

On some nonlinear partial differential equations involving the 1–Laplacian

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Accépté pour publication aux Annales de la Faculté des Sciences de Toulouse

Abstract

In this paper we present an approximation result concerning the first eigenvalue of the 1-Laplacian operator. More precisely, for Ω a bounded regular open domain, we consider a minimisation of the functional $\int_{\Omega} |\nabla u| + n \left(\int_{\Omega} |u| - 1 \right)^2$ over the space $W_0^{1,1}(\Omega)$. For n large enough, the infimum is achieved in some sense on $BV(\Omega)$, and letting n go to infinity this provides an approximation of the first eigenfunction for the first eigenvalue, since the term $n \left(\int_{\Omega} |u|^2 - 1 \right)^2$ “tends” to the constraint $\|u\|_1 = 1$.

1 Introduction: the first eigenvalue for the 1-Laplacian

In recent fields, several authors were interested on the study of the “first eigenvalue” for the 1-Laplacian operator, that we shall denote as the not everywhere defined $u \mapsto -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$.

Due to the singularity of this operator, the definition of the first eigenvalue can be correctly defined with the aid of a variational formulation: let λ_1 be defined as

$$\lambda_1 := \inf_{\substack{u \in W_0^{1,1}(\Omega) \\ \|u\|_1 = 1}} \int_{\Omega} |\nabla u|. \quad (1.1)$$

Notice that λ_1 is well defined and is positive, due to Poincaré’s inequality.

In order to justify the term “eigenvalue” for λ_1 , one must prove the existence of an associated “eigenfunction”. As in the p -Laplacian case, an eigenfunction will be a solution of (1.1). Unfortunately, since $W^{1,1}(\Omega)$ is not a reflexif space, one cannot hope to obtain a solution for (1.1) by classical arguments.

This difficulty can be overcome by introducing the space $BV(\Omega)$, which is the weak closure of $W^{1,1}(\Omega)$, and by extending the infimum to that space, using the features of $BV(\Omega)$: Density of regular maps in BV , existence of the trace map on the boundary... However, these properties are not sufficient to obtain solutions by classical methods, since the trace map –which is well defined on

$BV(\Omega)$ – is not continuous for the weak topology. This new difficulty can be “solved” by introducing –as it is the case in the theory of minimal surfaces and in plasticity and also for related problems– a “relaxed” formulation for (1.1). This relaxed formulation consists in replacing the condition $\{u = 0\}$ on the boundary by the addition of a term $\int_{\partial\Omega} |u|$ in the functional to minimize. The new formulation is then

$$\inf_{\substack{u \in BV(\Omega) \\ \|u\|_1 = 1}} \int_{\Omega} |\nabla u| + \int_{\partial\Omega} |u|. \quad (1.2)$$

This problem has an infimum equal to λ_1 . It can be seen by approximating function in $BV(\Omega)$ by functions in $W^{1,1}(\Omega)$ for a topology related to the narrow topology of measures. This topology is precised in section 2.

Then the existence of a minimizer of (1.2) in $BV(\Omega)$ can be proved, using classical arguments, arguments which will be precised later in this paper.

To obtain the partial differential equation satisfied by a minimizer of (1.2), equation which can be seen as an eigenvalue’s equation, the author used in [20] an approximation of (1.1) by the following problem on $W_0^{1,1+\varepsilon}(\Omega)$:

$$\lambda_{1+\varepsilon} := \inf_{\substack{u \in W_0^{1,1+\varepsilon}(\Omega) \\ \|u\|_1 = 1}} \int_{\Omega} |\nabla u|^{1+\varepsilon}, \quad (1.3)$$

and proves that $\lambda_{1+\varepsilon}$ converges to λ_1 . Moreover, if u_ε is a positive solution of the minimizing problem defined in (1.3), u_ε converges weakly in $BV(\Omega)$ to some u which satisfies

$$-\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) = \lambda_1,$$

in a sense which needs to be precised, and is detailed in the present paper.

Let us note that it is also proved in [18] that there are characteristic functions of sets which are solutions. These sets are therefore called eigensets.

Another approach is used in [2], [3], where the authors use the concept of Cheeger sets [14]. In these papers, the authors present a remarkable construction of eigensets for 2–dimensional convex sets Ω . Among their results, there is the uniqueness of eigensets in the case $N = 2$.

Our aim in the present article is to propose an approach of the first eigenvalue and the first eigenfunction of the 1-Laplacian operator, using a penalization method, which consists in replacing the condition $\int_{\Omega} |u| = 1$ in (1.2) by the introduction of the term $n \left(\int_{\Omega} |u| - 1 \right)^2$ inside the fonctionnal to minimize. This provides in the same time, a new proof of the existence and uniqueness of a positive eigenfunction.

2 Survey on known results about the space $BV(\Omega)$

We begin to recall the definition of the space of functions with bounded variation. Let Ω be an open regular domain in \mathbb{R}^N , $N > 1$, and let $\mathcal{M}^1(\Omega)$ be the

space of bounded measures in Ω . We define

$$BV(\Omega) = \{u \in L^1(\Omega), \nabla u \in \mathcal{M}^1(\Omega)\}.$$

Endowed with the norm $\int_{\Omega} |\nabla u| + \int_{\Omega} |u|$, the space $BV(\Omega)$ is a Banach space.

More useful is the weak topology for variational technics :

We define *the weak topology* with the aid of sequences, as follows: we say that a sequence $u_n \rightarrow u$ weakly in $BV(\Omega)$ if the following two conditions are fulfilled:

- $\int_{\Omega} |u_n - u| \rightarrow 0$ in $L^1(\Omega)$ when $n \rightarrow \infty$,
- $\int_{\Omega} \partial_i u_n \phi \rightarrow \int_{\Omega} \partial_i u \phi, \quad \forall i = 1, 2, \dots, N \quad \forall \phi \in \mathcal{C}_c(\Omega)$ when $n \rightarrow \infty$.

Let us note that the second convergence is also denoted as *the vague convergence* of ∇u_n towards ∇u .

We shall also use the concept of *tight convergence* in $BV(\Omega)$: we say that a sequence u_n converges tightly to u in $BV(\Omega)$ if the following two conditions are fulfilled:

- $u_n \rightarrow u$, weakly in $BV(\Omega)$ when $n \rightarrow \infty$,
- $\int_{\Omega} |\nabla u_n| \rightarrow \int_{\Omega} |\nabla u|$ when $n \rightarrow \infty$.

