A fast precipitation and dissolution reaction for a reaction diffusion system arising in a porous medium ⋆

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Abstract

This paper is devoted to the study of a fast reaction diffusion system arising in reactive transport. It extends the articles [8,10] since a precipitation and dissolution reaction is considered so that the reaction term is not sign-definite and is moreover discontinuous. Energy type methods allow us to prove uniform estimates and then to study the limiting behavior of the solution as the kinetic rate tends to infinity in the special situation of one aqueous species and one solid species.

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1 Introduction

In this paper we consider the reaction-diffusion system,

$$(P^\lambda)\begin{cases} u_t = \Delta u - \lambda G(u,w) & \text{in } \Omega\times(0,T) & (1.1)\\ w_t = \lambda G(u,w) & \text{in } \Omega\times(0,T) & (1.2)\\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T) & (1.3)\\ u(x,0)=u_0(x),\quad w(x,0)=w_0(x) & \text{for } x\in\Omega & (1.4)\end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and T is a positive constant. We suppose that λ is a positive constant and that the function $G(\cdot,\cdot)$ is given by

$$G(u,w) = (u-\overline{u})^+ - sign^+(w)(u-\overline{u})^-, \tag{1.5}$$

where $\overline{u}$ is a given positive constant and

$$s^+ = \max(0,s),\quad s^- = \max(0,-s),\ \text{ and } sign(s) = \begin{cases} 1 & \text{if } s>0,\\ -1 & \text{if } s<0,\\ 0 & \text{if } s=0.\end{cases}$$

The above system (P^λ) is a simplified adimensional model of reactive transport in a porous medium at the Darcy scale, where u stands for a concentration of an aqueous species, therefore mobile, and w stands for a concentration of a mineral species. The term $\lambda G(u,w)$ is a reaction rate that models either a precipitation if $u-\overline{u}\geq 0$, or a dissolution otherwise. The positive constant $\overline{u}$ is the thermodynamic constant of the dissolution reaction and λ is a constant rate. Reactive transport problems arise in the field of radioactive waste storage, oil industry or CO_2 storage. Indeed, water rocks interactions like precipitation and dissolution reactions have a strong impact both on flow and solute transport.

We focus on reactions which are very fast compared with the diffusion process so that λ is a large parameter. In this paper we extend a result of Eymard, Hilhorst, van der Hout and Peletier [6], which they obtained in the special case of a function $G(\cdot,\cdot)$ assumed to be nonnegative and nondecreasing in both arguments. The Stefan problem obtained when $\lambda\to+\infty$ is the same as that of [8,10] but the problem (P^λ) considered in this paper has an additional precipitation term. In [8], the main tool is a finite volume method used in any space dimension. In [10], a Legendre function (associated with the liquid concentration) is used in one space dimension to deal with discontinuities. Note that in [3], the existence of a solution to the same problem with two aqueous species instead of one is proven; however, the study of the singular limit in this more complex case is still an open problem, since the techniques presented here do not seem to be easily adaptable. We suppose that the initial functions u_0 and w_0 satisfy the hypotheses:

$$
(H_0) \qquad \begin{cases} u_0, w_0 \in L^2(\Omega) \;\; 0 \le u_0 \le M_1 \text{ and } 0 \le w_0 \le M_2 \text{ a.e in } \Omega, \\ \text{for some positive constants } M_1 \text{ and } M_2 \text{ such that } M_1 > \overline{u}. \end{cases}
$$

We set $Q_T := \Omega \times (0,T)$ and denote by $W_2^{2,1}(Q_T) = \{u \in L^2(Q_T), \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial u}{\partial t} \in L^2(Q_T), i,j = 1,\cdots,N\}$ and by $C^{0,1}([0,T];L^\infty(\Omega))$ the space of Lipschitz continuous functions with values in $L^\infty(\Omega)$. Next we define a notion of weak solution for Problem (P^λ).

Definition 1.1 (u^λ, w^λ) *is a weak solution of Problem* (P^λ) *if for all* $T > 0$

(i) $\quad u^\lambda \in W_2^{2,1}(Q_T) \;\; , \;\; w^\lambda \in C^{0,1}([0,T];L^\infty(\Omega));$

(ii) $$\int_\Omega u^\lambda(T)\xi(T) - \int_\Omega u_0\xi(0) - \int_{Q_T} \left\{u^\lambda \xi_t - \nabla u^\lambda \nabla \xi - \lambda G(u^\lambda, w^\lambda)\xi\right\} = 0,$$
$$\int_\Omega w^\lambda(T)\xi(T) - \int_\Omega w_0\xi(0) - \int_{Q_T} \left\{w^\lambda \xi_t + \lambda G(u^\lambda, w^\lambda)\xi\right\} = 0,$$

for all $\xi \in H^1(Q_T)$.

The existence of a nonnegative solution pair is proven in [3] for the case of two aqueous species; the system studied in this latter case fully contains the one studied here. In view of its regularity, we remark that it satisfies the differential equations in Problem (P^λ) a.e. in Q_T. The purpose of this paper is to prove the following result.

