

# On an Extension of Min-Semistable Distributions

M. BEN ALAYA <sup>(1)</sup>, T. HUILLET <sup>(2)</sup>, A. PORZIO <sup>(1,3)</sup>

<sup>(1)</sup> LAGA, Université de Paris XIII,  
Institut Galilée, 93430, Villetaneuse, France

<sup>(2)</sup> LPTM, Université de Cergy-Pontoise et CNRS (UMR 8089),  
2, Avenue A. Chauvin, 95032 Cergy-Pontoise Cedex, France

<sup>(3)</sup> CPTH, Ecole Polytechnique, 91128 Palaiseau, France.

## Abstract

This work focuses on a functional equation which extends the notion of min-semistable distributions. Our main results are an existence theorem and a characterization theorem for its solutions. The first establishes the existence of a class of solutions of this equation under a condition on the first zero on the positive axis of the associated structure function. The second shows that solutions belonging to a subclass of complementary distribution function can be identified by their behavior at the origin. Our constructed solutions are in this subclass. The uniqueness question is also discussed.

*AMS Classification:* 62E10, 60E05.

*Key Words and Phrases:* stable and semistable laws, functional equation.

*Running title:* random min-semistable distributions.

mba@math.univ-paris13.fr, huillet@u-cergy.fr, Anna.Porzio@cpht.polytechnique.fr

## 1 INTRODUCTION

In this paper we shall consider the functional equation defined on the space of complementary cumulative probability distribution functions (for short ccdf)  $\bar{F}$  with support  $[0, \infty]$  :

$$(E) : \bar{F}(x) = \mathbb{E} \left[ \prod_{i=1}^M \bar{F}(C_i x)^{\Gamma_i} \right]. \quad (1)$$

Here  $M \in \mathbb{N}^*$  is a integer-valued random variable and  $(C_i, i \geq 1)$  and  $(\Gamma_i, i \geq 1)$  are sequences of random variables such that  $C_i > 0$ ,  $\Gamma_i \geq 1$ . In the statistical literature, the function,  $\bar{F}$  is also called the survival or survivor function. The solution  $\bar{F}$  of (E) can be regarded as a fixed point of the transformation  $T$  defined on the set of complementary cumulative distribution functions by

$$T\bar{F}(x) = \mathbb{E} \left[ \prod_{i=1}^M \bar{F}(C_i x)^{\Gamma_i} \right].$$



Let  $X$  be the random variable with ccdf  $\bar{F}$  satisfying (E). When  $\Gamma_i$  are integral-valued random variables, equation (E) reads in terms of random variables  $X \stackrel{d}{=} \min_{1 \leq i \leq M} \min_{1 \leq j \leq \Gamma_i} C_{i,j} X_{i,j}$ . Here the  $X_{i,j}$  are i.i.d. copies of  $X$ , for each  $i$ ,  $C_{i,j}$  are i.i.d. copies of  $C_i$  and  $X_{i,j}$  are independent of  $C_{i,j}$ ,  $\Gamma_i$  and  $M$ . After a suitable identification of variables, this distributional equality can be put into the simpler form

$$X \stackrel{d}{=} \min_{1 \leq i \leq N} A_i X_i \quad (2)$$

in terms of new random variables  $N \in \mathbb{N}^*$  and  $\{A_i, i \geq 1\}$  positive. Here,  $X_i$  are i.i.d. copies of  $X \geq 0$  and independent of the random variables  $\{N, A_i, i \geq 1\}$ . This identity in law expresses the invariance property under weighted minima considered by Alsmeyer and Rösler [1].

Let again  $\Gamma_i$  be integral-valued random variables. Equation (1), on the space of Laplace-Stieltjes transforms instead of space of ccdf yields an equation similar to (2), namely

$$X \stackrel{d}{=} \sum_{i=1}^N A_i X_i. \quad (3)$$

Under this form, it has been intensively studied by several authors.

Initially, the functional equation associated to (3) was introduced in Mandelbrot [19] and [20] in the context of a model for turbulence. Later, Kahane and Peyrière [16] obtained necessary and sufficient conditions for the existence of solutions of (3), when the  $A_i$  are independent and identically distributed and  $N$  is a constant. Holley and Liggett [14] obtained the same kind of results when  $A_i$  are fixed multiple of a given random variable.

On physical grounds, such distributions provided examples of invariant measures for infinite interacting particle systems. Motivated by questions raised by these works on the nature of such invariant measures, their ergodic behavior, notably the possible display of phase transitions, Durrett and Liggett [11] studied (3) in a quite general setting. More precisely, taking  $N$  constant and  $A_i$  non negative with arbitrary law, they gave necessary and sufficient conditions for the existence of solutions under a sole condition on the moments of the  $A_i$ . Moreover, they characterized all these solutions and proved some convergence results.

Random variables satisfying (3) can also be viewed as a generalization of semistable laws, in that they are stable under random weighted means. In this view, Guivarc'h [13] discussed equation (3) when the  $A_i$  are independent identically distributed variables and  $N$  is constant. He gave theorems of existence and uniqueness of solutions and analyzed particularly their behavior at infinity.

More recently, Liu [17] [18] extended the results of [11] on equation (3) allowing  $N$  to be an almost surely finite random variable, finding the optimal conditions for the existence of its solutions. As it is reviewed in [17], equation (3) or some variants of it, arises in several other application fields: for instance, it defines distributions appearing as limiting distribution of some branching processes (either of the Bellman-Harris or of the Crump-Mode types) or Hausdorff measures of some random fractal sets [17]. See also Caliebe [7], [8] for recent results and references.

Coming back to equation (1), the idea of taking non-integral powers  $\Gamma_i > 1$  in a similar equation is due initially to Barral [2]. Considering the following functional equation

$$f(x) = (\mathbb{E}(f(Cx)))^\gamma$$

where  $C$  is a positive random variable and  $\gamma \geq 1$  is non-random, he was able to obtain analogue results as in [11] and [18] by studying it in a space containing the space of Laplace-Stieltjes transforms and included in the space of complementary distribution functions.

On the other hand, in [3], the problem of characterizing the cumulative distribution functions (for short cdf) with support  $[0, \infty]$ , say  $G$ , satisfying the functional equation

$$G(x) = \prod_{i=1}^m G(x/c_i)^{\gamma_i} \quad (4)$$

for some integer  $m > 1$ , and real numbers  $c_i > 0$ ,  $\gamma_i > 0$ ,  $i = 1, \dots, m$  was considered. These have been called multiscaling max-semistable distributions. Functional equation (4) may be viewed as a version of the integrated Cauchy functional equation whose solution can be defined by appealing to Corollary 2.3.2 of [21]. This constitutes a by-product of a Deny's theorem (see [21]).

Setting  $\bar{F}(x) = G(1/x)$  when  $x > 0$  and  $\bar{F}(x) = 1$  for  $x \leq 0$ , the complementary cumulative distribution function  $\bar{F}$ , with support  $[0, \infty]$ , is solution to

$$\bar{F}(x) = \prod_{i=1}^m \bar{F}(c_i x)^{\gamma_i} \quad (5)$$

and we can deduce similarly the class of the so called multiscaling min-semistable distributions.

In [3], the physical meaning of functional equation (4) has been discussed to some extent. Essentially, it was emphasized that any strictly positive random variable, interpreted as some observable, can be viewed as the maximum of a Poisson number of "micro-events". The model (4) expresses that the observable under concern might as well result from the aggregation of  $m > 1$  independent observations of statistically similar events, each with its specific intensity  $\gamma_i$  and scale  $c_i$  (in other words, it might as well result from more frequent micro-events but with smaller reduced amplitudes); it translates an amplitude and scale invariance principle for the observable. Such fixed point equation also appears in discrete scale invariance in Renormalization Group theory in Physics. This model exhibits log-periodic features, whose empirical evidence was underlined in diverse application fields such as finance, turbulence, rupture theory, DLA growth, geophysics and frustrated systems' statistics. (see [15] and references therein). In a concrete physical situation, it seems natural to imagine that the intensity and scale parameters are unknown, or, more realistically, modelled by some random variables. This motivates the randomization of this model.

The functional equation (E) given by (1) can indeed be viewed as a randomization of the equation (4). By putting  $G(x) = \bar{F}(1/x)$  when  $x > 0$  and  $G(x) = 0$  when  $x \leq 0$ , conclusions drawn from (E) can readily be translated to the randomization of the equation (4) namely

$$G(x) = \mathbb{E} \left[ \prod_{i=1}^M G(x/C_i)^{\Gamma_i} \right]. \quad (6)$$

Central to the solution of the functional equation (4) was the Kahane-Peyrière-Mandelbrot (KPM) real valued structure function defined by  $q \rightarrow \sum_{i=1}^m \gamma_i c_i^q$ ,  $q \in \mathbb{R}$ .

