On vertex algebra representations of the Schrödinger-Virasoro Lie algebra

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Abstract

The Schrödinger-Virasoro Lie algebra $\mathfrak{sv}$ is an extension of the Virasoro Lie algebra by a nilpotent Lie algebra formed with a bosonic current of weight $\frac{3}{2}$ and a bosonic current of weight 1. It is also a natural infinite-dimensional extension of the Schrödinger Lie algebra, which – leaving aside the invariance under time-translation – has been proved to be a symmetry algebra for many statistical physics models undergoing a dynamics with dynamical exponent $z = 2$; it should consequently play a role akin to that of the Virasoro Lie algebra in two-dimensional equilibrium statistical physics.

We define in this article general Schrödinger-Virasoro primary fields by analogy with conformal field theory, characterized by a 'spin' index and a (non-relativistic) mass, and construct vertex algebra representations of $\mathfrak{sv}$ out of a charged symplectic boson and a free boson and its associated vertex operators. We also compute two- and three-point functions of still conjectural massive fields that are defined by an analytic continuation with respect to a formal parameter.

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0 Introduction

The Schrödinger-Virasoro algebra $\mathfrak{sv}$ is defined in [21, 39] as the infinite-dimensional Lie algebra generated by L_n, Y_m, M_p, $n,p \in \mathbb{Z}$, $m \in \frac{1}{2} + \mathbb{Z}$, with Lie brackets

$$[L_n, L_p] = (n-p)L_{n+p},\ [L_n, Y_m] = (\frac{n}{2} - m)Y_{n+m},\ [L_n, M_p] = -pM_{n+p}$$

$$[Y_m, Y_{m'}] = (m-m')M_{m+m'},\ [Y_m, M_p] = 0,\ [M_n, M_p] = 0 \tag{0.1}$$

where $n,p \in \mathbb{Z}, m, m' \in \frac{1}{2} + \mathbb{Z}$. It is a semi-direct product of the non centrally extended Virasoro algebra

$$\mathfrak{g} = \mathfrak{vir}_0 := \langle L_n \rangle_{n \in \mathbb{Z}} \tag{0.2}$$

by the two-step nilpotent infinite dimensional Lie algebra

$$\mathfrak{h} = \langle Y_m \rangle_{m \in \frac{1}{2} + \mathbb{Z}} \oplus \langle M_p \rangle_{p \in \mathbb{Z}}. \tag{0.3}$$

The Y_m ($m \in \mathbb{Z} + \frac{1}{2}$), resp. M_p ($p \in \mathbb{Z}$), may be seen as the components of L-conformal currents with conformal weight $\frac{3}{2}$, resp. 1. Note that the current Y is *bosonic* although its weight is a half-integer. The supersymmetric partner G of the Virasoro field appearing in the Neveu-Schwarz algebra (see [33] or [29], §5.9) is also of weight $\frac{3}{2}$, but it is *odd*, which changes drastically the representation theory and the range of applications, the 'bosonicity' of Y accounting for the appearance of a *space-dependence* which is absent from usual (super)conformal field theory.

This infinite-dimensional Lie algebra was originally introduced in [21] by looking at the invariance of the free Schrödinger equation in (1+1)-dimensions

$$(2\mathcal{M}\partial_t - \partial_r^2)\psi = 0. \tag{0.4}$$

Its maximal subalgebra of Lie symmetries (acting projectively on the wave function ψ) is known under the name of Schrödinger Lie algebra, $\mathfrak{sch}_1$ (see [32, 34, 35]), and can be embedded into $\mathfrak{sv}$ as

$$\mathfrak{sch}_1 = \langle L_{-1}, L_0, L_1 \rangle \ltimes \langle Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}, M_0 \rangle = \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{gal},$$

where $\mathfrak{gal}$ - isomorphic to the three-dimensional nilpotent Heisenberg Lie algebra - contains the generators of Galilei transformations; the generators of $\mathfrak{sch}_1$ act on ψ as follows:

$$L_{-1} = -\partial_t, \quad L_0 = -t\partial_t - \frac{1}{2}r\partial_r - \lambda, \quad L_1 = -t^2\partial_t - tr\partial_r - \frac{\mathcal{M}}{2}r^2 - 2\lambda t \tag{0.5}$$

(generators of time translation, scaling transformation - with *scaling exponent* $\lambda = \frac{1}{4}$ in this case - and 'special' transformation);

$$Y_{-\frac{1}{2}} = -\partial_r, \quad Y_{\frac{1}{2}} = -t\partial_r - \mathcal{M}r, \quad M_0 = -\mathcal{M} \tag{0.6}$$

(generators of space translation, special Galilei transformation and phase shift). All together, these generate the following finite transformations [23]:

$$\psi(t,r) \to \dot{\beta}(t')^{-\lambda} \exp\left(-\frac{\mathcal{M}}{4}\frac{\ddot{\beta}(t')}{\dot{\beta}(t')}r'^2\right)\psi'(t',r') \tag{0.7}$$

where $t = \beta(t') = \frac{at'+b}{ct'+d}$, $r = r'\sqrt{\dot\beta(t')}$ for the Möbius transformations in $SL(2,\mathbb{R})$;

$$\psi(t,r) \to \exp\left(\mathcal{M}\left(\frac{1}{2}\alpha(t')\dot\alpha(t') - r'\dot\alpha(t')\right)\right)\psi'(t',r') \tag{0.8}$$

where

$$t = t', \quad r = r' - \alpha(t') = r' - at' - b \tag{0.9}$$

for the Galilei transformations;

$$\psi(t,r) \to \exp(\mathcal{M}\gamma)\psi'(t,r) \tag{0.10}$$

(γ constant) for the phase shifts.

By a straightforward extrapolation of these formulas to Lie generators of arbitrary integer of half-integer indices, or – in other words – to arbitrary functions of time $\alpha(t), \beta(t), \gamma(t)$, one finds a realization of the Lie algebra $\mathfrak{sv}$ or of the Schrödinger-Virasoro group (defined in [39]) which exponentiates $\mathfrak{sv}$.

