

A BOREL-CANTELLI LEMMA FOR INTERMITTENT INTERVAL MAPS

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ABSTRACT. We consider intermittent maps T of the interval, with an absolutely continuous invariant probability measure μ . Kim showed that there exists a sequence of intervals A_n such that $\sum \mu(A_n) = \infty$, but $\{A_n\}$ does not satisfy the dynamical Borel-Cantelli lemma, i.e., for almost every x , the set $\{n : T^n(x) \in A_n\}$ is finite. If $\sum \text{Leb}(A_n) = \infty$, we prove that $\{A_n\}$ satisfies the Borel-Cantelli lemma. Our results apply in particular to some maps T whose correlations are not summable.

1. INTRODUCTION

Let T be an ergodic probability preserving transformation of a space (X, μ) , and let A_n be a sequence of subsets of X with $\sum \mu(A_n) = +\infty$. It is an interesting question to know whether, for almost every point x , $T^n(x)$ belongs to A_n infinitely often. By the classical Borel-Cantelli lemma, this holds if the sets $T^{-n}A_n$ are pairwise independent, but this condition is almost never satisfied for dynamical systems, so one is led to looking for weaker conditions.

If T is invertible, taking $A_n = T^n(A)$ for some fixed set A gives a trivial counterexample (and similar counterexamples also exist for noninvertible maps). Hence, some regularity conditions on the sets A_n are necessary. For uniformly hyperbolic dynamical systems, Chernov and Kleinbock have solved the problem for lots of families of balls in [CK01] (see also [Mau06]). The partially hyperbolic case is dealt with in [Dol04]. Concerning non-uniformly hyperbolic (or expanding) systems, Kim has considered in [Kim07] a family of interval maps with a neutral fixed points and obtained partial results. Our goal in this note is to complete these results (for the same family of maps) and obtain a full description of the situation.

Consider some parameter $\alpha > 0$ and let $T_\alpha : (0, 1] \rightarrow (0, 1]$ be given by

$$(1.1) \quad T_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } x \in (0, 1/2], \\ 2x - 1 & \text{if } x \in (1/2, 1]. \end{cases}$$

It preserves a unique (up to multiplication by a scalar) absolutely continuous measure μ , and this measure has finite mass if and only if $\alpha < 1$. Henceforth, we will only consider this case, and assume that μ is normalized to be a probability measure. We will also denote by Leb the Lebesgue measure on $(0, 1]$.

In [Kim07], Kim proves the following result: for any $\alpha < 1$, there exist intervals A_n such that $\sum \mu(A_n) = \infty$ but, for almost every x , $T_\alpha^n(x) \in A_n$ occurs only

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finitely many times. In other words, the answer to the Borel-Cantelli problem in this setting is not always positive. On the other hand, he proves that, if A_n is a sequence of intervals in $(d, 1]$ for some $d > 0$, with $\sum \mu(A_n) = \infty$, and

- either $A_{n+1} \subset A_n$ for all n
- or $\alpha < (3 - \sqrt{5})/2$

then, for almost every x , $T_\alpha^n(x)$ belongs to A_n infinitely many times. In this note, we prove the following theorem:

Theorem 1.1. *Let $\alpha < 1$, and let A_n be a sequence of intervals with $\sum \text{Leb}(A_n) = \infty$. Then, for almost every x , $T_\alpha^n(x)$ belongs to A_n infinitely many times.*

The measures μ and Leb are uniformly equivalent on every interval $(d, 1]$ (more precisely, on every interval $(d, 1]$, the density h of μ with respect to Leb is Lipschitz continuous and bounded from above and below). Hence, this theorem implies the aforementioned result of Kim.

