Endomorphisms of Kleinian Groups.

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1 Introduction.

A group G is *cohopfian* (or has the *co-Hopf property*) if any injective endomorphism $f : G \to G$ is surjective.

Answering a question of E. Rips, Z. Sela showed in [Se] that a torsion-free, non virtually cyclic word-hyperbolic group (in Gromov's sense) is cohopfian if and only if it is not a non-trivial free product. The cohopficity of 3-manifold groups was studied by many authors; see [PW] and [OP] where a more complete list of references on this subject is given.

A non-trivial free product $A * B$ is never cohopfian, as it contains the proper subgroup $A * mBm^{-1}$ isomorphic to $A * B$ if $m \notin (A \cup B)$. More generally, let the group G split as an HNN-extension, $G = A*_C = \langle A, t \mid tCt^{-1} = \varphi(C)\rangle$, and suppose that t centralizes C. Then G is not cohopfian (set $f : G \to G$ be the identity on A and $f(t) = t^2$; then f is injective, not surjective). It is shown in [OP] that this example can be realized as a Kleinian group. Note that in this case, the group G splits over a parabolic subgroup C which is of infinite index in the unique maximal parabolic subgroup $\tilde{C}$ of G containing C (where $\tilde{C} =< C, t >$), and $\tilde{C}$ is not conjugate into A. In such case we will refer to the group C and the corresponding splitting of G over C as *essentially non-maximal.* On the other hand it is also shown in [OP] that G is cohopfian if it does not split over an elementary subgroup. A natural question is whether all non cohopfian torsion free one-ended Kleinian groups arise *only* in this way, in other words is G non cohopfian if and only if G has essentially non-maximal splittings over parabolic subgroups ? The main result of the paper (Theorem A below) is a criterion showing that essentially this is the case.

Let G be a one-ended, non-elementary, geometrically finite Kleinian group. Instead of studying directly the "absolute" cohopfian property of G, we extend this notion to the "relative" case. Let $\mathcal{E} = \{E_1, ..., E_n\}$ be a fixed set of elementary subgroups of G (a "peripheral system") and suppose that $f : G \to G$ is an endomorphism which sends each E_i into itself. Then Theorem B below guarantees that f is a surjective if G has no essentially non-maximal splittings over elementary subgroups *relatively* to the system $\mathcal{E}$ (i.e. a splitting in which every $\mathcal{E}_i$ is elliptic).

The notion of "relative cohopficity" can be easily illustrated by the example of a surface with boundary. Let S be a compact surface of genus $g > 1$ whose boundary is a finite collection of loops α_i $(i = 1, ...n)$. Let E_i be the cyclic peripheral subgroup of $G = \pi_1(S)$ generated by α_i and $\mathcal{E} = \{E_1, ..., E_n\}$. The group G is a free group and is not cohopfian ; however it is cohopfian relatively to $\mathcal{E}$, i.e. if $f : (G, \mathcal{E}) \to (G, \mathcal{E})$ is an injective endomorphism sending each group E_i into itself then f is surjective.

The proof of the co-hopficity criterion goes as follows. Let $f : (G, \mathcal{E}) \to (G, \mathcal{E})$ be an injective, non-surjective endomorphism of a one-ended Kleinian group G. In Section 3, refining the main result of the paper [OP], we prove using the theory of groups acting on real trees that the group G splits over elementary subgroups relatively to the system $\mathcal{E}$ (Proposition 3.1). Our further goal is to find among all the trees T_n, a $(G, \mathcal{E})$-tree T and another injective, non-surjective map $F : (G, \mathcal{E}) \to (G, \mathcal{E})$ so that F sends all vertex and edge stabilizers of T into themselves. In the simplest case, when the tree T is dual to a splitting of G as an amalgamated product $G = A *_C B$, we obtain that $F(A, C) \subset (A, C)$ and $F(B, C) \subset (B, C)$. An argument based upon M. Bestvina-M. Feighn's accessibility theorem [BeF2] will then show (Section 6) that the pairs $(A, (C \cup \mathcal{E}))$ and $(B, (B \cup \mathcal{E}))$ are "simpler" than $(G, \mathcal{E})$. The general case will follow by induction.

In Section 7, we prove that if a group G admits an essentially non maximal splitting over a parabolic group, then it is not cohopfian.

In Section 8 we treat the case of infinitely ended groups. The proofs here are based on the techniques developed in the previous Sections.

Let us point out that the methods of Z. Sela's paper [Se] do not work for geometrically finite Kleinian groups containing parabolic subgroups of rank bigger than one. The main reason is that the crucial point of many considerations in [Se] is so called "shortening argument" which does not work if the injectivity radius of the space tends to zero. In the present paper we apply different methods. We also note that most of our arguments do not require constant negative sectional curvature, what we really use is strict negativity of the curvature and two purely algebraic facts: elementary groups are virtually abelian and geometrically finite groups are finitely presentable. We believe that our results can be generalized to the case of geometrically finite subgroups of the isometry group of Hadamard manifolds of pinched negative curvature as well as to relatively hyperbolic groups with respect to a system of elementary virtually nilpotent subgroups.

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2 Preliminaries and Formulations of the Results.

Let $\mathbb{H}^n$ be the real hyperbolic space of dimension n. A group G is *Kleinian* if G is a discrete subgroup of the orientation preserving part of the isometry group $\mathrm{Isom}_+\mathbb{H}^n$ of $\mathbb{H}^n$. The limit set $\Lambda(G)$ of G is the set of accumulation points of some (any) orbit $G(z)$ $(z \in \mathbb{H}^n)$.

Recall that a Kleinian group $H \subset \mathrm{Isom}_+\mathbb{H}^n$ is *elementary* if its limit set $\Lambda(H) \subset \mathcal{S}^{n-1}_\infty$ is a finite set, and H is a finite elementary group if and only if $\Lambda(H) = \emptyset$. An infinite elementary group H is *loxodromic* (resp. *parabolic*) if the limit set $\Lambda(H)$ contains two points (resp. one point). By Bieberbach's theorems (see e.g. [Ra]) every elementary subgroup H of $\mathrm{Isom}_+\mathbb{H}^n$ is a finitely generated virtually abelian group, i.e. contains a free abelian subgroup A of finite index. The rank of the group A is called the *rank* of H. A loxodromic elementary group is always virtually cyclic (2-ended). A parabolic subgroup of rank bigger than one is a one-ended group.

Notation. *If C is elementary and infinite, it is contained in a unique maximal elementary subgroup of G. This subgroup will be denoted $\tilde{C}$ throughout the paper.*

A finitely generated Kleinian group G is *geometrically finite* if there exists an $\varepsilon > 0$ so that the hyperbolic volume of an ε-neighborhood of $C(\Lambda(G))/G$ is finite, where $C(\Lambda(G)) \subset \mathbb{H}^n$ is the convex hull of the limit set of G (i.e. the smallest convex subset of $\mathbb{H}^n$ invariant under the G-action) is finite.

We say that G *splits* as a graph of groups $X_* = (X, (C_e)_{e\in X^1}, (G_v)_{v\in X^0})$ (where C_e and G_v denote respectively edge and vertex groups of the graph X) if G is isomorphic to the fundamental group $\pi_1(X_*)$ in the sense of Serre [Se]. The Bass-Serre tree T is the universal cover in the sense of Serre of the graph $X = T/G$. When X has only one edge, we will say that G splits as an amalgamated free product (resp. an HNN-extension) if X has two vertices (resp. one vertex).

We will need the following definitions:

Definition 2.1. *Let G act on a tree T. A subset H of G is called elliptic (resp. hyperbolic) in T (and in the graph T/G) if H fixes a point in T (resp. does not fix a point in T). If T is the Bass-Serre tree of a splitting of G as a graph of groups, H is elliptic if and only if it is conjugate into a vertex group of this graph.*
We say that G splits relatively to a family of subgroups $(E_1, ...E_n)$, or that the pair $(G, \mathcal{E})$ splits as a graph of groups, if G splits as a graph of groups such that all the groups E_i are elliptic. A $(G, \mathcal{E})$-tree is a $G-$tree in which E_i are elliptic for all i.

Definition 2.2. *Suppose G splits as a graph of groups*

$$G = \pi_1(X, C_e, G_v) \tag{1}$$

and suppose that edge groups (i.e. the groups C_e) of this graph are elementary. We say that the edge stabilizer C_e is essentially non-maximal if the maximal elementary subgroup $\tilde{C}_e$ is not elliptic in the splitting (1). The splitting (1) is essentially non-maximal if there exists at least one such an edge. Otherwise we say that the splitting (1) is essentially maximal.

Theorem A. *Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a non-elementary, geometrically finite, one-ended Kleinian group without 2-torsion. Then G is cohopfian if and only if the following two conditions are satisfied:*

1) *G has no essentially non-maximal splittings.*

2) *G does not split as an amalgamated free product $G = A *_C \tilde{C}$, with $\tilde{C}$ maximal elementary, such that the normal closure of the subgroup C in $\tilde{C}$ is of infinite index in $\tilde{C}$.*

■

Note that if C is a non-trivial essentially non-maximal elementary subgroup of G, then $|\tilde{C} : C| = \infty$. Therefore C is a parabolic subgroup of G, and rank $(C) <$ rank $\tilde{C}$.

Corollary 2.3. *Let G be a non-elementary, geometrically finite, one-ended Kleinian group without 2-torsion. Suppose that every elementary subgroup C over which G splits has a finite index in the maximal elementary subgroup $\tilde{C}$, then G is cohopfian.*

As explained in the Introduction, the proof of Theorem A is based on the study of the relative case.

Definition 2.4. *Let G be a group, and $\mathcal{E} = (E_1, ..., E_n)$ a family of elementary subgroups. An endomorphism of G is called endomorphism of the pair $(G, \mathcal{E})$ if it sends each E_i into itself.*

The pair $(G, \mathcal{E})$ is cohopfian, if any injective endomorphism of $(G, \mathcal{E})$ is surjective. We say that the pair $(G, \mathcal{E})$ is one ended if $(G, \mathcal{E})$ does not split over finite subgroups.

Theorem B. *Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a non-elementary, geometrically finite, Kleinian group without 2-torsion and $\mathcal{E} = \{E_1, ..., E_k\}$ be a family of elementary subgroups of G. Suppose that the pair $(G, \mathcal{E})$ is one-ended. Then $(G, \mathcal{E})$ is cohopfian if the following two conditions are satisfied:*

1) *The pair $(G, \mathcal{E})$ has no essentially non-maximal splitting over elementary subgroups.*

2) *The pair $(G, \mathcal{E})$ does not split as an amalgamated free poduct $G = A *_C \tilde{C}$, with $\tilde{C}$ maximal elementary and the normal closure of C in $\tilde{C}$ is a subgroup of infinite index of $\tilde{C}$.* ■

Remark: *Theorem A is the special case of Theorem B where the family $\mathcal{E}$ is empty.*

We will need the following definition of *acylindrical* splittings introduced by Z. Sela in the torsion free case and in [De] in the general case:

Definition 2.5. *Let G split as a graph of groups $G = \pi_1(X)$ with elementary edge stabilizers and T be the Bass-Serre tree dual to this splitting.*

a) *Torsion free case: The splitting (and the tree T) is K-acylindrical if the stabilizer of each segment of T of diameter at least K is trivial.*

b) *General case: The G-tree T is called (K, Φ)-acylindrical if the stabilizer of each segment on T of the diameter at least K is a finite group*[1].

If G splits as a graph of groups $G = \pi_1(X)$, one says that this splitting is $(K, \Phi)-$acylindrical if the Bass-Serre tree - the universal cover of X - is $(K, \Phi)-$acylindrical.

■

Recall also (see e.g. [BeF1]) that a G-tree is called *irreducible* if it is minimal (i.e. there is no proper invariant subtree) and if the label of every vertex of valence two properly contains the labels of both edges incident to it (if the two edges are distinct). The relationship between Definitions 2.2 and 2.5 is established in the following lemma.

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[1] Here Φ stands for "finite".

Lemma 2.6. *Let G be a finitely presented Kleinian group, $\mathcal{E} = \{E_1, ..., E_k\}$ be a family of elementary subgroups of G, and suppose that the pair $(G, \mathcal{E})$ is one-ended. The pair $(G, \mathcal{E})$ has no essentially non-maximal splittings iff there exists a constant K such that each irreducible $(G, \mathcal{E})$-splitting over elementary subgroups is (K, Φ)-acylindrical. In this case, every essentially non-maximal splitting of $(G, \mathcal{E})$ as an amalgamated free product or an HNN-extension is $(3, \Phi)$-acylindrical.*

Proof: Suppose that the pair $(G, \mathcal{E})$ has no essentially non-maximal splittings and let G act on a simplicial tree T with elementary edge stabilizers. Then G splits as the graph of groups $X = T/G$. Let m denote the number of edges of X. We will first show that the tree T is $2m+1$-acylindrical. To this end, suppose that l is an embedded path in T consisting of n successive edges such that $n \geq 2m+1$. We want to show that the stabilizer C of l is a finite group. Arguing by contradiction suppose that the group C is infinite. Since $n \geq 2m+1$ the path l contains at least three distinct edges e_1, e_2, e_3 which are in the same G-orbit. Let C_i be the stabilizer of the edge e_i and let α_i and α'_i be its vertices ($i = 1, 2, 3$). Let $e_2 = g(e_1)$ and $e_2 = h(e_3)$ for some g and h not belonging to C_2. We have $C \subset \bigcap_{i=1}^{3} C_i$ and $C_2 = gC_1g^{-1},\ C_2 = hC_3h^{-1}$.

As $g^{-1}C_2g \cap C_2 \supset C$ and C is infinite, we deduce that $g^{-1}\tilde{C}_2g = \tilde{C}_2$ where $\tilde{C}_2$ is the unique maximal elementary subgroup of G containing C_2. The same property holds for h. Thus the elements g and h belong to $\tilde{C}_2$ which also contains C. As G does not have essentially non-maximal splittings, it follows that $\tilde{C}_2$ fixes a point on the tree T and so there is a vertex $v \in T$ whose stabilizer D contains $\tilde{C}_2$.

Let $[\alpha'_i, \alpha_{i+1}]$ denote the segment of the path l between the vertices α'_i and α_{i+1}. A standard argument [S, I-6.4] shows that either the element g fixes a point x in $[\alpha'_1, \alpha_2]$ or g acts on T without fixed points. We have already shown that the latter case is impossible. Similarly, the element h fixes a point $y \in [\alpha'_2, \alpha_3]$. Now their common fixed point v belongs to the same connected component of $T \setminus e_2$ as one of the vertices x or y, say x. Thus h fixes the path between y and v in T. This path contains the edge e_2, and so $h \in C_2$ which is impossible. Thus the group C must be finite. In particular, if the graph X contains only one edge the splitting X (i.e. amalgam or an HNN-extension) is $(3, \Phi)$-acylindrical.