In its randomized version, the KPM structure function now reads

$$\tau(q) = \mathbb{E} \left[ \sum_{i=1}^M \Gamma_i C_i^q \right], \quad q \in \mathbb{R}. \quad (7)$$

We shall assume that  $\tau(q) < \infty$  whenever  $q \geq 0$ . Essentially this function is convex. We note that  $\tau(0) \geq 1$  and  $\tau(0) = 1$  corresponds to the case  $M = \Gamma_1 = 1$  and equation (E) admits a non-degenerate solution if and only if  $C_1 = 1$ . This trivial situation will be avoided in the sequel by assuming  $\tau(0) > 1$ .

The first main result is an existence theorem, which establishes the existence of solutions under a condition on the first zero on the positive axis of the structure function (7). Following [11], [13], [17] and [18], we first prove the existence of solutions of (E) in the special case, where  $\tau(1) = 1$  and  $\tau'(1) < 0$ . Then, the general case is investigated by introducing a transport operator. Our techniques follow the lines of Durrett and Liggett [11], and Liu [17] [18].

Next, we exhibit a large space of complementary distribution functions containing the given solutions, namely, with  $F := 1 - \bar{F}$

$$\mathcal{F} = \{ \bar{F} \in C^0(\mathbb{R}_+, [0, 1]) : \exists \lambda > 0, c > 0, \text{ satisfying } \frac{F(ax)}{F(x)} \leq ca^\lambda, \forall a > 1, x > 0 \}.$$

Then we show a characterization theorem, which tells that the solutions of (E) belonging to  $\mathcal{F}$  can be identified by their behavior at the origin.

The paper is organized as follows. In Section 2 existence of solutions of equation (E) in the special and general case is studied. In section 3, the main characterization theorem is first stated. The core of section 3 is devoted to the proof of some technical results, which will contribute to elucidate the behavior at the origin of the solutions belonging to space  $\mathcal{F}$ . In section 4, we discuss the uniqueness of the solution.

## 2 EXISTENCE of SOLUTIONS

### 2.1 The special case: existence of a solution

In this section we suppose that, with  $\log_+ x := 0 \vee \log x$ ,  $x > 0$ ,

$$(i) \quad \mathbb{E} \left[ \sum_{i=1}^M \Gamma_i C_i \log_+ \left( \sum_{i=1}^M \Gamma_i C_i \right) \right] < \infty$$

$$(ii) \quad \tau(1) = \mathbb{E} \left[ \sum_{i=1}^M \Gamma_i C_i \right] = 1 \text{ and } (iii) \quad \tau'(1) < 0.$$

We note that  $\tau(0) > 1$ . If conditions (ii) and (iii) are fulfilled, we shall refer to the special case. Define

$$\mathcal{E} = \left\{ \bar{F} \text{ ccdf} : \bar{F} \text{ convex with } -\infty < \bar{F}'(0) < 0 \right\},$$

and let  $\mathcal{E}_1 := \left\{ \bar{F} \text{ ccdf} : \bar{F} \text{ convex with } \bar{F}'(0) = -1 \right\}$ . Note that if  $\bar{F} \in \mathcal{E}$ , then  $F$  is absolutely continuous with respect to Lebesgue measure. In the following theorem we give sufficient conditions which guarantee the existence of a non-degenerate solution to the functional equation (E). This result is obtained by adapting the proof of

theorem 3.1. of Liu [17]. Liu himself used techniques developed in Durrett and Liggett [11] and some ideas of Doney and Biggins (see [9], [10], [4]). For the reader convenience the proof of some technical arguments used in the theorem 1 below will be postponed to section 3.

**Theorem 1** *Under the above conditions (i), (ii) and (iii), there exists a solution of (E) in  $\mathcal{E}_1$ , implying, in particular,  $F(x)/x \rightarrow 1$  as  $x \downarrow 0$ .*

**Proof :** For a complementary cumulative distribution function (ccdf)  $\bar{F}$ , we define non-negative functions  $D$  and  $G$  on  $\mathbb{R}$  by

$$D(z) = \frac{1 - \bar{F}(e^{-z})}{e^{-z}} \quad (8)$$

and

$$G(z) = e^z \mathbb{E} \left[ \left( \prod_{i=1}^M \bar{F}(e^{-z} C_i)^{\Gamma_i} \right) - 1 + \sum_{i=1}^M \Gamma_i (1 - \bar{F}(e^{-z} C_i)) \right]. \quad (9)$$

Let  $Z$  be a random variable with distribution determined by

$$\mathbb{E}(\Psi(Z)) = \mathbb{E} \left( \sum_{i=1}^M \Gamma_i C_i \Psi(-\log(C_i)) \right), \quad (10)$$

for all bounded measurable functions  $\Psi$ . Since  $\tau(1)$  is finite  $\Psi(Z)$  is integrable.

Let  $\bar{F}_0(x) = e^{-x} \mathbf{1}_{(x \geq 0)} + \mathbf{1}_{(x < 0)}$  and  $\bar{F}_{n+1} = T\bar{F}_n$ ,  $n \geq 1$ . Replacing  $\bar{F}$  by  $\bar{F}_n$  in equations (8) and (9) we obtain the associated functions noted by  $D_n$  and  $G_n$  in place of  $D$  and  $G$  for all  $n \in \mathbb{N}$ . Noticing that, for  $x \geq 0$ ,

$$\bar{F}_1(x) = \mathbb{E} \left[ \exp(-x \sum_{i=1}^M \Gamma_i C_i) \right] \geq \exp \left[ -x \mathbb{E} \left( \sum_{i=1}^M \Gamma_i C_i \right) \right] = \exp(-x) = \bar{F}_0(x) \quad (11)$$

and  $\bar{F}_1(x) = \bar{F}_0(x) = 1$  for  $x < 0$  we deduce, by the monotony of  $T$ , that  $\bar{F}_{n+1} \geq \bar{F}_n$ . From lemma 9 (iii),  $G_{n+1} \leq G_n$  and from lemma 8 we have

$$D_{n+1}(z) = \mathbb{E}(D_n(z + Z)) - G_n(z) \geq \mathbb{E}(D_n(z + Z)) - G_0(z). \quad (12)$$

Thus,

$$D_n(z) \geq \mathbb{E}(D_0(z + S_n)) - \sum_{k=0}^{n-1} \mathbb{E}(G_0(z + S_k)), \quad (13)$$

for all  $n \geq 1$ . Here,  $S_n := \sum_{k=0}^n Z_k$  where  $(Z_k)_{k \geq 1}$  is a sequence of independent random variables with the same distribution as  $Z$ , and  $S_0 = 0$ . As

$$\mathbb{E}(Z) = -\mathbb{E} \left( \sum_{i=1}^M \Gamma_i C_i \log(C_i) \right) > 0, \quad (14)$$

$S_n$  goes almost surely to  $+\infty$  when  $n$  tends to infinity. Since  $D_0(z)$  is bounded and  $\lim_{z \rightarrow +\infty} D_0(z) = 1$ , we get

$$\lim_{n \rightarrow +\infty} \mathbb{E}(D_0(z + S_n)) = 1. \quad (15)$$

The function  $f(z) := \sum_{k=0}^{\infty} \mathbb{E}(G_0(z + S_k))$  satisfies the renewal equation  $f = G_0 + F_{-Z} * f$ , where  $F_{-Z}$  is the cdf of the random variable  $-Z$ , with  $-\infty < \mathbb{E}(-Z) < 0$ . When  $G_0$  is direct Riemann integrable, as we will show below, the renewal theorem yields  $\lim_{z \uparrow \infty} f(z) = 0$  ([12], page 381). This result, together with (13, 15) implies

$$\lim_{z \uparrow \infty} \lim_{n \uparrow \infty} D_n(z) \geq 1.$$

But using  $D_{n+1} \leq D_n \leq \dots \leq D_0$  we obtain  $\lim_{z \uparrow \infty} \lim_{n \uparrow \infty} D_n(z) \leq \lim_{z \uparrow \infty} D_0(z) = 1$ . This shows that  $\lim_{z \uparrow \infty} \lim_{n \uparrow \infty} D_n(z) = 1$ . Calling  $\bar{F}_\infty(x)$  the limiting cdf of  $\bar{F}_n(x)$ , we obtain that  $\bar{F}_\infty(x)$  is derivable at point 0 with  $\bar{F}'_\infty(0) = -1$ .