Let us note that the last assertion is equivalent to say that, for all $\phi \in \mathcal{C}(\bar{\Omega}, \mathbb{R}^N)$,

$$\int_{\Omega} \nabla u_n \cdot \phi \rightarrow \int_{\Omega} \nabla u \cdot \phi, \quad \text{when } n \rightarrow \infty.$$

We now recall some facts about embedding and compact embedding from $BV(\Omega)$ to other L^q spaces :

- If Ω is an open \mathcal{C}^1 set, then $BV(\Omega)$ is continuously embedded in $L^p(\Omega)$ for all $p \leq \frac{N}{N-1}$.
- If Ω is also bounded and smooth, the embedding is compact in $L^p(\Omega)$ for every $p < \frac{N}{N-1}$.

Finally we recall the existence of a map, called *trace map*, defined on $BV(\Omega)$, which coincides with the restriction on $\partial\Omega$ of u when u belongs to $\mathcal{C}(\bar{\Omega}) \cap BV(\Omega)$ or less classically when $u \in W^{1,1}(\Omega)$. This map is continuous for the strong topology, and is not continuous under the weak topology. However the following property holds: if $u_n \rightarrow u$ tightly in $BV(\Omega)$, then

$$\int_{\partial\Omega} |u_n - u| \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

We end this section by enouncing a generalization of the Green's formula : this will allow us to give sense to the product $\sigma \cdot \nabla u$ when σ is in $L^\infty(\Omega, \mathbb{R}^N)$, $\text{div } \sigma \in L^N(\Omega)$ and $u \in BV(\Omega)$, and will be useful to give sense to the partial differential equation associated to the eigenvalue.

Let us recall that $\mathcal{D}(\Omega)$ is the space of \mathcal{C}^∞ -functions, with support on Ω .

Proposition 2.1. *Let $\sigma \in L^\infty(\Omega, \mathbb{R}^N)$, $\operatorname{div} \sigma \in L^N(\Omega)$ and $u \in BV(\Omega)$. Define the distribution $\sigma \cdot \nabla u$ by the following formula : for any $\varphi \in \mathcal{D}(\Omega)$,*

$$\langle \sigma \cdot \nabla u, \varphi \rangle = - \int_{\Omega} (\operatorname{div} \sigma) u \varphi - \int_{\Omega} (\sigma \cdot \nabla \varphi) u. \quad (2.1)$$

Then

$$|\langle \sigma \cdot \nabla u, \varphi \rangle| \leq \|\sigma\|_\infty \langle |\nabla u|, |\varphi| \rangle.$$

In particular, $\sigma \cdot \nabla u$ is a bounded measure which satisfies:

$$|\sigma \cdot \nabla u| \leq \|\sigma\|_\infty |\nabla u|.$$

In addition, if $\varphi \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)$, the following Green's Formula holds:

$$\langle \sigma \cdot \nabla u, \varphi \rangle = - \int_{\Omega} (\operatorname{div} \sigma) u \varphi - \int_{\Omega} (\sigma \cdot \nabla \varphi) u + \int_{\partial\Omega} \sigma \cdot \vec{n} u \varphi, \quad (2.2)$$

where \vec{n} is the unit outer normal to $\partial\Omega$.

Suppose that $U \in BV(\mathbb{R}^N \setminus \overline{\Omega})$, that $u \in BV(\Omega)$ and define the function \tilde{u} as:

$$\tilde{u} = \begin{cases} u & \text{in } \Omega, \\ U & \text{in } \mathbb{R}^N \setminus \overline{\Omega}. \end{cases}$$

Then $\tilde{u} \in BV(\mathbb{R}^N)$ and

$$\nabla \tilde{u} = \nabla u \chi_\Omega + \nabla U \chi_{(\mathbb{R}^N \setminus \overline{\Omega})} + (U - u) \delta_{\partial\Omega},$$

where in the last term, U and u denote the trace of U and u on $\partial\Omega$ and $\delta_{\partial\Omega}$ denotes the uniform Dirac measure on $\partial\Omega$. Finally, we introduce the measure $\sigma \cdot \nabla \tilde{u}$ on $\overline{\Omega}$ by the formula

$$(\sigma \cdot \nabla \tilde{u}) = (\sigma \cdot \nabla u) \chi_\Omega + \sigma \cdot \vec{n} (U - u) \delta_{\partial\Omega}$$

where $(\sigma \cdot \nabla u) \chi_\Omega$ has been defined in 2.1. Then $\sigma \cdot \nabla \tilde{u}$ is absolutely continuous with respect to $|\nabla \tilde{u}|$, with the inequality

$$|\sigma \cdot \nabla \tilde{u}| \leq \|\sigma\|_\infty |\nabla \tilde{u}|.$$

For a proof the reader can consult [17], [35], [43].

3 Presentation of the results

We now describe the approximation result here enclosed. For $n \in \mathbb{N}^*$, let us consider the following minimization problem:

$$\lambda_{1,n} = \inf_{u \in W_0^{1,1}(\Omega)} \left\{ \int_{\Omega} |\nabla u| + n \left(\int_{\Omega} |u| - 1 \right)^2 \right\}. \quad (3.1)$$

As it is done for analogous problem in [], let us introduce the relaxed formulation associated :

$$\tilde{\lambda}_{1,n} = \inf_{u \in BV(\Omega)} \left\{ \int_{\Omega} |\nabla u| + \int_{\partial\Omega} |u| + n \left(\int_{\Omega} |u| - 1 \right)^2 \right\}. \quad (3.2)$$

We shall prove in the following section the result :

Theorem 3.1. *Let Ω be a piecewise C^1 bounded domain in \mathbb{R}^N , $N > 1$. For every $n \in \mathbb{N}^*$, the problem (3.2) possesses a solution u_n in $BV(\Omega)$ which can be chosen nonnegative. Moreover, u_n satisfies the following partial differential equation:*

$$\begin{cases} -\operatorname{div} \sigma_n + 2n \left(\int_{\Omega} u_n - 1 \right) \operatorname{sign}^+(u_n) = 0 & \text{in } \Omega, \\ \sigma_n \in L^\infty(\Omega, \mathbb{R}^N), \quad \|\sigma_n\|_\infty \leq 1, \\ \sigma_n \cdot \nabla u_n = |\nabla u_n| & \text{in } \Omega, \\ u_n \text{ is not identically zero, } -\sigma_n \cdot \vec{n}(u_n) = u_n & \text{on } \partial\Omega, \end{cases} \quad (3.3)$$

where \vec{n} denotes the unit outer normal to $\partial\Omega$, $\sigma_n \cdot \nabla u_n$ is the measure defined in Proposition 2.1 and $\operatorname{sign}^+(u_n)$ is some function in $L^\infty(\Omega)$ such that $\operatorname{sign}^+(u_n)u_n = u_n$ in Ω .