By the result of M. Bestvina-M. Feighn [BeF2] there is a uniform upper bound $\nu(G) < \infty$ for the number of edges of all irreducible splittings of G with elementary edge stabilizers. Thus, setting $K = 2\nu(G) + 1$ we obtain the result. The necessary condition is proved.

Conversely, suppose that the group has an essentially non-maximal splitting $G = \pi_1(X, C_e, G_v)$ relatively to the system $\mathcal{E}$. As the pair $(G, \mathcal{E})$ is one ended, every edge group $C_e = C$ of X is an infinite elementary subgroup of infinite index in $\tilde{C}$. Therefore $\tilde{C}$ is an infinite parabolic subgroup of G, and $\tilde{C}$ does not fix a point in T - universal cover of X. Since the group $\tilde{C}$ is a finitely generated virtually abelian group, it then follows from [S, 6.5, Proposition 27] that there is an element t in $\tilde{C}$ acting hyperbolically on T. The group $\tilde{C}$ contains an abelian subgroup of finite index C' and, so there exists $k \in \mathbb{N}$ that $t^k \in C'$, and t^k centralizes the group $C_0 = C \cap C'$. Therefore, the group C_0 fixes also the edges $e, t^k(e), ..., t^{nk}(e), ...$, and, hence a segment of arbitrary big length. We see that the graph X is not (K, Φ)-acylindrical for any $K \in \mathbb{N}$. The lemma follows. *QED.*

In the final Section we will need somewhat different notion of acylindricity for splittings of an infinitely ended group G over finite subgroups. We call such a splitting **strictly** K-acylindrical if the stabilizer of each segment of the corresponding Bass-Serre tree T of the diameter at least K is a proper subgroup of some edge stabilizer of T. We prove in Section 8 the following theorem:

Theorem C. *Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a non-elementary, geometrically finite Kleinian group without 2-torsion. Then G is cohopfian if and only if the following three conditions are satisfied:*

1) *G does not have essentially non-maximal splittings over infinite elementary subgroups.*

2) *G does not split as an amalgamated free product $G = A *_C \tilde{C}$, so that the normal closure of the subgroup C in $\tilde{C}$ is of infinite index in $\tilde{C}$.*

3) *Every splitting of G over finite groups is strictly M-acylindrical for a uniform constant M.*

■

Remark 2.7. *By Lemma 2.6 Condition 1) can be replaced by the following*

1) There exists a constant K such that each irreducible splitting of G over an infinite elementary subgroup is (K, Φ)-acylindrical. ■

We now introduce some terminology which will be used in the suite .

A G-tree $\hat{T}$ is called a *resolution* of a G-tree T if there exists G-equivariant simplicial map $\rho : \hat{T} \to T$.

Suppose that T is a $(G, \mathcal{E})$-tree and $\varphi : (G, \mathcal{E}) \to (G, \mathcal{E})$ is a monomorphism. Let φ^*T denote the G-tree defined as follows: as metric space, φ^*T is T, but the action of G on T is obtained from the original action by composing with φ:

$$g_{\varphi^*T}(x) = \varphi(g)_T(x).$$

The stabilizer of a vertex v (edge e) of the tree φ^*T is equal to $\varphi^{-1}(G_v)$ (respectively $\varphi^{-1}(C_e)$) where G_v (respectively C_e) is the stabilizer of v (respectively e) on T.

A *marking* of the G-tree T is a subtree t of T which is a fundamental domain for the action of the group G on T. A pair (T, t) will be called *marked* tree where t is a marking of T. If t is a marking of T and $f : G \to G$ is an injective endomorphism we denote $\tilde{t}$ a marking of the tree f^*T containing t setwise. Two markings t, t' of the tree T are isomorphic if there exists an automorphism φ of G and a G-equivariant isometry $I : \varphi^*T \to T$ sending t to t'. Note that if the graph T/G is finite there are at most finitely many different markings of T up to isomorphism. We say that the G-tree T *dominates* the G-tree T' if there exists a resolution $\rho : T \to \varphi^*T'$ for some automorphism φ of G. Similarly, we say that the marked tree (T, t) *dominates* the marked G-tree (T', t') if there exists a resolution $\rho : (T, t) \to (\varphi^*T', t')$ sending the marking t to the marking t'.

3 Finding a splitting of a non-cohopfian pair $(G, \mathcal{E})$.

Let G be a non cohopfian Kleinian group, and $f : G \to G$ is an injective non surjective endomorphism, then the result of [OP] implies that G admits a non-trivial action on a simplicial tree with elementary edge stabilizers. The following proposition provides a relative version of this result:

Proposition 3.1. *Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a non-elementary, geometrically finite Kleinian group without 2-torsion and $\mathcal{E} = \{E_1, ..., E_k\}$ is a finite family of elementary subgroups of G. Suppose that the pair $(G, \mathcal{E})$ is non-cohopfian and let $f : (G, \mathcal{E}) \to (G, \mathcal{E})$ be an injective endomorphism which is not surjective. Then $(G, \mathcal{E})$ has a non-trivial splitting over elementary subgroups.*

Proof: We may assume (w.l.o.g.) that all the subgroups E_i are infinite maximal elementary subgroups of G and E_i are loxodromic for the first s subgroups from $\mathcal{E}$ $(0 \le s \le k)$. Suppose also that the elements γ_i generate the infinite cyclic subgroup $< \gamma_i >$ of finite index of E_i $(i = 1, ..., s)$. Let A_{γ_i} denote the invariant axis of the element γ_i and $\mathrm{dist}_{\mathbb{H}^n}(\cdot)$ be the hyperbolic distance between subsets of $\mathbb{H}^n (i = 1, ..., s)$. We start with the following:

Lemma 3.2. *Suppose that there exists $i \in \{1, ..., s\}$ such that for all $g \in G$ the quantity $\mathrm{dist}_{\mathbb{H}^n}(A_{\gamma_i}, f^m(b)(A_{\gamma_i}))$ is bounded. Then, there exists natural numbers $m_0,\ n_0 \in \mathbb{N}$ and elements $\alpha_m \in G$ such that for all $m > m_0$:*

$$f^m(\gamma_i^{n_0}) = \alpha_m f^{m_0}(\gamma_i^{n_0})\alpha_m^{-1},\ k_m \in \mathbb{Z}$$

Proof of the lemma: We will need the following result:

Uniform Klein Combination (UKC) Theorem. (M. Gromov [Gro], T. Delzant [De2], G. Noskov [N]) *Suppose G is geometrically finite group and γ is a loxodromic element and E be its maximal elementary subgroup. Then there exists N such that for any element $a \in G \setminus E$ the elements γ^N and $a\gamma^N a^{-1}$ freely generate the free group F_2.*

Assuming this Theorem we shall prove the Lemma. Let $\gamma_i = \gamma$ and $E = E_i$. As the group E does not have 2-torsion it is well known [DD, 6.12] that $E = K \rtimes C$ where $C =< \gamma > \cong \mathbb{Z}$ and K is a finite group of order l. There exists $k \in \mathbb{N}$ such that γ^k centralizes E. It is then easy to check that there exists $s \in \mathbb{N}$ so that $f(\gamma^{kl}) = \gamma^{kls}$. Setting $\tilde{\gamma} = \gamma^{klN}$, where N is given by the above UKC Theorem, we have $f(\tilde{\gamma}) = \tilde{\gamma}^s$.

By hypothesis, for every element $g \in G$ there exists a constant $K < \infty$ such that

$$\mathrm{dist}_{\mathbb{H}^n}(A_\gamma, f^m(g)(A_\gamma)) \le K\ (m \in \mathbb{N}).$$

Set $g_m = f^m(g)$, and choose points $w_m \in A_\gamma$ and $y_m \in g_m(A_\gamma)$ so that $d_{\mathbb{H}^n}(w_m, y_m) = \mathrm{dist}_{\mathbb{H}^n}(A_\gamma, g_m(A_\gamma))$ $(j = 1, 2;\ m \in \mathbb{N})$. Let w'_m be the point $g_m^{-1}(y_m) \in A_\gamma$, then $d_{\mathbb{H}^n}(w_m, g_m(w'_m)) \le K$. As the group $< \tilde{\gamma} >$ is a finite index subgroup of E, it acts co-compactly on the axis A_γ. So there exist integers $k_m,\ r_m$ such that $w_m = \tilde{\gamma}^{k_m}(z_m),\ w'_m = \tilde{\gamma}^{r_m}(z'_m),\ z_m,\ z'_m \in A_i$ and $d_{\mathbb{H}^n}(z_m, z'_m) \le K_1 < +\infty$ for some K_1. We obtain

$$d_{\mathbb{H}^n}(z_m, \tilde{\gamma}^{-k_m} g_m \tilde{\gamma}^{r_m}(z_m)) \le K + K_1 < +\infty.$$

As the group G is discrete and $\tilde{\gamma}^{-k_m} g_m \tilde{\gamma}^{r_m} \in G$ ($m \in \mathbb{N}$), it follows that $\exists m_0$ such that $\forall m > m_0\ : \tilde{\gamma}^{-k_m} g_m \tilde{\gamma}^{r_m} = \tilde{\gamma}^{-k_{m_0}} g_{m_0} \tilde{\gamma}^{-r_{m_0}}$.

We deduce that for every $g \in G$ there exists $m_0 \in \mathbb{N}$ such that $\forall m > m_0$ there exist integers k_m and r_m such that

$$f^m(g) = \tilde{\gamma}^{k_m} f^{m_0}(g) \tilde{\gamma}^{r_m} \quad (j = 1, 2), \tag{$*$}$$

where $k_m := k_m - k_{m_0},\ r_m := -r_m - r_{m_0}$. Now pick any element $a \in G \setminus E$ and set $g = a\tilde{\gamma}a^{-1}$.

We can also choose m_0 so that (*) holds not only for g but also for g^2 (after replacing k_m (resp. r_m) by t_m (resp. s_m)). We obtain

$$f^m(g^2) = \tilde{\gamma}^{t_m} f^{m_0}(g^2) \tilde{\gamma}^{s_m} = \tilde{\gamma}^{k_m} f^{m_0}(g) \tilde{\gamma}^{r_m + k_m} f^{m_0}(g) \tilde{\gamma}^{r_m}. \tag{$**$}$$

As $f^{m_0}(E) \subset E$, the subgroup $f^{-m_0}(E)$ is elementary (being isomorphic to $f^{m_0}(f^{-m_0}(E))$) and contains E. By the maximality of the latter, we get $f^{-m_0}(E) = E$. So $f^m(a)$ is an element which does not belong to E ($\forall\ m \in \mathbb{N}$).

The **UKC** Theorem now yields that the elements γ^N and $h_{m_0} = f^{m_0}(a)\gamma^N f^{m_0}(a^{-1})$ freely generate the free group F_2. As $\tilde{\gamma} = \gamma^{klN}$ and $f(\tilde{\gamma}) = \tilde{\gamma}^s$, we obtain that $f^{m_0}(g) = (h_{m_0})^{s^{m_0}kl}$. Thus, the elements $\tilde{\gamma}$ and $f^{m_0}(g)$ also generate a free group. Then it follows from (**) that $r_m = -k_m$ and so:

$$f^m(g) = \tilde{\gamma}^{k_m} f^{m_0}(g) \tilde{\gamma}^{-k_m} = \tilde{\gamma}^{k_m} f^{m_0}(a\gamma^{klN}a^{-1}) \tilde{\gamma}^{-k_m}, \quad m > m_0.$$

proving the lemma. ■

Proof of the proposition: Let us choose a generating system $S = \{\gamma_1, ..., \gamma_r, a_1, ..., a_l\}$ of G where γ_i are generators of subgroups $E_i \in \mathcal{E}$ and the elements a_j do not belong to $\mathcal{E}$ ($1 \le i \le l$). If for some $i \in \{1, ..., s\}$ there exists an element $b_i \in G$ such that the function $\mathrm{dist}_{\mathbb{H}^n}(A_{\gamma_i}, f^m(b_i)(A_{\gamma_i}))$ is not bounded we add the elements b_i and $b_i\gamma_i b_i^{-1}$ to the system S and retain the same notation S for it. Consider now the following displacement function:

$$d_m(f, S, G) = \min_{x \in \mathbb{H}^n} \max_{s \in S}\ d_{\mathbb{H}^n}(x, f^m(s)(x)). \tag{5}$$

It is proved [OP] that if the map f is not surjective then for any generating system S the function $d_m(f, S, G)$ is not bounded ($m \in \mathbb{N}$). In this case by the theorem of Bestvina-Paulin [Be], [Pa], the group G acts stably and non-trivially on a real tree $T_{\mathbb{R}}$ with elementary edge stabilizers. Furthermore, it is proven in [Be], [Pa] that

$$\overline{\lim}_{m \to \infty} \frac{l(f^m(g))}{d_m(f, S, G)} = L_{\mathbb{R}}(g), \tag{6}$$

where $l(g) = \inf\ d_{\mathbb{H}^n}(x, g(x))$ and $L_{\mathbb{R}}(g) = \inf\ d_{T_{\mathbb{R}}}(x, g(x))$ are the translation lengths in the hyperbolic space $\mathbb{H}^n$ and in the tree $T_{\mathbb{R}}$ respectively. By Rips' theorem [BeF1] there exists a non-trivial simplicial G-tree with elementary edge stabilizers.

Arguing by contradiction suppose that for every simplicial G-tree one of the subgroups E_i acts hyperbolically on it ($i = 1, ..., k$). By the relative version of Rips' theorem [BF1, Theorem 9.6] there exists an element $\gamma \in \mathcal{E}$, which acts hyperbolically on the real tree $T_{\mathbb{R}}$ too, implying that the quantity $L_{\mathbb{R}}(\gamma)$ is strictly positive.

After passing to a subsequence, we may choose an element $g \in S$ and a point $x_m \in \mathbb{H}^n$ which realizes the mini-max in (5):

$$d_m(f, S, G) = d_{\mathbb{H}^n}(x_m, f^m(g)(x_m)),$$

and such that the following inequality holds:

$$0 < \lim_{m\to\infty} \frac{l(f^m(\gamma))}{d_{\mathbb{H}^n}(x_m, f^m(g)(x_m))} \leq 1.$$

Note that up to passing to a further subsequence we may suppose that for every $m \in \mathbb{N}$ the group $f^m(\gamma)$ generates infinite virtually cyclic loxodromic group. Indeed if $f^m(\gamma)$ is parabolic ($\forall\ m > m_0$) then (6) yields that γ fixes a point in the tree $T_{\mathbb{R}}$ which is impossible. So we may assume (w.l.o.g.) that $\gamma \in E_1$. As $f^m(E_1) \subset E_1$ the group $f^m(E_1)$ is an infinite virtually cyclic loxodromic subgroup of G leaving the axis A_γ invariant ($m \in \mathbb{N}$).