The proof involves a measurement of how sets $T_\alpha^{-i}A_i$ and $T_\alpha^{-j}A_j$ are “close to be independent”. For the following informal description of the proof, assume for the sake of simplicity that the intervals A_n are all contained in $(1/2, 1]$. The speed of decay of correlations of the map T_α is exactly $1/n^{\beta-1}$ for $\beta = 1/\alpha$, which means that the best estimate we could hope for is of the form

$$(1.2) \quad |\mu(T_\alpha^{-i}A_i \cap T_\alpha^{-j}A_j) - \mu(A_i)\mu(A_j)| \leq \frac{C\mu(A_j)}{(j-i)^{\beta-1}},$$

for $j > i$. This estimate indeed holds, and implies Theorem 1.1 when the sequence $1/n^{\beta-1}$ is summable, that is, when $\alpha < 1/2$. However, it is not sufficient when $1/2 \leq \alpha < 1$, and we need to know further terms in the asymptotics of the correlations. Here comes into play our main technical tool, the renewal sequence of transfer operators, studied by Sarig in [Sar02]. Using the results in [Gou04a], we will prove the existence of a sequence c_n converging to 1 such that

$$(1.3) \quad |\mu(T_\alpha^{-i}A_i \cap T_\alpha^{-j}A_j) - c_{j-i}\mu(A_i)\mu(A_j)| \leq \frac{C\mu(A_j)}{(j-i)^\beta}.$$

This sequence is of the form $c_n = 1 + c/n^{\beta-1} + o(1/n^{\beta-1})$ for some nonzero constant c , which shows that (1.2) is indeed optimal. For the purposes of the Borel-Cantelli problem, (1.3) is sufficient and will imply Theorem 1.1 in all cases, since the sequence $1/n^\beta$ is summable whenever $\alpha < 1$.

On the technical level, the results in [Sar02, Gou04a] deal with spaces of Lipschitz functions. However, the essential results are formulated in an abstract Banach spaces framework. They can therefore also be applied to spaces of functions with bounded variation, which is what is needed here to deal with the characteristic functions of intervals.

Remark 1.2. *Theorem 1.1 still holds for transformations with an even more neutral fixed point, as soon as there is still an absolutely continuous invariant probability measure. This is for example the case if the fixed point is of the form $x + x^2(\log x)^2$, or more generally for the class of maps introduced by Holland in [Hol05]. However, the results of [Gou04a] are not sufficient to prove this, and one needs to use results in the unpublished thesis [Gou04b], for example Remark 2.4.8 or Remark 2.4.11.*

2. ABSTRACT TOOLS

First of all, let us recall a criterion implying the Borel-Cantelli property (proved e.g. in [Spi64, Proposition 6.26.3]):

Theorem 2.1. *Let B_n be sets of a probability space (X, μ) with $\sum \mu(B_n) = \infty$. Assume that*

$$(2.1) \quad \limsup_{n \rightarrow \infty} \frac{\sum_{0 \leq i < j < n} \mu(B_i \cap B_j)}{\left(\sum_{j=0}^{n-1} \mu(B_j)\right)^2} \leq \frac{1}{2}.$$

Then almost every point of X belongs to infinitely many B_n 's.

We will apply this result to $B_n = T_\alpha^{-n}(A_n)$. Hence, we need a good quantitative estimate on $\mu(T_\alpha^{-i}A_i \cap T_\alpha^{-j}A_j)$. This estimate will be provided by *renewal sequences of transfer operators*, as used by Sarig in [Sar02]. For our purpose, the following abstract result will be most useful. Let \mathbb{D} be the closed unit disk in \mathbb{C} .

Theorem 2.2. *Let \mathcal{BV} be a Banach space, and let $(R_n)_{n \geq 1}$ be a sequence of continuous linear operators on \mathcal{BV} . Assume that, for some $\beta > 1$, $\sum_{k > n} \|R_k\| = O(1/n^\beta)$. Hence, $R(z) = \sum R_n z^n$ and $R'(z) = \sum n R_n z^{n-1}$ are well defined operators on \mathcal{BV} , for $z \in \mathbb{D}$. Assume moreover that 1 is a simple isolated eigenvalue of $R(1)$, and that the corresponding eigenprojector P satisfies $PR'(1)P = \gamma P$ for some $\gamma \neq 0$. Assume also that, for any $z \in \mathbb{D} \setminus \{1\}$, $I - R(z)$ is invertible on \mathcal{BV} .*

Let $T_n = \sum_{l=1}^{\infty} \sum_{k_1 + \dots + k_l = n} R_{k_1} \dots R_{k_l}$. This operator acts continuously on \mathcal{BV} . Then there exists a sequence $c_n \in \mathbb{C}$ converging to 1 such that $T_n - c_n P / \gamma = O(1/n^\beta)$.