Moreover $\lambda_{1,n}$ converges towards λ_1 and u_n converges towards the first eigenfunction u .

Remark 3.1. Clearly, u_n is not identically zero for n large enough as soon as $n > \lambda_1$.

Remark 3.2. From Proposition 2.1 (with $U = 0$), the conditions

$$\sigma_n \cdot \nabla u_n = |\nabla u_n| \quad \text{in } \Omega, \quad -\sigma_n \cdot \vec{n}(u_n) = u_n \quad \text{on } \partial\Omega,$$

are equivalent to

$$\sigma_n \cdot \nabla \tilde{u}_n = |\nabla \tilde{u}_n| \quad \text{on } \Omega \cup \partial\Omega.$$

Remark 3.3. The identity $\sigma_n \cdot \nabla u_n = |\nabla u_n|$ makes sense since

$$-\operatorname{div} \sigma_n = -2n \left(\int_{\Omega} u_n - 1 \right) \operatorname{sign}^+(u_n),$$

which implies that $\operatorname{div} \sigma_n \in L^\infty(\Omega)$, therefore $\sigma_n \cdot \nabla u_n$ is well-defined by Proposition 2.1.

We subdivide the proof of Theorem 3.1 into several steps :

- First step: We use some kind of regularization of the minimization problem by introducing for some $\varepsilon > 0$ and small

$$\inf_{u \in W_0^{1,1+\varepsilon}(\Omega)} \left\{ \int_{\Omega} |\nabla u|^{1+\varepsilon} + n \left(\int_{\Omega} |u|^{1+\varepsilon} - 1 \right)^2 \right\}.$$

We prove that for n large enough, this problem possesses a solution which can be chosen nonnegative and denoted by $u_{n,\varepsilon}$, which satisfies

$$\left\{ -\operatorname{div}(|\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon}) + 2n \left(\int_{\Omega} u_{n,\varepsilon}^{1+\varepsilon} - 1 \right) u_{n,\varepsilon}^\varepsilon = 0, \quad \text{in } \Omega, \right.$$

- Second step: We extend $u_{n,\varepsilon}$ by zero outside of Ω and observe that the sequence still denoted $(u_{n,\varepsilon})$ is uniformly bounded in $BV(\mathbb{R}^N)$, more precisely

$$\int_{\mathbb{R}^N} |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \leq C.$$

Then we can extract from $u_{n,\varepsilon}$ a subsequence, such that $u_{n,\varepsilon} \rightharpoonup u_n$ weakly in $BV(\mathbb{R}^N)$. The limit function belongs to $BV(\mathbb{R}^N)$ and is zero outside of $\bar{\Omega}$.

- Third step: we prove that $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon}$ is uniformly bounded in $L^q(\Omega) \forall q < \infty$. Then we can extract from $\sigma_{n,\varepsilon}$ a subsequence, such that $\sigma_{n,\varepsilon} \rightharpoonup \sigma_n$ weakly in $L^q(\Omega) \forall q < \infty$, such that $\|\sigma\|_\infty \leq 1$ and $\sigma_n \cdot \nabla u_n = |\nabla u_n|$ in $\Omega \cup \partial\Omega$.
- Fourth step: we prove that u_n is a solution of the minimizing problems (3.2) and (3.3). We also prove that σ_n satisfies the problem (3.3).
- Fifth step: we establish that $\lambda_{1,n}$ converges strongly to λ_1 when n goes to ∞ and that u_n converges strongly to the first eigenfunction associated to λ_1 .

4 Proof of the main result

We provide here the proof of Theorem 3.1, outlined as above.

Step 1: We prove here the existence and uniqueness of a positive solution for the following approximation problem

$$\lambda_{1+\varepsilon,n} = \inf_{u \in W_0^{1,1+\varepsilon}(\Omega)} I_{1+\varepsilon,n}(u), \quad (4.1)$$

where $I_{1+\varepsilon,n}$ is the following functional

$$I_{1+\varepsilon,n}(u) = \int_{\Omega} |\nabla u|^{1+\varepsilon} + n \left(\int_{\Omega} |u|^{1+\varepsilon} - 1 \right)^2, \quad (4.2)$$

for some positive ε given.

We first prove that $\lambda_{1+\varepsilon,n}$ is achieved, using standard variational technics: Let $(u_i)_i$ be a minimizing sequence for $\lambda_{1+\varepsilon,n}$. Without loss of generality, up to replace u_i by $|u_i|$, one may assume that u_i is nonnegative. Since $I_{1+\varepsilon,n}$ is coercive, (u_i) is bounded in $W_0^{1,1+\varepsilon}(\Omega)$.

As a consequence, we may extract from it a subsequence, still denoted $(u_i)_i$, which converges weakly in $W_0^{1,1+\varepsilon}(\Omega)$ to some function $u_{n,\varepsilon} \in W_0^{1,1+\varepsilon}(\Omega)$. Furthermore, by the Rellich-Kondrakov Theorem [10], [9], [1], $(u_i)_i$ converges to $u_{n,\varepsilon}$ in $L^{1+\varepsilon}(\Omega)$.

Using the weak lower semicontinuity of the semi-norm $\int_{\Omega} |\nabla u|^{1+\varepsilon}$ for the weak topology of $W_0^{1,1+\varepsilon}(\Omega)$, one has:

$$\begin{aligned} \lambda_{1+\varepsilon,n} &\leq \int_{\Omega} |\nabla u_{n,\varepsilon}|^{1+\varepsilon} + n \left(\int_{\Omega} |u_{n,\varepsilon}|^{1+\varepsilon} - 1 \right)^2 \\ &\leq \liminf_{i \rightarrow +\infty} \left[\int_{\Omega} |\nabla u_i|^{1+\varepsilon} + n \left(\int_{\Omega} |u_i|^{1+\varepsilon} - 1 \right)^2 \right] = \lambda_{1+\varepsilon,n}. \end{aligned}$$

Hence, $u_{n,\varepsilon}$ is a solution of the minimization problem (4.1).