Next, we show that the sequence  $\bar{F}_n$  remains in  $\mathcal{E}_1 \cap C^1(\mathbb{R}_+, [0, 1])$ . In other words, suppose  $\bar{F}_n \in C^1(\mathbb{R}_+, [0, 1])$  with  $\bar{F}_n$  convex and  $\bar{F}'_n(0) = -1$ ; let us show that this also holds for  $\bar{F}_{n+1} = T\bar{F}_n$ . By the dominated convergence theorem

$$\bar{F}'_{n+1}(x) = -\mathbb{E} \left[ \sum_{i=1}^M \Gamma_i C_i \prod_{j \neq i} \bar{F}_n(C_j x)^{\Gamma_j} \bar{F}_n(C_i x)^{\Gamma_i - 1} \left( -\bar{F}'_n(C_i x) \right) \right]$$

because the term in the bracket is bounded from above by  $\sum_{i=1}^M \Gamma_i C_i$ , which is integrable. Hence  $\bar{F}_n \in \mathcal{E}_1 \cap C^1(\mathbb{R}_+, [0, 1])$ . By passing to the limit, the convexity property is preserved.

Now, it remains to prove direct Riemann integrability of  $G_0$ . By lemma 9 (ii),  $e^{-z} G_0(z)$  is a decreasing function of  $z$  and following ([11], page 287) it suffices to show that  $G_0$  is Lebesgue integrable. Using  $u \leq e^{-(1-u)}$ , when  $u \in [0, 1]$ , we get

$$G_0(z) \leq e^z \left[ \mathbb{E} \phi \left( \sum_{i=1}^M \Gamma_i (1 - \bar{F}_0(C_i e^{-z})) \right) \right] \tag{16}$$

where  $\phi(x) := e^{-x} - 1 + x$ ,  $x \geq 0$ . We shall split  $\int_{\mathbb{R}} G_0(z) dz$  into two parts.

- For  $z < 0$ , we note that  $\phi$  is decreasing and  $\phi(x) < x$ . Therefore,

$$\int_{-\infty}^0 G_0(z) dz < \mathbb{E} \left( \sum_{i=1}^M \Gamma_i \right) \int_{-\infty}^0 e^z dz < \infty.$$

- For  $z > 0$ , using the inequality  $1 - e^{-x} \leq x$ ,  $x \geq 0$  and recalling that  $\bar{F}_0(x) = e^{-x}$ ,  $x > 0$ , we obtain  $G_0(z) \leq e^z \mathbb{E} \phi \left( \sum_{i=1}^M \Gamma_i C_i e^{-z} \right)$ . As a result,

$$\int_0^{\infty} G_0(z) dz \leq \int_0^{\infty} e^z \mathbb{E} \phi \left( \sum_{i=1}^M \Gamma_i C_i e^{-z} \right) dz.$$

Introducing the random variable  $S = \sum_{i=1}^M \Gamma_i C_i$  and letting  $u = e^{-z}$ , we get

$$\int_0^{\infty} G_0(z) dz \leq \int_0^1 \frac{1}{u^2} \mathbb{E} \phi(Su) du$$

which by theorem B of Bingham and Doney, 1974 ([5], page 718), is finite if and only if  $\mathbb{E} S \log_+ S < \infty$ . This condition has been imposed.  $\square$

## 2.2 Behavior of solutions in the special case

Let us distinguish the *lattice* and the *non-lattice* cases.

**Definition 2** We will speak of the *lattice case* when a common span of  $-\log C_i$ ,  $i \geq 1$  exists and is  $-\log c$ ,  $c > 0$ .

We consider the random walk previously introduced by  $S_n = \sum_{k=0}^n Z_k$  where  $(Z_k)_{k \geq 1}$  is i.i.d. random variables with the same distribution as  $Z$  given in equation (10), and  $S_0 = 0$ . It is easy to check that

**Proposition 3** The random variables  $-\log C_i$  have a common span  $-\log c$  if and only if the random walk  $S_n$  is arithmetic in the sense that the support of the distribution of  $S_n$  is  $\{-k \log c\}_{k \in \mathbb{Z}}$ .

Let us give the following definition.

**Definition 4** We note by  $\mathcal{S}_c$  the set of functions  $s(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_+$  satisfying:

→ In the *lattice case* with common span  $-\log c$ ,  $c > 0$  :  $s(z) := e^{-\nu(z)}$  for some right-continuous bounded periodic function  $\nu(\cdot)$  on  $\mathbb{R}$  with period  $-\log c$ , such that  $z - \nu(z)$  is non-decreasing function.

→ In the *non-lattice case*:  $s(z) := s > 0$ , the constant function for all  $z \in \mathbb{R}$ .

The following corollary is easily obtained from theorem 1.

**Corollary 5** In the special case, if  $\bar{F} \in \mathcal{E}_1$  is a solution to the functional equation (E), then  $\bar{F}_s(x) := \bar{F}(xs(-\log x))$ , where  $s \in \mathcal{S}_c$ , is also a solution to the same equation. The solution  $\bar{F}_s(x)$  now satisfies the property  $\frac{\bar{F}_s(x)}{xs(-\log x)} \rightarrow_{x \downarrow 0} 1$ .

This means that in the special case the solutions to (E) are determined modulo a scaling factor  $s$  which can be a log-periodic function in the lattice case.

## 2.3 Existence of a solution in the general case

Consider the functional equation (E). We recall that  $\tau(0) > 1$  and  $\tau$  is convex. Under a condition on  $\tau$ , we obtain the following existence theorem.

**Theorem 6** Suppose that there exists  $0 < \alpha < \infty$  such that  $\tau(\alpha) = 1$  and  $\tau'(\alpha) \leq 0$ . Two cases arise

(i) Case  $\tau'(\alpha) < 0$ : if  $\mathbb{E} \left[ \sum_{i=1}^M \Gamma_i C_i^\alpha \log_+ \left( \sum_{i=1}^M \Gamma_i C_i^\alpha \right) \right] < \infty$ , there exists a non trivial cdf  $\bar{F}$  solution to (E).

(ii) Case  $\tau'(\alpha) = 0$ : if  $\mathbb{E} \left[ \sum_{i=1}^M \Gamma_i C_i^\beta \log_+ \left( \sum_{i=1}^M \Gamma_i C_i^\beta \right) \right] < \infty$  for all  $\beta < \alpha$ , there exists a non trivial cdf  $\bar{F}$  solution to (E).

**Proof:** (i) Suppose  $\tau'(\alpha) < 0$ . Consider the cdf  $\bar{F}_\alpha$ , as a solution to the functional equation

$$(E_\alpha) : \bar{F}_\alpha(x) = \mathbb{E} \left[ \prod_{i=1}^M \bar{F}_\alpha(C_i^\alpha x)^{\Gamma_i} \right] \tag{17}$$

The associated structure function is  $\tau_\alpha(q) = \tau(\alpha q)$  with  $\tau_\alpha(1) = 1$ ,  $\tau'_\alpha(1) = \alpha\tau'(\alpha) < 0$ . The existence of  $\bar{F}_\alpha$  in  $\mathcal{E}_1$  is given by the theorem 1, in the special case, substituting  $C_i^\alpha$  to  $C_i$ . Finally, the cdf  $\bar{F}(x) = \bar{F}_\alpha(x^\alpha)$  solves the functional equation (E).

(ii) Suppose  $\tau'(\alpha) = 0$ . Let  $0 < \beta < \alpha$ . Consider the random variables  $C_i(\beta) = C_i^\beta \tau(\beta)^{-1}$  and introduce the functional equation

$$(E_\beta) : \bar{F}_\beta(x) = \mathbb{E} \left[ \prod_{i=1}^M \bar{F}_\beta(C_i(\beta)x)^{\Gamma_i} \right]. \tag{18}$$

Its associated structure function is  $\tau_\beta(q) = \tau(\beta q) / \tau(\beta)^q$ . We have  $\tau_\beta(1) = 1$ . As  $\tau(0) > 1$  and  $\tau$  is convex,  $\tau(\beta) > 1$  and  $\tau'(\beta) < 0$ , for each  $\beta < \alpha$ . We have  $\tau'_\beta(1) = \frac{\beta\tau'(\beta) - \tau(\beta)\log\tau(\beta)}{\tau(\beta)} < 0$ . Consider now a sequence  $\beta_n$  with  $0 < \beta_n < \alpha$ , and  $\beta_n \rightarrow \alpha$  as  $n \rightarrow \infty$ . From theorem 1 and corollary 5,  $(E_{\beta_n})$  has a solution, say  $\bar{F}_{\beta_n}$ , in  $\mathcal{E}$  satisfying  $\bar{F}_{\beta_n}(1) = 1/2$ . The sequence  $\bar{F}_{\beta_n} \in \mathcal{E}$  is an equi-continuous sequence of functions  $[0, \infty) \rightarrow [0, 1]$ , because, for all  $x > 0$ ,  $\bar{F}_{\beta_n}(x)/x$  is non-increasing. By an extended version of Arzelà's theorem [6], one can extract a convergent subsequence. By the same transformation as in (i), the cdf  $\bar{F}(x) = \bar{F}_\alpha(x^\alpha)$  also solves the functional equation (E) in this case.  $\square$