Proof. [Gou04a, Theorem 5.4] (for large enough N) shows that T_n converges to P/γ , and that there exists a sequence of operators Q_n such that $T_n - PQ_nP = O(1/n^\beta)$. This theorem even gives a closed form expression for Q_n , but we will not need it.

Since P is a one-dimensional projection, there exists a complex number d_n such that $PQ_nP = d_n P$. The convergence of T_n to P/γ shows that d_n converges to $1/\gamma$. We obtain the theorem for $c_n = \gamma d_n$. \square

In [Sar02, Gou04a], this theorem is applied by taking R_n to be the ‘‘first return transfer operators’’ to $Y = (1/2, 1]$, acting on the space of Lipschitz continuous functions on Y . Here, we will use the same operators R_n , but we will use for \mathcal{BV} the space of functions of bounded variation on Y .

3. PROOF OF THE MAIN THEOREM

In all this section, we fix $\alpha \in (0, 1)$ and write T for T_α . Let also $\beta = 1/\alpha$.

Let $Y = (1/2, 1]$, let $\varphi : Y \rightarrow \mathbb{N}^*$ denote the first return time from Y to itself. Let also \widehat{T} be the transfer operator associated to T , given for $f \in L^1(\text{Leb})$ by

$$(3.1) \quad \widehat{T}f(x) = \sum_{Ty=x} f(y)/T'(y).$$

Let $R_n f = \widehat{T}^n(1_{Y \cap \{\varphi=n\}} f)$, and $T_n f = 1_Y \widehat{T}^n(1_Y f)$. These operators act on $L^1(Y)$. Moreover, R_n corresponds to considering the first returns at time n , while T_n

considers all returns at time n . It is therefore easy to check the following *renewal equation* (see e.g. [Sar02, Proposition 1]):

$$(3.2) \quad T_n = \sum_{l=1}^{\infty} \sum_{k_1 + \dots + k_l = n} R_{k_1} \dots R_{k_l}.$$

Let \mathcal{BV} be the space of functions of bounded variation on Y . An element f of \mathcal{BV} is a bounded function on \mathbb{R} , supported in Y , and its norm is

$$(3.3) \quad \text{Var}(f) := \sup_{N \in \mathbb{N}} \sup_{x_0 < \dots < x_N} \sum_{i=0}^{N-1} |f(x_{i+1}) - f(x_i)|,$$

where the x_i 's are real numbers (not necessarily in Y). In particular, $\|f\|_{L^\infty} \leq \text{Var}(f)/2$.

Lemma 3.1. *The operators R_n acting on \mathcal{BV} satisfy the assumptions of Theorem 2.2. The spectral projection P corresponding to the eigenvalue 1 of $R(1)$ is given by*

$$(3.4) \quad Pf = \frac{(\int_Y f \, d\text{Leb})}{\mu(Y)} h_Y$$

where h_Y is the restriction to Y of the density h of the invariant probability measure μ . Additionally, $PR'(1)P = P/\mu(Y)$.

Proof. This lemma is proved in [Gou04a] for the action of R_n on the space \mathcal{L} of Lipschitz functions on Y . We will adapt this proof to the space \mathcal{BV} .

The set $\{\varphi = n\}$ is a subinterval I_n of Y , and T^n is a diffeomorphism between I_n and Y . Moreover, $|I_n| \sim c/n^{\beta+1}$ for some constant $c > 0$, and the distortion of T^n on I_n is uniformly bounded, independently of n , in the following sense: there exists $C > 0$ such that, for all $x, y \in I_n$,

$$(3.5) \quad \left| 1 - \frac{(T^n)'(x)}{(T^n)'(y)} \right| \leq C |T^n x - T^n y|.$$

See e.g. [You99, Section 6] for a proof of these facts. Let $\psi_n : Y \rightarrow I_n$ be the inverse of T^n on I_n , so that

$$(3.6) \quad R_n f(x) = \psi_n'(x) f(\psi_n x).$$

Then

$$(3.7) \quad \text{Var}(R_n f) \leq \|\psi_n'\|_{L^\infty} \text{Var}(f \circ \psi_n) + \|f\|_{L^\infty} \text{Var}(\psi_n') \leq C |I_n| \text{Var}(f).$$

In particular,

$$(3.8) \quad \|R_n\|_{\mathcal{BV} \rightarrow \mathcal{BV}} \leq \frac{C}{n^{\beta+1}}.$$