We now prove that this weak solution solves the following partial differential equation:

$$\begin{cases} -\operatorname{div}\sigma_{n,\varepsilon} + 2n \left(\int_{\Omega} u_{n,\varepsilon}^{1+\varepsilon} - 1 \right) u_{n,\varepsilon}^{\varepsilon} = 0 & \text{in } \Omega, \\ \sigma_{n,\varepsilon} \cdot \nabla u_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{1+\varepsilon} & \text{in } \Omega, \\ u_{n,\varepsilon} > 0 & \text{in } \Omega, \quad u_{n,\varepsilon} = 0 \quad \text{on } \partial\Omega. \end{cases} \quad (4.3)$$

Indeed, for every $h \in \mathcal{D}(\Omega)$, we have:

$$\begin{aligned} & DI_{1+\varepsilon,n}(u_{n,\varepsilon}) \cdot h \\ &= (1 + \varepsilon) \left[\int_{\Omega} |\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon} \cdot \nabla h + 2n \left(\int_{\Omega} u_{n,\varepsilon}^{1+\varepsilon} - 1 \right) \int_{\Omega} u_{n,\varepsilon}^{\varepsilon} h \right] \\ &= (1 + \varepsilon) \int_{\Omega} \left[-\operatorname{div} (|\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon}) + 2n \left(\int_{\Omega} u_{n,\varepsilon}^{1+\varepsilon} - 1 \right) u_{n,\varepsilon}^{\varepsilon} \right] h \\ &= 0. \end{aligned}$$

Thus, we get:

$$-\operatorname{div} (|\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon}) + 2n \left(\int_{\Omega} u_{n,\varepsilon}^{1+\varepsilon} - 1 \right) u_{n,\varepsilon}^{\varepsilon} = 0, \quad (4.4)$$

in a distribution sense.

Since $u_{n,\varepsilon}$ is a weak solution of equation (4.4), by regularity results (as developed by Guedda-Veron [33], see also Tolksdorf [47]), one gets that $u_{n,\varepsilon} \in \mathcal{C}^{1,\alpha}(\bar{\Omega}) \forall \alpha \in (0, 1)$. Moreover, since $u_{n,\varepsilon}$ is a nonnegative weak solution of the equation (4.4), by the strict maximum principle of Vazquez (see [48]), $u_{n,\varepsilon}$ is positive everywhere. Hence, setting $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon}$, we have shown that $u_{n,\varepsilon} \in \mathcal{C}^{1,\alpha}(\bar{\Omega}) \cap W_0^{1,1+\varepsilon}(\Omega)$ is a positive solution of (4.3).

Lemma 4.1. *The problem (4.3) has a unique positive solution*

Proof of Lemma 4.1. Let u and v be two positive solutions of (4.3). Then we have:

$$-\operatorname{div} [\sigma_{n,\varepsilon}(u) - \sigma_{n,\varepsilon}(v)] + 2n [\alpha(u) - \alpha(v)] u^{\varepsilon} + 2n \alpha(v) (u^{\varepsilon} - v^{\varepsilon}) = 0, \quad (4.5)$$

where $\alpha(u) = \int_{\Omega} u^{1+\varepsilon} - 1$.

Case 1: $\|u\|_{1+\varepsilon} \geq \|v\|_{1+\varepsilon}$.

Let us multiply (4.5) by $(u - v)^+$ then integrate. It is clear that

$$2n [\alpha(u) - \alpha(v)] \int_{\Omega} u^{\varepsilon} (u - v)^+ \geq 0.$$

So we get that:

$$\int_{\Omega} [\sigma_{n,\varepsilon}(u) - \sigma_{n,\varepsilon}(v)] \cdot \nabla (u - v)^+ + 2n \alpha(v) \int_{\Omega} (u^{\varepsilon} - v^{\varepsilon}) (u - v)^+ \leq 0. \quad (4.6)$$

We know that

$$\int_{\Omega} [\sigma_{n,\varepsilon}(u) - \sigma_{n,\varepsilon}(v)] \cdot \nabla (u - v) \geq 0. \quad (4.7)$$

On the other hand it is clear that

$$\int_{\Omega} (u^\varepsilon - v^\varepsilon) (u - v) \geq 0. \quad (4.8)$$

So, we can conclude that:

$$\int_{\Omega} [\sigma_{n,\varepsilon}(u) - \sigma_{n,\varepsilon}(v)] \cdot \nabla(u - v)^+ + 2n \alpha(v) \int_{\Omega} (u^\varepsilon - v^\varepsilon) (u - v)^+ \geq 0. \quad (4.9)$$

So from (4.6) and (4.9), we obtain that

$$\int_{\Omega} [\sigma_{n,\varepsilon}(u) - \sigma_{n,\varepsilon}(v)] \cdot \nabla(u - v)^+ + 2n \alpha(v) \int_{\Omega} (u^\varepsilon - v^\varepsilon) (u - v)^+ = 0.$$

Then $\int_{\Omega} (u^\varepsilon - v^\varepsilon) (u - v)^+ = 0$, which implies $(u - v)^+ = 0$, i.e. $u \leq v$. Using $\|u\|_{1+\varepsilon} \geq \|v\|_{1+\varepsilon}$, one finally gets $u = v$ a.e.

Case 2: $\|u\|_{1+\varepsilon} \leq \|v\|_{1+\varepsilon}$.

We use the same arguments as in the Case 1, just replacing $(u - v)^+$ by $(v - u)^+$. \square

Thus, we have proved the existence and uniqueness of a positive solution to the problem (4.1).

Step 2: We prove here that $\lim_{\varepsilon \rightarrow 0} \lambda_{1+\varepsilon,n} = \lambda_{1,n}$.

Proposition 4.1. *For every $n \in \mathbb{N}^*$, we have:*

$$\limsup_{\varepsilon \rightarrow 0} \lambda_{1+\varepsilon,n} \leq \lambda_{1,n}$$

Proof of Proposition 4.1. Let $\delta > 0$ be given and $\varphi \in \mathcal{D}(\Omega)$ such that

$$I_{1,n}(\varphi) = \int_{\Omega} |\nabla \varphi| + n \left(\int_{\Omega} |\varphi| - 1 \right)^2 \leq \lambda_{1,n} + \delta.$$

But $\lim_{\varepsilon \rightarrow 0} I_{1+\varepsilon,n}(\varphi) = I_{1,n}(\varphi)$, hence,

$$\limsup_{\varepsilon \rightarrow 0} \lambda_{1+\varepsilon,n} \leq \lambda_{1,n} + \delta.$$

δ being arbitrary, we get $\limsup_{\varepsilon \rightarrow 0} \lambda_{1+\varepsilon,n} \leq \lambda_{1,n}$. \square

Let now $u_{n,\varepsilon}$ be the positive solution of the minimizing problem (4.1). Using Poincaré's and Hölder's inequalities, we get

$$\begin{aligned} \int_{\Omega} u_{n,\varepsilon} dx &\leq C \int_{\Omega} |\nabla u_{n,\varepsilon}| dx \leq C \left(\int_{\Omega} |\nabla u_{n,\varepsilon}|^{1+\varepsilon} dx \right)^{\frac{1}{1+\varepsilon}} |\Omega|^{\frac{\varepsilon}{1+\varepsilon}} \\ &\leq C' \lambda_{1+\varepsilon,n}. \end{aligned}$$

These inequalities show that $(u_{n,\varepsilon})_{\varepsilon>0}$ and $(\nabla u_{n,\varepsilon})_{\varepsilon>0}$ are both bounded in $L^1(\Omega)$. This means that $(u_{n,\varepsilon})_{\varepsilon>0}$ is bounded in $BV(\Omega)$. We denote by u_n the limit of some subsequence in BV for the weak topology.