It follows from Lemma 3.2 that there exists an element $b \in G$ such that the distance $\mathrm{dist}_{\mathbb{H}^n}(A_{\gamma_i}, f^m(b)(A_\gamma))$ is unbounded; for otherwise the element γ would act elliptically on the tree $T_{\mathbb{R}}$ as $f^m(\gamma^{n_0})$ is conjugate to the element $f^{m_0}(\gamma^{n_0})$ ($\forall\ m > m_0$). Furthermore we may assume by construction, that the system S contains the elements b and $h = b\gamma b^{-1}$. Set $h_m = f^m(h) = b_m f^m(\gamma) b_m^{-1}$. Notice that $l(h_m) = l(f^m(\gamma))$. To finish the proof of the Proposition we will show that γ can not act hyperbolically on $T_{\mathbb{R}}$. There are two cases according to whether or not the quantity $D_m = \mathrm{dist}_{\mathbb{H}^n}(x_m, A_\gamma)$ remains bounded .

Case 1. *D_m is unbounded.*

As $\gamma \in S$, so $d_{\mathbb{H}^n}(f^m(\gamma)(x_m), x_m) < d_m(f, S, G)$. Let us choose a point $w_m \in A_\gamma$ which realizes the distance D_m. Since $l(h_m) = l(f^m(\gamma))$ we obtain:

$$\frac{l(h_m)}{d_m(f,S,G)} = \frac{l(f^m(\gamma))}{d_m(f,S,G)} = \frac{d_{\mathbb{H}^n}(w_m, f^m(\gamma)(w_m))}{d_m(f,S,G)} \leq e^{-D_m}\frac{d_{\mathbb{H}^n}(x_m, f^m(\gamma)(x_m))}{d_m(f,S,G)} \leq e^{-D_m} \to 0.$$

implying that the element $h = b\gamma b^{-1}$ acts elliptically on $T_{\mathbb{R}}$ and, so is γ. A contradiction.

Case 2. *D_m is bounded.*

Since $h \in S$, so $d_{\mathbb{H}^n}(h_m(x_m), x_m) < d_m(f, S, G)$. Choose $z_m \in A_{h_m} = b_m(A_\gamma)$ such that $d_{\mathbb{H}^n}(x_m, z_m) = \mathrm{dist}_{\mathbb{H}^n}(x_m, A_{h_m})$ and denote this distance M_m. As $\mathrm{dist}_{\mathbb{H}^n}(A_\gamma, b_m(A_\gamma)) \to \infty$ we obtain that up to a subsequence $M_m \to +\infty$ ($m \to +\infty$). Then

$$\frac{l(h_m)}{d_m(f,S,G)} \leq e^{-M_m}\frac{d_{\mathbb{H}^n}(h_m(x_m), x_m)}{d_m(f,S,G)} \to 0,$$

As before it follows that the element γ acts elliptically on the tree $T_{\mathbb{R}}$ contradicting our hypothesis. Therefore, we have shown that there exists a non-trivial $(G, \mathcal{E})$-tree. The proposition is proved.■

4 Accessibility of Finitely Presented Groups

In this section we collect some results about different versions of accessibility (acylindrical and hierarchical) for finitely presented groups. Let G denote an abstract (not necessarily Kleinian) group.

We will consider decompositions of finitely presented groups over so called elementary subgroups which we now define axiomatically:

Definition 4.1. *Let G be a finitely presented group and $\mathcal{C}$ a fixed family of subgroups of G. We call the family $\mathcal{C}$ and every element $C \in \mathcal{C}$ elementary if the following axioms are satisfied:*

(1) If $C \in \mathcal{C}$ then every subgroup and every conjugate of C is in $\mathcal{C}$.

(2) Every infinite subgroup belonging to $\mathcal{C}$ is contained in a unique maximal subgroup $\tilde{C}$ so that $\tilde{C} \in \mathcal{C}$. The union of an ascending sequence of finite elementary groups is elementary.

(3) Every subgroup of $\mathcal{C}$ satisfies the following fixed-point condition: whenever C acts on a simplicial tree τ, C preserves a point in τ, or a point on its ideal boundary $\partial\tau$ or a pair of points on $\partial\tau$ (possibly permuting them).

(4) If $C \in \mathcal{C}$ is an infinite maximal elementary subgroup then its normalizer in G is contained in $\mathcal{C}$, i.e. $gCg^{-1} = C$ implies that $g \in C$ for all $g \in G$.

Examples of elementary families are well-known in the geometry of negatively curved spaces. Namely, discrete subgroups of the hyperbolic space $\mathbb{H}^n$ or, more generally Hadamard spaces with a pinched negative curvature, are elementary in the classical sense if their limit set is a finite set. In this case they are also elementary according to our axioms $(1-4)$. Indeed, the properties (1), (2) and (4) are easy exercises, the only property which is non-trivial is the axiom (3) which follows from Margulis' lemma saying that every such group is virtually nilpotent (abelian in the constant curvature case) and from Tits' theorem [Ti] implying that every virtually nilpotent group satisfies (3). Another important example one obtains by considering elementary subgroups (i.e. virtually cyclic) of word-hyperbolic (Gromov) groups which are also elementary according to the axioms (1-4).

A finite hierarchy of length k of the group G over elementary subgroups is defined inductively (on k) as follows ([De-Po]):

Definition 4.2. *Let G be a group and $\mathcal{C}$ a family of elementary subgroups of G. If G do not split as an amalgamated free product or an HNN-extension over a subgroup in $\mathcal{C}$ we say that G admits a hierarchy (of length 0). We say that G admits a finite hierarchy of length k if G splits as $G^0 = G_1^1 *_C G_2^1$ or $G = G_1^1 *_C$ $(C \in \mathcal{C})$, and one of the groups G_1^1 or G_2^1 admits a finite hierarchy of length $k-1$ and the other admits a finite hierarchy of length at most $k-1$. We say that G admits a hierarchy if this holds for some integer k (which we call the length of the hierarchy.)*

We define then the number $l(G)$ to be the minimal number of the lengths among all hierarchies of G. Similarly $l(G, \mathcal{E})$ denotes the minimal number of the lengths of all hierarchies of G such that all the subgroups in $\mathcal{E}$ are elliptic in every decomposition appearing in this hierarchy.

Hierarchical Accessibility Theorem. *Let G be a finitely presented group without 2-torsion and $\mathcal{C} \subset G$ an elementary family of subgroups. Let $\mathcal{E} = \{E_1, ...E_k\}$ be a fixed finite subset of $\mathcal{C}$. Then $(G, \mathcal{E})$ has a finite hierarchy over elementary subgroups.*

In other words, either $l(G, \mathcal{E}) = 0$, or there exists a decomposition of $(G, \mathcal{E})$ as an amalgamated free product (or an HNN-extension)

$$G = A *_C B,\ \ (G = A*_C),$$

such that

$$\max\{l(A, A \cap \mathcal{E}), l(B, B \cap \mathcal{E})\} < l(G, \mathcal{E}). \tag{3}$$

Proof: The proof of the main Theorem 3.6 of the paper [De-Po], can easily be adapted to the relative case, by keeping track of the peripheral system $\mathcal{E}$. Let us sketch this proof. Recall that in order to prove Theorem 3.2 in [De-Po] we used a version of an invariant $c(\cdot)$ (called *complexity*) of finitely presented groups appeared first in [De1]. Consider a simplicial developable orbihedron Π of dimension 2 whose fundamental group is G (see [Ha]) such that the vertex stabilizers of Π are in $\mathcal{C}$ and every subgroup E_i fixes a vertex $x_i \in \Pi$ $(i = 1, ..., k)$. We define first $c(\Pi, \mathcal{E})$ to be the pair $(T(\Pi), b_1(\Pi))$, where $T(\Pi)$ is the number of 2-dimensional faces of Π and $b_1(\Pi)$ is the first Betti number of the underlying topological space of Π. Then $c(G, \mathcal{E})$ is defined to be the infimum (for the lexicographical order) over all such G-orbihedra Π.

If T is a $(G, \mathcal{E})$-tree, the main result of [De-Po, Theorem 3.2] produces a simplicial tree $\hat{T}$ and a resolution $f : \hat{T} \to T$ so that the invariant $c(\cdot)$ of the vertex stabilizers of $\hat{T}$ strictly decreases. All we need to check is that the groups E_i are still elliptic on the tree $\hat{T}$. To see this consider the orbihedron universal cover P of the complex Π. The Axioms of Definition 4.1 allow one to construct a G-equivariant map $\rho : P \to T \cup \partial T$ (see [De-Po, 4.1]). Recall that the tree $\hat{T}$ is constructed to be the dual tree to the lamination $\Lambda \subset P$ whose leaves are pre-images under ρ of the midpoints of the edges of T. Let $E_i \in \mathcal{E}$ be an elementary subgroup which fixes a vertex $x_i \in P$. By hypothesis it also fixes a vertex v_i in the tree T. As the map f is equivariant every element $g \in E_i$ stabilizes a component Ω_i of $P \setminus \Lambda$ which contains x_i. Thus the group E_i is contained in the stabilizer $G_{\hat{v}_i}$ of the vertex $\hat{v}_i$ corresponding to the component Ω_i which is a vertex stabilizer of $\hat{T}$. The result now follows by the argument of [De-Po, Theorem 3.6]. ■

Acylindrical Superaccessibility Theorem (relative to a subset). *Let G be a finitely presented group and $E_1, ..., E_q$ a fixed finite set of infinite elementary subgroups of G. Suppose that the pair $(G, \mathcal{E})$ is one-ended and there is a finite bound for orders of finite subgroups of G. Then for each $K \in \mathbb{R}$ there exists a finite number of G-trees $T_1, ..., T_M$ such that all subgroups E_i are elliptic on T_j, and for every minimal (K, Φ)-acylindrical $(G, \mathcal{E})$-tree T, there exist an automorphism φ of G sending each group E_i into itself and a resolution $\varphi^*(T_i) \to T$ $(i \in \{1, ..., M\})$.*

This theorem in the torsion-free case (i.e. for K-acylindrical splittings) in the absolute form (i.e. without the claim about subgroups E_i) was proved by Z. Sela [Se1]. The absolute form of the case with torsion is given in [De]. The argument of [De] can be adapted to the relative case along the following lines.

Proof: Let Π be a finite 2-dimensional CW-complex with $\pi_1(\Pi) \cong G$ all of whose 2-faces are either bigons or triangles. Suppose also that Π contains subcomplexes B_i $(i = 1, ..., q)$ whose fundamental groups are isomorphic to E_i. One can construct a G-equivariant simplicial map $\rho : P \to T$ where P is the universal cover of Π. Let $\tilde{\Lambda}$ denote a lamination of P whose leaves are preimages under ρ of the midpoints of the edges of T. By construction, $\tilde{\Lambda}$ is a G-equivariant lamination and let Λ denote $\tilde{\Lambda}/G$. One defines a subgraph Λ_k of Λ by describing its intersection with each face Δ of Π. Namely $\Delta \cap \Lambda_k$ are those leaves of Λ in Δ whose image under ρ is situated within a distance at least k from the images of the vertices or the center of Δ. It is proven in [De] (see Lemma 1.5) that the action on T of the fundamental group of each connected component of Λ_k pointwise fixes a segment of the length k. It follows from the hypothesis that the fundamental group of each connected component of Λ_k is finite.

One can collapse all the leaves of Λ_k and all sub-complexes B_i to points. As the number of faces of Π and leaves in $\Lambda \backslash \Lambda_k$ is uniformly bounded, we note that the number of faces and edges of the resulting orbihedron Π' is uniformly bounded (here one uses the minimality of the tree T [De, Lemmas 2.1, 2.2]). Each vertex stabilizer of Π' is either finite or is one of the groups E_i. As the orders of finite subgroups of G are uniformly bounded, there are only finitely many orbihedrons with all these properties, so Π' must belong to a finite set of orbihedrons $\{\Omega_1, ..., \Omega_M\}$, with M depending only on the group G and the system of its subgroups E_i $(i = 1, ..., q)$.

There exists a simplicial map θ_k between the complexes Π' and Ω_k (for some $k \in \{1, ..., M\}$). This map induces an isomorphism $(\theta_k)_* : G \to \pi_1^o(\Omega_k)$ where $\pi_1^o(\Omega_k)$ is the fundamental group of Ω_k (in the sense of orbihedra). Notice that θ_k lifts to an equivariant map $\tilde{\theta}_k$ between P' and $\tilde{\Omega}_k$ which are the orbihedron universal covers of Π' and Ω_k correspondingly. If $\tilde{\theta}_k(x_i) = y_j$ where the stabilizer of the point $x_i \in P'$ is E_i and $y_j \in \Omega_k$ $(i, j \in \{1, ..., q\}, k \in \{1, ..., M\})$ then we have $(\theta_k)_*(C_j) \subset \mathrm{Stab}(y_j) = C_j$. After possibly replacing θ_k by a power we may suppose that $(\theta_k)_*(E_i) \subset E_i$. Following [De, Thm 3.1] let us consider the dual tree $\hat{\tau}$ to the lamination which is the image of the lamination Λ in P' and let T_k be the same for the orbihedron Ω_k. Arguing as in the proof of the Hierarchical Accessibility Theorem, we obtain that the groups E_i are elliptic in the tree $\hat{\tau}$ and there is an equivariant simplicial map $\hat{\tau} \to T_k$. The actions of the groups G and $\pi_1^o(\Omega_k)$ on the trees $\hat{\tau}$ and T_k respectively are conjugate by the map θ_k. Thus we have $\hat{\tau} = {\theta_k}^*(\tau_k)$ and the theorem follows. ■

Definition 4.3. *Let $\mathcal{F}$ be graph of groups decomposition of the pair $(G, \mathcal{E})$. We say that the graph $\mathcal{F}_1$ refines $\mathcal{F}$ if it is obtained from $\mathcal{F}$ by replacing of a vertex $v \in \mathcal{F}^0$ by a* **non-trivial** *graph of groups decomposition $\mathcal{F}_v$ of the pair $(G_v, \mathcal{E} \cap S)$, where S is the set of edge groups of $\mathcal{F}$.*

A sequence $\{\mathcal{F}_n\}$ of graphs of groups decompositions of $(G, \mathcal{E})$ is called a refining sequence if for every n the graph $\mathcal{F}_{n+1}$ refines $\mathcal{F}_n$. We call the refining sequence $\{\mathcal{F}_n\}$ stabilizing if there exists n_0 such that $\mathcal{F}_n = \mathcal{F}_{n_0}$ for all $n > n_0$; and non- stabilizing otherwise.

We need another accessibility result, which we are now going to prove, for refined sequences of splittings of finitely presented groups. Let G denote a finitely presented group equipped with the family $\mathcal{E}$ of elementary subgroups.