The original physical motivation for introducing these algebras is the following. In the statistical physics of many-body systems far from equilibrium, it is well-established that a dynamical, time-dependent scale-invariance frequently arises, even in cases where the stationary state does *not* have a static, time-independent scale invariance. The scaling generator L_0 describes a dynamics with dynamical exponent $z = 2$, characteristic of a diffusion-like evolution; a signature of this behaviour is the existence of scaling functions $\mathcal{G}_R$, $\mathcal{G}_C$ for the two-time response and correlation functions defined as (see [36])

$$R(t,s) := \frac{\partial\langle\phi(t_2,r_2)\rangle}{\partial h(t_1,r_1)}|_{h=0} = s^{-a-1}\mathcal{G}_R(\frac{t_2}{t_1}, \frac{(r_2-r_1)^2}{t_2-t_1}), \tag{0.11}$$

$$C(t,s) := \langle\phi(t_1,r_1)\phi(t_2,r_2)\rangle = s^{-b}\mathcal{G}_C(\frac{t_2}{t_1}, \frac{(r_2-r_1)^2}{t_2-t_1}) \tag{0.12}$$

for some scaling exponents a, b (at least in the scaling limit $t_2 \gg t_1$, $r_2 - r_1 \to \infty$), so that, loosely speaking, the time coordinate scales as the square of the space coordinate(s). For a simple illustration, consider the phase-ordering kinetics of a simple magnet (described in terms of an Ising model) with a completely random initial state, which at the initial time $t = 0$ is brought into contact with a thermal bath at a sufficiently low temperature so that more than one stable thermodynamic state exists. Then indeed one observes a $z = 2$ dynamical scaling, as reviewed in [9]. This is also the case for many different models at criticality, described for instance by a stochastic Langevin equation or a master equation, for which an equilibrium state does not even exist, see [21, 22]. Actually, much more can be said: in all these models, there is evidence for the existence of a dynamical invariance under the subalgebra $\mathfrak{age}_1 = \langle L_0, L_1\rangle \ltimes \langle Y_{\pm\frac{1}{2}}, M_0\rangle \subset \mathfrak{sch}_1$ where the time-invariance generator has been omitted, allowing for an ageing behaviour. Note for the sake of completeness that the interest has shifted very recently to the case $z \neq 2$, which is the general law for systems quenched exactly onto their critical temperature, or else for equilibrium critical dynamics, and may also apply to the physically completely different situation of Lifschitz points in equilibrium spin systems with uniaxial competing interactions (for a recent review on the available evidence for this, see [22, 22, 37, 12]); however, the symmetry algebras seem to be much more complicated in this case, and they are not directly related to the Schrödinger algebra.

Coming back to algebra, let us rephrase the physical consequences of symmetry in a mathematical way. Let

$$\rho : g \to (\Phi(t,r) \to \rho(g)(\Phi(t,r)) = \Phi_g(g.(t,r)))$$

be any realization of the Schrödinger Lie group Sch_1 exponentiating $\mathfrak{sch}_1$ as coordinate transformations acting projectively on a wave-function $\Phi(t,r)$: the statistical field $\Phi(t,r)$ is called *quasi-primary* if its n-point functions or correlators $\langle \Phi(t_1,r_1)\ldots\Phi(t_n,r_n)\rangle$ transform covariantly under ρ, namely:

$$\langle \Phi_g(g.(t_1,r_1))\ldots\Phi_g(g.(t_n,r_n))\rangle = \langle \Phi(t_1,r_1)\ldots\Phi(t_n,r_n)\rangle. \tag{0.13}$$

The predictions of this invariance principle have been extensively developed for different types of realizations of $\mathfrak{sch}_1$, including the *mass $\mathcal{M}$ realization* given by formulas (0.5,0.6) above which define *scalar massive fields*, and tested with success for relevant physical systems - see for instance [26], [36] or [27] for a review. A prominent feature of this type of covariance is the Bargmann superselection rule with respect to the mass: n-point functions of fields $\Phi_1,\ldots,\Phi_n$ with respective masses $\mathcal{M}_1,\ldots,\mathcal{M}_n$ cancel except if $\mathcal{M}_1+\ldots+\mathcal{M}_n = 0$.

The reader *should be aware that the mass* plays here a very different role by comparison with relativistic physics or with critical phenomena at equilibrium: it is the central charge of the Galilei algebra, and massless fields have in general no physical interest. Also, it has absolutely nothing to do with a parameter measuring the distance away from criticality (actually, some kinetic models at criticality have been proved to exhibit an $\mathfrak{age}$-invariance!).

The original project was to build the infinite-dimensional Lie algebra $\mathfrak{sv}$ into the cornerstone of a 'Schrödinger-field theory' with applications to $z = 2$ dynamical scaling, by analogy with the role played by the Virasoro algebra in the systematic study of two-dimensional statistical physics at equilibrium near the critical temperature. The 'coinduced' representations of $\mathfrak{sv}$ introduced in [39] and extensively used here are undoubtedly the natural *Schrödinger-Virasoro primary* (classical) fields to look at, and extend the tensor-density modules $\mathcal{F}_\lambda$ or classical primary fields (or weight currents) of $\mathfrak{vir}$. However, something fails right from the start since no interesting (even linear!) wave equation exhibiting this infinite-dimensional Lie algebra of symmetries has been found. It seems difficult or impossible to find such wave equations (at least scalar wave equations), see [11]. There may be a way to escape this problem, see [41], but it requires the use of a doubly-infinite Lie algebra of invariance (actually, a 'double' extension of the pseudodifferential algebra on the line) of which $\mathfrak{sv}$ appears to be a quotient. This complementary approach is currently under investigation.