In step 4 we shall precise this limit. In particular we shall obtain u_n as the restriction to Ω of some limit of extended functions $u_{n,\varepsilon}$ by zero outside of Ω .

Step 3: we obtain $\sigma_n = \frac{|\nabla u_n|}{|\nabla u_n|}$ as the weak limit of $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon}$.

Let $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon}$, one sees that $\sigma_{n,\varepsilon}$ is uniformly bounded in $L^{\frac{1+\varepsilon}{\varepsilon}}(\Omega)$. Let us prove that $\sigma_{n,\varepsilon}$ is uniformly bounded in every $L^q(\Omega)$, for all $q < \infty$. Indeed, let $q > 1$ be given and let ε be such that $q < \frac{1+\varepsilon}{\varepsilon}$. Then

$$\left(\int_{\Omega} |\sigma_{n,\varepsilon}|^q \right)^{\frac{1}{q}} \leq \left(\int_{\Omega} |\sigma_{n,\varepsilon}|^{\frac{1+\varepsilon}{\varepsilon}} \right)^{\frac{\varepsilon}{1+\varepsilon}} |\Omega|^{\frac{1+\varepsilon(1-q)}{(1+\varepsilon)q}} \leq C.$$

Then we may extract from it a subsequence, still denoted by $\sigma_{n,\varepsilon}$, such that $\sigma_{n,\varepsilon}$ tends to some σ_n weakly in $L^q(\Omega)$, for all $q < \infty$ and $\sigma_{n,\varepsilon}$ tends to σ_n a.e., when ε tends to 0.

We observe now that $\|\sigma_n\|_{\infty} \leq 1$. For that aim, let η be in $\mathcal{D}(\Omega, \mathbb{R}^N)$. Then

$$\begin{aligned} \left| \int_{\Omega} \sigma_n \cdot \eta \right| &\leq \liminf_{\varepsilon \rightarrow 0} \left| \int_{\Omega} \sigma_{n,\varepsilon} \cdot \eta \right| \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_{n,\varepsilon}|^{\varepsilon} |\eta| \\ &\leq \liminf_{\varepsilon \rightarrow 0} \left(\int_{\Omega} |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \right)^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_{\Omega} |\eta|^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \\ &\leq \liminf_{\varepsilon \rightarrow 0} (\lambda_{1+\varepsilon,n})^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_{\Omega} |\eta|^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \\ &\leq \int_{\Omega} |\eta|. \end{aligned}$$

This implies that $\|\sigma_n\|_{\infty} \leq 1$.

Let us now observe that $u_{n,\varepsilon}^{\varepsilon}$ is uniformly bounded in every $L^q(\Omega)$, $q < \infty$. Indeed, let q be given and let ε be small enough, such that $q < \frac{1+\varepsilon}{\varepsilon}$, then

$$\left(\int_{\Omega} |u_{n,\varepsilon}^{\varepsilon}|^q \right)^{\frac{1}{q}} \leq \left(\int_{\Omega} |u_{n,\varepsilon}|^{1+\varepsilon} \right)^{\frac{\varepsilon}{1+\varepsilon}} |\Omega|^{\frac{1+\varepsilon(1-q)}{q(1+\varepsilon)}} \leq C.$$

Then $w_{n,\varepsilon} = u_{n,\varepsilon}^{\varepsilon}$ converges weakly, in every $L^q(\Omega)$, $q < \infty$, up to a subsequence, to some w_n , when ε tends to 0.

Let us prove that $0 \leq w_n \leq 1$ and $(w_n - 1)u_n = 0$. For the first assertion, let $\eta \in \mathcal{D}(\Omega)$,

$$\begin{aligned} \left| \int_{\Omega} w_n \cdot \eta \right| &\leq \left(\int_{\Omega} |w_n|^{\frac{1+\varepsilon}{\varepsilon}} \right)^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_{\Omega} |\eta|^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \\ &\leq \liminf_{\varepsilon \rightarrow 0} (\lambda_{1+\varepsilon,n})^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_{\Omega} |\eta|^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \\ &\leq \int_{\Omega} |\eta|. \end{aligned}$$

Hence $0 \leq w_n \leq 1$, $\forall n \in \mathbb{N}^*$.

To prove that $(w_n - 1)u_n = 0$, let us observe that $u_{n,\varepsilon} \rightarrow u_n$ in $L^k(\Omega)$ strongly for all $k < \frac{N}{N-1}$ and $w_{n,\varepsilon} \rightarrow w_n$ in $L^{N+1}(\Omega)$ weakly, therefore

$$\int_{\Omega} w_{n,\varepsilon} u_{n,\varepsilon} \rightarrow \int_{\Omega} w_n u_n \quad \text{when } \varepsilon \rightarrow 0$$

Finally,

$$\int_{\Omega} w_n u_n = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} w_{n,\varepsilon} u_{n,\varepsilon} = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_{n,\varepsilon}^{1+\varepsilon} = \int_{\Omega} u_n.$$

Using the fact that $0 \leq w_n \leq 1$, one gets the result.

Passing to the limit in (4.4), one gets:

$$-\operatorname{div} \sigma_n + 2n \left(\int_{\Omega} u_n - 1 \right) w_n = 0. \quad (4.10)$$

Step 4: Extension of $u_{n,\varepsilon}$ outside Ω and convergence towards a solution of (4.3).