Suppose $\{\mathcal{F}_n\}$ is a sequence of decompositions of the pair $(G, \mathcal{E})$ so that the graph $\mathcal{F}_{n+1}$ is obtained from $\mathcal{F}_n$ by making an elementary refinement; i.e. the label of some vertex v of $\mathcal{F}_n$ is replaced by an elementary splitting $A *_C B$ or $A *_C$, in which all the edge groups of the graph $\mathcal{F}_n$ are elliptic. Collapsing a vertex is the inverse operation to the refinement. We call an edge e of a graph of groups of $(G, \mathcal{E})$ *non-trivial* if it is a loop or if the label of both of its vertices do not coincide with the label of e, otherwise we call e *trivial*. Likewise, we call a vertex v of valence two *trivial* if its label coincides with the label of one of the edges incident to it. Note that the label of a trivial vertex is necessarily an elementary subgroup of G.

Bestvina-Feighn's Accessibility Theorem [BeF2] ensures that there exists m such that all edges (and vertices) in $\mathcal{F}_n \setminus \mathcal{F}_m$ are trivial for $n > m$. Indeed, if it is not so then collapsing all the edges of the graph $\mathcal{F}_n$ whose labels coincide with the label of one of its vertices, we will obtain an irreducible graph of groups decomposition of $(G, \mathcal{E})$ with elementary edge stabilizers having an unbounded number of edges (when $n \to +\infty$); this is prohibited by [BeF2].

Suppose now that the group G admits a non-stabilizing sequence $\{\mathcal{F}_n\}$ then for some vertex v_m whose label is A_m we will have an infinite chain of elementary refinements :

$$A_m = A_{m+1} *_{C_{m+1}} C_m,\ A_{m+1} = A_{m+2} *_{C_{m+2}} C_{m+1},\ \dots\ A_{m+k} = A_{m+k+1} *_{C_{m+k+1}} C_{m+k}\ \dots\ , \tag{4}$$

where C_{m+k} is an *infinite* elementary subgroup of G (as G is one ended and splits over C_{m+k}). By Definition 4.3 each splitting in (4) is non-trivial, so we have $C_{m+k} \supsetneqq C_{m+k+1}$. It also follows that for all but finitely many indices $|C_{m+k} : C_{m+k+1}| < \infty$ as the rank of the maximal elementary group $\tilde{C}_{m+k}$ is finite. We obtain from (4) the following splitting in which all edges are trivial:

$$A_m = ((\dots((C_m *_{C_{m+1}} C_{m+1}) *_{C_{m+2}} C_{m+2}) *\ \dots\ *_{C_{m+k}} C_{m+k}) *_{C_{m+k+1}} A_{m+k}),\ \forall k \in \mathbb{N} \tag{4'}$$

Let $\mathcal{E}_m$ denote the union of $\mathcal{E}$ and the labels of the edges incident to the vertex v_m. We will need the following lemma.

Lemma 4.4. *Suppose that the pair $(G, \mathcal{E})$ is essentially non-maximal and admits a non-stabilizing sequence (4). Then the pair $(A_m, \mathcal{E}_m)$ splits as*

$$A_m = A *_C \tilde{C}_m, \quad \text{rank } C < \text{rank } \tilde{C},$$

where $C \subset \bigcap_{i \geq 1} C_{m+i},\ A \subset \bigcap_{i \geq 1} A_{m+i}$; and $\tilde{C}_m$ is maximal elementary subgroup of A_m containing C.

Remark 4.5. *We thank M. Bestvina for a suggestion how to prove this lemma. In the paper [Bo] a similar statement is proven (Theorem 6.1).*

Before we give the proof of the lemma we first provide an example of infinite non-stabilizing sequence of splittings which we borrow from [BeF1, p. 450].

Example. Take the free group $F_2 = F(x, y)$. Then we have the sequence of non-trivial splittings (compare with (4)):

$$F_2 = < x > *_{<x^2>} < x^2, y >;\ < x^2, y > = < x^2 > *_{<x^4>} < x^4, y >;\ < x^4, y > = < x^4 > *_{<x^8>} < x^8, y >\ \ \dots$$

Note that each of these splittings is non-trivial but all together they give a non-trivial splitting of F_2 where all edges are trivial:

$$F_2 = (\dots((< x > *_{<x^2>} < x^2 >) *_{<x^4>}) * \dots *_{<} x^{2k} >< x^{2k}, y >)\ \ \forall k \in N.$$

The group F_2 splits as $< x > * < y >$ where the edge group is obtained as $\mathrm{id} = \bigcap\limits_k < x^{2k} >$ and the other vertex group is

$$< y >= \bigcap_k < x^{2k}, y > .$$

Proof of Lemma 4.4. Note that, since all edge groups of the graph $\mathcal{F}_n$ are quasi-convex subgroups of G, it follows from the proof of [IKa, Lemma 3.5] that every vertex group of $\mathcal{F}_n$ is a quasi-convex subgroup of G. Then by [Sw], we have that A_m is a geometrically finite group, in particular, it is a finitely presented group.

For every $k \in \mathbb{N}$ let T_k denote the Bass Serre A_m-tree corresponding to the splitting (4') (m is fixed). Let P be a simply connected complex on which A_m acts co-compactly so that every subgroup $E_i \in \mathcal{E}$ fixes a point $p_i \in P$ ($i = 1, ..., q$).

Proceeding now as in the proof of the Acylindricity Theorem, we construct a A_m-equivariant simplicial map $f_k : P \to T_k$. To this end for a point $p_0 \in P$ we set $f_k(p_0) = x_0$ (e.g. the vertex whose stabilizer is C_m). Then extend this equivariantly by setting $f_k(gp_0) = gx_0$ ($g \in A_m$). Consider now the lamination Λ_k of the complex P which is the pullback by f_k of the midpoints of the tree T_k. The components of Λ_k are called tracks. Note that the tree T_k is obtained from T_{k+1} by collapsing the orbit of one edge. It follows from this construction that Λ_{k+1} is obtained by adding the A_m-orbit of one track (dual to the added edge in T_{k+1}) to the lamination Λ_k.

As the complex $\Pi = P/G$ is finite, there exists a natural number k_0 such that for all $k \geq k_0$ the tracks in $\Lambda_k \setminus \Lambda_{k_0}$ fall into finitely many families of non-equivalent under A_m mutually parallel tracks [Du]. Let C be the stabilizer of an increasing sequence (when $k \to \infty$) of such parallel tracks. The map f_k is equivariant so for every k there exists $n_k > k$ such that C_{n_k} contains C. As $C_k \subset C_{k-1}$ ($\forall\ k$) we obtain that $C \subset \bigcap\limits_k C_k$. The dual tree to this system of tracks gives rise to a splitting of A_m over C. Since the sequence C_k is strictly decreasing we also have $|C_m : C| = \infty$. Similarly, by the equivariance of f_k it follows that the stabilizers of the complementary components to the tracks are either subgroups of C_k or $\bigcap\limits_k A_k$.

Let X_m denote the corresponding graph of groups decomposition of A_m. The splitting given by X_m is non trivial (as the decomposition (4') is non trivial for every k) and is relative to the system of subgroups $\mathcal{E}_m$. So it refines the decomposition $\mathcal{F}_m$ of G. As the pair $(G, \mathcal{E})$ does not have essentially non maximal splittings, the splitting X_m is also essentially non maximal. Furthermore, by the choice of m, the graph X_m may only contain trivial edges and vertices. Thus, there is only one non-elementary vertex group $A \subset \bigcap A_i$, and all the other vertex groups are subgroups of C_m. Now the maximal elementary subgroup $\tilde{C}_m$ of A_m containing C is elliptic in X_m, so $\tilde{C}_m$ is conjugate either into A or into C_m. The former case is impossible by the non-triviality of the splitting, so $\tilde{C}_m = C_m$. Collapsing all vertices of X_m whose labels are elementary, we get the non-trivial one edge splitting $A_m = A *_C C_m$ (note that we cannot get HNN-extension which would be essentially non maximal in this case). The lemma is proved. ■

It follows from the Lemma that the group C is infinite as the pair $(G, \mathcal{E})$ is one-ended.

5 Finding a G-tree invariant under endomorphism

Let G be a Kleinian group and $\mathcal{E} = \{E_1, ..., E_k\}$ be a fixed finite family of elementary subgroups of G.

Suppose that the pair $(G, \mathcal{E})$ is not cohopfian. Then Proposition 3.1 tells us that $(G, \mathcal{E})$ splits as an amalgamated free product (or an HNN-extension) over an elementary subgroup. We get infinitely many such splittings in the following Proposition.

Proposition 5.1. *Let G be a non-elementary, geometrically finite, Kleinian group without 2-torsion endowed with the system $\mathcal{E}$. Then the following assertions are true:*

1) *If $f : (G, \mathcal{E}) \to (G, \mathcal{E})$ is a non-surjective monomorphism, then there exists a $(G, \mathcal{E})$-tree τ so that for every $n \in \mathbb{N}$ the tree $f^{n*}(\tau)$ is a non-trivial $(G, \mathcal{E})$-tree.*

2) *If in addition, the pair $(G, \mathcal{E})$ is one-ended and has no essentially non-maximal splittings then there exists a $(G, \mathcal{E})$-tree J such that for all $n \in \mathbb{N}$, the tree $f^{n*}(J)$ is a non-trivial, $\underline{(K, \Phi)\text{-acylindrical}}$ $(G, \mathcal{E})$-tree for some uniform constant K.*

Proof: We prove the first part of the proposition by induction on the length $l(\cdot, \mathcal{E})$ of a hierarchy of $(G, \mathcal{E})$. Note that by proposition 3.1 we have $l(G, \mathcal{E}) \geq 1$. By the Hierarchical Accessibility Theorem, $(G, \mathcal{E})$ splits as an amalgamated free product or HNN, $G = A *_C B$ or $G = A *_C$ $(C \in \mathcal{C})$ with

$$\max\{l(A, A \cap \mathcal{E}), l(B, B \cap \mathcal{E})\} < l(G, \mathcal{E}).$$

Let T denote the Bass-Serre tree dual to this splitting. If for all $n \in \mathbb{N}$ the trees $f^{n*}(T)$ are non-trivial, let $T = \tau$. If not, there exists $m \in \mathbb{N}$ such that up to conjugation $f^m(G)$ is a subgroup of A or B, say A. As f is injective, the subgroup A is a non-elementary group and $f^m(A) \subsetneqq A$.

As we have noticed in the previous chapter from [IKa], [Sw], it follows that A and B are finitely presented groups.

So let us first check the statement of the proposition when $l(G, \mathcal{E}) = 1$. Then $l(A, A \cap \mathcal{E}) = 0$ and a contradiction : the pair $(A, \mathcal{E} \cap A)$ is not cohopfian, so proposition 3.1 applied to A implies that A splits non-trivially relatively to $A \cap \mathcal{E}$.

Suppose now that $l(G, \mathcal{E}) > 1$; as f is a non-surjective monomorphism of $(A, A \cap \mathcal{E})$ and $l(A, \mathcal{E} \cap A) < l(G, \mathcal{E})$ we can apply the induction hypothesis to A. So there exists a non-trivial $(A, \mathcal{E} \cap A)$-tree T_A such that $f^{n*}(T_A)$ is a non-trivial $(A, \mathcal{E} \cap A)$-tree for all $n \in \mathbb{N}$. Let τ denote $f^*(T_A)$ which can be also considered as a G-tree (as $f(G) \subset A$). We get a sequence of G-trees $f^{n*}(\tau)$ $(n \in \mathbb{N})$, which are all non-trivial A-trees when restricted to A. Whence $f^{n*}(\tau)$ is a non-trivial G-tree ($\forall$ $n \in \mathbb{N}$).

If $E \in \mathcal{E}$ then by the induction hypothesis, the group $E \cap A$ is a subgroup of a vertex stabilizer of τ, say G_v. Thus, $f^{-n}(E \cap A)$ is contained in the vertex stabilizer $f^{-n}(G_v)$ of the tree $f^{n*}\tau$. Since $f(E_i) \subset E_i$ we obtain $E_i \subset f^{-n}(E_i \cap A) \subset f^{-n}(G_v)$. We have shown that the system $\mathcal{E}$ is elliptic in the trees $f^{n*}\tau$, and therefore $f^{n*}\tau$ is a non-trivial $(G, \mathcal{E})$-tree for every $n \in \mathbb{N}$. The first part of the proposition is proved.

The graph τ/G may be reducible. In this case we collapse in τ/G every edge whose label is equal to the label of one of its vertices. Denote J the universal cover (in the sense of Serre) of this new graph of groups decomposition of G. As G acts on τ without global fixed points, then obviously, G also acts without global fixed points on J, and the system $\mathcal{E}$ is elliptic on J. Furthermore, as the set of the non-elementary vertex stabilizers of the trees J and τ is the same, it follows from Part 1) that the trees $f^{n*}J$ are also non-trivial $(G,\mathcal{E})$-trees ($\forall\ n \in N$). The pair $(G,\mathcal{E})$ is one-ended and has no essentially non-maximal splittings, so Lemma 2.6 now yields that the reduced $(G,\mathcal{E})$-tree J is (K,Φ)-acylindrical for some uniform constant K. Then the trees $f^{n*}J$ are also (K,Φ)-acylindrical for the *same* constant K. Indeed, otherwise there is a segment l of length K on the tree $f^{n*}(J)$ whose pointwise stabilizer is an infinite subgroup C. Thus $f^n(C)$ is an infinite subgroup fixing pointwise the segment l on J too. This is impossible.

■

Proposition 5.2. *Suppose that the pair $(G,\mathcal{E})$ is one-ended and does not have essentially non-maximal splittings. Let $f:(G,\mathcal{E})\to(G,\mathcal{E})$ be a non-surjective monomorphism. Then there exist a non-trivial $(G,\mathcal{E})$-tree T with elementary edge stabilizers and a non-surjective monomorphism $F:(G,\mathcal{E})\to(G,\mathcal{E})$ such that $F(G_s)\subset G_s$ for every vertex (resp. edge) stabilizer G_s.*

Proof: Suppose $f:(G,\mathcal{E})\to(G,\mathcal{E})$ is an injective non-surjective endomorphism. We first claim that there exist a non-surjective monomorphism F of $(G,\mathcal{E})$ and a marked $(G,\mathcal{E})$-tree (T,t) which dominates $(F^*T,\tilde{t}\ )$ (see Preliminaries for the terminology).

To be able to apply the Acylindrical Superaccessibility Theorem, we consider the minimal G-subtree J_n of $f^{n*}J$. Let j be a marking of J and let $\tilde{j}_n$ be the marking of J_n containing j. By Proposition 5.1 there exists a tree J so that $f^{n*}(J)$ is a non-trivial (K,Φ)-acylindrical $(G,\mathcal{E})$-tree for some uniform constant K. Clearly, the same is true for the minimal subtree J_n.