The purpose of this paper is to construct explicit non-trivial vertex algebra representations of $\mathfrak{sv}$. We hope that this is only a first step towards a deeper understanding of Schrödinger-invariant fields, and that a connection with actual physical models can eventually be established. Indeed, these representations open the road to an explicit computation of n-point functions from the knowledge of the symmetries. In particular, some three-point functions (which are known to depend on an arbitrary scaling function for massive $\mathfrak{sch}_1$-covariant fields) are computed here for a conjectural $\mathfrak{sv}$-covariant massive field which must still be spelled out completely.

The paper is organized as follows.

Section 1 is introductory on the Schrödinger-Virasoro Lie algebra and its representations. Most of the material contained here is adapted from [39]. However, after developing the theory a while, it appeared necessary to deal with an *extended Schrödinger-Virasoro Lie algebra* denoted by $\tilde{\mathfrak{sv}}$ that is defined here for the first time. The extension of the results of [39] to $\tilde{\mathfrak{sv}}$ is more or less straightforward. The Lie algebra $\tilde{\mathfrak{sv}}$ appears to have *three* independent central extensions (in other terms, three central charges), whereas $\mathfrak{sv}$ admits only one central extension. The centrally extended Lie algebra is denoted by $\tilde{\mathfrak{sv}}_{c,\kappa,\alpha}$ (see Lemma 1.2).

Section 2 deals mostly with the definition of $\mathfrak{sv}$- and $\tilde{\mathfrak{sv}}$-primary fields, see Definition 2.1.1. They depend on the choice of a 'spin representation' ρ of $\mathfrak{sv}_0 \cong \langle L_0 \rangle \ltimes \langle Y_{\frac{1}{2}}, M_1 \rangle \subset \mathfrak{sv}$ or $\tilde{\mathfrak{sv}}_0 \cong \langle N_0 \rangle \ltimes \mathfrak{sv}_0 \subset \tilde{\mathfrak{sv}}$ (see below for a definition of N_0). It appears from the examples that $\tilde{\mathfrak{sv}}$-primary fields are also characterized by a matrix Ω acting on the representation space of ρ, which is unexpected from a mathematical point of view.

Section 3 is devoted to the construction of the *$a\bar{b}$-theory*. The name refers to the fact that the $\tilde{\mathfrak{sv}}$ fields (see Definition 3.1.3) are built out of two independent fields of conformal field theory, namely a free boson $a(z)$ and a charged symplectic boson $\bar{b}(z) = (\bar{b}^+(z), \bar{b}^-(z))$. Note that the complex variable z becomes the real time variable t in this theory and the conjugate variable $\bar{z}$ apparently leaves the picture. The so-called *polynomial fields* $\Phi_{j,k}$ and *generalized polynomial fields* ${}_\alpha\Phi_{j,k}$, $j,k \in \mathbb{N}, \alpha \in \mathbb{R}$ – all of them $\tilde{\mathfrak{sv}}$-primary fields – are constructed (see Theorems 3.2.4 and 3.2.5) as polynomials in the fields a, $\bar{b}$, the ${}_\alpha\Phi_{j,k}$ involving furthermore the vertex operator V_α built from a. The space-dependence of the fields appears from the repeated application of the generator $Y_{-\frac{1}{2}}$, interpreted as a space-translation.

In Section 4, we compute the two- and three-point functions of the polynomial and generalized polynomial fields introduced in Section 3.

Finally, Section 5 conjectures the existence of *massive* fields, see Theorem 5.1 and Theorem 5.2 for a definition, whose two-point and (at least in one case) three-point functions are explicitly computed.

1 On the extended Schrödinger Lie algebra $\tilde{\mathfrak{sv}}$ and its coinduced representations

Recall from the Introduction the realization of $\mathfrak{sch}_1$ as Lie symmetries of the free Schrödinger equation

$$(2\mathcal{M}\partial_t - \partial_r^2)\psi(t,r) = 0 \tag{1.1}$$

(see formulas (0.5) and (0.6) above). Suppose now that the wave-function $\psi = \psi_{\mathcal{M}}(t,r)$ is indexed by the mass parameter. Then a 'trick' first used in [23] (see also [24] for an application to the Dirac-Lévy-Leblond equation and [40] for other invariant equations), with far-reaching consequences, is to consider (formally) a Laplace transform of the Schrödinger equation with respect to the mass: the Laplace transformed field

$$\tilde{\psi}(t,r,\zeta) := \int \psi_{\mathcal{M}}(t,r) e^{\mathcal{M}\zeta}\, d\mathcal{M} \tag{1.2}$$

satisfies the field equation $\Delta\tilde{\psi}(t,r,\zeta) = 0$, where

$$\Delta := 2\partial_t\partial_\zeta - \partial_r^2 \tag{1.3}$$

is formally equivalent to a Laplacian in three dimensions. Transforming accordingly the Lie symmetry generators in $\mathfrak{sch}_1$ is equivalent to 'replacing' $\mathcal{M}$ by ∂_ζ in (0.5, 0.6).

The difference with the usual fixed mass setting is that the new wave equation has more symmetries (as well-known, the Laplacian in three dimensions is $\mathfrak{conf}_3$-invariant, where $\mathfrak{conf}_3 \cong \mathfrak{so}(4,1)$ is the Lie algebra of infinitesimal conformal transformations), including in particular $N_0 = -r\partial_r - 2\zeta\partial_\zeta$. This new generator of $\mathfrak{conf}_3$ acts as a *derivation* on $\mathfrak{sch}_1$ in the above realization, namely

$$[N_0, L_{0,\pm 1}] = 0, \quad [N_0, Y_{\pm\frac{1}{2}}] = Y_{\pm\frac{1}{2}}, \quad [N_0, M_0] = 2M_0. \tag{1.4}$$

One obtains thus a 7-dimensional maximal parabolic Lie subalgebra of $\mathfrak{conf}_3$ (see [23]), $\tilde{\mathfrak{sch}}_1 = \langle N_0\rangle \ltimes \mathfrak{sch}_1$. Note that an embedding of the Schrödinger algebra into the conformal algebra (in $d = 3$ space dimensions) had been defined in a different context in [10].