Theorem 1 *Suppose that* u_0 *and* w_0 *satisfy the hypotheses* (H_0). *Then for every* $\lambda > 0$, *Problem* (P^λ) *has a unique nonnegative weak solution* (u^λ, w^λ). *Moreover there exist functions* $U \in L^2(Q_T)$, $W \in L^2(Q_T)$ *such that* u^λ *and* w^λ *converge strongly in* $L^2(Q_T)$ *to* U *and* W *respectively, as* λ *tends to* ∞. *The function* $Z := -(U + W) + \overline{u}$ *is the unique weak solution of the Stefan problem*

$$
(SP) \begin{cases} Z_t = \Delta(Z^+) & \text{in } \Omega \times (0,T) \qquad (1.6) \\ \dfrac{\partial Z^+}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T) \qquad (1.7) \\ Z(x,0) = -(u_0(x) + w_0(x)) + \overline{u} \text{ for } x \in \Omega. & \qquad (1.8) \end{cases}
$$

Conversely the limit pair (U, W) *is given by* $(U, W) = (-Z^+ + \overline{u}, Z^-)$.

Let us first present the results of numerical simulations which show the behaviour of the solution to Problem (P^λ) as λ becomes larger, and which were obtained by solving a nonlinear system obtained from a discretization of (P^λ) by the finite volume method [7], for finite values of λ. For $\lambda = \infty$, we discretized by the finite volume method the Problem (SP). The physical domain Ω is the unit square $(0,1) \times (0,1)$. The initial condition is taken to be $u_0 = 0$, except in two squares where $u_0 = 2.10^{-3}$, $w_0 = 0$ and $\overline{u} = 10^{-3}$. In these squares, since $u_0 > \overline{u}$, then at initial time, the mineral precipitates, and therefore w increases. In the sequel of the simulation, we observe the dissolution of w. In figures 1 and 2, we display concentration maps of w after 1 time step (precipitation has occured) and 6 time steps (dissolution has occured). We select the values $\lambda = 10^3$(a), 10^6(b) and $\lambda = +\infty$(c). When λ increases the precipitation front gets stiffer. Indeed in Figure 2, $\max(w)$ is 6.5 10^{-4}(a), 9.8 10^{-4}(b), $\overline{u} = 10^{-3}$(c). Therefore, we observe the expected behaviour of the solution as λ increases.

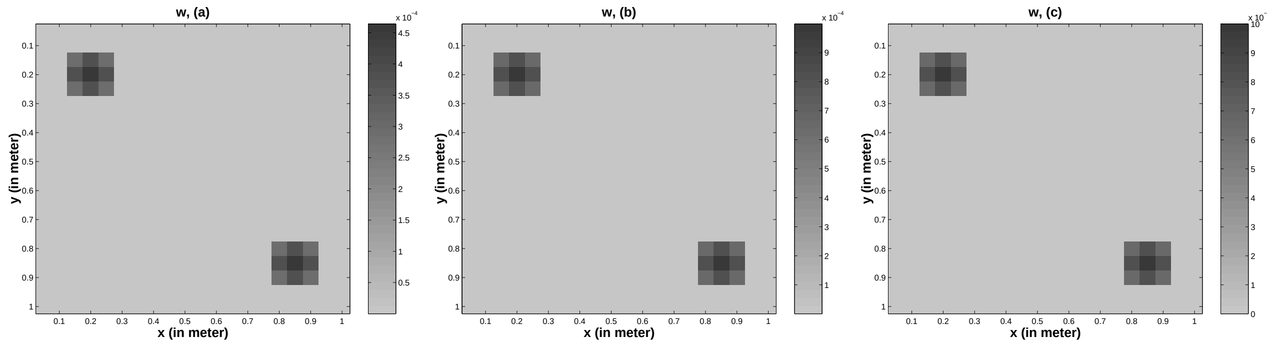

Fig. 1. Concentration of w after 1 time step for 10^3(a), 10^6(b), $\lambda = \infty$(c).

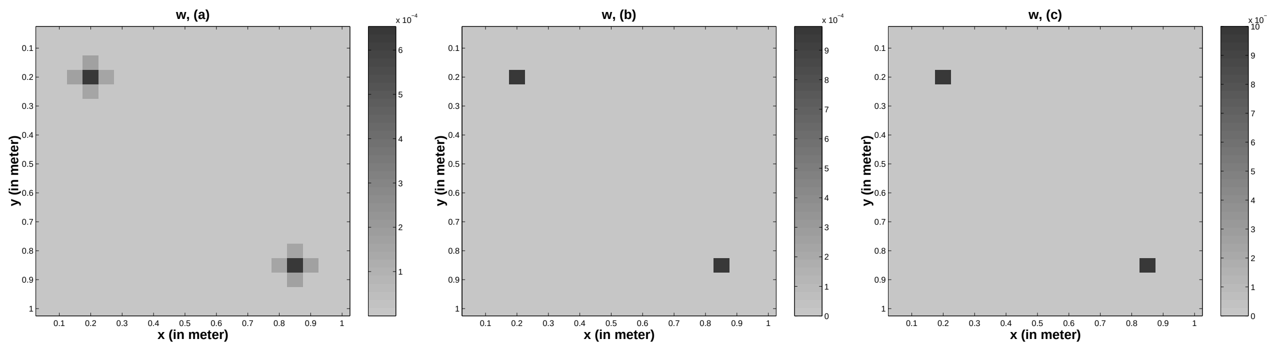

Fig. 2. Concentration of w after 6 time steps for 10^3(a), 10^6(b), $\lambda = \infty$(c).