Let $\tilde{u}_{n,\varepsilon}$ be the extension of $u_{n,\varepsilon}$ by 0 in $\mathbb{R}^N \setminus \bar{\Omega}$. Since $u_{n,\varepsilon} = 0$ on $\partial\Omega$, then $\tilde{u}_{n,\varepsilon} \in W^{1,1+\varepsilon}(\mathbb{R}^N)$ and $(\tilde{u}_{n,\varepsilon})$ is bounded in $BV(\mathbb{R}^N)$. Then one may extract from it a subsequence, still denoted $(\tilde{u}_{n,\varepsilon})$, such that

$$\tilde{u}_{n,\varepsilon} \longrightarrow v_n \quad \text{in } L^k(\mathbb{R}^N), \quad \forall k < \frac{N}{N-1} \quad \text{when } \varepsilon \longrightarrow 0,$$

with $v_n = 0$ outside of $\bar{\Omega}$ and

$$\nabla \tilde{u}_{n,\varepsilon} \rightharpoonup \nabla v_n \quad \text{weakly in } \mathcal{M}^1(\mathbb{R}^N) \quad \text{when } \varepsilon \longrightarrow 0,$$

We denote by u_n the restriction of v_n to Ω . We use in the above some limit σ_n of $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1} \nabla u_{n,\varepsilon}$ obtained in the third step.

Multiplying the equation (4.4) by $\tilde{u}_{n,\varepsilon} \varphi$, where $\varphi \in \mathcal{D}(\mathbb{R}^N)$, and integrating by parts, one obtains:

$$\int_{\Omega} \sigma_{n,\varepsilon} \cdot \nabla(\tilde{u}_{n,\varepsilon} \varphi) + 2n \left(\int_{\Omega} \tilde{u}_{n,\varepsilon}^{1+\varepsilon} - 1 \right) \int_{\Omega} \tilde{u}_{n,\varepsilon}^{1+\varepsilon} \varphi = 0,$$

or equivalently

$$\int_{\mathbb{R}^N} |\nabla(\tilde{u}_{n,\varepsilon})|^{1+\varepsilon} \varphi + \int_{\Omega} \sigma_{n,\varepsilon} u_{n,\varepsilon} \cdot \nabla \varphi + 2n \left(\int_{\mathbb{R}^N} \tilde{u}_{n,\varepsilon}^{1+\varepsilon} - 1 \right) \int_{\mathbb{R}^N} \tilde{u}_{n,\varepsilon}^{1+\varepsilon} \varphi = 0. \quad (4.11)$$

Since $\sigma_{n,\varepsilon} \rightharpoonup \sigma_n$ in $L^q(\Omega)$ for all $q < \infty$, in particular for any $\alpha > 0$, $\sigma_{n,\varepsilon}$ tends weakly towards σ_n in $L^{N+\alpha}(\Omega)$. Since $\tilde{u}_{n,\varepsilon}$ tends strongly towards v_n in $L^k(\Omega)$, $k < \frac{N}{N-1}$, one obtains that:

$$\int_{\Omega} \sigma_{n,\varepsilon} u_{n,\varepsilon} \cdot \nabla \varphi \longrightarrow \int_{\Omega} \sigma_n u_n \cdot \nabla \varphi, \quad \text{when } \varepsilon \rightarrow 0.$$

By passing to the limit in the equation (4.11) and defining, up to extracting a subsequence, the measure μ on \mathbb{R}^N by: $\lim_{\varepsilon \rightarrow 0} |\nabla(\tilde{u}_{n,\varepsilon})|^{1+\varepsilon} = \mu$, one obtains:

$$\langle \mu, \varphi \rangle + \int_{\Omega} \sigma_n u_n \cdot \nabla \varphi + 2n \left(\int_{\mathbb{R}^N} v_n - 1 \right) \int_{\mathbb{R}^N} v_n \varphi = 0. \quad (4.12)$$

On the other hand, multiplying equation (4.10) by $v_n \varphi$ where $\varphi \in \mathcal{D}(\mathbb{R}^N)$, one gets

$$\int_{\Omega \cup \partial\Omega} \sigma_n \cdot (\nabla v_n) \varphi + \int_{\Omega} \sigma_n u_n \cdot \nabla \varphi + 2n \left(\int_{\Omega} u_n - 1 \right) \int_{\Omega} u_n \varphi = 0. \quad (4.13)$$

Substracting (4.13) from (4.12), one gets

$$\mu = \sigma_n \cdot \nabla v_n \quad \text{in } \Omega \cup \partial\Omega. \quad (4.14)$$

This implies in particular, according to Proposition 2.1, that

$$|\mu| \leq |\nabla v_n| \quad \text{in } \Omega \cup \partial\Omega,$$

and

$$\int_{\mathbb{R}^N} |\nabla(\tilde{u}_{n,\varepsilon})|^{1+\varepsilon} \longrightarrow \int_{\mathbb{R}^N} |\nabla v_n| \quad \text{when } \varepsilon \rightarrow 0.$$

Finally recalling that according to proposition 2.1 , one has $\nabla v_n \cdot \sigma_n \leq |\nabla v_n|$ on $\Omega \cup \partial\Omega$ one derives that

$$|\nabla v_n| = \sigma_n \cdot \nabla v_n \quad \text{in } \Omega \cup \partial\Omega.$$

Recall that from Proposition 2.1

$$\begin{aligned} \nabla v_n &= \nabla u_n \chi_\Omega - u_n \delta_{\partial\Omega} \vec{n}, \\ \sigma_n \cdot \nabla v_n &= \sigma_n \cdot \nabla u_n \chi_\Omega - \sigma_n \cdot \vec{n} u_n \delta_{\partial\Omega}, \end{aligned}$$

we have obtained

$$\begin{cases} \sigma_n \cdot \nabla u_n = |\nabla u_n| & \text{in } \Omega, \\ \sigma_n \cdot \vec{n} u_n = -u_n & \text{on } \partial\Omega. \end{cases}$$

Then u_n is a nonnegative solution of (3.3). Moreover, the convergence of $|\nabla \tilde{u}_{n,\varepsilon}|$ is tight on $\overline{\Omega}$, i.e.

$$\int_{\Omega} |\nabla u_{n,\varepsilon}| \longrightarrow \int_{\Omega} |\nabla u_n| + \int_{\partial\Omega} u_n, \quad \text{when } \varepsilon \rightarrow 0.$$

Indeed, one has $\int_{\Omega} |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \longrightarrow \int_{\Omega} |\nabla u_n| + \int_{\partial\Omega} u_n$ when $\varepsilon \rightarrow 0$. Using the lower semicontinuity for the extension $u_{n,\varepsilon}$ and Hölder's inequality, we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_{n,\varepsilon}|^{1+\varepsilon} &= \int_{\Omega} |\nabla u_n| + \int_{\partial\Omega} u_n \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_{n,\varepsilon}| \\ &\leq \liminf_{\varepsilon \rightarrow 0} \left(\int_{\Omega} |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} |\Omega|^{\frac{\varepsilon}{1+\varepsilon}} \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \end{aligned}$$

The result is proved.