Definition 1.1

Let $\tilde{\mathfrak{sv}} \supset \mathfrak{sv}$ be the (abstract) Lie algebra generated by L_n, M_n, N_n ($n \in \mathbb{Z}$) and Y_m ($m \in \frac{1}{2}+\mathbb{Z}$) with the following additional brackets:

$$[L_n, N_p] = -pN_{n+p}, [N_n, N_p] = 0, \quad [N_n, Y_p] = Y_{n+p}, \ [N_n, M_p] = 2M_{n+p} \tag{1.5}$$

Note that the N_n, $n \in \mathbb{Z}$, may be interpreted as a second L-conformal current with conformal weight 1.

Lemma 1.2

1. *Let*

$$\mathfrak{h} = \langle Y_m \mid m \in \frac{1}{2} + \mathbb{Z}\rangle \oplus \langle M_p \mid p \in \mathbb{Z}\rangle \tag{1.6}$$

and

$$\tilde{\mathfrak{h}} = \langle N_n \mid n \in \mathbb{Z}\rangle \oplus \mathfrak{h}. \tag{1.7}$$

Then $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ are Lie subalgebras of $\tilde{\mathfrak{sv}}$ and one has the following double semi-direct product structure:

$$\tilde{\mathfrak{h}} = \langle N_n \mid n \in \mathbb{Z}\rangle \ltimes \mathfrak{h}, \quad \tilde{\mathfrak{sv}} = \mathfrak{vir}_0 \ltimes \tilde{\mathfrak{h}}. \tag{1.8}$$

The Lie algebra $\tilde{\mathfrak{h}}$ is solvable.

2. *The Lie algebra* $\tilde{\mathfrak{sch}}_1 = \langle N_0 \rangle \ltimes \mathfrak{sch}_1$ *is a maximal Lie subalgebra of* $\tilde{\mathfrak{sv}}$.

3. *The Lie algebra* $\tilde{\mathfrak{sv}}$ *has three independent classes of central extensions given by the cocycles*

$$c_1(L_n, L_m) = \frac{1}{12} n(n^2 - 1)\delta_{n+m,0}; \tag{1.9}$$

$$c_2(N_n, N_m) = n\delta_{n+m,0}; \tag{1.10}$$

$$c_3(L_n, N_m) = n^2\delta_{n+m,0} \tag{1.11}$$

(the zero components of the cocycles have been omitted).

Proof.

Points 1 and 2 are straightforward. Let us turn to the proof of point 3.

The Lie subalgebra $\mathfrak{sv}$ is known (see [21] or [39]) to have only one class of central extensions given by the multiples of the Virasoro cocycle c_1; it extends straightforwardly by zero to $\tilde{\mathfrak{sv}}$. Then any central cocycle c of $\tilde{\mathfrak{sv}}$ which is non-trivial on the N-generators may be decomposed by L_0-homogeneity (see [17]) into the following components

$$c(N_m, N_p) = a_m\delta_{m+p},\ c(N_m, M_p) = b_m\delta_{m+p},\ c(L_m, N_p) = c_m\delta_{m+p} \tag{1.12}$$

The b_m are easily seen to vanish by applying the Jacobi relation to $[N_n, [Y_m, Y_p]]$ where $n+m+p = 0$. The same relation applied to $[L_n, [N_m, N_p]]$, respectively $[L_n, [L_m, N_p]]$, yields $pa_m = ma_p$, viz. $(n+m)(c_n - c_m) = (n-m)c_{n+m}$, hence $a_m = \kappa m$ and $c_m = \alpha m^2 + \beta m$ for some coefficients κ, α, β. The coefficient β may be set to zero by adding a constant to N_0. Finally, the two remaining cocycles are easily seen to be non-trivial and independent. □

Definition 1.3

Let $\tilde{\mathfrak{sv}}_{c,\kappa,\alpha}$ *be the central extension of* $\mathfrak{sv}$ *corresponding to the cocycle* $cc_1 + \kappa c_2 + \alpha c_3$, *i.e. such that*

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}cn(n^2-1)\delta_{n+m,0}; \quad [N_n, N_m] = \kappa n\delta_{n+m,0}; \quad [L_n, N_m] = -mN_{n+m} + \alpha n^2\delta_{n+m,0}. \tag{1.13}$$

We shall now define a series of representations $\tilde{\rho}$ of $\tilde{\mathfrak{sv}}$, that we call *coinduced representations*, which are the analogues of the density modules or conformal currents of the Virasoro representation theory. They are indexed by a 'spin' parameter ρ corresponding to the choice of a class of equivalence of representations of the subalgebra $\tilde{\mathfrak{sv}}_0 \subset \tilde{\mathfrak{sv}}$ (see below for a definition of $\tilde{\mathfrak{sv}}_0$).

The Lie algebra $\tilde{\mathfrak{sv}}$ is provided with a graduation δ defined by

$$\delta(L_n) = nL_n,\ \delta(N_n) = n,\ \delta(Y_m) = (m - \frac{1}{2})Y_m,\ \delta(M_n) = (n-1)M_n \quad (n \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z}) \tag{1.14}$$

Note that $\delta = \mathrm{ad}\,(-\frac{1}{2}N_0 - L_0) = -\frac{1}{2}[N_0, .] - [L_0, .]$.