The mathematical justification of this asymptotic behaviour is obtained by proving Theorem 1: this is the aim of the remainder of this article. This article is organized as follows : In Section 2 we prove a comparison principle for Problem (P^λ), which implies the uniqueness of its weak solution. This result is quite natural since the monotonicity properties of G in u and in w make it a cooperative system [1]. In Section 3 we present some a priori estimates, which imply that as λ tends to ∞, the sum $-(u^\lambda + w^\lambda) + \overline{u}$ tends to the unique weak solution of the Stefan problem (SP).

2 Comparison principle and uniqueness

We first prove the following comparison principle.

Theorem 2 *Let* (u, w) *and* (ϕ, ψ) *be such that* $u, \phi \in W_2^{2,1}(Q_T)$ *and* $w, \psi \in C^{0,1}([0, T]; L^\infty(\Omega))$ *and suppose that they satisfy*

$$u_t \geq \Delta u - \lambda G(u,w) \qquad a.e \ in \ Q_T \tag{2.9}$$
$$w_t \geq \lambda G(u,w) \qquad a.e \ in \ Q_T \tag{2.10}$$
$$\phi_t \leq \Delta \phi - \lambda G(\phi,\psi) \qquad a.e \ in \ Q_T \tag{2.11}$$
$$\psi_t \leq \lambda G(\phi,\psi) \qquad a.e \ in \ Q_T \tag{2.12}$$
$$\frac{\partial u}{\partial n} = \frac{\partial \phi}{\partial n} = 0 \ \ a.e \ on \ \partial\Omega \times (0,T) \tag{2.13}$$
$$u(x,0) \geq u_0(x) \geq \phi(x,0) \qquad for \ x \in \Omega \tag{2.14}$$
$$w(x,0) \geq w_0(x) \geq \psi(x,0) \qquad for \ x \in \Omega \tag{2.15}$$

Then

$$u(x,t) \geq \phi(x,t) \ \ a.e \ in \ Q_T \tag{2.16}$$
$$w(x,t) \geq \psi(x,t) \ \ a.e \ in \ Q_T. \tag{2.17}$$

Before presenting the proof, we recall a technical result stated by Crandall and Pierre [5].

Lemma 2.1 *Let p: $\mathbf{R} \to \mathbf{R}$ be Lebesgue measurable and bounded and define q by $q(r) = \int_0^r p(s)ds$. Let $\omega \in W^{1,1}(0,T,L^1(\Omega))$. Then $q(\omega) \in W^{1,1}(0,T,L^1(\Omega))$ and*

$$\frac{d}{dt}q(\omega) = p(\omega)\frac{d}{dt}\omega \ \ a.e.$$

This Lemma will be used several times in this article either with $p(s) = sign(s)$ and thus $q(s) = |s|$ or with $p(s) = sign^+(s)$ and thus $q(s) = s^+$.

Proof. We subtract the inequality for u (2.9) from that for ϕ (2.11), and multiply the result by $\mathrm{sign}^{\delta,+}(\phi - u)$; similarly we subtract the inequality for ψ (2.12) from that for w (2.10), and multiply the result by $\mathrm{sign}^{\delta,+}(\psi - w)$, where $\mathrm{sign}^{\delta,+}$ is a smooth nondecreasing regularization of $sign^+$ which converges pointwise. This approximation can be built as in Lemma 3.1 below. Adding both inequalities and integrating the result on Ω we deduce that

$$\int_\Omega \frac{\partial}{\partial t}a_\delta(\phi - u) + \int_\Omega \frac{\partial}{\partial t}a_\delta(\psi - w) \leq \int_\Omega \Delta(\phi - u)\,\mathrm{sign}^{\delta,+}(\phi - u) + \lambda\int_\Omega \tau_\delta(u,w,\phi,\psi)$$

with $a_\delta(s) = \int_0^s \mathrm{sign}^{\delta,+}(r)dr$ which converges to s^+ and

$$\tau_\delta(u,w,\phi,\psi) = \Big(G(u,w) - G(\phi,\psi)\Big)\,\mathrm{sign}^{\delta,+}(\phi - u) + \Big(G(\phi,\psi) - G(u,w)\Big)\,\mathrm{sign}^{\delta,+}(\psi - w).$$

Then,

$$\frac{d}{dt}\int_\Omega \{a_\delta(\phi - u) + a_\delta(\psi - w)\} \leq -\int_\Omega |\nabla(\phi - u)|^2\{\mathrm{sign}^{\delta,+}(\phi - u)\}' + \lambda\int_\Omega \tau_\delta(u,w,\phi,\psi).$$

This implies, in view of Lemma 2.2, Lebesgue's dominated convergence theorem and the hypotheses on the initial data, that for all $t \in [0, T]$,

$$\int\int_{Q_T} (\phi - u)^+(x,t)dx + \int_\Omega (\psi - w)^+(x,t)dx \leq 0,$$

so that $\phi \leq u$ and $\psi \leq w$ on Q_T. □

Lemma 2.2 *Let the function* τ_δ *be defined by*

$$\tau_\delta(u,w,\phi,\psi) = \Big(G(u,w) - G(\phi,\psi)\Big)\,\mathrm{sign}^{\delta,+}(\phi - u) + \Big(G(\phi,\psi) - G(u,w)\Big)\,\mathrm{sign}^{\delta,+}(\psi - w),$$

with $\mathrm{sign}^{\delta,+}$ *a smooth approximation of sign*$^+$*. Then* $\lim_{\delta\downarrow 0} \tau_\delta \leq 0$*.*