**Remark 1** *From the above proof, we note that when  $\alpha \leq 1$  the constructed solution is convex.*

### 3 CHARACTERIZATION of SOLUTIONS

**The space of solutions:** We will look for a solution of equation (E) in the space  $\mathcal{F}$ . We recall that  $F = 1 - \bar{F}$  and

$$\mathcal{F} = \{ \bar{F} \in C^0(\mathbb{R}_+, [0, 1]) : \exists \lambda > 0, c > 0, \text{ satisfying } \frac{F(ax)}{F(x)} \leq ca^\lambda, \forall a > 1, x > 0 \}.$$

We note that this space contains the space of all absolutely continuous distributions with density  $f$  such that  $\frac{xf}{F}$  is bounded which itself contains  $\mathcal{E}$ . For the first inclusion, there exists  $\lambda > 0$  such that  $\frac{xf(x)}{F(x)} < \lambda$ . Then for  $a > 1$  and  $x > 0$ , integrating on the interval  $[x, ax]$ , we get  $\frac{F(ax)}{F(x)} \leq a^\lambda$ . For the second inclusion, as  $\bar{F}$  is convex  $\frac{1-\bar{F}(x)}{x}$  is decreasing. By differentiating, we obtain  $\frac{xf(x)}{F(x)} < 1$ . Moreover, We have  $\mathcal{E} \subset \mathcal{F}$ .

As recalled in the introduction, Barral (in his paper [2]) studied a similar equation and found out a space of continuous functions possessing some key properties. We go further along this way, defining a space  $\mathcal{F}$  containing the constructed solutions given by theorem 1 and theorem 6.

#### 3.1 Behavior of the solutions in $\mathcal{F}$

We now come to the behavior at the origin of the solutions to (E) belonging to  $\mathcal{F}$ . In [11], Durrett and Liggett characterize the behavior at the origin of the solutions of the functional equation for Laplace transforms corresponding to the identity in law given in (3). This is found in theorem 2.18 of ([11] pages 288-291) and is based

on several technical results, namely lemma 2.3, corollary 2.17 and theorem 2.12. Replacing them respectively by our lemma 8, corollary 13 and theorem 12, we can adapt their proof and obtain the following theorem. For the reader convenience the statement and proof of the quoted technical results are postponed to a subsequent sub-section.

**Theorem 7** *Suppose the following condition  $(H_\delta)$  holds,*

$$\exists \delta > 0 : \forall q \in \mathbb{R}_+, \sum_{i=1}^M \Gamma_i C_i^q \in \mathbb{L}_{1+\delta}, \quad (H_\delta).$$

*Suppose also that there is an  $\alpha > 0$  such that  $\tau(\alpha) = 1$ ,  $\tau'(\alpha) \leq 0$ . Then, if  $\bar{F}$  is solution to (E) and if  $\bar{F} \in \mathcal{F}$ , there exists  $s(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_+$ , continuous periodic with period  $-\log c$ ,  $c > 0$ , in the lattice case and constant in the non-lattice case, such that  $x \rightarrow x^\alpha s(-\log x)$  is increasing, with*

$$(i) \quad \frac{F(x)}{x^\alpha s(-\log x)} \rightarrow_{x \downarrow 0} 1, \text{ if } \tau'(\alpha) < 0$$

and

$$(ii) \quad \frac{F(x)}{x^\alpha |\log x| s(-\log x)} \rightarrow_{x \downarrow 0} 1, \text{ if } \tau'(\alpha) = 0.$$

### 3.2 Technical results

In order to adapt the techniques developed in Durrett and Liggett [11] and Liu [17] we start by giving several technical lemmas which are essential to obtain theorem 12 and corollary 13. Finally, we derive our main theorem 7. We recall that  $\tau(q) < \infty$  whenever  $q \geq 0$ . Let us define a random variable  $Z_\alpha$ ,  $\alpha > 0$ , by the equality

$$\mathbb{E}\Psi(Z_\alpha) = \tau(\alpha)^{-1} \mathbb{E} \left( \sum_{i=1}^M \Gamma_i C_i^\alpha \Psi(-\log C_i) \right), \quad (19)$$

for all bounded measurable function  $\Psi$ .

For an arbitrary cdf  $\bar{F}$ , we define the functions  $D_\alpha$  and  $G_\alpha$  by

$$D_\alpha(z) = \frac{1 - \bar{F}(e^{-z})}{e^{-\alpha z}} \quad (20)$$

and

$$G_\alpha(z) = e^{\alpha z} \mathbb{E} \left[ \left( \prod_{i=1}^M \bar{F}(e^{-z} C_i)^{\Gamma_i} \right) - 1 + \sum_{i=1}^M \Gamma_i (1 - \bar{F}(e^{-z} C_i)) \right]. \quad (21)$$

We let  $\bar{F}_1$  an arbitrary cdf and  $\bar{F}_2 = T\bar{F}_1$ . We denote by  $D_{\alpha,i}$  and  $G_{\alpha,i}$  the corresponding functions associated to  $\bar{F}_i$ ,  $i = 1, 2$ . We first give a series of lemmas

**Lemma 8** *We have*

$$D_{\alpha,2}(z) = \tau(\alpha) \mathbb{E} D_{\alpha,1}(z + Z_\alpha) - G_{\alpha,1}(z).$$

**Proof.** We have

$$D_{\alpha,2}(z) = e^{\alpha z} (1 - \bar{F}_2(e^{-z})) = e^{\alpha z} \mathbb{E} \left[ \sum_{i=1}^M \Gamma_i (1 - \bar{F}_1(C_i e^{-z})) \right] - G_{\alpha,1}(z) =$$

$$\mathbb{E} \left[ \sum_{i=1}^M \Gamma_i C_i^\alpha D_{\alpha,1}(z - \log C_i) \right] - G_{\alpha,1}(z) = \tau(\alpha) \mathbb{E} D_{\alpha,1}(z + Z_\alpha) - G_{\alpha,1}(z). \quad \square$$

**Lemma 9** *We have*

- (i)  $G_\alpha(z) \geq 0$ .
- (ii)  $e^{-\alpha z} G_\alpha(z)$  is a decreasing function of  $z$ .
- (iii) If  $\bar{F}_2 \geq \bar{F}_1$  then for all  $z$ :  $G_{\alpha,2}(z) \leq G_{\alpha,1}(z)$ .

**Proof:** From the inequality

$$\left( \prod_{i=1}^M u_i^{\Gamma_i} \right) - 1 + \sum_{i=1}^M \Gamma_i (1 - u_i) \geq \left( \prod_{i=1}^M v_i^{\Gamma_i} \right) - 1 + \sum_{i=1}^M \Gamma_i (1 - v_i), \quad (22)$$

$0 \leq u_i \leq v_i \leq 1$ , we deduce the monotone decreasing feature of the function  $e^{-\alpha z} G_\alpha(z)$ . Let  $\bar{F}_1$  and  $\bar{F}_2$  be two cdfs with  $\bar{F}_1 \leq \bar{F}_2$ . Replacing  $\bar{F}$  by  $\bar{F}_1$ , respectively by  $\bar{F}_2$ , in equation (21), we obtain their associated functions  $G_{\alpha,1}$  and  $G_{\alpha,2}$ . From the above inequality we have  $G_{\alpha,2} \leq G_{\alpha,1}$ . Finally, inequality (22) can be checked by remarking

$$\partial_{u_j} \left[ \left( \prod_{i=1}^M u_i^{\Gamma_i} \right) - 1 + \sum_{i=1}^M \Gamma_i (1 - u_i) \right] = \Gamma_j \left( \left( \prod_{i \neq j} u_i^{\Gamma_i} u_j^{\Gamma_j - 1} \right) - 1 \right) \leq 0. \quad \square$$

**Lemma 10** *With  $\phi(u) := e^{-u} - 1 + u$  and a cdf  $\bar{F} \in \mathcal{F}$ , we have*

$$(i) \quad G_\alpha(z) \leq e^{\alpha z} \mathbb{E} [\phi(W D_\alpha(z) e^{-\alpha z})]$$

where  $W := \sum_{i=1}^M \Gamma_i \max(c C_i^\lambda, 1)$ .