Set $\widetilde{\mathfrak{sv}}_n = \{X \in \widetilde{\mathfrak{sv}} \mid \delta(X) = nX\} = \langle L_n, N_n, Y_{n+\frac{1}{2}}, M_{n+1}\rangle$ for $n = 0, 1, 2, \ldots$ and $\widetilde{\mathfrak{sv}}_{-1} = \langle L_{-1}, Y_{-\frac{1}{2}}, M_0\rangle$. Note that we choose to exclude N_{-1} from $\widetilde{\mathfrak{sv}}_{-1}$ although $\delta(N_{-1}) = -N_{-1}$. Then $\widetilde{\mathfrak{fsv}} := \oplus_{n\geq -1}\widetilde{\mathfrak{sv}}_n$ is a Lie subalgebra of $\widetilde{\mathfrak{sv}}$. The subspace $\widetilde{\mathfrak{sv}}_{-1}$ is commutative and the Lie subalgebra $\widetilde{\mathfrak{sv}}_0 := \{X \in \widetilde{\mathfrak{sv}} \mid \delta(X) = 0\}$ is a double extension of the commutative Lie algebra $\langle Y_{\frac{1}{2}}, M_1\rangle \cong \mathbb{R}^2$ by L_0 and N_0 as follows:

$$\widetilde{\mathfrak{sv}}_0 = (\langle L_0\rangle \oplus \langle N_0\rangle) \ltimes \langle Y_{\frac{1}{2}}, M_1\rangle \qquad (1.15)$$

Namely, one has

$$[L_0, Y_{\frac{1}{2}}] = -\frac{1}{2}Y_{\frac{1}{2}},\ [L_0, M_1] = -M_1; \quad [N_0, L_0] = 0,\ [N_0, Y_{\frac{1}{2}}] = Y_{\frac{1}{2}},\ [N_0, M_1] = 2M_1. \qquad (1.16)$$

Note that N_0 acts by conjugation as $-2L_0$ on $\widetilde{\mathfrak{sv}}_0$. Also, the adjoint action of $\widetilde{\mathfrak{sv}}_0$ preserves $\widetilde{\mathfrak{sv}}_{-1}$, so that $\widetilde{\mathfrak{sv}}_0 \oplus \widetilde{\mathfrak{sv}}_{-1} = \widetilde{\mathfrak{sv}}_0 \ltimes \widetilde{\mathfrak{sv}}_{-1}$ is a Lie algebra too. Actually, $\widetilde{\mathfrak{fsv}}$ appears to be the *Cartan prolongation* of $\widetilde{\mathfrak{sv}}_0 \ltimes \widetilde{\mathfrak{sv}}_{-1}$ (see [1]): if one realizes $\widetilde{\mathfrak{sv}}_0 \ltimes \widetilde{\mathfrak{sv}}_{-1}$ as the following polynomial vector fields [2]

$$L_{-1} = -\partial_t,\ Y_{-\frac{1}{2}} = -\partial_r,\ M_0 = -\partial_\zeta \qquad (1.17)$$

$$L_0 = -t\partial_t - \frac{1}{2}r\partial_r,\ N_0 = -r\partial_r - 2\zeta\partial_\zeta,\ Y_{\frac{1}{2}} = -t\partial_r - r\partial_\zeta,\ M_1 = -t\partial_\zeta \qquad (1.18)$$

then the Lie algebra $\widetilde{\mathfrak{sv}}_{-1} \oplus \widetilde{\mathfrak{sv}}_0 \oplus \widetilde{\mathfrak{sv}}_1 \oplus \ldots$ defined inductively by

$$\widetilde{\mathfrak{sv}}_n := \{\mathcal{X} \in \mathcal{P}_n \mid [\mathcal{X}, \widetilde{\mathfrak{sv}}_{-1}] \subset \widetilde{\mathfrak{sv}}_{n-1}\}, \quad n \geq 1 \qquad (1.19)$$

(where $\mathcal{P}_n$ stands for the vector space of homogeneous polynomial vector fields on $\mathbb{R}^3$ of degree $n+1$) defines a vector field realization of $\widetilde{\mathfrak{fsv}}$ which extends straightforwardly into a representation of $\widetilde{\mathfrak{sv}}$. Namely, let $f \in \mathbb{C}[t, t^{-1}]$: then

$$L_f = -f(t)\partial_t - \frac{1}{2}f'(t)r\partial_r - \frac{1}{4}f''(t)r^2\partial_\zeta \qquad (1.20)$$

$$N_f = -f(t)(r\partial_r + 2\zeta\partial_\zeta) - \frac{1}{2}f'(t)r^2\partial_\zeta \qquad (1.21)$$

$$Y_f = -f(t)\partial_r - f'(t)r\partial_\zeta \qquad (1.22)$$

$$M_f = -f(t)\partial_\zeta \qquad (1.23)$$

The restriction to $\mathfrak{sv}$ of the above realization of $\widetilde{\mathfrak{sv}}$ extends (after a Laplace transform) the mass $\mathcal{M}$ realization of $\mathfrak{sch}_1$, see formulas (0.5), (0.6) and was originally obtained in [21].

Let us now find out the *coinduced representations* of $\widetilde{\mathfrak{fsv}}$. The work was done in [39] for the Lie algebra $\mathfrak{sv}$. The generalization to $\widetilde{\mathfrak{sv}}$ is only a matter of easy computations. Hence we merely recall the definition and give the results.

[2]Note that this realization was originally obtained in [23], where the generator denoted by N coincides with $L_0 - \frac{N_0}{2} = -t\partial_t + \zeta\partial_\zeta$.

Let ρ be a representation of $\tilde{\mathfrak{sv}}_0 = (\langle L_0\rangle \oplus \langle N_0\rangle) \ltimes \langle Y_{\frac{1}{2}}, M_1\rangle$ into a vector space $\mathcal{H}_\rho$. Then ρ can be trivially extended to $\tilde{\mathfrak{sv}}_+ = \oplus_{i\geq 0}\tilde{\mathfrak{sv}}_i$ by setting $\rho(\sum_{i>0}\tilde{\mathfrak{sv}}_i) = 0$. Standard examples are provided:

(i) either by choosing a representation ρ of the $(ax+b)$-Lie algebra $\langle L_0, Y_{\frac{1}{2}}\rangle$ and extending it to $\tilde{\mathfrak{sv}}_0$ by setting

$$\rho(N_0) = -2\rho(L_0) + \mu \mathrm{Id}\ (\mu \in \mathbb{R}), \quad \rho(M_1) = C\rho(Y_{\frac{1}{2}})^2\ (C \in \mathbb{R}); \tag{1.24}$$

(ii) or by choosing a representation ρ of the $(ax+b)$-Lie algebra $\langle L_0, M_1\rangle$ and extending it to $\tilde{\mathfrak{sv}}_0$ by setting

$$\rho(N_0) = -2\rho(L_0) + \mu \mathrm{Id}\ (\mu \in \mathbb{R}), \quad \rho(Y_{\frac{1}{2}}) = 0. \tag{1.25}$$

Actually, one may show easily that finite-dimensional indecomposable representations of $\langle L_0, Y_{\frac{1}{2}}\rangle$ or $\langle L_0, M_1\rangle$ are given (up to the addition of a constant to L_0) by restricting any finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$ to its Borel subalgebra or traceless upper-triangular matrices. (On the other hand, the classification of all indecomposable finite-dimensional representations of $\tilde{\mathfrak{sv}}_0$ is probably a very difficult task). It happens so that all examples considered in this article are obtained as in (i) or (ii).