Proof. We use the monotonicity of $G(u,w)$ ($\nearrow u$, $\searrow w$). We rewrite τ_δ in the form,

$$\begin{aligned}
\tau_\delta &= \{G(u,w) - G(\phi,w)\}\,\mathrm{sign}^{\delta,+}(\phi - u) + \{G(\phi,w) - G(\phi,\psi)\}\,\mathrm{sign}^{\delta,+}(\phi - u)\\
&\quad +\{G(\phi,\psi) - G(\phi,w)\}\,\mathrm{sign}^{\delta,+}(\psi - w) + \{G(\phi,w) - G(u,w)\}\,\mathrm{sign}^{\delta,+}(\psi - w)\\
&= \quad -\{G(\phi,w) - G(u,w))\}\,\mathrm{sign}^{\delta,+}(\phi - u) + \{G(\phi,w) - G(u,w))\}\,\mathrm{sign}^{\delta,+}(\psi - w)\\
&\quad -\{G(\phi,w) - G(\phi,\psi)\}\,\mathrm{sign}^{\delta,+}(\psi - w) + \{G(\phi,w) - G(\phi,\psi)\}\,\mathrm{sign}^{\delta,+}(\phi - u)
\end{aligned}$$

Applying Lebesgue's dominated convergence theorem we deduce that

$$\int_\Omega \tau_\delta(u,w,\phi,\psi) \to \int_\Omega \tau(u,w,\phi,\psi) \quad \text{as} \quad \delta \downarrow 0,$$

where

$$\begin{aligned}
\tau = -\{G(\phi,w) - G(u,w)\}^+ + \{G(\phi,w) - G(u,w)\}sign^+(\psi - w)\\
-\{G(\phi,w) - G(\phi,\psi)\}^+ + \{G(\phi,w) - G(\phi,\psi)\}sign^+(\phi - u) \leq 0.
\end{aligned}$$

□

Corollary 2.3 *Under hypotheses* (H_0)*, let* (u^λ, w^λ) *be a weak solution of Problem* (P^λ)*, then:*

$$u^\lambda(x,t) \leq \tilde{u}(t) := \overline{u} + (M_1 - \overline{u})\,\mathrm{e}^{\lambda t}, \;\; a.e \text{ in } Q_T \tag{2.18}$$

$$w^\lambda(x,t) \leq \tilde{w}(t) := M_2 + (M_1 - \overline{u})(1 - \mathrm{e}^{\lambda t}), \;\; a.e \text{ in } Q_T. \tag{2.19}$$

Proof. We check that $(\tilde{u}, \tilde{w})$ is the weak solution of Problem (P^λ) with the constant initial data $(\tilde{u}(x,0), \tilde{w}(x,0)) = (M_1, M_2)$. The result then follows from the comparison principle given in Theorem 2. □

3 A priori estimates

The purpose of this section is to prove the convergence Theorem 1. We first introduce some notations and give technical lemmas. To begin with we construct a smooth nondecreasing approximation of the sign function which we denote by sign^δ and which converges pointwise to the function sign; we then define by $\mathcal{H}^\delta$ the regularization of the Heaviside function $\mathcal{H}^\delta(s) := \int_0^s \text{sign}^\delta(\tau)d\tau$.

Lemma 3.1 *There exist a sequence of smooth nondecreasing functions* (sign^δ) *which converges pointwise to the function sign.*

Proof. First we introduce two smooth functions f_0 and f_1 defined respectively by

$$f_1(x) = \begin{cases} e^{-\frac{1}{(x-1)^2}} & \text{for } x < 1 \\ 0 & \text{for } x \geq 1 \end{cases} \quad \text{and } f_0(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

The function $f_0 f_1 \in C^\infty(\mathbf{R})$ and is equal to zero on $\mathbf{R} \setminus [0,1]$. Moreover defined F by

$$F(x) = \frac{1}{\displaystyle\int_0^1 f_0(x)f_1(x)dx} \int_0^x f_0(x)f_1(x)dx$$

one can check that $F \in C^\infty(\mathbf{R})$ and satisfies $F(0) = 0$, $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x \geq 1$. Finally, setting $\text{sign}^\delta(x) = F(\frac{x}{\delta})$ for $\delta > 0$, we deduce that $\text{sign}^\delta(x)$ converges to $\text{sign}(x)$ for all $x \in \mathbf{R}$ as δ tends to 0. $\square$

We now prove some a priori estimates.

Lemma 3.2 *Let* (u^λ, w^λ) *be the solution of* (P^λ)*. Then there exists* $C_1 > 0$ *only depending on* T, Ω *and* $\overline{u}$, *such that*

$$\int\int_{Q_T} |w_t^\lambda| dxdt = \lambda \int\int_{Q_T} |G(u^\lambda, w^\lambda)| dxdt \leq C_1, \tag{3.20}$$

and

$$\int\int_{Q_T} |\nabla u^\lambda|^2 dxdt \leq C_1. \tag{3.21}$$

Proof. We first prove (3.20). Multiplying (1.1) by $\text{sign}^\delta(u - \overline{u})$ and integrating the result on Ω we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left\{\mathcal{H}^\delta(u^\lambda - \overline{u})\right\} = -\int_\Omega \left\{\nabla(u^\lambda - \overline{u})\right\}^2 \left\{\text{sign}^\delta\right\}'(u^\lambda - \overline{u}) - \lambda \int_\Omega G(u^\lambda, w^\lambda)\,\text{sign}^\delta(u^\lambda - \overline{u}).$$