$$(ii) \quad \lim_{z \uparrow \infty} \frac{G_\alpha(z)}{D_\alpha(z)} = 0.$$

**Proof:** (i) Using  $u \leq e^{-(1-u)}$  if  $0 < u < 1$ , we get

$$G_\alpha(z) \leq e^{\alpha z} \mathbb{E} \left[ \left( \prod_{i=1}^M \left[ e^{-(1-\bar{F}(C_i e^{-z}))} \right]^{\Gamma_i} \right) - 1 + \sum_{i=1}^M \Gamma_i (1 - \bar{F}(C_i e^{-z})) \right)$$

$$\leq e^{\alpha z} \mathbb{E} \left[ e^{-\sum_{i=1}^M \Gamma_i (1 - \bar{F}(C_i e^{-z}))} - 1 + \sum_{i=1}^M \Gamma_i (1 - \bar{F}(C_i e^{-z})) \right]$$

$$\leq e^{\alpha z} \mathbb{E} \left[ \phi \left( \sum_{i=1}^M \Gamma_i (1 - \bar{F}(C_i e^{-z})) \right) \right].$$

Now, two cases arise

- if  $C_i \leq 1$ , then  $C_i e^{-z} \leq e^{-z}$  and  $1 - \bar{F}(C_i e^{-z}) \leq 1 - \bar{F}(e^{-z})$ .
- if  $C_i \geq 1$ , then

$$F(C_i e^{-z}) \leq c C_i^\lambda F(e^{-z}) \text{ and so, } 1 - \bar{F}(C_i e^{-z}) \leq c C_i^\lambda (1 - \bar{F}(e^{-z})).$$

As a result,  $1 - \bar{F}(C_i e^{-z}) \leq (c C_i^\lambda \vee 1) (1 - \bar{F}(e^{-z}))$  and function  $u \rightarrow \phi(u)$  being monotone increasing

$$\mathbb{E} \left[ \phi \left( \sum_{i=1}^M \Gamma_i (1 - \bar{F}(C_i e^{-z})) \right) \right] \leq \mathbb{E} \left[ \phi \left( \sum_{i=1}^M \Gamma_i (c C_i^\lambda \vee 1) (1 - \bar{F}(e^{-z})) \right) \right].$$

Finally,  $G_\alpha(z) \leq e^{\alpha z} \mathbb{E}[\phi(W D_\alpha(z) e^{-\alpha z})]$ .

(ii) We first note that  $e^{-\alpha z} D_\alpha(z) \rightarrow_{z \uparrow \infty} 0$ . To prove (ii), we need to check  $\lim_{t \downarrow 0} \mathbb{E} \{ \phi(Wt) / t \} = 0$ . Now,  $\phi(u) / u$  is bounded and so  $|\phi(Wt) / t| < K \cdot W$ , for a suitable constant  $K > 0$ . Further,  $W$  is integrable since  $W \leq \sum_{i=1}^M \Gamma_i + \sum_{i=1}^M c \Gamma_i C_i^\lambda$  and  $\mathbb{E}W \leq \tau(0) + c \tau(\lambda) < \infty$ .  $\square$

**Lemma 11** *Let  $\alpha \in \mathbb{R}$  and  $Z_\alpha$  be defined by (19). Let  $g$  be a non-negative function on  $\mathbb{R}$ . If  $g(y) = \tau(\alpha) \mathbb{E}g(y + Z_\alpha)$ , then*

$$g(y) = \sum_{\beta \in \mathcal{S}} \zeta_\beta(y) \exp(-(\beta - \alpha)y)$$

where  $\zeta_\beta(y) \geq 0$  with:  $\zeta_\beta(x + y) = \zeta_\beta(y)$  for all  $x \in \text{Supp}(Z_\alpha)$  and  $\mathcal{S} := \{\beta : \tau(\beta) = 1\}$ , with  $|\mathcal{S}| \in \{0, 1, 2\}$ . If  $|\mathcal{S}| = 0$ , we use the convention  $g = 0$ .

**Proof:** Following Lau-Rao-Shanbhag theorem [21]: if  $|\mathcal{S}| = 0$ , then  $g = 0$ ; if  $|\mathcal{S}| \in \{1, 2\}$ , we get  $g(y) = \sum_{\beta \in \mathcal{S}} \zeta_\beta(y) \exp(-\eta_\beta y)$  where  $\eta_\beta$  satisfies

$$\tau(\alpha) \mathbb{E}(\exp(-\eta_\beta Z_\alpha)) = 1.$$

Clearly  $\mathbb{E} \left( \sum_{i=1}^M \Gamma_i C_i^\alpha \exp(\eta_\beta \log C_i) \right) = \mathbb{E} \left( \sum_{i=1}^M \Gamma_i C_i^{\alpha + \eta_\beta} \right) = 1$ , leading to  $\beta = \alpha + \eta_\beta$  where  $\beta \in \mathcal{S}$ .  $\square$

**Remark 2** *In the lattice case,  $\zeta_\beta$  are periodic functions and in the non-lattice case,  $\zeta_\beta$  are constants. Under the additional hypothesis  $g(0) = 1$ , when  $|\mathcal{S}| = 2$ , necessarily  $\zeta_{\beta_2}(0) = 1 - \zeta_{\beta_1}(0)$  where  $(\beta_1, \beta_2)$  are the two solutions to  $\tau(\beta) = 1$ .*

**Theorem 12** *Assume  $\bar{F} \in \mathcal{F}$  and that  $\bar{F} \in \mathcal{F}$  is a solution of (E). Then*

- (i) *there exists  $\alpha > 0$ :  $\tau(\alpha) = 1$ .*
- (ii) *Let  $\alpha > 0$  satisfy  $\tau(\alpha) = 1$  and  $\tau'(\alpha) \leq 0$ , then*

$$\limsup_{x \uparrow \infty} \frac{D_\alpha(x + y)}{D_\alpha(x)} \leq 1 \text{ if } \tau'(\alpha) < 0$$

and

$$\lim_{x \uparrow \infty} \frac{D_\alpha(x + y)}{D_\alpha(x)} = 1 \text{ if } \tau'(\alpha) = 0$$

where  $y$  is any non negative multiple of  $-\log c$  in the lattice case and  $y \in \mathbb{R}_+$  in the non-lattice case.

**Proof:** Following [11], let  $\alpha > 0$  and  $h_x(y) = \frac{D_\alpha(x+y)}{D_\alpha(x)}$ . We have

$$h_x(y) = \tau(\alpha) \mathbb{E} h_x(y + Z_\alpha) - \frac{G_\alpha(x+y)}{D_\alpha(x+y)} h_x(y).$$

Note that  $\bar{F}$  is not necessarily convex as in the Laplace transform context. Nevertheless we can adapt the proof of theorem 2.12 in [11] to cdfs  $\bar{F} \in \mathcal{F}$ . When  $\bar{F} \in \mathcal{F}$ , there exists  $\lambda > 0$  and  $c > 0$  such that

$$h_x(y) \leq e^{\alpha y} \mathbf{1}_{y \geq 0} + ce^{(\alpha-\lambda)y} \mathbf{1}_{y < 0}.$$

As a result, the set  $\{h_x(\cdot), x \in \mathbb{R}\}$  is uniformly bounded and equi-continuous on the bounded subsets of  $\mathbb{R}$ . We can therefore extract a subsequence  $h_{x_n}$  converging uniformly on the bounded subsets of  $\mathbb{R}$  to some function  $h$ . The sequence  $(x_n)_{n \in \mathbb{N}}$  converges to infinity when  $n$  tends to  $\infty$ . From the inequality above  $h_{x_n}(y + Z_\alpha)$  is dominated by  $e^{\alpha(y+Z_\alpha)} \mathbf{1}_{(y+Z_\alpha) \geq 0} + ce^{(\alpha-\lambda)(y+Z_\alpha)} \mathbf{1}_{(y+Z_\alpha) < 0}$  and

$$\mathbb{E} \left[ e^{\alpha(y+Z_\alpha)} \mathbf{1}_{(y+Z_\alpha) \geq 0} + ce^{(\alpha-\lambda)(y+Z_\alpha)} \mathbf{1}_{(y+Z_\alpha) < 0} \right] \leq e^{\alpha y} \frac{\tau(0)}{\tau(\alpha)} + ce^{(\alpha-\lambda)y} \frac{\tau(\lambda)}{\tau(\alpha)} < \infty.$$

By dominated convergence theorem and lemma 10 (ii), we obtain

$$h(y) = \tau(\alpha) \mathbb{E} h(y + Z_\alpha).$$

Consider an  $\alpha > 0$  satisfying  $\tau(\alpha) > 1$ . From lemma 11, there exists  $\beta \in \mathbb{R}$  such that  $\tau(\beta) = 1$ ; equivalently,  $\beta$  satisfies  $\mathbb{E} \exp\{-(\beta - \alpha)Z_\alpha\} = 1/\tau(\alpha)$  leading to  $\beta > \alpha > 0$ . This proves (i). In the non-lattice case, assuming  $|\mathcal{S}| = 2$  with  $\mathcal{S} := \{\beta : \tau(\beta) = 1\}$ , with  $\zeta_\beta > 0$

$$h(y) = \sum_{\beta \in \mathcal{S}} \zeta_\beta \exp(-(\beta - \alpha)y).$$

Assuming  $\beta_1 < \beta_2$ , taking  $\alpha = \beta_1$  and recalling  $h(0) = 1$ , we have for  $y > 0$

$$h(y) = \zeta_{\beta_1} + (1 - \zeta_{\beta_1}) \exp(-(\beta_2 - \alpha)y) \leq 1.$$

In the case  $\beta_1 = \beta_2$ , with  $\tau'(\beta_1) = 0$ ,  $h(y) = \zeta_{\beta_1} = 1$ . In the lattice case, for  $y$  a multiple of  $-\log c$ ,

$$h(-k \log c) = \sum_{\beta \in \mathcal{S}} \zeta_\beta(0) \exp(-(\beta - \alpha)y)$$

and using similar arguments,  $h(y) \leq 1$  if  $\tau'(\beta_1) < 0$ ,  $h(y) = 1$  if  $\tau'(\beta_1) = 0$ .