Let us now define the representation of $\widetilde{\mathfrak{fsv}}$ coinduced from ρ.

Definition 1.4 (see [39])

The ρ-formal density module $(\tilde{\mathcal{H}}_\rho, \tilde\rho)$ *is the coinduced module*

$$\begin{aligned}\tilde{\mathcal{H}}_\rho &= \mathrm{Hom}_{\mathcal{U}(\tilde{\mathfrak{sv}}_+)}(\mathcal{U}(\widetilde{\mathfrak{fsv}}), \mathcal{H}_\rho) \\ &= \{\phi:\ \mathcal{U}(\widetilde{\mathfrak{fsv}}) \to \mathcal{H}_\rho \text{ linear} \mid \phi(U_0V) = \rho(U_0).\phi(V),\quad U_0 \in \mathcal{U}(\tilde{\mathfrak{sv}}_+), V \in \mathcal{U}(\widetilde{\mathfrak{fsv}})\}\end{aligned} \tag{1.26}$$

with the natural action of $\mathcal{U}(\widetilde{\mathfrak{fsv}})$ *on the right*

$$(d\tilde\rho(U).\phi)(V) = \phi(VU), \quad U, V \in \mathcal{U}(\widetilde{\mathfrak{fsv}}). \tag{1.27}$$

These abstract-looking formal density modules may be identified with the following representations by matrix first-order differential operators.

Theorem 1.5

The $\widetilde{\mathfrak{fsv}}$*-module* $(\widetilde{\mathcal{H}}_\rho, \tilde\rho)$ *of* $\widetilde{\mathfrak{fsv}}$ *is isomorphic to the action of the following matrix differential operators on functions:*

$$\tilde\rho(L_f) = \left(-f(t)\partial_t - \frac{1}{2}f'(t)r\partial_r - \frac{1}{4}f''(t)r^2\partial_\zeta\right) \otimes \mathrm{Id}_{\mathcal{H}_\rho} + f'(t)\rho(L_0) + \frac{1}{2}f''(t)r\rho(Y_{\frac{1}{2}}) + \frac{1}{4}f'''(t)r^2\rho(M_1);$$

$$\tilde\rho(N_f) = \left(-f(t)(r\partial_r + 2\zeta\partial_\zeta) - \frac{1}{2}f'(t)r^2\partial_\zeta\right) \otimes \mathrm{Id}_{\mathcal{H}_\rho} + f(t)\rho(N_0)$$
$$+f'(t)r\rho(Y_{\frac{1}{2}}) + (\frac{1}{2}f''(t)r^2 + 2\zeta f'(t))\rho(M_1);$$
$$\tilde\rho(Y_f) = \big(-f(t)\partial_r - f'(t)r\partial_\zeta\big) \otimes \mathrm{Id}_{\mathcal{H}_\rho} + f'(t)\rho(Y_{\frac{1}{2}}) + f''(t)r\ \rho(M_1);$$
$$\tilde\rho(M_f) = -f(t)\partial_\zeta \otimes \mathrm{Id}_{\mathcal{H}_\rho} + f'(t)\ \rho(M_1). \tag{1.28}$$

It may be extended into a representation of $\widetilde{\mathfrak{sv}}$ *by simply extrapolating the above formulas to* $f \in \mathbb{R}[t, t^{-1}]$.

The representations of $\widetilde{\mathfrak{sv}}$ thus obtained will be called *coinduced representations.*

2 The Schrödinger-Virasoro primary fields and the superfield interpretation of $\widetilde{\mathfrak{sv}}$

Just as conformal fields are given by quantizing density modules in the Virasoro representation theory, we shall define in this section $\tilde{\mathfrak{sv}}$-primary fields by quantizing the coinduced representations $\tilde\rho$ introduced in the previous section.

2.1 Definition of the Schrödinger-Virasoro primary fields

Our foundamental hypothesis is that correlators of $\tilde{\mathfrak{sv}}$-primary fields $\langle \Phi_1(t_1, r_1, \zeta_1) \ldots \Phi_n(t_n, r_n, \zeta_n)\rangle$ should be singular only when some of the time coordinates coincide; this is confirmed by the computations of two- and three-point functions for scalar massive Schrödinger-covariant fields (see [21] or [23], or also Appendix A). Hence one is led to the following assumption:

A $\tilde{\mathfrak{sv}}$-primary field $\Phi(t, r, \zeta)$ may be written as $\Phi(t, r, \zeta) = \sum_\mu \Phi^{(\mu)}(t, r, \zeta)e_\mu$, where $(e_\mu)_{\mu=1,\ldots,\dim \mathcal{H}_\rho}$ is a basis of the representation space $\mathcal{H}_\rho$ (see Section 1) and

$$\Phi^{(\mu)}(t, r, \zeta) := \sum_\xi \Phi^{(\mu),\xi}(t, \zeta)r^\xi \tag{2.1}$$

where ξ varies in a denumerable set of real values which is bounded below (so that it is possible to multiply two such formal series) and stable with respect to translations by positive integers. It may have been more logical to decompose further $\Phi^{(\mu),\xi}(t, \zeta)$ as $\sum_\sigma \Phi^{(\mu),\xi,\sigma}(t)\zeta^\sigma$, as we shall occasionally do (see subsection 3.2), but this leads to unncessarily complicated notations and turns out to be mostly counter-productive. In any case, $\Phi^{(\mu),\xi}(t, \zeta)$ is to be seen as a ζ-indexed quantum field in the variable t, the latter playing the same role as the complex variable z of conformal field theory, implying the possibility of defining normal ordering, operator product expansions and so on. Note that the $\mathcal{H}_\rho$-components of the field Φ are written systematically *inside parentheses* in order to avoid any possible confusion with other indices.

