epsilon \in (0, 1/2)$  and define

$$R_n = \inf \left\{ s \geq n : \frac{\xi_s^\dagger}{kh(s)} \geq (1-\epsilon) \right\}.$$

From the above definition, it is clear that  $R_n \geq n$  and that  $R_n$  diverge a.s. as  $n$  goes to  $+\infty$ . From Theorem 2, we deduce that  $R_n$  is finite a.s.

Now, by (1.2) and since  $(\xi, \mathbb{P}^\dagger)$  is a strong Markov process with no positive jumps, we have that

$$\begin{aligned} \mathbb{P}^\dagger \left( \frac{J_{R_n}}{kh(R_n)} \geq (1-2\epsilon) \right) &= \mathbb{P}^\dagger \left( J_{R_n} \geq \frac{(1-2\epsilon)\xi_{R_n}}{(1-\epsilon)} \right) \\ &= \mathbb{E}^\dagger \left( \mathbb{P}^\dagger \left( J_{R_n} \geq \frac{(1-2\epsilon)\xi_{R_n}}{(1-\epsilon)} \middle| \xi_{R_n} \right) \right) \\ &= \mathbb{E}^\dagger \left( \frac{W(\epsilon \xi_{R_n})}{W(\xi_{R_n})} \right). \end{aligned}$$

Now applying (3.7), we have that

$$\mathbb{E}^\uparrow \left( \frac{W(\epsilon \xi_{R_n})}{W(\xi_{R_n})} \right) \geq K_1^2 \epsilon^{-1} \mathbb{E}^\uparrow \left( \frac{\psi(1/\xi_{R_n})}{\psi(\epsilon^{-1}/\xi_{R_n})} \right)$$

and since the hypothesis (H2) is satisfied, an application of the Fatou-Lebesgue Theorem shows that

$$\liminf_{n \rightarrow +\infty} \mathbb{E}^\uparrow \left( \frac{\psi(1/\xi_{R_n})}{\psi(\epsilon^{-1}/\xi_{R_n})} \right) \geq \mathbb{E}^\uparrow \left( \liminf_{n \rightarrow +\infty} \frac{\psi(1/\xi_{R_n})}{\psi(\epsilon^{-1}/\xi_{R_n})} \right) > 0,$$

which implies that

$$\lim_{n \rightarrow +\infty} \mathbb{P}^\uparrow \left( \frac{J_{R_n}}{kh(R_n)} \geq (1 - 2\epsilon) \right) > 0.$$

Since  $R_n \geq n$ ,

$$\mathbb{P}^\uparrow \left( \frac{J_t}{kh(t)} \geq (1 - 2\epsilon), \text{ for some } t \geq n \right) \geq \mathbb{P}^\uparrow \left( \frac{J_{R_n}}{kh(R_n)} \geq (1 - 2\epsilon) \right).$$

Therefore, for all  $\epsilon \in (0, 1/2)$

$$\mathbb{P}^\uparrow \left( \frac{J_t}{kh(t)} \geq (1 - 2\epsilon), \text{ i.o., as } t \rightarrow +\infty \right) \geq \lim_{n \rightarrow +\infty} \mathbb{P}^\uparrow \left( \frac{J_{R_n}}{kh(R_n)} \geq (1 - 2\epsilon) \right) > 0.$$

The event on the left hand side is in the upper-tail sigma-field  $\cap_t \sigma\{\xi_s^\uparrow : s \geq t\}$  which is trivial from Bertoin's construction of  $(\xi, \mathbb{P}^\uparrow)$  (see Theorem VII.20 in [5]). Hence

$$\limsup_{t \rightarrow +\infty} \frac{J_t}{h(t)} \geq k(1 - 2\epsilon), \quad \mathbb{P}^\uparrow - \text{a.s.},$$

and since  $\epsilon$  can be chosen arbitrarily small, the result for large times is proved. In order to prove the law of the iterated logarithm for small times, we now define the following stopping time

$$R_n = \inf \left\{ \frac{1}{n} < s : \frac{\xi_s^\uparrow}{kh(s)} \geq (1 - \epsilon) \right\}.$$