Fix  $y > 0$ . Let  $(x_n)_{n \in \mathbb{N}}$  a sequence converging to infinity when  $n$  tends to  $\infty$  such that

$$\limsup_{x \uparrow \infty} \frac{D_\alpha(x+y)}{D_\alpha(x)} = \lim_{n \uparrow \infty} h_{x_n}(y).$$

Extracting a convergent subsequence from  $\{h_{x_n}\}_{n \geq 1}$ , converging uniformly to some  $h$  on bounded subsets of  $\mathbb{R}$ , this function  $h$  fulfills the above conditions and similar arguments apply, completing the proof.  $\square$

**Corollary 13** Let  $\bar{F} \in \mathcal{F}$  and assume there is an  $\alpha > 0$  satisfying  $\tau(\alpha) = 1$  and  $\tau'(\alpha) \leq 0$ . Then, under the condition  $(H_\delta)$  that

$$\exists \delta > 0 : \forall q \in \mathbb{R}_+, \sum_{i=1}^M \Gamma_i C_i^q \in \mathbb{L}_{1+\delta},$$

$G_\alpha(x)$  is direct Riemann integrable on  $\mathbb{R}$ .

**Proof:** We first note, following [11], that if  $G_\alpha(x)$  is integrable, and if, as follows from lemma 9,  $e^{-\alpha x} G_\alpha(x)$  is a decreasing function of  $x$ , then  $G_\alpha(x)$  is direct Riemann integrable. Hence it suffices to show that  $G_\alpha$  is integrable. By lemma 10, for  $\bar{F} \in \mathcal{F}$  we have  $0 \leq G_\alpha(z) \leq e^{\alpha z} \mathbb{E}[\phi(WD_\alpha(z)e^{-\alpha z})]$  where  $\phi(u) = e^{-u} - 1 + u$  and  $W = \sum_{i=1}^M \Gamma_i \max(cC_i^\lambda, 1)$ .

At  $z = -\infty$  : from monotonicity of  $\phi$  and recalling  $\phi(x) \leq x$ ,

$$\phi(WD_\alpha(z)e^{-\alpha z}) \leq \phi(W) \leq W \leq \sum_{i=1}^M \Gamma_i + \sum_{i=1}^M c\Gamma_i C_i^\lambda.$$

Hence,  $G_\alpha(z) \leq e^{\alpha z} (\tau(0) + c\tau(\lambda))$ , which is integrable at  $z = -\infty$ .

At  $z = +\infty$  : as  $e^{-\alpha z} D_\alpha(z) \rightarrow_{z \uparrow \infty} 0$ , there exists  $z_0 : \forall z \geq z_0, D_\alpha(z) \leq e^{\beta z}$  for some  $0 < \beta < \alpha$ . By lemma 10,

$$\begin{aligned} \int_{z_0}^{+\infty} G_\alpha(x) dx &\leq \int_{z_0}^{+\infty} e^{\alpha x} \mathbb{E} \phi(WD_\alpha(x)e^{-\alpha x}) dx \\ &\leq \int_{z_0}^{+\infty} e^{\alpha x} \mathbb{E} \phi(We^{(\beta-\alpha)x}) dx. \end{aligned}$$

Passing to the variable  $u = e^{(\beta-\alpha)x}$ , letting  $u_0 = e^{(\beta-\alpha)z_0}$ , we obtain

$$\int_{z_0}^{+\infty} G_\alpha(x) dx \leq \frac{1}{\alpha - \beta} \int_0^{u_0} \frac{\mathbb{E} \phi(Wu)}{u^{2+\beta/(\alpha-\beta)}} du = \frac{1}{\alpha - \beta} \int_0^{u_0} \frac{\varphi_W(u) - 1 + u\mathbb{E}W}{u^{2+\beta/(\alpha-\beta)}} du$$

where  $\varphi_W(u)$  is the Laplace-Stieltjes transform of  $W$ . By theorem *B* of Bingham and Doney, 1974 ([5], page 718), this integral is finite if and only if  $\mathbb{E} \phi(W^{1+\beta/(\alpha-\beta)}) < \infty$ . Assuming  $0 < \beta/(\alpha - \beta) < 1$ , this condition holds as soon as  $0 < \frac{\beta}{\alpha - \beta} < 1 \wedge \delta$  or when  $0 < \beta < \frac{\alpha\delta}{\delta+1} \wedge \frac{\alpha}{2}$ .  $\square$

### 3.3 Characterization of the constructed solutions

We now give a more precise statement on the constructed solutions of equation (E) as given by theorem 1 and theorem 6. Using theorem 7 we are able to describe more precisely their behavior at the origin. The definition 4 of the space  $\mathcal{S}_c$  is adapted to the special case. We introduce a more general space  $\mathcal{S}_{\alpha,c}$  which will be used in the general case and for which  $\mathcal{S}_{1,c} := \mathcal{S}_c$ .

**Definition 14** Define the space  $\mathcal{S}_{\alpha,c}$  as the set of functions  $s(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_+$  satisfying

→ In the lattice case with common span  $-\log c$ ,  $c > 0$  :  $s(z) := e^{-\alpha\nu(z)}$  for some right-continuous bounded periodic function  $\nu(\cdot)$  on  $\mathbb{R}$  with period  $-\log c$ , such that  $z - \nu(z)$  is non-decreasing function.

→ In the non-lattice case:  $s(z) := s > 0$ , the constant function for all  $z \in \mathbb{R}$ .

**Theorem 15** *Suppose that there exists  $0 < \alpha < \infty$  such that  $\tau(\alpha) = 1$  and  $\tau'(\alpha) \leq 0$ . Two cases arise*

(i) *If  $\tau'(\alpha) < 0$  and  $\mathbb{E} \left[ \sum_{i=1}^M \Gamma_i C_i^\alpha \log_+ \left( \sum_{i=1}^M \Gamma_i C_i^\alpha \right) \right] < \infty$ , then for each  $s \in \mathcal{S}_{\alpha,c}$  there exists  $\bar{F}$ , solution of (E), satisfying*

$$\frac{F(x)}{x^\alpha s(-\log x)} \xrightarrow{x \downarrow 0} 1.$$

(ii) *If  $\tau'(\alpha) = 0$  and  $\mathbb{E} \left[ \left( \sum_{i=1}^M \Gamma_i C_i^\beta \right)^{1+\delta} \right] < \infty$  for all  $0 < \beta < \alpha$ , then, for each  $s \in \mathcal{S}_{\alpha,c}$ , there exists  $\bar{F}$ , solution of (E), satisfying*

$$\frac{F(x)}{x^\alpha |\log x| s(-\log x)} \xrightarrow{x \downarrow 0} 1.$$

**Proof:** Consider the cdf  $\bar{F}_\alpha$ , as a solution to the functional equation

$$(E_\alpha) : \bar{F}_\alpha(x) = \mathbb{E} \left[ \prod_{i=1}^M \bar{F}_\alpha(C_i^\alpha x)^{\Gamma_i} \right]. \quad (23)$$

(i) If  $\tau'(\alpha) < 0$ , reconsidering the proof of theorem 6 concerning this case, there exists  $\bar{F}_\alpha$  in  $\mathcal{E}_1$ , solution to  $(E_\alpha)$ . In particular we have

$$\lim_{x \downarrow 0} \frac{F_\alpha(x)}{x} = 1.$$

Now for each  $s \in \mathcal{S}_{\alpha,c}$  the cdf  $\bar{F}(x) = \bar{F}_\alpha(x^\alpha s(-\log x))$  solves the functional equation (E) with the claimed behavior at 0.