Suppose now that $\tilde{\mathfrak{sv}}$ (or any of its central extensions) acts on Φ by the coinduced representation $\tilde\rho$ of Theorem 1.5. This action decomposes naturally as an action on each field component $\Phi^{(\mu),\xi}$ as follows (where Einstein's summation convention is implied):

$$\begin{aligned}[L_m,\Phi^{(\mu),\xi}(t,\zeta)] &= -t^{m+1}\partial_t\Phi^{(\mu),\xi}(t,\zeta)-\frac{\xi}{2}(m+1)t^m\Phi^{(\mu),\xi}(t,\zeta)-\frac14(m+1)mt^{m-1}\partial_\zeta\Phi^{(\mu),\xi-2}\\ &+(m+1)t^m\rho(L_0)^\mu_\nu\Phi^{(\nu),\xi}(t,\zeta)\\ &+\frac12(m+1)mt^{m-1}\rho(Y_{\frac12})^\mu_\nu\Phi^{(\nu),\xi-1}(t,\zeta)\\ &+\frac14(m+1)m(m-1)t^{m-2}\rho(M_1)^\mu_\nu\Phi^{(\nu),\xi-2}(t,\zeta);\end{aligned}\tag{2.2}$$

$$\begin{aligned}[N_m,\Phi^{(\mu),\xi}(t,\zeta)] &= -t^m(\xi+2\zeta\partial_\zeta)\Phi^{(\mu),\xi}(t,\zeta)-\frac{m}{2}t^{m-1}\partial_\zeta\Phi^{(\mu),\xi-2}(t,\zeta)+t^m\rho(N_0)^\mu_\nu\Phi^{(\nu),\xi}(t,\zeta)\\ &+\ mt^{m-1}\rho(Y_{\frac12})^\mu_\nu\Phi^{(\nu),\xi-1}(t,\zeta)+\frac{m(m-1)}{2}t^{m-2}\rho(M_1)^\mu_\nu\Phi^{(\nu),\xi-2}(t,\zeta)\\ &+\ 2mt^{m-1}\zeta\rho(M_1)^\mu_\nu\Phi^{(\nu),\xi}(t,\zeta);\end{aligned}\tag{2.3}$$

$$\begin{aligned}[Y_m,\Phi^{(\mu),\xi}(t,\zeta)] &= -t^{m+\frac12}(\xi+1)\Phi^{(\mu),\xi+1}(t,\zeta)-(m+\frac12)t^{m-\frac12}\partial_\zeta\Phi^{(\mu),\xi-1}(t,\zeta)\\ &+(m+\frac12)t^{m-\frac12}\rho(Y_{\frac12})^\mu_\nu\Phi^{(\nu),\xi}(t,\zeta)\\ &+(m+\frac12)(m-\frac12)t^{m-3/2}\rho(M_1)^\mu_\nu\Phi^{(\nu),\xi-1}(t,\zeta);\end{aligned}\tag{2.4}$$

$$[M_m,\Phi^{(\mu),\xi}(t,\zeta)]=-t^m\partial_\zeta\Phi^{(\mu),\xi}(t,\zeta)+mt^{m-1}\rho(M_1)^\mu_\nu\Phi^{(\nu),\xi}(t,\zeta).\tag{2.5}$$

In order to define $\tilde{\mathfrak{sv}}_{c,\kappa,\alpha}$-primary fields, one needs first the following assumption: there exist four mutually local fields

$$L(t)=\sum_{n\in\mathbb{Z}}L_nt^{-n-2}\ ,\ Y(t)=\sum_{n\in\mathbb{Z}+\frac12}Y_nt^{-n-3/2},\ M(t)=\sum_{n\in\mathbb{Z}}M_nt^{-n-1},\quad N(t)=\sum_{n\in\mathbb{Z}}N_nt^{-n-1}$$

with the following OPE's:

$$L(t_1)L(t_2)\sim\frac{\partial L(t_1)}{t_1-t_2}+\frac{2L(t_2)}{(t_1-t_2)^2}+\frac{c/2}{(t_1-t_2)^4},\quad c\in\mathbb{R}\tag{2.6}$$

so that L is a Virasoro field with central charge c;

$$L(t_1)Y(t_2)\sim\frac{\partial Y(t_2)}{t_1-t_2}+\frac{\frac32 Y(t_2)}{(t_1-t_2)^2},\quad L(t_1)M(t_2)\sim\frac{\partial M(t_2)}{t_1-t_2}+\frac{M(t_2)}{(t_1-t_2)^2}\tag{2.7}$$

and

$$L(t_1)N(t_2)\sim\frac{\partial M(t_2)}{t_1-t_2}+\frac{M(t_2)}{(t_1-t_2)^2}+\frac{\alpha}{(t_1-t_2)^3}\tag{2.8}$$

so that Y (resp. M) is an L-primary field with conformal weight $\frac{3}{2}$ (resp. 1) and N is primary with conformal weight 1 up to the term $\frac{\alpha}{(t_1-t_2)^3}$ due to the central extension;

$$Y(t_1)Y(t_2) \sim \frac{\partial M}{t_1 - t_2} + \frac{2M(t_2)}{(t_1-t_2)^2}, \quad Y(t_1)M(t_2) \sim 0, \quad M(t_1)M(t_2) \sim 0 \tag{2.9}$$

and

$$N(t_1)M(t_2) \sim \frac{2M(t_2)}{t_1 - t_2}, \quad N(t_1)Y(t_2) \sim \frac{Y(t_2)}{t_1-t_2}, \quad N(t_1)N(t_2) \sim \frac{\kappa}{(t_1-t_2)^2} \tag{2.10}$$

which all together yield in mode decomposition the centrally extended Lie algebra $\widetilde{\mathfrak{sv}}_{c,\kappa,\alpha}$.