Step 5: The convergence of $\lambda_{1+\varepsilon,n}$ towards λ_1

In this step we explicit the relation between the values $\lambda_{1+\varepsilon,n}$ when n is large, and the first eigenvalue λ_1 defined in the first part.

Theorem 4.1. *Let u_n be a nonnegative solution of 4.4, then, up to a subsequence, as $n \rightarrow \infty$, (u_n) converges to $u \in BV(\Omega)$, $u \geq 0$, $u \not\equiv 0$, which realizes the minimum defined in (1.2). Moreover*

$$\lim_{n \rightarrow \infty} \lambda_{1,n} = \lambda_1.$$

Proof of the Theorem 4.1. For $\lambda_{1,n}$ and λ_1 defined as above, it is clear that we have:

$$\limsup_{n \rightarrow \infty} \lambda_{1,n} \leq \lambda_1. \quad (4.15)$$

Let $(u_n)_n$ be a sequence of positive solutions of the relaxed problem defined in (3.2). We begin to prove that $(u_n)_n$ is bounded in $BV(\Omega)$. For that aim let us note that by (4.15), one gets that $n \left(\int_{\Omega} u_n - 1 \right)^2$ is bounded by λ_1 , which implies that $\lim_{n \rightarrow \infty} \left(\int_{\Omega} u_n - 1 \right)^2 = 0$. Then

$$\lim_{n \rightarrow \infty} \int_{\Omega} u_n = 1,$$

Hence, $(u_n)_n$ is bounded in $L^1(\Omega)$.

Using once more (4.15), one can conclude that $(u_n)_n$ is bounded in $BV(\Omega)$. Then, the extension of each u_n by zero outside of $\overline{\Omega}$ is bounded in $BV(\mathbb{R}^N)$. One can then extract from it a subsequence, still denoted u_n , such that

$$u_n \rightharpoonup u \quad \text{weakly in } BV(\mathbb{R}^N) \quad \text{when } n \rightarrow \infty,$$

obviously $u = 0$ outside of $\overline{\Omega}$ and $u > 0$ in Ω . By the compactness of the Sobolev embedding from $BV(\Omega)$ into $L^1(\Omega)$, one has $\|u\|_{L^1(\Omega)} = 1$. Using the lower semi continuity of the total variation $\int_{\mathbb{R}^N} |\nabla u|$ with respect to the weak topology, one has (since $u_n \rightarrow u$ in $L^1(\Omega)$)

$$\begin{aligned} \lambda_1 &\leq \int_{\mathbb{R}^N} |\nabla u| \leq \int_{\mathbb{R}^N} |\nabla u| + n \left(\int_{\mathbb{R}^N} u - 1 \right)^2 \\ &\leq \liminf_{n \rightarrow \infty} \left[\int_{\mathbb{R}^N} |\nabla u_n| + n \left(\int_{\mathbb{R}^N} u_n - 1 \right)^2 \right] \\ &\leq \limsup_{n \rightarrow \infty} \lambda_{1,n} \leq \lambda_1. \end{aligned}$$

Then one gets that

$$\lim_{n \rightarrow \infty} \lambda_{1,n} = \lambda_1.$$

Since $u = 0$ outside of $\overline{\Omega}$, one has on \mathbb{R}^N $\nabla u = \nabla u \chi_{\Omega} - u \vec{n} \delta_{\partial\Omega}$ and then

$$\int_{\mathbb{R}^N} |\nabla u| = \int_{\Omega} |\nabla u| + \int_{\partial\Omega} u.$$

Moreover, one obtains that:

$$\lim_{n \rightarrow \infty} n \left(\int_{\Omega} u_n - 1 \right)^2 = 0,$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n| = \int_{\Omega} |\nabla u| + \int_{\partial\Omega} |u|.$$

Then, we get the tight convergence of u_n to u in $BV(\overline{\Omega})$.

Let us observe that $\text{sign}^+(u_n)$ converges to some w , $0 \leq w \leq 1$ in every $L^q(\Omega)$, $\forall q < \infty$. Using the convergence of u_n to u in $L^q(\Omega)$, $\forall q < \frac{N}{N-1}$, one gets

$$\int_{\Omega} u_n = \int_{\Omega} u_n \text{sign}^+(u_n) \longrightarrow \int_{\Omega} u = 1 \quad \text{when } n \rightarrow \infty.$$

As a consequence

$$-2n \left(\int_{\Omega} u_n - 1 \right) \int_{\Omega} u_n \longrightarrow \lambda_1 \quad \text{when } n \rightarrow \infty,$$

and then also

$$-2n \left(\int_{\Omega} u_n - 1 \right) \longrightarrow \lambda_1 \quad \text{when } n \rightarrow \infty.$$

This ends the proof of the main result. \square

The author thanks the referee for its remarks and advices which permit to improve this paper.

References

- [1] **Adams, R.** *Sobolev spaces*, Academic Press, New York, (1975).
- [2] **Alter, F. ; Cazelles, V. ; Chambolle, A.** *A characterization of convex calibrable sets in \mathbb{R}^N* , preprint.
- [3] **Alter, F. ; Cazelles, V. ; Chambolle, A.** *Evolution of convex sets in the plane by the minimizing total variation flow*, preprint.
- [4] **Andreu, F. ; Caselles, V. ; Mazón, J. M.** *A strongly degenerate quasilinear elliptic equation*, *Nonlinear Anal.* 61(4) (2005), 637–669.
- [5] **Adams, D. R. ; Lenhart, S. M.** *An Obstacle control problem with a source term*, *Appl. Math. Optim.* 47 (2003), 79–95.
- [6] **Adams, D. R. ; Lenhart, S. M. ; Yong, J.** *Optimal control of the obstacle for an elliptic variational inequality*, *Appl. Math. Optim.* 38 (1998), 121–140.
- [7] **Attouch, H. ; Picard, C.** *Inéquations variationnelles avec obstacles et espaces fonctionnels en théorie du potentiel. (French) [Variational inequalities with obstacles and function spaces in potential theory]*, *Applicable Anal.* 12 (1981), no. 4, 287–306.
- [8] **Attouch, H.** *Variational convergence for functions and operators* *Applicable Mathematics Series*. Pitman.
- [9] **Aubin, T.** *Problèmes isoperimétriques et espaces de Sobolev*, *J. Differential Geometry*, 11 (1976), 573–598.
- [10] **Aubin, T.** *Some Nonlinear Problems in Riemannian Geometry*, *Springer Monographs in Mathematics*, (1998).
- [11] **Barbu, V.** *Optimal control of variational inequality*, Pitman, London (1984).