Using the nonnegativity of $s \mapsto \left\{\text{sign}^\delta\right\}'$, we have the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left\{\mathcal{H}^\delta(u^\lambda-\overline{u})\right\}+\lambda\int_\Omega G(u^\lambda,w^\lambda)\,\mathrm{sign}^\delta(u^\lambda-\overline{u})\le 0,$$

which we integrate on $(0,t)$, $0<t\le T$ to obtain

$$\int_\Omega \left\{\mathcal{H}^\delta(u^\lambda-\overline{u})\right\}(t)+\lambda\int_{Q_t} G(u^\lambda,w^\lambda)\,\mathrm{sign}^\delta(u^\lambda-\overline{u})\le \int_\Omega\left\{\mathcal{H}^\delta(u_0-\overline{u})\right\}.$$

It then follows from Lebesgue's dominated convergence theorem that

$$\int_\Omega |u^\lambda-\overline{u}|(t)+\lambda\int_{Q_t} G(u^\lambda,w^\lambda)sign(u^\lambda-\overline{u})\le C.$$

In view of the special expression of $G(\cdot,\cdot)$, we remark that

$$G(u^\lambda,w^\lambda)sign(u^\lambda-\overline{u})=|G(u^\lambda,w^\lambda)|,$$

so that finally

$$\int_\Omega |u^\lambda-\overline{u}|(t)+\lambda\int_{Q_t}|G(u^\lambda,w^\lambda)|\le C$$

holds. Moreover in view of (1.2) we deduce that

$$\int\int_{Q_T}|w_t^\lambda|=\lambda\int\int_{Q_T}|G(u^\lambda,w^\lambda)|\le C_1,$$

which coincides with (3.20). Next we prove (3.21). Multiplying (1.1) by $u^\lambda-\overline{u}$ and integrating the result on Q_T we obtain

$$\int\int_{Q_T} u_t^\lambda(u^\lambda-\overline{u})=-\int\int_{Q_T}|\nabla(u^\lambda-\overline{u})|^2-\lambda\int\int_{Q_T}G(u^\lambda,w^\lambda)(u^\lambda-\overline{u})$$

which implies that

$$\int\int_{Q_T}|\nabla(u^\lambda-\overline{u})|^2=-\frac{1}{2}\int\int_{Q_T}\frac{\partial}{\partial t}|u^\lambda-\overline{u}|^2-\lambda\int\int_{Q_T}G(u^\lambda,w^\lambda)(u^\lambda-\overline{u})\le\lambda\int\int_{Q_T}|G(u^\lambda,w^\lambda)(u^\lambda-\overline{u})|.$$

Using (3.20) and the fact that u^λ is bounded we deduce (3.21). □

Next we prove estimates of differences of space and time translates of $\{u^\lambda\}$. We set for $r\in\mathbf{R}^+$:

$$\Omega_r=\{x\in\Omega,\quad B(x,2r)\subset\Omega\}.$$

Lemma 3.3 *There exists $C_2>0$ only depending on T, Ω and $\overline{u}$ such that*

$$\int_0^T\int_{\Omega_r}|u^\lambda(x+\xi,t)-u^\lambda(x,t)|^2dxdt\le C_2|\xi|^2,\tag{3.22}$$

and

$$\int_0^{T-\tau}\int_\Omega|u^\lambda(x,t+\tau)-u^\lambda(x,t)|^2dxdt\le C_2\tau\tag{3.23}$$

for all $\xi\in\mathbf{R}^N$, $|\xi|\le 2r$ and $\tau\in(0,T)$.

Proof. We first prove (3.22). We have that

$$\begin{aligned}
\int_0^T \int_{\Omega_r} |u^\lambda(x+\xi,t) - u^\lambda(x,t)|^2 dxdt &= \int_0^T \int_{\Omega_r} \left| \int_0^1 \nabla u^\lambda(x+\sigma\xi,t).\xi d\sigma \right|^2 dxdt \\
&\le \int_0^T \int_{\Omega_r} \left[\int_0^1 |\nabla u^\lambda(x+\sigma\xi,t)|^2 d\sigma \int_0^1 |\xi|^2 d\sigma \right] dxdt \\
&\le |\xi|^2 \int_0^1 \left[\int_0^T \left(\int_{\Omega_r} |\nabla u^\lambda(x+\sigma\xi,t)|^2 dx \right) dt \right] d\sigma \\
&\le |\xi|^2 \int_0^1 \left[\int_0^T \left(\int_\Omega |\nabla u^\lambda(y,t)|^2 dy \right) dt \right] d\sigma \\
&\le |\xi|^2 \int_0^T \int_\Omega |\nabla u^\lambda(y,t)|^2 dydt.
\end{aligned}$$

Using (3.21) we deduce (3.22). Next we prove (3.23).

$$\begin{aligned}
&\int_0^{T-\tau} \int_\Omega |u^\lambda(x,t+\tau) - u^\lambda(x,t)|^2 dxdt \\
&= \int_0^{T-\tau} \int_\Omega \Big(u^\lambda(x,t+\tau) - u^\lambda(x,t) \Big) \Big(\int_0^\tau \partial_t u^\lambda(x,t+\sigma) d\sigma \Big) dxdt \\
&= \int_0^{T-\tau} \int_\Omega \Big(u^\lambda(x,t+\tau) - u^\lambda(x,t) \Big) \Big(\int_0^\tau (\Delta u^\lambda - \lambda G(u^\lambda, w^\lambda))(x,t+\sigma) d\sigma \Big) dxdt \\
&= \int_0^{T-\tau} \int_\Omega \Big(u^\lambda(x,t+\tau) - u^\lambda(x,t) \Big) \Big(\int_0^\tau \Delta u^\lambda(x,t+\sigma) d\sigma \Big) dxdt \\
&\quad -\lambda \int_0^{T-\tau} \int_\Omega \Big(u^\lambda(x,t+\tau) - u^\lambda(x,t) \Big) \Big(\int_0^\tau G(u^\lambda, w^\lambda)(x,t+\sigma) d\sigma \Big) dxdt \\
&=: I + II
\end{aligned} \tag{3.24}$$