As in Theorem 2.2, we define for $z \in \overline{\mathbb{D}}$ an operator $R(z) = \sum R_n z^n$. By (3.8), this operator is well defined on \mathcal{BV} . Moreover, by [Gou04a, Paragraph 6.3], $R(z)$ also acts continuously on the space \mathcal{L} of Lipschitz continuous functions on Y , and satisfies the following properties. First of all, $R(z)$ satisfies a Lasota-Yorke inequality between \mathcal{L} and L^1 . Hence, by the theorem of Ionescu-Tulcea and Marinescu, any eigenfunction of $R(z)$ (for an eigenvalue of modulus 1) which belongs to L^1 belongs in fact to \mathcal{L} . Moreover, for $z \in \overline{\mathbb{D}} \setminus \{1\}$, $I - R(z)$ is invertible on \mathcal{L} , while $R(1)$ has a simple eigenvalue at 1, the corresponding eigenfunction being h_Y

Let us now prove that, for any $z \in \overline{\mathbb{D}}$, the essential spectral radius of $R(z)$ acting on \mathcal{BV} is < 1 . This could be proved by mimicking the arguments in [Ryc83], but it is easier to refer to [Rue96, Theorem B.1]. Indeed, this theorem shows that the essential spectral radius of $R(z)$ is bounded by $\|z^\varphi/(T^\varphi)'\|_{L^\infty} < 1$.

Let $z \in \overline{\mathbb{D}} \setminus \{1\}$. If $I - R(z)$ were not invertible on \mathcal{BV} , then there would exist a function $f \in \mathcal{BV}$ such that $R(z)f = f$. The function f would in particular belong to L^1 , hence, by the above argument, it would belong to \mathcal{L} . This is a contradiction since $I - R(z)$ is invertible on \mathcal{L} . In the same way, we check that $R(1)$ has a simple eigenvalue at 1, the eigenfunction still being the density of the invariant measure. Moreover, the eigenprojection is given by (3.4).

We compute finally $PR'(1)P$. The formula for Pf gives

$$(3.9) \quad PR'(1)Pf = \frac{\left(\int_Y R'(1)h_Y \, d\text{Leb}\right)}{\mu(Y)} \frac{\left(\int_Y f \, d\text{Leb}\right)}{\mu(Y)} h_Y = \gamma Pf,$$

for $\gamma = \left(\int_Y R'(1)h_Y \, d\text{Leb}\right) / \mu(Y)$. Moreover,

$$(3.10) \quad \int R_n h_Y \, d\text{Leb} = \int \widehat{T}^n(1_{\{\varphi=n\}} h_Y) \, d\text{Leb} = \int 1_{\{\varphi=n\}} h_Y \, d\text{Leb} = \mu\{\varphi = n\}.$$

Summing these formulas over n gives

$$(3.11) \quad \int R'(1)h_Y \, d\text{Leb} = \sum n\mu\{\varphi = n\} = \int_Y \varphi \, d\mu = 1$$

by Kac Formula. Hence, $\gamma = 1/\mu(Y)$. \square

Corollary 3.2. *There exist $C > 0$, and a sequence c_n of complex numbers converging to 1 when n tends to infinity, such that, for any functions f, g supported in Y , for any $n > 0$,*

$$(3.12) \quad \left| \int f \cdot g \circ T^n \, d\text{Leb} - c_n \left(\int f \, d\text{Leb} \right) \left(\int g \, d\mu \right) \right| \leq \frac{C \|f\|_{\mathcal{BV}} \|g\|_{L^1(\text{Leb})}}{n^\beta}.$$

Proof. We have

$$(3.13) \quad \int f \cdot g \circ T^n \, d\text{Leb} = \int 1_Y \widehat{T}^n(1_Y f) g \, d\text{Leb} = \int T_n f \cdot g \, d\text{Leb}.$$

Moreover, by (3.2), Lemma 3.1 and Theorem 2.2, there exist a sequence c_n converging to 1 and a constant C such that

$$\begin{aligned} \left\| T_n f - c_n \left(\int_Y f \, d\text{Leb} \right) h_Y \right\|_{\mathcal{BV}} &= \|T_n f - c_n \mu(Y) P f\|_{\mathcal{BV}} \leq \|f\|_{\mathcal{BV}} \|T_n - c_n \mu(Y) P\| \\ &\leq \frac{C \|f\|_{\mathcal{BV}}}{n^\beta}. \end{aligned}$$