- [12] **Bellettini, G. ; Caselles, V. ; Novaga, M.** *Explicit solutions of the eigenvalue problem $-\operatorname{div} \left(\frac{Du}{|Du|} \right) = \lambda u$.*
- [13] **Brézis, H.** *Analyse fonctionnelle, théorie et applications*, Paris N. Y. Barcelone, Milano, Mexico, Sao paulo, Masson, (1983).
- [14] **Cheeger, J.** *A lower bound for the smallest eigenvalue of the Laplacian* in Problems in Analysis, Symposium in honor of Salomon Bochner, Princeton Univ. Press . Princeton (1970), 195-199.
- [15] **De Giorgi ; Carriero, M. ; Leaci, A.** *Existence theorem for a minimum problem with a Free dicontinuity set*, A.R.M.A. , (1989) 108, 195-218.
- [16] **Demengel, F.** *On Some Nonlinear Partial Differential Equations Involving The 1-Laplacian and Critical Sobolev exponent*, ESAIM: Control, Optimisation and Calculus of Variations, 4 (1999), 667-686.
- [17] **Demengel, F.** *Some compactness result for some spaces of functions with bounded derivatives*, A.R.M.A. 105(2) (1989), 123-161.
- [18] **Demengel, F.** *Theoremes d'existence pour des equations avec l'operateur 1-Laplacien, premiere valeur propre pour $-\Delta_1$* , C.R Acad. Sci. Paris, Ser. I334 (2002), 1071-1076.
- [19] **Demengel, F.** *Some existence's results for noncoercive 1-Laplacian operator*, Asymptot. Anal. 43(4) (2005), 287-322.
- [20] **Demengel, F.** *Functions locally almost 1-harmonic*, Applicable Analysis, Vol.83(9), September (2004), 865-896.
- [21] **Demengel, F.** *On some nonlinear partial differential equations involving the "1-Laplacian" and critical Sobolev exponent*, ESAIM Control Optim. Calc. Var.4 (1999), 667-686.
- [22] **Demengel, F.** *Convexité dans les espaces fonctionnels*, Ellipses (2004), 1ère edition.
- [23] **Demengel, F.** *Espaces fonctionnels pour la théorie des équations aux dérivées partielles elliptiques*, EDP Sciences, Editions du CNRS, A paraître en (2007).
- [24] **Demengel, F.** *Mesures et distributions, théorie et illustration par les exemples : Mesures de radon, distributions, convolutions, transformations de Fourier, distributions périodiques* Editions Ellipses, (2000).
- [25] **Demengel, F. ; Hebey, E.** *On Some nonlinear equations involving the "p-Laplacian" with critical Sobolev growth*, Adv. Differential Equations 3 (1998), n. 4, 533-574.
- [26] **Demengel, F. ; Nazaret, B.** *On Some nonlinear partial differential equations involving the "p-Laplacian" and critical Sobolev trace maps*, Asymptotic Analysis 23 (2000), 135-156.
- [27] **Dieudonné, J.** *Eléments d'analyse*, Tome II, Editions Gauthier-villars, (1974).

- [28] **Ekeland, I. ; Temam, R.** *Convex analysis and variational problems*, North-Holland, Amsterdam, New York, (1976).
- [29] **Friedman, A.** *Optimal control for variational inequality*, SIAM J. Control Optim., 24 (1986), 439-451.
- [30] **Gilbarg, D. ; Trudinger, N.S.** *Elliptic Partial Differential Equations of Second Order*, 2nd Edition, Springer Verlag, Berlin, (1983).
- [31] **Giusti, E.** *Minimal surfaces and functions of bounded variation*, Notes de cours redigés par G.H. Williams. Departement of Mathematics Australian National University, Canberra (1977), et Birkhäuser (1984).
- [32] **Giaquinta, M. ; Modica, G. ; Soucek, J.** *Cartesian Currents in the Calculus of Variations I*, LNM, Vol 37, Springer, (1997).
- [33] **Guedda, M. ; Veron, L.** *Quasilinear elliptic equations involving critical sobolev exponents*, Nonlinear Analysis, Theory, Methods and Applications, 13 (1989), 879-902.
- [34] **Kinderlehrer, D. ; Stampacchia, G.** *Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, (1980).
- [35] **Kohn, R.V. ; Temam, R.** *Dual spaces of stress and strains with applications to Hencky plasticity*, Appl.Math.Optim 10 (1983), 1-35.
- [36] **Lions, P.L.** *The concentration-compactness principle in the calculus of variations. The limit case*, I et II. Rev. Mat. Iberoamericana 1 (1985), 145-201.
- [37] **Lions, J. L.** *Quelques méthodes de résolution des problèmes aux limites non linéaires* Eudes Mathématiques, Paris (1969).
- [38] **Lindqvist, P.** *On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$* , Proceedings of the American Mathematic Society Volume 91 (1), (1990).
- [39] **Murat, F. ; Cioranescu, D.** *Un terme étrange venu d'ailleurs*, Non-Linear Partial Differential Equations and Their Applications, Collège de France Seminar, Vols II and III (ed. H. Brezis and J.L. Lions), Research Notes in Mathematics, Vols 60 and 70, Pitman, Boston, MA, (1982), pp. 98-138 and 154-178.
- [40] **Rodrigues, J.** *Obstacle Problems in Mathematical Physics*, North Holland, Amsterdam, (1987).
- [41] **Rockafellar, R. ; Tyrrell** *Convex analysis*, Princeton Mathematical Series, No. 28 Princeton University Press, Princeton, N.J. (1970).
- [42] **Rockafellar, R.** *Convex analysis in the calculus of variations*, Advances in convex analysis and global optimization (Pythagorion, 2000), 135-151, Nonconvex Optim. Appl., 54, Kluwer Acad. Publ., Dordrecht, (2001).
- [43] **Strang, G. ; Temam, R.** *Functions with bounded derivatives*, A.R.M.A.(1980), 493-527

- [44] **Strang, G. ; Temam, R.** *Functions with bounded deformation*, A.R.M.A., 75 (1980), 7-21.
- [45] **Temam, R.** *Problèmes mathématiques en plasticité*, Gauthiers-Villars, (1981).
- [46] **Talenti, G.** *Best Constants in Sobolev inequality*, Ann. di Matem. Pura ed Appl., 110 (1976), 353-372.
- [47] **Tolksdorf, P.** *Regularity for a more general class of quasilinear elliptic equations*, Journal of Differential Equations, 51 (1984), 126-150.
- [48] **Vazquez, J. L.** *A Strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., 12 (1984), 191-202.
- [49] **Yosida, Functional analysis**, Springer-Verlag, Berlin, (1965).