Following same argument as above and assuming that (H1) is satisfied, we get that for a fixed  $\epsilon \in (0, 1/2)$  and  $n$  sufficiently large

$$\mathbb{P}^\uparrow \left( \frac{J_{R_n}}{kh(R_n)} \geq (1 - 2\epsilon) \right) > 0.$$

Next, we note that

$$\mathbb{P}^\uparrow \left( \frac{J_{R_p}}{kh(R_p)} \geq (1 - 2\epsilon), \text{ for some } p \geq n \right) \geq \mathbb{P}^\uparrow \left( \frac{J_{R_n}}{kh(R_n)} \geq (1 - 2\epsilon) \right).$$

Since  $R_n$  converge a.s. to 0 as  $n$  goes to  $\infty$ , the conclusion follows taking the limit when  $n$  goes towards to  $+\infty$ . ■

*Proof of Theorem 4:* The proof of this theorem is very similar to the proof of the previous result. Following same arguments, we first prove the law of the iterated logarithm for large times. Assume that the hypothesis (H2) and (H3) are satisfied. Since  $\xi_t^\uparrow - J_t \leq \xi_t^\uparrow$  for every  $t$  and Theorem 2, it is clear that

$$\limsup_{t \rightarrow +\infty} \frac{\xi_t - J_t}{h(t)} \leq \limsup_{t \rightarrow +\infty} \frac{\xi_t}{h(t)} = k \quad \mathbb{P}^\uparrow\text{-a.s.},$$

then the upper bound is proved.

Now, fix  $\epsilon \in (0, 1/2)$  and similarly as the last proof we define

$$R_n = \inf \left\{ s \geq n : \frac{\xi_s^\uparrow}{kh(s)} \geq (1 - \epsilon) \right\}.$$

Now, by (1.2) and since  $(\xi, \mathbb{P}^\uparrow)$  is a strong Markov process with no positive jumps, we have that

$$\begin{aligned} \mathbb{P}^\uparrow \left( \frac{\xi_{R_n} - J_{R_n}}{kh(R_n)} \geq (1 - 2\epsilon) \right) &= \mathbb{P}^\uparrow \left( J_{R_n} \leq \frac{\epsilon}{(1 - \epsilon)} \xi_{R_n} \right) \\ &= \mathbb{E}^\uparrow \left( \mathbb{P}^\uparrow \left( J_{R_n} \leq \frac{\epsilon}{(1 - \epsilon)} \xi_{R_n} \mid \xi_{R_n} \right) \right) \\ &= 1 - \mathbb{E}^\uparrow \left( \frac{W(k(\epsilon)\xi_{R_n})}{W(\xi_{R_n})} \right), \end{aligned}$$

where  $k(\epsilon) = (1 - 2\epsilon)/(1 - \epsilon)$ .

Since the hypothesis (H3) is satisfied, an application of the Fatou-Lebesgue Theorem shows that

$$\limsup_{n \rightarrow +\infty} \mathbb{E}^\uparrow \left( \frac{W(k(\epsilon)\xi_{R_n})}{W(\xi_{R_n})} \right) \leq \mathbb{E}^\uparrow \left( \limsup_{n \rightarrow +\infty} \frac{W(k(\epsilon)\xi_{R_n})}{W(\xi_{R_n})} \right) < 1,$$

which implies that

$$\lim_{n \rightarrow +\infty} \mathbb{P}^\uparrow \left( \frac{\xi_{R_n} - J_{R_n}}{kh(R_n)} \geq (1 - 2\epsilon) \right) > 0.$$

Again, since  $R_n \geq n$ ,

$$\mathbb{P}^\uparrow \left( \frac{\xi_t - J_t}{kh(t)} \geq (1 - 2\epsilon), \text{ for some } t \geq n \right) \geq \mathbb{P}^\uparrow \left( \frac{\xi_{R_n} - J_{R_n}}{kh(R_n)} \geq (1 - 2\epsilon) \right).$$