By the Acylindrical Superaccessibility Theorem, there exists a family of $(G,\mathcal{E})$-trees $\tau_1,...,\tau_m$ such that for every *minimal* (K,Φ)-acylindrical $(G,\mathcal{E})$-tree τ, the tree τ_i dominates τ for some $i\in\{1,...,m\}$. Furthermore, the number of possible markings of the trees τ_i ($i\in\{1,...,m\}$) is finite (up to automorphism of $(G,\mathcal{E})$). So for a given resolution $\rho_k:\tau_k\to\tau$ we can find a marking $t_k\subset\rho_k^{-1}(t)$ of the tree τ_k such that $\rho_k:(\tau_k,t_k)\to(\tau,t)$. Thus, we obtain a finite number of marked trees $(\tau_1,t_1),...,(\tau_M,t_M)$ ($M\geq m$) such that for every minimal marked $(G,\mathcal{E})$-tree (τ,t) there exists a marked tree (τ_k,t_k) dominating (τ,t).

Passing to a subsequence, we can assume that there is a marked tree (τ_i,t_i) which dominates all the trees $(J_n,\tilde{j}_n)$. Note that, for every $k\in\mathbb{N}$ the tree $f^{k^*}\tau_i$ is a non trivial $(G,\mathcal{E})$-tree, as it dominates the non-trivial tree $(f^{k+n})^*J$ for some $n\in\mathbb{N}$.

Assuming (w.l.o.g.) that the above set of marked trees $\{(\tau_1,t_1),...,(\tau_M,t_M)\}$ contains (J,j), consider the following order relation on the set of indices $\{1,2,...,M\}$. We say that $i\geq k$ if there exists an injective endomorphism F of $(G,\mathcal{E})$ such that F is surjective iff f is, and the marked tree (τ_i,t_i) dominates the marked tree $(F^*(\tau_k),\tilde{t}_k)$. Note that this relation is transitive. Indeed, if $i\geq k$ then there is a resolution from the marked tree (τ_i,t_i) to the marked tree $(\varphi_1^*F_1^*\tau_k,\tilde{t}_k)$ where $\tilde{t}_k$ is a marking of the tree $\varphi_1^*F_1^*\tau_k$ containing t_k. If also $j\geq i$ then there is a resolution $(\tau_j,t_j)\to(\varphi_2^*F_2^*\tau_i,\tilde{t}_i)$ implying that (τ_j,t_j) resolves $(\varphi_2^*F_2^*\varphi_1^*F_1^*\tau_k,\tilde{t}_k)$. As each map F_i is surjective iff f is surjective, the transitivity of this relation follows.

As $M<+\infty$, we must have $l\geq l$ for some index $l\in\{1,2,...,M\}$. Therefore, there exists an injective endomorphism F of $(G,\mathcal{E})$ and a resolution $\rho_m:\tau_l\to F^*\tau_l$ sending the marking t_l to the marking $\tilde{t}_l$. Furthermore, the map F is surjective iff f is. Setting $T=\tau_l,\ t=t_l,\ \tilde{t}=\tilde{t}_l$ we obtained the marked $(G,\mathcal{E})$-tree (T,t) which dominates $(F^*T,\tilde{t}\ )$. This proves our claim.

We have $\rho_m(t) = \tilde{t}$. The resolution ρ_m is a composition of finitely many folds [BeF1], so it does not increase the number of G-orbites of edges of T. As $t \subseteq \tilde{t} = \rho_m(t)$ we obtain $\rho_m(t) = t = \tilde{t}$. Whence $G_s \subset F^{-1}(G_s)$ and so $F(G_s) \subset G_s$ for every vertex (resp. edge) stabilizer G_s of the tree T. The map F and the tree T satisfy the conclusion of the proposition. ■

6 Proof of Theorem B.

Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a non-elementary Kleinian group without 2-torsion equipped with a finite system $\mathcal{E} = \{E_1, ..., E_k\}$ of elementary subgroups and suppose that $F : (G, \mathcal{E}) \to (G, \mathcal{E})$ is a monomorphism of G sending each subgroup E_i into itself. Let $\tilde{E}_i$ denote E_i if E_i is finite and the maximal elementary subgroup of G containing E_i if E_i is infinite.

The aim of this section is to prove :

Theorem B. *Suppose that the pair $(G, \mathcal{E})$ is one-ended. Then $(G, \mathcal{E})$ is cohopfian if the following two conditions are satisfied:*

1) *The pair $(G, \mathcal{E})$ has no essentially non-maximal splittings over elementary subgroups.*

2) *The pair $(G, \mathcal{E})$ does not split as an amalgamated free product $G = A *_C \tilde{C}$, with $\tilde{C}$ maximal elementary such that the normal closure of the subgroup C in $\tilde{C}$ is of infinite index in $\tilde{C}$.*

Proof: Suppose $f : (G, \mathcal{E}) \to (G, \mathcal{E})$ is an injective endomorphism. If the pair $(G, \mathcal{E})$ is indecomposable over elementary subgroups then Proposition 3.1 implies that f is surjective. So we may assume that $(G, \mathcal{E})$ splits non-trivially over elementary subgroups. By Proposition 5.2 we can find a non-trivial $(G, \mathcal{E})$-tree T and an injective endomorphism F of $(G, \mathcal{E})$ sending each vertex (edge) stabilizer of T into itself. We will need the following two lemmas:

Lemma 6.1. *Suppose that there exists a graph of groups Y decomposition of $(G, \mathcal{E})$ and an endomorphism F of $(G, \mathcal{E})$ sending all vertex and edge groups of Y into themselves. If $F|_{G_v}$ is surjective for every vertex group G_v of Y then $F : G \to G$ is surjective too.*

Proof: If the graph Y is a tree of groups then the vertex groups G_v $(v \in Y^0)$ generate the whole group G, and so the map F is surjective. Assume then that Y is not a tree, fix a maximal subtree of Y, and let e be an edge which is not in the maximal subtree. By Proposition 5.2 there exists a resolution ρ and marking t, so that ρ sends the marked tree (T, t) to the marked tree (F^*T, t). Let a be a vertex of e. As e does not separate T it follows that there exists an element $g \in G$ and lifts $\tilde{a}_1$ and $\tilde{a}_2$ of a to the marking t such that $g(\tilde{a}_1) = \tilde{a}_2$. We want to show that g is in the image of F. As the subtree t is also a marking of the tree F^*T, there exists $g_1 \in G$ so that $F(g_1)(\tilde{a}_1) = \tilde{a}_2$ by definition of the G-action on the tree F^*T. This implies that the element $F(g_1) \cdot g^{-1}$ belongs to the stabilizer $G_{\tilde{a}_2}$ of the vertex $\tilde{a}_2$. By hypothesis F restricted to $G_{\tilde{a}_2}$ is surjective. So there exists $g_2 \in G$ for which $F(g_1) \cdot g^{-1} = F(g_2)$. It follows that the element g is in the image of F. The lemma is proved. ■

The next lemma shows that we have only to worry about non-elementary vertex groups.

Lemma 6.2. *Suppose that Y is a splitting of the pair $(G, \mathcal{E})$ which is essentially non-maximal and the pair $(G, \mathcal{E})$ satisfies condition 2) of Theorem B. Suppose also that F is an injective endomorphism sending every vertex (edge) group of Y into itself. If the map $F|_{G_v}$ is surjective for every non-elementary vertex group G_v of Y then $F : G \to G$ is surjective.*

Proof: Collapsing each edge of the graph $Y = T/G$ whose label is equal to the label of the vertex incident to it, we may assume that the splitting Y is irreducible (as this operation does not modify the non-elementary vertex stabilizers, so all the assumptions of the lemma remain valid for the new splitting). Let us now consider edge groups and elementary vertex groups of the graph $Y = T/G$. Since our group G is non-elementary, after collapsing all pairs of adjacent neighboring vertices v_1 and v_2 whose labels are elementary groups we still get a non-trivial splitting of G satisfying all the above properties. Similarly, if there is a vertex v whose vertex group G_v is elementary and such that there is a loop e emanating from v, then we collapse this loop to v. The resulting vertex group will be still elementary and the map F sends it into itself. So we may assume that every edge $e \in Y^1$ which is not a loop has at least one vertex $v \in \partial e$ whose label is a non-elementary group G_v. Moreover there is no loop of Y emanating from a vertex whose label is elementary.

We can also suppose that our graph Y does not have vertex groups which are *non-maximal* elementary groups. Indeed, if G_v is such a group then, since Y is an essentially non-maximal splitting, the maximal elementary subgroup $\widetilde{G}_v$ containing G_v is contained in some other vertex group, say $G_{v'}$. However, the group G_v is contained in the stabilizer of the edge belonging to the path between v' and v. This contradicts the irreducibility of the graph Y.

Let us now prove that the restriction $F|_{G_e}$ on every edge group G_e is surjective. Indeed, as F is injective the group $F^{-1}(G_e)$ is elementary (being isomorphic to $F(F^{-1}(G_e))$ which is a subgroup of G_e) and we have $F^{-1}(G_e) \supset G_e$ since $F(G_e) \subset G_e$. Let $v_1 \in \partial e$ be one of the vertices of e whose stabilizer G_{v_1} is not elementary. Since $F|_{G_{v_1}}$ is surjective, for any $y \in G_e$ there exists $x \in G_{v_1}$ so that $F(x) = y$. If another vertex $v_2 \in \partial e$ has also a non-elementary stabilizer then by the same reason and the injectivity of F we obtain that $x \in G_{v_2}$ and so $x \in G_e$. Now if, G_{v_2} is a maximal elementary subgroup of G we have $F^{-1}(G_{v_2}) \supset F^{-1}(G_e) \supset G_e$. Thus $F^{-1}(G_{v_2}) = G_{v_2}$ since G_e is infinite and is contained in the unique maximal elementary subgroup G_{v_2}. This shows that $x \in G_{v_2}$ and again $x \in G_e$.

Let us prove that F is surjective on every elementary maximal vertex group E_v ($v \in Y^0$) of the graph Y. Let e_i ($i = 1, ..., l$) be the edges incident to v and C_i be their labels. As there is no loop emanating from the vertex v we get a decomposition $G = A *_{C_v} E_v$, where C_v is the elementary group generated by C_i ($i = 1, ..., l$) and A is the group generated by labels of the vertices of $Y^0 \setminus \{v\}$. We already know that our map F is surjective on every edge group C_i and so it is surjective on C_v. Moreover applying the previous argument to each edge stabilizer of the tree T, we conclude that the restriction of F on every G-conjugate of C_v is also surjective. It now follows that F is surjective on the normal closure N_v of C_v in E_v. Since F maps the group E_v into itself, and N_v onto itself, it induces an injective map $\Phi : E_v/N_v \to E_v/N_v$. By the hypothesis 2) of the Theorem, the group E_v/N_v is finite, whence the map Φ is surjective, and so F is surjective on E_v. We have proved that F is surjective on every edge and every elementary vertex group of the graph Y. The conclusion now follows from the previous lemma. ■

The remaining part of the proof of Theorem B consists of two steps.

Step 1. Decomposition procedure.

Let Y be the graph of groups decomposition given by Proposition 5.2, and F the injective endomorphism of $(G, \mathcal{E})$, such that F sends every vertex and edge group of Y into itself. If $F|_{G_v}$ is surjective for every non-elementary vertex group G_v of Y then Lemma 6.2 implies that F (and so f) is surjective. We may therefore assume that there exists a non-elementary vertex group G_v of the graph Y such that $F|_{G_v}$ is not surjective. Then by Proposition 3.1 the pair $(G_v, \mathcal{E} \cup \mathcal{C}_v)$ splits non-trivially over elementary subgroups, where $\mathcal{C}_v$ is the set of labels of the edges of Y incident to the vertex v.

We claim now that every splitting Y_v of $(G_v, \mathcal{E} \cup \mathcal{C}_v)$ is essentially non-maximal and the pair $(G_v, (\mathcal{E} \cup \mathcal{C}_v))$ satisfies Condition 2) of the Theorem. Let C denote an edge stabilizer of Y_v and $\tilde{C}$ be the maximal elementary subgroup of G containing C. We want to show that the group $\tilde{C}_v = \tilde{C} \cap G_v$ is conjugate into some vertex group of the splitting Y_v. Let T_v and T denote respectively the corresponding Bass-Serre tree of the splittings Y_v and Y. As the splitting $Y_v = T_v/G_v$ refines the graph Y, it gives rise to a new splitting $\mathcal{Y}_v$ of $(G, \mathcal{E})$. Let $\mathcal{T}_v$ be the corresponding tree. The splitting $\mathcal{Y}_v$ of G is essentially non-maximal, so the group $\tilde{C}$ is conjugate into the label of some vertex v_1 of $\mathcal{T}_v$. If v_1 belongs to T_v there is nothing to prove. If not, $v_1 \in (\mathcal{T}_v \setminus \{v\})$, and so the group $\tilde{C}_v$ fixing the vertices v_1 and v of the tree T, also fixes a path between them pointwise. Thus $\tilde{C}_v$ is a subgroup of an edge group of the graph Y and by hypothesis is elliptic in the splitting Y_v as was promised. Similarly the pair $(G_v, \mathcal{E} \cup \mathcal{C}_v)$ is one-ended. As every splitting of the pair $(G_v, \mathcal{E} \cup \mathcal{C}_v)$ over elementary subgroups refines Y we obtain that $(G_v, \mathcal{E} \cup \mathcal{C}_v)$ does not split as $G_v = A *_C \tilde{C}$ where the normal closure of C in $\tilde{C}$ is a subgroup of infinite index of $\tilde{C}$.

All the edge groups of the graph $Y = T/G$ are quasi-convex subgroups of G. The results [IKa, Lemma 3.5] and [Sw] imply that every vertex stabilizer G_v is a geometrically finite group, and so is a finitely presented group [Ra]. Proposition 5.2 applies to the vertex group G_v giving an injective endomorphism F_v of G_v which sends all vertex (edge) groups of X_v to themselves and which is surjective iff F is. We now decompose relatively to the edge groups all other non-elementary vertex groups of the graph Y, then pass to all non-elementary vertex groups obtained further etc. The following Lemma guarantees that the decomposition procedure stops.