Together with (3.13), this concludes the proof. \square

Proof of Theorem 1.1. Let first A_n be a sequence of intervals contained in $(1/2, 1]$, with $\sum \text{Leb}(A_n) = \infty$ (or, equivalently, $\sum \mu(A_n) = \infty$). Let $B_n = T^{-n} A_n$. Let

$j > i$. Applying Corollary 3.2 to $f = 1_{A_i} h_Y$, $g = 1_{A_j}$ and $n = j - i$, we get

$$\begin{aligned} & |\mu(B_i \cap B_j) - c_{j-i} \mu(B_i) \mu(B_j)| \\ &= \left| \int 1_{A_i} h_Y \cdot 1_{A_j} \circ T^{j-i} d\text{Leb} - c_{j-i} \left(\int 1_{A_i} h_Y d\text{Leb} \right) \left(\int 1_{A_j} d\mu \right) \right| \\ &\leq \frac{C \text{Var}(1_{A_i} h_Y) \text{Leb}(A_j)}{(j-i)^\beta}. \end{aligned}$$

The function h is Lipschitz continuous on Y , and bounded from below. In particular, $\text{Leb}(A_j) \leq C\mu(A_j) = C\mu(B_j)$. We conclude

$$(3.14) \quad |\mu(B_i \cap B_j) - c_{j-i} \mu(B_i) \mu(B_j)| \leq \frac{C\mu(B_j)}{(j-i)^\beta}.$$

Let $\varepsilon > 0$. Let K be such that, for $n \geq K$, $|c_n| \leq 1 + \varepsilon$. Then

$$\begin{aligned} \sum_{0 \leq i < j < n} \mu(B_i \cap B_j) &\leq \sum_{0 \leq i < j < n} |c_{j-i}| \mu(B_i) \mu(B_j) + \sum_{j=1}^{n-1} \left(\sum_{i=0}^{j-1} \frac{C'}{(j-i)^\beta} \right) \mu(B_j) \\ &\leq \sum_{0 \leq i < j < n} (1 + \varepsilon) \mu(B_i) \mu(B_j) + \sum_{j=1}^{n-1} \left(K \sup_{p \in \mathbb{N}} |c_p| + \sum_{p=1}^{\infty} \frac{C'}{p^\beta} \right) \mu(B_j). \end{aligned}$$

Therefore,

$$(3.15) \quad \frac{\sum_{0 \leq i < j < n} \mu(B_i \cap B_j)}{\left(\sum_{j=0}^{n-1} \mu(B_j) \right)^2} \leq \frac{1 + \varepsilon}{2} + \left(K \sup_{p \in \mathbb{N}} |c_p| + \sum_{p=1}^{\infty} \frac{C'}{p^\beta} \right) \frac{1}{\sum_{j=0}^{n-1} \mu(B_j)}.$$

Since $\sum_{j \in \mathbb{N}} \mu(B_j) = \infty$, this upper bound is at most $1/2 + \varepsilon$ for large enough n . We have proved that

$$(3.16) \quad \limsup_{n \rightarrow \infty} \frac{\sum_{0 \leq i < j < n} \mu(B_i \cap B_j)}{\left(\sum_{j=0}^{n-1} \mu(B_j) \right)^2} \leq \frac{1}{2}.$$

By Theorem 2.1, this concludes the proof in this case.

Consider now A_n an arbitrary sequence of intervals in $(0, 1]$ with $\sum \text{Leb}(A_n) = \infty$. Let $A'_n = T^{-1}(A_{n+1}) \cap (1/2, 1]$. Since $\text{Leb}(A'_n) = \text{Leb}(A_n)/2$, this sequence of intervals satisfies $\sum \text{Leb}(A'_n) = \infty$, and A'_n is a subinterval of $(1/2, 1]$. The first part of the proof shows that, for almost every x , $T^n x$ belongs to A'_n infinitely often. However, if $T^n(x) \in A'_n$, then $T^{n+1}(x) \in A_{n+1}$. This concludes the proof. \square

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