Therefore, for all  $\epsilon \in (0, 1/2)$

$$\mathbb{P}^\uparrow \left( \frac{\xi_t - J_t}{kh(t)} \geq (1 - 2\epsilon), \text{ i.o., as } t \rightarrow +\infty \right) \geq \lim_{n \rightarrow +\infty} \mathbb{P}^\uparrow \left( \frac{\xi_{R_n} - J_{R_n}}{kh(R_n)} \geq (1 - 2\epsilon) \right) > 0.$$

The event on the left hand side is in the upper-tail sigma-field  $\cap_t \sigma\{\xi_s^\uparrow : s \geq t\}$  which is trivial, then

$$\limsup_{t \rightarrow +\infty} \frac{\xi_t - J_t}{h(t)} \geq k(1 - 2\epsilon), \quad \mathbb{P}^\uparrow\text{-a.s.},$$

and since  $\epsilon$  can be chosen arbitrarily small, the result for large times is proved. Similarly as in the proof of the previous result, we can prove the result for small times using the following stopping time

$$R_n = \inf \left\{ \frac{1}{n} < s : \frac{\xi_s^\uparrow}{kh(s)} \geq (1 - \epsilon) \right\}.$$

Following same argument as above and assuming that (H1) and (H4) are satisfied, we get that for a fixed  $\epsilon \in (0, 1/2)$  and  $n$  sufficiently large

$$\mathbb{P}^\uparrow \left( \frac{\xi_{R_n} - J_{R_n}}{kh(R_n)} \geq (1 - 2\epsilon) \right) > 0.$$

Next, we note that

$$\mathbb{P}^\uparrow \left( \frac{\xi_{R_n} - J_{R_p}}{kh(R_p)} \geq (1 - 2\epsilon), \text{ for some } p \geq n \right) \geq \mathbb{P}^\uparrow \left( \frac{\xi_{R_n} - J_{R_n}}{kh(R_n)} \geq (1 - 2\epsilon) \right).$$

Again, since  $R_n$  converge a.s. to 0 as  $n$  goes to  $\infty$ , the conclusion follows taking the limit when  $n$  goes towards to  $+\infty$ . ■

*Proof of Theorem 5:* Let  $(x_n)$  be a decreasing sequence such that  $\lim x_n = 0$ . We define the events

$$A_n = \left\{ \text{There exist } t \in [U_{x_{n+1}}^\uparrow, U_{x_n}^\uparrow] \text{ such that } \xi_t^\uparrow < f(t) \right\}.$$

Since  $U_{x_n}^\uparrow$  tends to 0, a.s. when  $n$  goes to  $+\infty$ , we have

$$\left\{ \xi_t^\uparrow < f(t), \text{ i.o., as } t \rightarrow 0 \right\} = \limsup_{n \rightarrow +\infty} A_n.$$

Let us chose  $x_n = r^n$ , for  $r < 1$ . Since  $f$  is increasing the following inclusions hold

$$A_n \subset \left\{ \text{There exist } t \in [r^{n+1}, r^n] \text{ such that } tr < f(U_t^\uparrow) \right\},$$

and

$$\left\{ \text{There exist } t \in [r^{n+1}, r^n] \text{ such that } tr^{-1} < f(U_t^\uparrow) \right\} \subset A_n$$

Then we prove the convergent part. Let us suppose that  $f$  satisfies

$$\int_0^\infty f(x)\nu(dx) < \infty.$$

Hence from Theorem VI.3.2 in [13] and the fact that  $U^\uparrow$  is a subordinator, we have that

$$\mathbb{P}^\uparrow \left( tr < f(U_t), \text{ i.o., as } t \rightarrow 0 \right) = 0,$$

which implies that

$$\lim_{t \rightarrow 0} \frac{\xi_t}{f(t)} = \infty \quad \mathbb{P}^\uparrow\text{-a.s.},$$

since we can replace  $f$  by  $cf$ , for any  $c > 1$ .  
Similarly, if  $f$  satisfies that

$$\int_0^\infty f(x)\nu(dx) = \infty,$$

again from Theorem VI.3.2 in [13], we have that

$$\mathbb{P}^\dagger\left(tr^{-1} < f(U_t), \text{ i.o., as } t \rightarrow 0\right) = 1,$$

which implies that

$$\liminf_{t \rightarrow 0} \frac{\xi_t}{f(t)} = 0 \quad \mathbb{P}^\dagger\text{-a.s.},$$

since we can replace  $f$  by  $cf$ , for any  $c < 1$ .