Lemma 6.3. *This refining decomposition procedure stops after finitely many steps.*

Proof: By [BeF2] there exists a constant $\nu(G)$ such that every graph of groups decomposition of G with elementary edge groups can contain at most $\nu(G)$ non-trivial edges and vertices. Denote by Y_n the graph of groups decomposition of $(G, \mathcal{E})$ which we obtain after n refining decompositions described above, and let T_n denote the corresponding Bass-Serre tree. Suppose that the sequence Y_n does not stabilize. Then there exists $n_0 > \nu(G)$ such that every component $Y_n \setminus Y_{n+1}$ $(n > n_0)$ can only contain trivial vertices and edges (see Chapter 4 for the definitions). So there exists a vertex v_{m_0} $(m_0 = m(n_0) > n_0)$ of the graph Y_{n_0} whose label is a non-elementary group G_{m_0} such that the above decomposition procedure gives us the following refining sequence:

$$G_{m_0} = A_1 *_{C_2} C_1, \ A_1 = A_2 *_{C_3} C_2, \ ...., \tag{5}$$

where A_i are non-elementary and C_i are elementary vertex groups and $C_i \subset C_{i-1}$ $(i = 1, 2, ...)$. Then Lemma 4.4 implies that pair $(G_{m_0}, (\mathcal{E} \cup \mathcal{C}_{m_0} \cap G_{m_0}))$ splits as

$$G_{m_0} = A *_C \tilde{C}_{m_0}, \tag{6}$$

where $\mathcal{C}_{m_0}$ is the set of the stabilizers of edges of T_{n_0} incident to the vertex v_{m_0}, and $\tilde{C}_{m_0}$ is the maximal elementary subgroup of G_{m_0} containing all C_i. Furthermore, by Lemma 4.4 $\mathrm{rank}(C) < \mathrm{rank}(\tilde{C}_{m_0})$. We are now going to replace the infinite refining sequence (5) by one splitting (6) over a subgroup of smaller rank. To this end, we apply our machinery described in Chapter 5 to the splitting (6). Let F_{m_0} be the endomorphism of G_{m_0} obtained according to this procedure. It preserves the first splitting $G_{m_0} = A_1 *_{C_2} C_1$ in (5), sending each vertex (edge) group of it into itself.

Let t_{m_0} denote the Bass-Serre tree corresponding to the splitting (6). We claim that $(F^l_{m_0})^* t_{m_0}$ is a non-trivial $(G_{m_0}, (\mathcal{E} \cup \mathcal{C}_{m_0} \cap G_{m_0}))$-tree for all $l \in \mathbb{N}$ (compare with Proposition 5.2). For otherwise, $F^k_{m_0}(G_{m_0}) \subset A$ for some $k \in \mathbb{N}$. We have also $F_{m_0}(\tilde{C}_{m_0}) \subset \tilde{C}_{m_0}$, and so $F^k_{m_0}(\tilde{C}_{m_0}) \subset (A \cap \tilde{C}_{m_0} = C)$ which is impossible since $\mathrm{rank}(C) < \mathrm{rank}(\tilde{C}_{m_0})$ and F is injective.

As the tree $(F^k_{m_0})^* t_{m_0}$ refines the graph Y_{n_0} of G and $n_0 > \nu(G)$, it may contain only one conjugacy class of non-elementary vertex stabilizers and all its edge stabilizers are conjugate into C. Collapsing all vertices in the graph $({F^k_{m_0}}^*(t_{m_0}))/G_{m_0}$ whose labels are elementary, we reduce it to an edge of groups such that the label of the edge is an infinite index subgroup of the label of one of the vertices which is a maximal elementary subgroup. So without changing the notations, we may assume (w.l.o.g.) that F_{m_0} sends vertex (edge) groups of the splitting (6) into themselves. We now refine the splitting given by the graph Y_{n_0} by replacing the vertex group G_{m_0} by the splitting (6) and retain the same notation Y_{n_0} for the new splitting.

Similarly, if the decomposition procedure for the pair $(A, (\mathcal{E} \cup \mathcal{C}_{m_0} \cap A))$ does not stop after finitely many steps there exists a decomposition $A = B *_K \tilde{K}_A$ where B is non-elementary and K is an infinite index subgroup of the maximal elementary subgroup $\tilde{K}_A$ of A containing K. We get a splitting of G_{m_0} which refines the splitting Y_{n_0} giving the new graph Y_{n_0+1} of groups decomposition of $(G, \mathcal{E})$. By the argument given before Lemma 6.3 all these splittings are essentially non-maximal relatively to the edge groups. Let T_{n_0+1} denote the Bass-Serre tree corresponding to the splitting Y_{n_0+1}.

We now claim that if $g\tilde{K}g^{-1} = \tilde{C}$ $(g \in G)$ then

$$\mathrm{rank}(K) < \mathrm{rank}(C).$$

Indeed, up to conjugation we may assume that either $C \subset B$ or $C \subset \tilde{K}_A$. In the latter case the claim follows from Lemma 4.4 (as $C = \tilde{K}_A$ in this case). If $C \subset B$ we have the decomposition $G_{m_0} = \tilde{K}_A *_K B *_C \tilde{C}_{m_0}$. Note that $\tilde{C}_{m_0}$ is the maximal elementary subgroup of G (i.e. $\tilde{C} = \tilde{C}_{m_0} \supset C$). Indeed for otherwise, $\tilde{C}_{m_0}$ fixes a path in T_{n_0+1} between two vertices which are stabilized by $\tilde{C}_{m_0}$ and $\tilde{C}$. As the vertex stabilized by $\tilde{C}_{m_0}$ is of valence one in the tree T_{n_0+1}, this path contains the edge stabilized by C, and so $C = \tilde{C}_{m_0}$ which is impossible. Thus $\tilde{C} = \tilde{C}_{m_0}$. Let w be the vertex of T_{n+1} fixed by $\tilde{K}$. Then $\tilde{C}$ fixes the vertex gw. By the above argument the stabilizer of gw is $\tilde{C}_{m_0} = \tilde{C}$. So we obtain that $\tilde{K} = \tilde{K}_A = g^{-1}\tilde{C}g$ with $g \in B$. This would imply that $\tilde{C}$ is conjugate into A contradicting the non-triviality of the splitting (6). This proves our claim.

It now follows that there are only finitely many decompositions of A refining the graph Y_{n_0} over subgroups which are conjugate into the **same** maximal elementary subgroup $\tilde{C}$ of G. On the other hand, the group G is geometrically finite, so it can contain only finitely many conjugacy classes of maximal parabolic subgroups [Ra]. We deduce that the above decomposition procedure will necessarily terminate after finitely many steps. The lemma is proved. ■

Step 2. Surjectivity of f.

Our process of decomposition of the group G has a structure of a rooted tree which we shall describe now. By Lemma 6.3 this tree $\mathcal{T}$ is finite, and can be written as $\mathcal{T} = \bigcup_{n=1}^{M} V_n$. The initial group G corresponds to the root vertex O. Each vertex x of $\mathcal{T}$ belongs to set V_n of vertices of level n for some $n \in \{1, ..., M\}$. Every vertex of level ≥ 2 has a unique *parent*. The parent vertex X corresponds to a group G_X with a fixed graph of groups decomposition for which G_x is one of the vertex groups (we borrow this family terminology from the paper [BiJo]). In its turn the vertex x will have a collection of "children" $V(x) \subset V_{n+1}$ which correspond to vertex groups of the graph of groups decomposition of the group G_x. Edges of the tree $\mathcal{T}$ indicate "family ties" between "parents" and "children". Furthermore, by Proposition 5.2 to each vertex $x \in V_n$ we associate an endomorphism $F_x : G_x \to G_x$ which preserves the splitting of G_x sending the labels of the "children" of x in $V(x) \subset V_{n+1}$ into themselves.

Those vertices $v \in V_n$ which are either elementary or indecomposable over elementary subgroups (relatively to the edge groups) will be terminal vertices of the tree $\mathcal{T}$. For every non-terminal vertex $x \in V_n$ we apply the decomposition procedure described on Step 1 to get vertex groups $V(x) \in V_{n+1}$ and the corresponding endomorphism F_x sending them to itself etc.

After descending along the tree $\mathcal{T}$ we reach the final level V_M all of whose vertices are terminal (of course there could be some terminal vertices of $\mathcal{T}$ belonging to other levels). Now we are going to go up in order to prove the surjectivity of the original map f. Each vertex $w \in V_{M-1}$ is either terminal or there is a set of its "children" $x_i \in V(w) \subset V_M$ which are all terminal. In the former case the map F_w is surjective. In the latter case by Proposition 3.1 it follows that $F_w|_{G_{x_i}}$ is surjective for every non-elementary vertex group $x_i \in V(w)$. Then by Lemma 6.2 we obtain that F_w is surjective for every $w \in V_{M-1}$. Similarly, $w \in V(u)$ for some vertex $u \in V_{M-2}$ (the "parent" of w). We have by Proposition 5.2 that the corresponding maps $F_u|_{G_w}$ and F_w are surjective or not simultaneously. Therefore, $F_u|_{G_w}$ is surjective for all $w \in V(u)$ whose labels are non-elementary. Again by Lemma 6.2 F_u is surjective and so on.

Applying this procedure finitely many times we finally arrive at the first level V_1 of $\mathcal{T}$ corresponding to the vertices of the graph Y. We have just shown that for all non-terminal vertices $v \in V_1$ the maps F_v are surjective and, so the map $F|_{G_v}$ is surjective. Similarly, Lemma 6.2 implies that the map $F : G \to G$ is an automorphism of G. Then our initial map $f : G \to G$ is an automorphism too.

To finish the proof we only need to show that $f|_E$ is surjective on every $E \in \mathcal{E}$. As $f(E) \subset E$ the conclusion is obvious when E is finite. If it is not the case then by the uniqueness of the maximal elementary subgroup $\tilde{E}$ of G containing E we have $f(\tilde{E}) = \tilde{E}$ as $f(E)$ is an infinite subgroup of both. So $f|_{\tilde{E}} : \tilde{E} \to \tilde{E}$ is an automorphism. Then using the fact that any increasing sequence of subgroups of a virtually abelian group of finite rank must stabilize, we deduce that $f(E) = E$. Theorem B is proved. ■

7 Necessary Condition in Theorem A.

The necessary condition in Theorem A follows directly from the following result:

Theorem D. *A finitely generated discrete group* $G \subset \mathrm{Isom}_+\mathbb{H}^n$ *is not cohopfian if one of the two conditions below is satisfied:*

1) *G has an essentially non-maximal splitting*

$$G = \pi_1(X, G_v, C_e), \text{ where each vertex group } C_e \text{ is elementary} \tag{1'}$$

2) *The group G splits as an amalgamated free product* $G = \Gamma *_C \tilde{C}$*, so that* $\tilde{C}$ *is a maximal elementary subgroup of G and the normal closure of C in* $\tilde{C}$ *is a subgroup of infinite index of* $\tilde{C}$*.*

We start with:

Remarks 7.1. *1) In particular infinite elementary subgroups of Kleinian groups are not co-hopfian (case 2 with* $\Gamma = C = 1$*).*

2) Examples of discrete geometrically finite groups in $\mathbb{H}^n$ *which are described in 1) and 2) of Theorem D, exist, see [OP].*

3) If every elementary group over which G splits is in fact abelian and if there exists a splitting X described in Condition 2) then there is another splitting over elementary subgroups which is essentially non-maximal: the maximal elementary vertex group can be written as a central HNN-extension with a base containing all corresponding edge stabilizers. However, this is not the case in general, as there exists (torsion-free) virtually abelian groups with finite abelianization.

We will first study essentially non-maximal splittings of G:

Proposition 7.2. *Suppose G splits as a graph of groups (1'), where one of the edge groups* $C_e = E$ *of the splitting (1') is essentially non-maximal, and let* $\tilde{E}$ *be the maximal elementary subgroup of G containing E with infinite index so that* $\tilde{E}$ *is hyperbolic in the splitting (1'). Then there exists an element* $g \in \tilde{E}$ *so that g centralizes E and* $g^n \notin E$ $(n \in \mathbb{N})$*.*

Proof of 7.2: Let T be the Bass-Serre tree corresponding to the splitting (1'). The group $\tilde{E}$ contains a normal free abelian subgroup $\tilde{A}$ of finite index. As $\tilde{E}$ acts on T hyperbolically it follows that the group $\tilde{A}$ also does. Hence by [S, I-6.5, Proposition 27] it follows that $\tilde{A}$ leaves a line $L \subset T$ invariant. As $\tilde{A}$ is normal in $\tilde{E}$ the group $\tilde{E}$ also leaves L invariant. Then either $\tilde{E}$ acts by translations on L; or it acts dihedrally on L (permuting the end points of L). So, there is a projection η of $\tilde{E}$ on $\mathbb{Z}$ or onto $\mathbb{Z}_2 * \mathbb{Z}_2$. Moreover since the subgroup E of $\tilde{E}$ fixes an edge e in T it fixes the axis L pointwise. So we may suppose that $e \subset L$ and that E is the kernel of η (which is the kernel of the action of $\tilde{E}$ on L). It follows that up to passing to a subgroup of index 2 and retaining the notation $\tilde{E}$ for it, we have the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] & \tilde{E} \arrow[r, "\eta"] & \mathbb{Z} \arrow[r] & 1, \end{tikzcd}$$

Let t denote the element of $\tilde{E}$ which is mapped on the generator of $\mathbb{Z}$, so we have $t^n \notin E$ $(\forall n \in \mathbb{N})$. There exits $m \in \mathbb{N}$ so that $t^m \in \tilde{A}$ and up to replacing t by t^m and passing to a further subgroup of finite index we may suppose that $t \in \tilde{A}$. Also $t^n \notin E$ $(\forall n \in \mathbb{N})$.

Let A denote the group $\tilde{A} \cap E$ which is a normal abelian subgroup of E of finite index. We have:

$$0 \longrightarrow A \longrightarrow E \xrightarrow{\xi} F \longrightarrow 1, \qquad (*),$$

where F is a finite group.

Definition. An automorphism of E will be called the *automorphism of the sequence (*)* if its restriction to A is trivial and if it induces the identity on F. The group of the automorphisms of (*) is denoted $\mathrm{Aut}(*)$.

Let $s : F \to E$ be a set theoretic cross-section of ξ and $\psi \in \mathrm{Aut}(*)$. Put $c_\psi(f) = \psi(s(f))s(f)^{-1}$, $\forall f \in F$.

Lemma 7.3. *The following assertions hold:*

a) *$c_\psi(f)$ is a 1-cocycle of F taking values in A.*

b) *For each $f \in F$ the map $\psi \to c_\psi$ determines a group homomorphism*

$$Aut(*) \to Z^1(F, A).$$

Proof: a) Notice first that $c_\psi(f) \in A$ since ψ induces the identity map on F. We have then: $s(f \cdot g) = s(f) \cdot s(g) \cdot \alpha(f, g)$, where $\alpha(f, g) \in A$.