(ii) If  $\tau'(\alpha) = 0$ , reconsidering the proof of theorem 6 concerning this case, there exists  $\bar{F}_\alpha$  solution to  $(E_\alpha)$ . By construction,  $\bar{F}_\alpha$  is convex, and a fortiori  $\bar{F}_\alpha \in \mathcal{F}$ . Hence we can deduce from theorem 7 that

$$\lim_{x \downarrow 0} \frac{F_\alpha(x)}{x |\log x| s_\alpha(-\log x)} = 1,$$

where  $s_\alpha(\cdot)$  is continuous and log-periodic. Following the arguments of ([11] page 290), using the convexity of  $\bar{F}_\alpha$  and the fact that the structure function  $\tau_\alpha$  associated to  $(E_\alpha)$  satisfies  $\tau_\alpha(q) = \tau(\alpha q) = 1$  and  $\tau'_\alpha(q) = 0$  at point  $q = 1$ , the function  $s_\alpha(\cdot)$  is monotone and periodic and so is a constant, say  $\kappa > 0$ .

Now for each function  $\tilde{s}(\cdot) \in \mathcal{S}_{\alpha,c}$ , the cdf  $\bar{F}(x) := \bar{F}_\alpha(x^\alpha \tilde{s}(-\log x))$  solves the functional equation (E) and:

$$\lim_{x \downarrow 0} \frac{F(x)}{x^\alpha |\log x| \tilde{s}(-\log x)} = \lim_{x \downarrow 0} \frac{F_\alpha(x^\alpha \tilde{s}(-\log x))}{x^\alpha |\log(x^\alpha \tilde{s}(-\log x))| \tilde{s}(-\log x)} \cdot \frac{|\log(x^\alpha \tilde{s}(-\log x))|}{|\log x|}.$$

Using the behavior at 0 of  $F_\alpha(x)$ , we obtain

$$\lim_{x \downarrow 0} \frac{F(x)}{x^\alpha |\log x| \tilde{s}(-\log x)} = \kappa \cdot \lim_{x \downarrow 0} \frac{|\log(x^\alpha \tilde{s}(-\log x))|}{|\log x|} = \kappa \alpha.$$

So  $F(x)$  has the claimed behavior at 0 with  $s(\cdot) = \kappa \alpha \cdot \tilde{s}(\cdot) \in \mathcal{S}_{\alpha,c}$ .  $\square$

**Remark 3** *From the proof, we note that for  $\alpha < 1$  and  $s(\cdot) = s > 0$  constant, the constructed solution is convex.*

## 4 UNIQUENESS of SOLUTIONS

In this section we discuss the uniqueness of solutions of equation (E).

As it was noticed in the introduction, when  $\Gamma_i$  are integral-valued random variables, a fortiori when  $\Gamma_i = 1$ , equation (E) yields the following equation in distribution

$$X \stackrel{d}{=} \min_{1 \leq i \leq N} A_i X_i \tag{24}$$

where  $A_i > 0$ . We can adapt the proof of the uniqueness theorem given in Liu ([18] page 105) to our ccdf context. We obtain the following result.

**Theorem 16** *Assume  $\Gamma_i$  are integral-valued random variables. Assume there is an  $\alpha > 0$  satisfying  $\tau(\alpha) = 1$  and  $\tau'(\alpha) \leq 0$ . Under condition  $(H_\delta)$  of theorem 7, the solution to (E) in the space  $\mathcal{F}$  is unique. By uniqueness, it is meant that: if  $\bar{F}_1$  and  $\bar{F}_2$  are solutions whose behaviors in a neighborhood of zero are both given by the same pair  $(\alpha, s(\cdot))$  in (i) and (ii) of theorem 7, then  $\bar{F}_1 = \bar{F}_2$ .*

**Sketch of proof:** For all sequences  $\sigma \in \cup_{i \geq 1} \mathbb{N}^*$  of positive integers, with  $|\sigma|$  the length of  $\sigma$ , let  $(A_{\sigma,1}, A_{\sigma,2}, \dots)$  be i.i.d. copies of  $(A_1, A_2, \dots)$ . For a ccdf  $\bar{F}$ ,  $T^n \bar{F}$  is the ccdf of  $\min_{|\sigma|=n} l_\sigma X_\sigma$ , where  $l_\sigma := A_{\sigma_1} A_{\sigma_1 \sigma_2} \dots A_{\sigma_1 \sigma_2 \dots \sigma_n}$  if  $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$ ,  $\{X_\sigma : |\sigma| = n\}$  are i.i.d. copies with ccdf  $\bar{F}$ , independent of  $\{A_\sigma : |\sigma| \leq n\}$ . The results of Lemmas 7.1 and 7.2 of ([18] page 104) still hold because we have the same tree structure. We are in the position to obtain a version of lemma 7.3 of ([18] page 104) while considering the quantity  $T^n \bar{F}(x) = \mathbb{E} \prod_{|\sigma|=n} \bar{F}(x l_\sigma)$  replacing Laplace-Stieltjes transforms by ccdfs. Under  $(H_\delta)$ , let  $\bar{F}_1$  and  $\bar{F}_2$  be two solutions in  $\mathcal{F}$  of (E) whose behaviors in a neighborhood of zero are both given by the same pair  $(\alpha, s(\cdot))$  in (i) and (ii) of theorem 7. Then,  $1 - \bar{F}_1 \sim 1 - \bar{F}_2$  in a neighborhood of 0 and following the steps of theorem 7.1. in ([18] page 105),  $\lim_{n \uparrow \infty} T^n \bar{F}_1 = \bar{F}_2$ .  $\square$

In the general case, when  $\Gamma_i \geq 1$  but not necessarily integral-valued, we obtain the uniqueness in the only case when we suppose that there is an  $\alpha$  satisfying  $\tau(\alpha) = 1$  and  $\tau'(\alpha) < 0$ .

**Theorem 17** *Assume there is an  $\alpha > 0$  satisfying  $\tau(\alpha) = 1$  and  $\tau'(\alpha) < 0$ . Under condition  $(H_\delta)$ , the solution to (E) in the space  $\mathcal{F}$  is unique: if  $\bar{F}_1$  and  $\bar{F}_2$  are solutions whose behaviors in a neighborhood of zero are both given by the same pair  $(\alpha, s(\cdot))$  in (i) of theorem 7, then  $\bar{F}_1 = \bar{F}_2$ .*

**Proof:** Let us now show that if there are two solutions in  $\mathcal{F}$ , with similar behavior close to 0, then they coincide. Let  $\bar{F}_1$  and  $\bar{F}_2$  be two distinct ccdfs in  $\mathcal{F}$  which are solutions to (E), with  $\bar{F}_1$  and  $\bar{F}_2$  both equivalent close to 0 to  $x^\alpha s(-\log x)$  with  $s(\cdot)$  continuous and periodic from theorem 7. Consider  $d(\bar{F}_1, \bar{F}_2) := \sup_{x > 0} \left| \frac{\bar{F}_1(x) - \bar{F}_2(x)}{x^\alpha s(-\log x)} \right|$ . As  $\bar{F}_1$  and  $\bar{F}_2$  are solutions in  $\mathcal{F}$ , the function  $x \rightarrow \left| \frac{\bar{F}_1(x) - \bar{F}_2(x)}{x^\alpha s(-\log x)} \right|$  is continuous on  $[0, \infty)$ , vanishes at  $\infty$ , so its supremum is attained at some point  $x_0$  in  $(0, \infty)$ . Clearly,  $d(\bar{F}_1, \bar{F}_2) = \left| \frac{T\bar{F}_1(x_0) - T\bar{F}_2(x_0)}{x_0^\alpha s(-\log x_0)} \right|$ . Now, from Jensen's inequality

$$\left| T\bar{F}_1(x_0) - T\bar{F}_2(x_0) \right| \leq \mathbb{E} \left| \prod_{i=1}^M \bar{F}_1(C_i x_0)^{\Gamma_i} - \prod_{i=1}^M \bar{F}_2(C_i x_0)^{\Gamma_i} \right|$$

Using the inequality  $\left| \prod_{i=1}^M a_i - \prod_{i=1}^M b_i \right| \leq \sum_{i=1}^M |a_i - b_i|$  for  $a_i \in [0, 1]$  and  $b_i \in [0, 1]$ ,  $i = 1, \dots, M$ , we obtain

$$|T\bar{F}_1(x_0) - T\bar{F}_2(x_0)| \leq \mathbb{E} \left[ \sum_{i=1}^M \left| \bar{F}_1(C_i x_0)^{\Gamma_i} - \bar{F}_2(C_i x_0)^{\Gamma_i} \right| \right]. \quad (25)$$

Let  $A := \{ \bar{F}_1(C_i x_0) \neq \bar{F}_2(C_i x_0), \Gamma_i > 1 \text{ for some } i \in \{1, \dots, M\} \}$ .