Using (3.21) we have that

$$\begin{aligned}
|I| &\le \int_0^\tau \Big(\int_0^{T-\tau} \int_\Omega \Big| \nabla u^\lambda(x,t+\tau) - \nabla u^\lambda(x,t) \Big| |\nabla u^\lambda(x,t+\sigma)| dxdt \Big) d\sigma \\
&\le 2\tau \int_0^T \int_\Omega \Big| \nabla u^\lambda(x,t)|^2 dxdt \le 2\tau C_1.
\end{aligned} \tag{3.25}$$

In view of (3.20) and Corollary 2.3 we obtain that

$$|II| \le 2KC_1\tau. \tag{3.26}$$

Finally substituting (3.25) and (3.26) into (3.24) we deduce (3.23). □

Next we prove estimates of differences of space and time translates of $\{w^\lambda\}$.

Lemma 3.4 *There exists a positive function h such that $h(\xi) \to 0$ as $\xi \to 0$ and:*

$$\int_0^T \int_{\Omega_r} |w^\lambda(x+\xi,t) - w^\lambda(x,t)| dx dt \le h(\xi), \tag{3.27}$$

and

$$\int_0^{T-\tau} \int_{\Omega} |w^\lambda(x,t+\tau) - w^\lambda(x,t)| dx dt \le C_1 \tau \tag{3.28}$$

for all $\xi \in \mathbf{R}^N$, $|\xi| \le 2r$ and $\tau \in (0,T)$.

Proof. We first show (3.28). We have in view of the ordinary differential equation (1.2) for w^λ, that

$$\begin{aligned}\int_0^{T-\tau} \int_{\Omega} |w^\lambda(x,t+\tau) - w^\lambda(x,t)| dx dt &= \int_0^{T-\tau} \int_{\Omega} \left| \int_0^\tau \partial_t w^\lambda(x,t+\sigma) d\sigma \right| dx dt \\ &\le \int_0^\tau \left(\int_0^T \int_{\Omega} \lambda |G(u^\lambda,w^\lambda)(x,t)| dx dt \right) d\sigma,\end{aligned} \tag{3.29}$$

which with (3.20) gives (3.28). Next we prove (3.27). We first introduce the set

$$\Omega'_r = \cup_{x\in\Omega_r} B(x,r)$$

and remark that by definition

$$\Omega_r \subset \Omega'_r \subset \Omega.$$

As it is done in [6], we introduce the function $\psi \in C_0^\infty(\Omega'_r)$, such that

$$\psi(x) = \frac{4}{r} \int_{\Omega_1} \rho\left(\frac{4(y-x)}{r}\right) dy \qquad \text{for all } x \in \Omega'_r,$$

where $\Omega_1 = \cup_{x\in\Omega_r} B(x, r/4)$ and ρ is the function defined by

$$\rho(x) = \begin{cases} \rho_0 exp\left(-\dfrac{1}{1-|x|^2}\right) & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and ρ_0 is a constant chosen such that $\int_{\mathbf{R}}^N \rho(x) dx = 1$. One can check that ψ satisfies that

$$0 \le \psi \le 1 \text{ in } \Omega'_r \text{ and } \psi = 1 \text{ in } \Omega_r.$$

Let $\xi \in \mathbf{R}^N$ with $|\xi| \le r$. For all $(x,t) \in \Omega'_r \times (0,T)$ we set

$$\tilde{u}^\lambda(x,t) = u^\lambda(x+\xi,t) \text{ and } \tilde{w}^\lambda(x,t) = w^\lambda(x+\xi,t).$$

Next we show that

$$E := \Big(G(u^\lambda, w^\lambda) - G(\tilde{u}^\lambda, \tilde{w}^\lambda)\Big)\Big(sign(u^\lambda - \tilde{u}^\lambda) - sign(w^\lambda - \tilde{w}^\lambda)\Big) \geq 0, \text{ a.e. in } \Omega'_r \times (0,T). \tag{3.30}$$

We consider 4 different cases :
If $u^\lambda > \tilde{u}^\lambda$ and $w^\lambda > \tilde{w}^\lambda$ or if $u^\lambda < \tilde{u}^\lambda$ and $w^\lambda < \tilde{w}^\lambda$ then $\Big(sign(u^\lambda - \tilde{u}^\lambda) - sign(w^\lambda - \tilde{w}^\lambda)\Big) = 0$ and thus $E = 0$.
If $u^\lambda \geq \tilde{u}^\lambda$ and $w^\lambda \leq \tilde{w}^\lambda$ since $u \mapsto G(u,v)$ is increasing and $v \mapsto G(u,v)$ is non increasing we have that

$$G(\tilde{u}^\lambda, \tilde{w}^\lambda) \leq G(u^\lambda, \tilde{w}^\lambda) \leq G(u^\lambda, w^\lambda).$$