$\psi(s(f \cdot g)) \cdot (s(f \cdot g))^{-1} = \psi(s(f)s(g)\alpha(f, g)) \cdot \alpha^{-1}(f, g)(s(f)s(g))^{-1} = \psi(s(f))\psi(s(g))s(g)^{-1}s(f)^{-1}$, since $\psi(a) = a$ ($\forall a \in A$). Further we derive :

$\psi(s(f \cdot g)) \cdot (s(f \cdot g))^{-1} = \psi(s(f))s(f)^{-1} + s(f)\psi(s(g))s(g)^{-1}s(f)^{-1} = \psi(s(f))s(f^{-1}) + \rho(f)c_\psi(f) = c_\psi(f) + \rho(f)c_\psi(f)$, where $\rho(f)$ denotes the action of $f \in F$ on A given by conjugation by $s(f)$. This proves a) by the definition of a cocycle (see [Br, p. 88]).

b) $c_{\psi_1\psi_2}(f) = \psi_1\psi_2(s(f))s(f)^{-1} = \psi_1[\psi_2(s(f))s(f)^{-1}s(f)]s(f)^{-1} = \psi_1(c_{\psi_2}(f) \cdot s(f))s(f)^{-1} = c_{\psi_2}(f) + c_{\psi_1}(f)$, here we used that $c_{\psi_2}(f) \in A$ and that ψ_1 keeps it unchanged. We have proved b). The lemma is proved. ■

Proof of the proposition. Recall that $t^n \in \tilde{A} \setminus E$ ($\forall\ n \in \mathbb{N}$). Let ψ be an inner automorphism of $\tilde{E}$ given by the conjugation via t. As t acts identically by conjugation on A it is easy to verify that it also induces the identity on F, i.e.:

$$\hat{t}f\hat{t}^{-1} = f,\ \forall\ f \in\ F,$$

where $\hat{t} = \xi(t)$. So ψ is an automorphism of the sequence $(*)$ and we get $c_\psi(f) = ts(f)t^{-1}s(f)^{-1}$. Since the group F is finite the first cohomology group $H^1(F, A)$ is finite too and, so there exists $p \in \mathbb{N}$ such that $c_{\psi^p}(f)$ is a coboundary. It follows that there exists $a \in A$ that $c_{\psi^p}(f) = a - \rho(s(f)) \cdot a = a + \rho(s(f))(-a)$. Writing this in the multiplicative form we have $c_{\psi^p}(f) = as(f)a^{-1}s(f)^{-1} = t^p s(f)t^{-p}s(f)^{-1}$ implying that $\forall\ s(f) \in E:\ a^{-1}t^p s(f)t^{-p}a = s(f)$. Putting $g = a^{-1}t^p$ we obtain that g is not trivial ($t^p \notin A$) and centralizes E. The proposition follows.

■

Note that above we also obtained the following fact which will be used further:

Remark 7.4. *The group of the automorphisms of the sequence (*) is finite modulo conjugation in E (in other words the subgroup of* $\mathrm{Out}(E)$ *which preserves (*) is finite)*

Indeed in the above proof for some power $p \in \mathbb{N}$ of $\psi \in \mathrm{Aut}(*)$ we will have:
$\psi^p(s(f))s(f)^{-1} = as(f)a^{-1}s(f)^{-1}$. Thus $\forall e \in E$ $\psi^p(e) = aea^{-1}$ since $e \cdot s(f^{-1}) \in A$ for some $f \in F$ and ψ^p is the identity on A. ■.

Lemma 7.5. *Suppose that the group G splits as a graph of groups $G = \pi_1(X, G_v, C_e)$ with elementary edge stabilizers such that one of the edge group $E = C_e$ is essentially non-maximal then G splits as an amalgamated free product or an HNN-extension:*

$$G = A *_K B, \text{ or } G = A*_K, \tag{6}$$

where K is essentially non-maximal and contains E.

Proof:

Let T denote the tree which is the universal covering of X and let $\tilde{E}$ be the maximal elementary subgroup of G containing E. The group $\tilde{E}$ acting on T without fixed points has an invariant line $L \subset T$ (see the beginning of the proof of Proposition 7.2). Since the subgroup E fixes a point in T it also fixes L pointwise. Let $\hat{e} \subset L$ be an edge of T and $\hat{\alpha}$ and $\hat{\beta}$ its vertices. We first claim that the stabilizer F of the edge $\hat{e}$ in G coincide with the stabilizer K of $\hat{e}$ in $\tilde{E}$ (i.e. the kernel of the action of $\tilde{E}$ on L). Indeed, both groups contain the group E which is an infinite group so by the uniqueness of the maximal elementary subgroup $\tilde{E}$ containing E it follows that $F \subset \tilde{E}$ which implies that $F = K$. Similar argument shows that the subgroup N of G leaving the line L invariant coincide with $\tilde{E}$. Indeed, the group N is elementary which follows from the fact that it has a projection to $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$ whose kernel is the elementary group K, consequently N is an elementary group containing $\tilde{E}$, and thus $\tilde{E} = N$.

Let α, β, e denote the images in X of $\hat{\alpha}$, $\hat{\beta}$, $\hat{e}$ respectively under the projection $p : T \to X$. Let us first consider the case when e does not separate the graph X. Then the group G is the HNN-extension $G = A*_K = \langle A,\ t \mid tKt^{-1} = \phi(K)\rangle$ where A is the fundamental group of the graph of groups $Y = X \setminus e$. Denote $\hat{Y}$ the component of the preimage $p^{-1}(Y)$ adjacent to the edge $\hat{e}$ at the point $\hat{\alpha} \in T$. Clearly $p(\hat{Y}) = Y$, so we may assume up to conjugation in G that the stabilizer of Y is A. As none G-translate of $\hat{e}$ is contained in $\hat{Y}$ and T is a tree we have $L \cap \hat{Y} = \{\hat{\alpha}\}$. Let $h \in \tilde{E} \setminus E$ be an element acting on L by translations. As $h(\hat{\alpha}) \in L \setminus \{\hat{\alpha}\}$, so h can not belong to the stabilizer of $\hat{Y}$. Consequently, the group $\tilde{E}$ is hyperbolic in the splitting $G = A*_K$.

The case when the edge e separates X is similar: we obtain the splitting $G = A *_K B$ where A and B are the fundamental groups of the graphs which are respectively connected components U and V of $X \setminus e$. Denote $\hat{U}$ and $\hat{V}$ the components of $p^{-1}(U)$ and $p^{-1}(V)$ which are adjacent along the edge $\hat{e}$ in T, in particular $\hat{\alpha} \in \hat{U}$ and $\hat{\beta} \in \hat{V}$. The stabilizers $\hat{U}$ and $\hat{V}$ are up to conjugation the groups A and B. Again we have $\hat{U} \cap L = \{\alpha\}$ and $\hat{V} \cap L = \{\beta\}$. There exists an element $h \in \tilde{E} \setminus K$ which acts by translations on L so $h(\hat{\alpha}) \in L \setminus \{\hat{\alpha}\}$, and the same for $\hat{\beta}$. Consequently, the element h does not belong to the stabilizers of $\hat{U}$ and $\hat{V}$. This shows that $\tilde{E}$ is not elliptic with respect to the splitting $G = A *_K B$. The lemma is proved. ■

Proof of Theorem D : Let us first consider **Condition 1)** of the theorem which is:

1) *G has an essentially non-maximal splittings.*

Then it follows from Lemma 7.5 that there is a splitting (6) of G as an amalgamated free product or an HNN-extension which is essentially non-maximal. The edge group K is an elementary essentially non-maximal subgroup and let $\tilde{K}$ be the maximal elementary subgroup of G containing K which is hyperbolic in the splitting (6). By Proposition 7.2 it follows that there exists an element $t \in \tilde{K} \setminus K$ which centralizes K.

Consider first the case of amalgamated product, i.e. $G = A *_K B$. Let us define the map $f : G \to G$ so that $f(a) = tat^{-1}$, and $f(b) = b$ ($\forall a \in A$, $\forall b \in B$). As t commutes with all elements from K, the map f is obviously a homomorphism. Furthermore, if $a \in A \setminus K$ then $tat^{-1} \in tAt^{-1} \setminus K$. So the group $G_1 = f(G)$ is isomorphic to the amalgamated free product $tAt^{-1} *_K B$, and every element $g \in G_1$ has the following normal form:

$$g = ta_1t^{-1} \cdot b_1 \cdot \ \dots \ \cdot ta_kt^{-1} \cdot b_k, \text{ or } g = b_1 \cdot ta_1t^{-1} \cdot \ \dots \ \cdot b_k \cdot ta_kt^{-1}, \ a_i \in A \setminus K, \ b_j \in B \setminus K. \quad (7)$$

If now $g = f(\gamma) = 1$ for some $\gamma \in G$, then using (7) it is easy to see that $\gamma \in A$ or $\gamma \in B$. So by injectivity of f on A and B we obtain $\gamma = 1$. Thus $f : G \to G_1$ is an isomorphism. We need only to show that $G_1 \subsetneqq G$.

The group G_1 being a subgroup of G acts on the Bass-Serre G-tree T corresponding to the splitting $G = A *_K B$. Denote by α and β the vertices of T whose stabilizers are A and B. Set $d = \min\limits_{g \in G} \ \mathrm{dist}_T(gtg^{-1}(\alpha), \alpha)$ then $d > 1$ since t is not conjugate into A and B. In particular, t cannot normalize A (since otherwise tAt^{-1} fixes the path between $t(\alpha)$ and α pointwise and so tAt^{-1} is conjugate into K which is impossible). An easy calculation shows that

$$\min_{\alpha} \mathrm{dist}_T(g(\alpha), \alpha) = 2k(d+1), \text{ where } \alpha \in \{a, b\}.$$

It follows that the set $\mathcal{L}(G_1)$ of translation lengths of elements of G_1 acting on T is equal to $\{2k(d+1) \mid k \in \mathbb{N}\}$ which is a proper subset of $\mathcal{L}(G)$ (e.g. $2 \in (\mathcal{L}(G) \setminus \mathcal{L}(G_1))$. This shows that $G_1 \subsetneqq G$.

Consider now the case of an HNN-extension $G = A*_K = \langle A, h \mid hKh^{-1} = \phi(K)\rangle$ and let T be the corresponding Bass-Serre tree. There are two more subcases: a) $h \in \tilde{K}$ and b) $h \notin \tilde{K}$. In the subcase a) we proceed as follows. By the proof of Proposition 7.2 there exist $p \in \mathbb{N}$ and $a \in \mathcal{A}(K)$ so that the element $g = h^p \cdot a$ commutes with every element of K, where $\mathcal{A}(K)$ is the maximal abelian subgroup of K. Now we define the map $f : G \to G$ to be the identity on A and set $f(h) = h^{p+1} \cdot a$. It is easy to check that f is an injective endomorphism (since h^{p+1} acts by conjugation on K in the same way as h does) which is not surjective.

In the subcase b) we proceed similarly to the case of an amalgamated free product, namely put $f(a) = tat^{-1}$, $\forall a \in A$ and $f(h) = h$, where t is an element in $\tilde{K} \setminus K$ acting hyperbolically on the tree T and centralizing K. Then any element g of the group $G_1 = f(G) = tAt^{-1}*_K = \langle tAt^{-1}, \ h | hKh^{-1} = \phi(K)\rangle$ can be written as:

$$g = ta_1t^{-1} \cdot h^{\varepsilon_1} \cdot ta_2t^{-1} \cdot \dots \cdot ta_kt^{-1} \cdot h^{\varepsilon_k}, \text{ or } g = h^{\varepsilon_1} \cdot ta_1t^{-1} \cdot h^{\varepsilon_2} ta_2t^{-1} \cdot \dots \cdot ta_kt^{-1}, \quad (7)$$

where $\varepsilon_i \in \mathbb{Z}$ and if $\varepsilon_i < 0$ and $a_i \in K$ then $\varepsilon_{i+1} \leq 0$, and if $\varepsilon_i > 0$ and $a_i \in K$ then $\varepsilon_{i+1} \geq 0$.

Let α be the vertex of T whose label is A and M be the line in T which is formed by the vertices $h^n(\alpha)$ $(n \in \mathbb{Z})$. As $h \notin \tilde{K}$ the element t does not belong to the maximal elementary subgroup containing h and so $t(\alpha) \notin M$. It is now straightforward that the translation length $l(g)$ of the element g in (7) is equal to $2kd + \sum_{i=1}^{k} |\varepsilon_i|$, where $d = \min_{g\in G}\ \mathrm{dist}_T(gtg^{-1}(\alpha), \alpha)$. Indeed each term ta_it^{-1} in (7) adds $2d$ to the expression of $l(g)$ and the term h^{ε_i} contributes $|\varepsilon_i|$ to it. For a fixed $d \geq 1$ the set $\{2kd + \sum_{i=1}^{k} |\varepsilon_i| \mid k \in \mathbb{N},\ \varepsilon_k \in \mathbb{Z}\}$ is a proper subset of $\mathbb{N}$ which is the set of translation lengths of G acting on T. This shows that in this case G_1 is a proper subgroup of G. Part 1) of Theorem D is proved.

Consider now **Condition 2)** of Theorem D which is:

The pair $(G, \mathcal{E})$ splits as an amalgamated free product $G = \Gamma *_C \tilde{C}$, so that $\tilde{C}$ is a maximal elementary subgroup of G and the normal subgroup of $\tilde{C}$ generated by C is of infinite index in $\tilde{C}$.

Suppose that G splits as an amalgamated free product $G = G *_C E_v$, where v is vertex whose label is a maximal elementary subgroup $E_v = \tilde{C}$. We are going to construct a proper monomorphism from G into G which is the identity on B and which sends E_v into itself being not surjective on it. We denote N_v the normal subgroup of E_v generated by C; by hypothesis $|E_v : N_v| = \infty$.

The group E_v is virtually abelian of finite rank, let A be a finite index normal free abelian subgroup of E_v. Denote $D = A \cap N_v$ and $F = E_v/A$. As N_v and A are normal in E_v the group D is normal in E_v too. Let also $s : F \to E_v$ be a normalized cross-section of the projection of E_v onto F. The group F acts on A by conjugation $a \to s(f)as^{-1}(f)$. Consider the vector space $A \otimes \mathbb{Q}$ which we equip with a scalar product invariant under the induced action of F. As the subspace $D \otimes \mathbb{Q}$ is invariant under the induced action of F there exists a subspace V in $A \otimes \mathbb{Q}$ complementary to $D \otimes \mathbb{Q}$ which is also invariant under this action. We can now find a subgroup B of A so that $V = B \otimes \mathbb{Q}$ and so $A \otimes \mathbb{Q} = (D \otimes \mathbb{Q}) \oplus (B \otimes \mathbb{Q})$. The group B has the following properties: $B \cap D = \{\mathrm{id}\}$ (since A is torsion free); B is normal in E_v; and the group $A' = D \oplus B$ is a normal free abelian subgroup of E_v of finite index. Denote $F' = E_v/A'$. Consider the map $h_n : A' \to A'$ defined as $h_n(d + b) = d + nb$ for every $d \in D$ and $b \in B$. One can now find a group H_n and a homomorphism $\varphi_n : E_v \to H_n$ so that the following diagram commutes :

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "i"] \arrow[d, "h_n"'] & E_v \arrow[r, "p"] \arrow[d, "\varphi_n"'] & F' \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & A' \arrow[r, "i_n"] & H_n \arrow[r, "p_n"] & F' \arrow[r] & 1.
\end{tikzcd}$$

In fact the group H_n is the largest quotient of $A' \rtimes E_v$ such that the left-hand square of the above diagram commutes (see e.g. [Br, page 94]). Note that φ_n is injective and not surjective since h_n is ($\forall n \in \mathbb{N}$). Furthermore, φ_n is the identity on D. We need now to show that H_n is isomorphic to E_v. Let us define a set-theoretic cross section $s_n : F' \to H_n$ of the projection p_n (see the diagram) to be $s_n = \varphi_n \circ s$. It is known (see e.g. [Br, III.3.12]) that the equivalence classes of extensions of A' by F' are in $1-\mathrm{to}-1$ correspondence with the elements of $H^2(F', A')$.