If  $\mathbb{P}(A) > 0$ , using the Hölderian character of  $u \rightarrow u^\gamma$ ,  $\gamma > 1$ ,  $u \in [0, 1]$ , that is  $|x^\gamma - y^\gamma| < \gamma|x - y|$ , for  $x, y \in [0, 1]$  and  $x \neq y$ , we get from (25)

$$|T\bar{F}_1(x_0) - T\bar{F}_2(x_0)| < \mathbb{E} \left[ \sum_{i=1}^M \Gamma_i \left| \bar{F}_1(C_i x_0) - \bar{F}_2(C_i x_0) \right| \right].$$

Now,

$$\begin{aligned} d(\bar{F}_1, \bar{F}_2) &= \left| \frac{T\bar{F}_1(x_0) - T\bar{F}_2(x_0)}{x_0^\alpha s(-\log x_0)} \right| \\ &< \mathbb{E} \left[ \sum_{i=1}^M \Gamma_i C_i^\alpha \left| \frac{\bar{F}_1(C_i x_0) - \bar{F}_2(C_i x_0)}{C_i^\alpha x_0^\alpha s(-\log(C_i x_0))} \right| \right] \leq \mathbb{E} \left( \sum_{i=1}^M \Gamma_i C_i^\alpha \right) d(\bar{F}_1, \bar{F}_2). \end{aligned}$$

For the first inequality, we use the fact that  $-\log C_i$ ,  $i \geq 1$  have a common span  $-\log c$ ,  $c > 0$ , and  $s(\cdot)$  is periodic with period  $-\log c$ . In this case,  $d(\bar{F}_1, \bar{F}_2) < d(\bar{F}_1, \bar{F}_2)$ , which is absurd.

If  $\mathbb{P}(A) = 0$ , we have almost surely, either  $\Gamma_i = 1$  or  $\bar{F}_1(C_i x_0) = \bar{F}_2(C_i x_0)$ , for all  $i \in \{1, \dots, M\}$ . From this, we get

$$\sum_{i=1}^M \left| \bar{F}_1(C_i x_0)^{\Gamma_i} - \bar{F}_2(C_i x_0)^{\Gamma_i} \right| = \sum_{i=1}^M \left| \bar{F}_1(C_i x_0) - \bar{F}_2(C_i x_0) \right|,$$

almost surely. In this case, using the argument above on the support of  $-\log C_i$ ,  $i \geq 1$ , and the periodicity of  $s(\cdot)$ , we obtain

$$\begin{aligned} d(\bar{F}_1, \bar{F}_2) &= \left| \frac{T\bar{F}_1(x_0) - T\bar{F}_2(x_0)}{x_0^\alpha s(-\log x_0)} \right| \\ &\leq \mathbb{E} \left[ \sum_{i=1}^M C_i^\alpha \left| \frac{\bar{F}_1(C_i x_0) - \bar{F}_2(C_i x_0)}{C_i^\alpha x_0^\alpha s(-\log(C_i x_0))} \right| \right] \leq \mathbb{E} \left( \sum_{i=1}^M C_i^\alpha \right) d(\bar{F}_1, \bar{F}_2). \end{aligned}$$

If  $d(\bar{F}_1, \bar{F}_2) \neq 0$ , then  $\mathbb{E} \left( \sum_{i=1}^M C_i^\alpha \right) \geq 1$ . Since  $\mathbb{E} \left( \sum_{i=1}^M C_i^\alpha \right) \leq \mathbb{E} \left( \sum_{i=1}^M \Gamma_i C_i^\alpha \right)$ , recalling that  $\mathbb{E} \left( \sum_{i=1}^M \Gamma_i C_i^\alpha \right) = 1$ , we obtain  $\mathbb{E} \left( \sum_{i=1}^M (\Gamma_i - 1) C_i^\alpha \right) = 0$ , which means  $\Gamma_i = 1$ , for all  $i \in \{1, \dots, M\}$ , almost surely. Hence, we recover the first case, which was dealt with by theorem 16.  $\square$

## 5 CONCLUDING REMARKS

In this paper, solutions to the functional equation (E), extending min-semistable distributions, are considered. The main extension with respect to previously studied functional equations of the same type is that it involves non-integral random powers.

The techniques employed to derive our results are largely inspired from the ones originally designed for Laplace-Stieltjes transforms in the semistable case for sums.

In a special case, we start constructing solutions from scratch in space  $\mathcal{E}_1$  involving convexity. When considering the general case, we need to introduce a larger space, namely space  $\mathcal{F}$ . It is the largest space within which solutions can be searched for, with the techniques we use to do so. The behavior at the origin of the solutions within  $\mathcal{F}$  is elucidated. The characterization theorem involving space  $\mathcal{S}_{\alpha,c}$  shows that there are solutions to (E) whose behaviors in a neighborhood of the origin are possibly far from regular. This suggests that, due to some restrictions imposed on the solutions (in particular continuity), we possibly miss some solutions with a wild behavior near zero. Nevertheless, despite some technical constraints that we feel not intrinsic to the solutions of the posed problem, we hope to have done a further step towards the comprehension of a widely explored functional equation.

## References

- [1] Alsmeyer, G. and Rösler, U. (2005). A stochastic fixed point equation for weighted minima and maxima. Universität Münster, Preprintreihe Angewandte Mathematik, 01/05-S Reference (2005), available at <http://www-computerlabor.math.uni-kiel.de/stochastik/roesler>.
- [2] Barral, J. (1998). Une extension de l'équation fonctionnelle de B. Mandelbrot. *C.R. Acad. Sc.*, Paris, tome **326**, Série 1, 421-426.
- [3] Ben Alaya, M. and Huillet, T. (2004). On Max-Multiscaling Distributions as extended Max-Semistable ones. *Stochastic Models*, **20**, Issue 4, 493-512.
- [4] Biggins, J.D. (1977). Martingale convergence in the branching random walk. *J. Appl. Prob.*, **14**, 25-37.
- [5] Bingham, N.H. and Doney, R.A. (1974). Asymptotic properties of supercritical branching processes. I. The Galton-Watson process. *Adv. Appl. Prob.*, **6**, 711–731.
- [6] Bourbaki N. (1961). *Topologie générale*. Livre III, Chapitre 10. Espaces fonctionnels. Deuxième édition, Actualités Sci. Indust., No. 1084., Hermann, Paris, 96 pp..
- [7] Caliebe, A. (2003). Symmetric fixed points of a smoothing transformation. *Adv. Appl. Prob.*, **35**, 377-394.
- [8] Caliebe, A. (2004). Representation of fixed points of a smoothing transformation. *Mathematics and computer science*. **III**, 311–324, Trends Math., Birkhäuser, Basel.
- [9] Doney, R.A. (1972). A limit theorem for a class of supercritical branching processes. *J. Appl. Prob.*, **9**, 707-724.
- [10] Doney, R.A. (1973). On a functional equation for general branching processes. *J. Appl. Prob.*, **10**, 198-205.
- [11] Durrett, R. and Liggett, T.M. (1983). Fixed points of the smoothing transformation. *Z. Wahrsch. Verw. Gebiete*, **64**, 275-301.
- [12] Feller, W. (1971). *An introduction to probability theory and its applications*, **2**, Wiley, New York.
- [13] Guivarc'h, Y. (1990). Sur une extension de la notion de loi semi-stable. *Ann. Inst. H. Poincaré Prob. and Statist.*, **26**, no. 2, 261–285.
- [14] Holley, R. and Liggett, T.M. (1981). Generalized potlach and smoothing processes. *Z. Wahrsch. Verw. Gebiete*, **55**, 165-195.
- [15] Huillet T., Porzio A. and Ben Alaya A. (2000). On the physical relevance of

- max- and log-max-selfsimilar distributions. *Eur. Phys. J. B*, **17**, 147-158.
- [16] Kahane, J.P. and Peyrière J. (1976). Sur certaines martingales de Benoit Mandelbrot. *Adv. in Math.*, **22**, 131-145.
- [17] Liu, Q. (1997). Sur une équation fonctionnelle et ses applications: une extension de théorème de Kesten-Stigum concernant des processus de branchement. *Adv. Appl. Prob.*, **29**, 353-373.
- [18] Liu, Q. (1998). Fixed points of a generalized smoothing transformation and applications to the branching random walk. *Adv. Appl. Prob.*, **30**, 85-112.
- [19] Mandelbrot, B.B. (1974a). Multiplications aléatoires et distributions invariantes par moyenne pondérée aléatoire. *C.R. Acad. Sc.*, Paris, tome **278**, Série 1, 289-292.
- [20] Mandelbrot, B.B. (1974b). Multiplications aléatoires et distributions invariantes par moyenne pondérée aléatoire: quelques extensions. *C.R. Acad. Sc.*, Paris, tome **278**, Série 1, 355-358.
- [21] Rao, C.R. and Shanbhag, D.N. (1994). *Choquet-Deny type functional equations with applications to stochastic models*. Wiley Series in Probability and Mathematical Statistics, John Wiley and Sons, Ltd, Chichester.