The integral test at  $+\infty$  is very similar to this of small times, it is enough to take  $x_n = r^n$ , for  $r > 1$  and follows the same arguments as in the proof for small times. ■

*Proof of Theorem 6:* The proof of parts (ii) and (iv) follows from the proof of Theorem 5, it is enough to note that we can replace  $\xi^\dagger$  by  $J^\dagger$  in the sets  $A_n$ . The proof of parts (i) and (iii) follows from Proposition 4.4 in [6]. ■

*Proof of Proposition 1:* Similarly as in the proof of Theorem 5, let  $(x_n)$  be a decreasing sequence such that  $\lim x_n = 0$  and  $c > 1$ . We define the events

$$A_n = \left\{ \text{There exist } t \in [U_{x_{n+1}}^\dagger, U_{x_n}^\dagger] \text{ such that } \xi_t^\dagger < cf(t) \right\}.$$

Since  $U_{x_n}^\dagger$  tends to 0, a.s. when  $n$  goes to  $+\infty$ , we have

$$\left\{ \xi_t^\dagger < cf(t), \text{ i.o., as } t \rightarrow 0 \right\} = \limsup_{n \rightarrow +\infty} A_n.$$

Since  $f$  is increasing the following inclusion holds

$$A_n \subset \left\{ x_{n+1} < cf(U_{x_n}^\dagger) \right\}.$$

On the other hand

$$\mathbb{P}^\dagger\left(x_{n+1} < cf(U_{x_n}^\dagger)\right) = \mathbb{P}^\dagger\left(f^{-1}(x_{n+1}/c) < U_{x_n}^\dagger\right),$$

where  $f^{-1}$  is the right-inverse of  $f$ .

Now, we take  $x_n = cf(r^n)$ , for  $r < 1$ . Since  $f$  is increasing and from the above equality, we get that

$$\mathbb{P}^\dagger\left(x_{n+1} < cf(U_{x_n}^\dagger)\right) \leq \mathbb{P}^\dagger\left(r^{n+1} < U_{cf(r^n)}\right)$$

The obvious inequality

$$\mathbb{P}^\dagger\left(a < U_t\right) \leq (1 - e^{-1})^{-1} \left(1 - \exp\{-t\Phi(1/a)\}\right),$$

applied for  $t = cf(r^n)$  and  $a = r^{n+1}$  entails that

$$\mathbb{P}^\uparrow\left(r^{n+1} < U_{cf(r^n)}\right) \leq (1 - e^{-1})^{-1} cf(r^n) \Phi(r^{-(n+1)}).$$

Since the mapping  $t \rightarrow t\Phi(1/t)$  increases, it is not difficult to deduce that the function  $\Phi$  satisfies that

$$\Phi(r^{-(n+1)}) \leq r^{-2} \Phi(r^{-(n-1)}).$$

Hence,

$$\begin{aligned} \sum_{n \geq k} \mathbb{P}^\uparrow\left(r^{n+1} < U_{cf(r^n)}\right) &\leq C(r) \sum_{n \geq k} \int_{n-1}^n f(r^t) \Phi(r^{-t}) dt \\ &\leq C(r) \int_0^{r^{k-1}} x^{-1} f(x) \Phi(1/x) dx. \end{aligned}$$

Since the last integral is finite, by the Borel-Cantelli lemma, we deduce that

$$\mathbb{P}^\uparrow\left(\xi_t < cf(t), \text{ i.o., as } t \rightarrow 0\right) = 0,$$

for all  $c \geq 1$ , hence

$$\lim_{t \rightarrow 0} \frac{\xi_t}{f(t)} = \infty, \quad \mathbb{P}^\uparrow\text{-a.s.}$$

In order to prove that the future infimum satisfies the same result, we note first that we can replace  $\xi^\uparrow$  by its future infimum in the sets  $A_n$ , and then the same arguments will give us the desired result. ■

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