If $\alpha \in H^2(F', A')$ is the element which corresponds to the upper row of the commutative diagram then it satisfies $i(\alpha(g,\gamma))s(g\gamma) = s(g)s(\gamma)$ $(g, \gamma \in F')$. We have

$$s_n(g)s_n(\gamma) = \varphi_n(s(g)\varphi_n(s(\gamma)) = \varphi_n(s(g)s(\gamma)) = \varphi_n(i(\alpha(g,\gamma))s(g\gamma)) = i_n(h_n(\alpha(g,\gamma)))s_n(g,\gamma).$$

Setting $\alpha_n(g,\gamma) = h_n(\alpha(g,\gamma))$ we obtain from the above identity that $\alpha_n(g,\gamma)$ is an element of $H^2(F', A')$. Since $\alpha(g,\gamma)$ takes its values in A' we can write $\alpha(g,\gamma) = d(g,\gamma) + b(g,\gamma)$ where $d(g,\gamma) \in D$ and $b(g,\gamma) \in B$. We also have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "i"] \arrow[d, "\pi_1"'] & E_v \arrow[r] \arrow[d, "\pi_2"'] & F' \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & B = A'/D \arrow[r, "j"] & E_v/D \arrow[r] & F' \arrow[r] & 1
\end{tikzcd}$$

where π_i are the natural projections and j is the natural inclusion. Define a section $\sigma : F' \to E_v/D$ to be $\sigma = \pi_2 \circ s$. Similarly we then show that $\sigma(g)\sigma(\gamma) = j(p_2(\alpha(g,\gamma)))\sigma(g,\gamma)$ $(g, \gamma \in F')$. As $b = \pi_1 \circ \alpha$ we obtain that b is also an element of $H^2(F', A')$. Since the group F' is finite it follows from [Br, III.10.2] that the group $H^2(F', A')$ is annihilated by $|F'|$. Choosing n to be $n \equiv 1 (\text{mod } |G|)$ we obtain that $b = nb$ and so $\alpha_n = d + b_n = d + b = \alpha$. This implies that α and α_n define the equivalent extensions and, so the groups E_v and H_n are isomorphic. Let $n_0 = |F'| + 1$ then the map φ_{n_0} is an endomorphism of E_v which is injective, non-surjective and is the identity on D. Since D is an abelian normal subgroup of N_v of finite index, it follows that φ_{n_0} is the identity on N_v too (because if h represents a coset in N_v/D then the element $\varphi_{n_0}(h) \cdot h^{-1}$ commutes with all elements in D and so $\varphi_{n_0}(h) \equiv h (\text{mod } D)$). By Remark 7.4 it follows that

$$\exists\ k \in \mathbb{N},\ \exists\ a \in D\ \forall h \in N_v : \varphi_{n_0}^k(h) = aha^{-1}.$$

Setting $F|_{E_v} = a^{-1} \cdot \varphi_{n_0}^k \cdot a$, we obtain an injective, not surjective endomorphism of E_v which is the identity on N_v. Extending now F by the identity to the fundamental group of the graph $X \setminus \{v\}$ we obtain a monomorphism $F : G \to G$ which is injective and not surjective. Theorem D is proved. ■

8 Cohopficity of Groups with Infinitely Many Ends.

In this section we provide a criterion establishing the co-Hopf property for multi-ended groups. We start with an abstract finitely presented group G. Let us recall that if G has infinitely many ends then the Dunwoody's accessibility Theorem [Du] states that there exists a graph of groups decomposition $G = \pi_1(X, G_v, C_e)$ such that all edge groups C_e are finite and all vertex groups G_v are one-ended. Furthermore the sets of vertex and edge groups of X are unique [DD, Proposition 7.4]. We will further call this graph of groups *DS-graph* of G (referring to Dunwoody-Stallings' theorems for splitting of groups with infinitely many ends [Du], [St]).

We denote $\mu(G)$ the number of edges of a DS-graph of G. If $G = A *_F B$ (resp. $G = A_F*$) and F is a finite group then $\max\{\mu(A), \mu(B)\} < \mu(G)$. Indeed as finite groups are always elliptic in any splitting we can always reach the terminal DS-graph of G by taking further decomposition of A and B over finite subgroups.

Before we state the main result of this section we give a more precise definition of an acylindrical splitting for a multi-ended group (compare with Definition 2.5):

Definition 8.1. *Let $G = \pi_1(Y)$ be a splitting of a group G as a graph of groups with finite edge stabilizers and T be the corresponding Bass-Serre tree. We call this splitting (and respectively the tree T)* **strictly K-acylindrical** *if the stabilizer of each segment of T of the diameter at least K is a proper subgroup of some edge stabilizer of T.* ■

We will prove the following.

Theorem E. *Let G be an infinitely ended finitely presented group and let $X^* = (X, G_v, C_e)$ denote its DS-graph. Suppose that every one-ended vertex group G_v is cohopfian. Then G is cohopfian if and only if every splitting of G over finite groups is strictly K-acylindrical for some uniform constant K.*

Proof of the sufficient condition: Assume that all splittings of G over finite groups are K-acylindrical for some fixed $K \in \mathbb{N}$. Note first that this property is then also true for each vertex group of any graph of groups decomposition of G over finite groups. Indeed, every splitting of such vertex group G_v over finite groups refines the splitting of G. Consequently, all splittings of G_v over finite groups are strictly K-acylindrical (for the same constant K). This remark will be constantly used in the argument which will mainly repeat the proof of Theorem B given in Sections 5 and 6. We will only indicate some modifications (and simplifications) which are to be done.

Suppose by contradiction that $f : G \to G$ is an injective endomorphism which is not surjective. Let us prove the statement by induction on the invariant $\mu(.)$. Note that if $\mu(G) = 0$ then G is one-ended which is impossible by our hypothesis. So let us assume that $\mu(G) > 0$ and the statement is true for all groups with the value of $\mu(\cdot)$ less than that of G.

Among all splittings of G over finite subgroups we choose one: $G = A *_E B$ or $G = A*_E$ for which E has a minimal order. Note that E cannot be trivial, as every free product decomposition is not strictly K-acylindrical for all K. Let T denote the Bass-Serre tree corresponding to this splitting. As in Section 5 we consider the sequence of G-trees $T_n = f^{n*}T$ with finite edge stabilizers. Note that if T_n is a trivial G-tree then arguing as in Proposition 5.1 we obtain that one of the groups A or B is not cohopfian. By the hypothesis it follows that it is not a one-ended group. We note that $\max(\mu(A), \mu(B)) < \mu(G)$ as every splitting of A and B over finite groups refines the above splitting of G. Using now the induction on the invariant $\mu(\cdot)$, we obtain similarly to 5.1 that the trees T_n are all non-trivial G-trees.

Let l denote a path of length K in the tree T_n. Then the stabilizer of l is a subgroup of the stabilizer l in the tree T. By the strict acylindricity of T it now follows that its order is strictly less than the order $o(E)$ of the group E. As G does not split over a subgroup of order less than $o(E)$, it does not split over the stabilizer of l. Then by Theorem 3.1 of [De] we obtain finitely many G-trees $\tau_1, ..., \tau_k$ such that every tree T_i is dominated by one of τ_j's where $i \in \mathbb{N},\ j \in \{1, ..., k\}$. Then applying the argument of Proposition 5.2 (which does not use the fact that the group G is Kleinian nor one-ended) we obtain a strictly K-acylindrical G-tree τ with finite edge stabilizers and a new monomorphism $F : G \to G$ which sends all vertex (resp. edge) stabilizers of τ into themselves. In addition, F is surjective if and only if f is.

The vertex groups of a DS-graph of G_v are cohopfian as they are vertex groups of a DS-graph of G. So by the induction hypothesis the map F restricted to every vertex stabilizer of τ is surjective. Furthermore, as every edge stabilizer of τ is finite and is preserved by F, F restricted on it, is surjective too. Thus to finish the proof we only need to consider the case when G is not generated by the vertex groups of the graph τ/G.

Following now the argument given in Lemma 6.1 we obtain a HNN-extension $G = A*_H = \langle A,\ H \mid tHt^{-1} = \varphi(H)\rangle$ so that $F(A) = A,\ F(H) = H,\ F(tHt^{-1}) = tHt^{-1}$. Then the element $a = t^{-1} \cdot F(t)$ normalizes H. Now if a is not conjugate into A then there is an infinite path in the Bass-Serre tree corresponding to the splitting $G = A*_H$ whose pointwise stabilizer is H. This is impossible as all splittings of G over finite subgroups are strictly acylindrical. Thus up to conjugation we obtain that $a \in A$ and there is an element $b \in A$ so that $F(b) = a$. This proves that t is in the image of F and so F is surjective. The sufficiency is proved.

To prove the necessary condition suppose that for every $K \in \mathbb{N}$ the group G admits a splitting over finite groups which is not strictly K-acylindrical. Set $K = 2\mu(G) + 1$ and let X denote such graph of groups decomposition of G and T its Bass-Serre tree. Then there is a path $l \subset T$ whose pointwise stabilizer H is equal to the edge stabilizer of every edge of l. The argument is now similar to the proof of Lemma 2.6. As the length of the path l is greater than $2\mu(G)$, it must contain at least three different edges e_1, e_2, e_3 belonging to the same G-orbit. So, $e_1 = g(e_2),\ e_3 = h(e_2)$ for some distinct elements g and h in $G \setminus H$. It follows that both elements g and h normalize H.

Suppose first that one of them, say g, acts hyperbolically on the tree T. Then replacing g by some power, we may assume that it centralizes the group H. Considering the corresponding splitting of G over H as an amalgamated free product $G = A *_H B$ or HNN-extension $G = A*_H$ we show that G is not cohopfian analogously to the proof of Theorem D (see the part concerning Condition 1).

If now both elements g and h act elliptically on T then the element $\gamma = gh$ also normalizes H and is hyperbolic. Indeed if not, g and h must have a common fixed point [S]. Then arguing as in Lemma 2.6 we would obtain that g and h fix the edge e_2 pointwise which is impossible. The proof now finishes similarly. The Theorem is proved. ■

The following is a slightly different version of the above Theorem.

Corollary 8.2. *Let G be an infinitely ended finitely presented group and let $X^* = (X, G_v, C_e)$ denote its DS-graph. Suppose that every splitting of G over finite groups is strictly K-acylindrical for a uniform constant K. Then G is cohopfian if and only if the pair $(G_v,\ \mathcal{C} \cap G_v)$ is cohopfian for every vertex v, where $\mathcal{C}$ is the set of edge groups of X^*.*

Proof: The proof of the *sufficiency* refines that of Theorem E by keeping track of edge groups. Indeed the map F sends all vertex and edge stabilizers of the tree τ into themselves. As edge stabilizers of τ are all finite, up to replacing F by some power we may assume that F is the identity on the set $\mathcal{C}$ of the edge stabilizers of the graph τ/G. If this graph is already DS-graph we stop; if not we repeat the above procedure for every vertex stabilizer G_v^1 of it. Then the Acylindricity Theorem of Section 4 allows us to find a new map $F_v^1 : G_v^1 \to G_v^1$ and a new decomposition of G_v^1 over finite subgroups such that F_v^1 sends all edge and vertex stabilizers of this decomposition and all the subgroups in $\mathcal{C}$ into themselves. Again by taking power, if necessary, we may assume F_v^1 to be the identity on each group in $\mathcal{C}$. Thus we have refined the graph τ/G by the decomposition of the vertex group G_v^1 and have found a new endomorphism of G which is equal to F on $(\tau/G)\setminus\{v\}$ and to F_v^1 on the above graph of groups decomposition of G_v^1. Continuing in this way we will arrive after finitely many steps to the DS-graph $X^* = (X, G_v, C_e)$

and a map $\Phi : G \to G$ which sends every vertex group G_v into itself and is the identity on every edge group. Furthermore, by construction Φ is surjective if and only if the map F is. As all pairs $(G_v,\ \mathcal{C} \cap G_v)$ are cohopfian the map Φ is surjective by the proof of Theorem E.

The necessary condition is easy. Indeed, suppose first that $f_v : (G_v, \mathcal{C} \cap G_v) \to (G_v, \mathcal{C} \cap G_v)$ is a non-surjective endomorphism. Up to taking power we may suppose that f_v is the identity on peripheral subgroups $\mathcal{C} \cap G_v$. Extending then f_v by the identity to the rest of the group G we get a non-surjective endomorphism of G which is impossible. ■

Theorems E and A allow us to get a criterion for the co-Hopf property of infinitely ended Kleinian groups.

Theorem C. *Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a non-elementary, geometrically finite Kleinian group without 2-torsion. Then G is cohopfian if and only if the following three conditions are satisfied:*

1) *G does not have essentially non-maximal splittings over infinite elementary subgroups.*

2) *G does not split as an amalgamated free product $G = A *_C \tilde{C}$, so that $\tilde{C}$ is a maximal elementary subgroup of G and the normal closure of the subgroup C in $\tilde{C}$ is of infinite index in $\tilde{C}$.*

3) *Every splitting of G over finite groups is strictly K-acylindrical for a uniform constant K.*

Remark 8.3. *By Lemma 2.6 the condition 1) can be replaced by the following:*

1') Each irreducible G-splitting over infinite elementary subgroups is (M, Φ)-acylindrical for some uniform constant $M > 0$.

Proof: The necessity of each of these conditions was already proved. To prove the sufficiency let us suppose that G is not cohopfian. Then by Theorem E there exists a one-ended vertex group G_v of a DS-graph of G which is not cohopfian. Then Theorem A implies that G_v admits a splitting described by one of the conditions 1) or 2) (where the group G is replaced by G_v). As all edge groups of DS-graph of G are finite this splitting of G_v refines a DS-graph of G. Obviously this gives a splitting of G which does not verify one of the conditions 1) or 2). Theorem C is proved. ■

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