Thus $E \geq 0$.
Similarly if $u^\lambda \leq \tilde{u}^\lambda$ and $w^\lambda \geq \tilde{w}^\lambda$ then $E \geq 0$. This concludes the proof of (3.30). In view of the ordinary differential equation for w^λ, (1.2), we deduce from (3.30) that

$$(w^\lambda - \tilde{w}^\lambda)_t\Big(sign(u^\lambda - \tilde{u}^\lambda) - sign(w^\lambda - \tilde{w}^\lambda)\Big) \geq 0, \text{ a.e. in } \Omega'_r \times (0,T). \tag{3.31}$$

Multiplying the equality

$$(u^\lambda - \tilde{u}^\lambda)_t + (w^\lambda - \tilde{w}^\lambda)_t - \Delta(u^\lambda - \tilde{u}^\lambda) = 0,$$

by $\Big[sign^\delta(u^\lambda - \tilde{u}^\lambda)\Big]\psi$ and integrating by part on $\Omega'_r \times (0,t)$ for $t \in (0,T)$ we obtain that

$$\int_0^t \int_{\Omega'_r} (u^\lambda - \tilde{u}^\lambda)_t \Big[sign^\delta(u^\lambda - \tilde{u}^\lambda)\Big]\psi dxds + \int_0^t \int_{\Omega'_r} (w^\lambda - \tilde{w}^\lambda)_t \Big[sign^\delta(u^\lambda - \tilde{u}^\lambda)\Big]\psi dxds + \int_0^t \int_{\Omega'_r} \nabla(u^\lambda - \tilde{u}^\lambda)\nabla\Big\{[sign^\delta(u^\lambda - \tilde{u}^\lambda)]\psi\Big\} dxds = 0 \tag{3.32}$$

which by Lebesgue's dominated convergence theorem implies that

$$\int_0^t \int_{\Omega'_r} (u^\lambda - \tilde{u}^\lambda)_t \Big[sign(u^\lambda - \tilde{u}^\lambda)\Big]\psi dxds + \int_0^t \int_{\Omega'_r} (w^\lambda - \tilde{w}^\lambda)_t \Big[sign(u^\lambda - \tilde{u}^\lambda)\Big]\psi dxds + \int_0^t \int_{\Omega'_r} \nabla(u^\lambda - \tilde{u}^\lambda) sign(u^\lambda - \tilde{u}^\lambda)\nabla\psi dxds \leq 0.$$

This with (3.31) gives that

$$\int_0^t \int_{\Omega'_r} |u^\lambda - \tilde{u}^\lambda|_t \psi dxds + \int_0^t \int_{\Omega'_r} |w^\lambda - \tilde{w}^\lambda|_t \psi dxds + \int_0^t \int_{\Omega'_r} \nabla|u^\lambda - \tilde{u}^\lambda|\nabla\psi dxds \leq 0,$$

which implies after integration in time in the two first terms that

$$\int_{\Omega'_r}\Big(|(u^\lambda-\tilde{u}^\lambda)(x,t)|+|(w^\lambda-\tilde{w}^\lambda)(x,t)|\Big)\psi(x)dx$$
$$\leq\int_{\Omega'_r}\Big(|u_0(x)-u_0(x+\xi)|+|w_0(x)-w_0(x+\xi)|\Big)\psi(x)dx+\int_0^t\int_{\Omega'_r}|(u^\lambda-\tilde{u}^\lambda)(x,t)||\Delta\psi(x)|dxds$$

Integrating this inequality with respect to t on $(0,T)$ and using the fact that $0\leq\psi\leq 1$ in Ω'_r and $\psi=1$ in Ω_r we deduce that

$$\int_0^T\int_{\Omega_r}\Big(|(u^\lambda-\tilde{u}^\lambda)(x,t)|+|(w^\lambda-\tilde{w}^\lambda)(x,t)|\Big)dxdt$$
$$\leq T\int_{\Omega'_r}\Big(|u_0(x)-u_0(x+\xi)|+|w_0(x)-w_0(x+\xi)|\Big)dx$$
$$+T\Big(\int_0^T\int_{\Omega'_r}|u^\lambda-\tilde{u}^\lambda|^2\Big)^{1/2}\Big(\int_0^T\int_{\Omega'_r}|\Delta\psi|^2\Big)^{1/2}.$$

Also using (3.22) with Ω_r replaced by Ω'_r we deduce that

$$\int_0^T\int_{\Omega_r}|(w^\lambda-\tilde{w}^\lambda)(x,t)|dxdt\leq T\int_{\Omega'_r}\Big(|u_0(x)-u_0(x+\xi)|+|w_0(x)-w_0(x+\xi)|\Big)dx$$
$$+C(T)|\xi|\Big(\int_{\Omega'_r}|\Delta\psi|^2\Big)^{1/2}. \tag{3.33}$$

Therefore we have proved (3.27) with $h(\xi)$ being equal to the right hand-side of (3.33).
□

Corollary 3.5 *Let (u^λ,w^λ) be the unique nonnegative solution of Problem (P^λ). There exist subsequences $\{u^{\lambda_m}\}$ and $\{w^{\lambda_m}\}$ and functions $U\in L^\infty(Q_T)$ and $W\in L^\infty(Q_T)$ such that*

$$u^{\lambda_m}\to U \text{ and } w^{\lambda_m}\to W$$

strongly in $L^2(Q_T)$ as λ_m tends to ∞. Moreover as λ_m tends to ∞, $u^{\lambda_m}\to U$ weakly in $L^2(0,T;H^1(\Omega))$.

Proof. The first part of Corollary 3.5 follows from Lemmas 3.3 and 3.4 and the Riesz-Fréchet-Kolmogorov theorem ([4] Theorem IV.25 and Corollary IV.26). The last assertion follows from (3.21). □

Lemma 3.6 *We have that $G(U,W)=0$ so that setting $Z:=-(U+W)+\overline{u}$ we obtain*

$$\begin{cases} \text{if } U\geq \overline{u} \text{ then } U=\overline{u} \text{ and } Z=-W \\ \text{if } U<\overline{u} \text{ then } W=0 \text{ and } Z=-U+\overline{u}. \end{cases}$$

Thus $Z^+=-U+\overline{u}$.

Proof. We set

$$G_\varepsilon(u,w):=(u-\overline{u})^+ - sign_\varepsilon^+(w)(u-\overline{u})^-,$$

where $sign_\varepsilon^+$ is the continuous function defined by

$$sign_\varepsilon^+(x)=\begin{cases} 1 & \text{if } x\geq\varepsilon \\ \frac{1}{\varepsilon}x & \text{if } 0\leq x\leq\varepsilon \\ 0 & \text{if } x\leq 0 \end{cases}$$

Next we check that

$$0\leq (u-\overline{u})G_\varepsilon(u,w)\leq (u-\overline{u})G(u,w), \tag{3.34}$$

for all $(u,w)\in\mathbf{R}^2$. We consider three cases :
If $w\geq\varepsilon$ then $G(u,w)=G_\varepsilon(u,w)=u-\overline{u}$ so that

$$0\leq (u-\overline{u})G_\varepsilon(u,w)\leq (u-\overline{u})G(u,w).$$

If $0\leq w\leq\varepsilon$ and $u-\overline{u}\geq 0$ then $(u-\overline{u})G_\varepsilon(u,w)=(u-\overline{u})^2=(u-\overline{u})G(u,w)\geq 0$.
If $0\leq w\leq\varepsilon$ and $u-\overline{u}\leq 0$ then we have $(u-\overline{u})G_\varepsilon(u,w)=\frac{1}{\varepsilon}w(u-\overline{u})^2\geq 0$ and moreover

$$(u-\overline{u})G_\varepsilon(u,w)\leq (u-\overline{u})G(u,w).$$

This concludes the proof of (3.34). Applying (3.34) at the point $(u^{\lambda_m},w^{\lambda_m})$ and integrating the result on Q_T we deduce that

$$0\leq\int_{Q_T}(u^{\lambda_m}-\overline{u})G_\varepsilon(u^{\lambda_m},w^{\lambda_m})\leq\int_{Q_T}(u^{\lambda_m}-\overline{u})G(u^{\lambda_m},w^{\lambda_m}).$$

In view of Corollary 3.5 and (3.20) we deduce that $\lim_{\lambda_m\uparrow\infty}\int_{Q_T}(u^{\lambda_m}-\overline{u})G(u^{\lambda_m},w^{\lambda_m})=0$ and thus since G_ε is continuous we have that

$$\int_{Q_T}(U-\overline{u})G_\varepsilon(U,W)=0.$$

Therefore $G_\varepsilon(U,W)=0$ or $U=\overline{u}$. Finally letting ε tend to 0 we obtain that $G(U,W)=0$ or $U=\overline{u}$, which concludes the proof of Lemma 3.6. □

Lemma 3.7 *The function $Z:=-(U+W)+\overline{u}$ is a weak solution of Problem (SP).*

Proof. Let $(u^{\lambda_m}, w^{\lambda_m})$ be the unique solution of Problem (P^{λ_m}). Then

$$\int_{Q_T} (u^{\lambda_m} + w^{\lambda_m})\xi_t - \int_{Q_T} \nabla u^{\lambda_m} \nabla \xi = -\int_\Omega (u_0 + w_0)\xi(0),$$

for all $\xi \in C^{2,1}(\overline{Q_T})$ such that $\xi(T) = 0$. Letting λ_m tends to ∞ we deduce that

$$\int_{Q_T} (U + W)\xi_t - \int_{Q_T} \nabla U.\nabla \xi = -\int_\Omega (u_0 + w_0)\xi(0) \tag{3.35}$$

for all $\xi \in C^{2,1}(\overline{Q_T})$ such that $\xi(T) = 0$.
We consider now the function $Z = -(U + W) + \overline{u}$, then by Lemma 3.6 we deduce that Z is a weak solution of Problem (SP) in the following sense : (i) $Z \in L^\infty(Q_T)$; (ii) Z satisfies the following equality

$$\int_{Q_T} Z\xi_t - \int_{Q_T} \nabla Z^+.\nabla \xi = \int_\Omega (u_0 + w_0 - \overline{u})\xi(0) \tag{3.36}$$

□

Lemma 3.8 *The function $Z := -(U + W) + \overline{u}$ is the weak solution of Problem (SP) and the whole sequence (u^λ, w^λ) converges to $(U, W) = (-Z^+ + \overline{u}, Z^-)$ as λ tends to ∞.*

Proof. Lemma 3.8 follows directly from the uniqueness of the weak solution of Problem (SP) (see [9] Proposition 5) and the fact that the functions $U = -Z^+ + \overline{u}$, $W = Z^-$ are uniquely defined as well. □

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