We may now define what a *ρ-$\tilde{\mathfrak{sv}}$-primary field* is. Note that we leave aside for the time being the essential condition which states that the values of the index ξ should be bounded from below; we shall actually see in subsection 3.2 that our free field construction works only for fields $\Phi^{(\mu)} = \sum_\xi \Phi^{(\mu),\xi} r^\xi$ such that $\Phi^{(\mu),\xi} = 0$ for all negative indices ξ. For technical reasons that will be explained below, we shall also define $\mathfrak{sv}$-primary fields and $\langle N_0 \rangle \ltimes \mathfrak{sv}$-primary fields.

In the following definition, we call (following [29]) *mutually local fields* a set $X_1, \ldots, X_n$ of operator-valued formal series in t whose commutators $[X_i(t_1), X_j(t_2)]$ are distributions of finite order supported on the diagonal $t_1 = t_2$. In other words, the fields $X_1, \ldots, X_n$ have meromorphic operator-product expansions (OPE).

Definition 2.1.1

1. *($\mathfrak{sv}$-primary fields)*

   *Let $\rho : \mathfrak{sv}_0 \to \mathcal{L}(\mathcal{H}_\rho)$ be a finite-dimensional representation of $\mathfrak{sv}_0$. A ρ-$\mathfrak{sv}$-*primary field *$\Phi(t,r,\zeta) = \sum_\mu \Phi^{(\mu)}(t,r,\zeta) e_\mu$ is given (at least in a formal sense) as an infinite series*

   $$\Phi^{(\mu)}(t,r,\zeta) = \sum_\xi \Phi^{(\mu),\xi}(t,\zeta) r^\xi$$

   where ξ varies in a denumerable set of real values which is stable with respect to integer translations, and the $\Phi^{(\mu),\xi}(t,\zeta)$ are mutually local fields with respect to the time variable t– which are also mutually local with the $\mathfrak{sv}$-fields $L(t), Y(t), M(t)$ – with the following OPE:

$$\begin{aligned} L(t_1)\Phi^{(\mu),\xi}(t_2,\zeta) \sim \frac{\partial\Phi^{(\mu),\xi}(t_2,\zeta)}{t_1-t_2} &+ \frac{(\frac{1}{2}\xi)\Phi^{(\mu),\xi}(t_2,\zeta) - \rho(L_0)^\mu_\nu \Phi^{(\nu),\xi}(t_2,\zeta)}{(t_1-t_2)^2} \\ &+ \frac{\frac{1}{2}\partial_\zeta \Phi^{(\mu),\xi-2}(t_2,\zeta) - \rho(Y_{\frac{1}{2}})^\mu_\nu \Phi^{(\nu),\xi-1}(t_2,\zeta)}{(t_1-t_2)^3} \\ &- \frac{\frac{3}{2}\rho(M_1)^\mu_\nu \Phi^{(\nu),\xi-2}(t_2)}{(t_1-t_2)^4} \end{aligned} \tag{2.11}$$

$$\begin{aligned} Y(t_1)\Phi^{(\mu),\xi}(t_2,\zeta) \sim \frac{(1+\xi)\Phi^{(\mu),\xi+1}(t_2,\zeta)}{t_1-t_2} &+ \frac{\partial_\zeta \Phi^{(\mu),\xi-1}(t_2,\zeta) - \rho(Y_{\frac{1}{2}})^\mu_\nu \Phi^{(\nu),\xi}(t_2,\zeta)}{(t_1-t_2)^2} \\ &- \frac{2\rho(M_1)^\mu_\nu \Phi^{(\nu),\xi-1}(t_2,\zeta)}{(t_1-t_2)^3} \end{aligned} \tag{2.12}$$

$$M(t_1)\Phi^{(\mu),\xi}(t_2,\zeta) \sim \frac{\partial_\zeta \Phi^{(\mu),\xi}(t_2,\zeta)}{t_1-t_2} - \frac{\rho(M_1)^\mu_\nu \Phi^{(\nu),\xi}(t_2,\zeta)}{(t_1-t_2)^2} \tag{2.13}$$

2. *($\langle N_0\rangle \ltimes \mathfrak{sv}$-primary fields) Let $\bar\rho : \tilde{\mathfrak{sv}}_0 = \langle N_0\rangle \ltimes \mathfrak{sv}_0 \to \mathcal{L}(\mathcal{H}_\rho)$ be a finite-dimensional representation of $\tilde{\mathfrak{sv}}_0$, and ρ be the restriction of $\bar\rho$ to $\mathfrak{sv}_0$. A $\bar\rho$-$\langle N_0\rangle \ltimes \mathfrak{sv}$-*primary field *$\Phi(t,r,\zeta)$ is a ρ-$\mathfrak{sv}$-primary field such that*

$$[N_0, \Phi^{(\mu),\xi}(t,\zeta)] = (\xi + 2\zeta\partial_\zeta)\Phi^{(\mu),\xi}(t,\zeta) - \bar\rho(N_0)^\mu_\nu \Phi^{(\nu),\xi}(t,\zeta). \tag{2.14}$$

3. *($\tilde{\mathfrak{sv}}$-primary fields) Let $\bar\rho : \tilde{\mathfrak{sv}}_0 = \langle N_0\rangle \ltimes \mathfrak{sv}_0 \to \mathcal{L}(\mathcal{H}_\rho)$ be a finite-dimensional representation of $\tilde{\mathfrak{sv}}_0$ and $\Omega : \mathcal{H}_\rho \to \mathcal{H}_\rho$ be a linear operator such that $[\rho(L_0),\Omega] = \Omega$, $[\rho(Y_{\frac{1}{2}}),\Omega] = [\rho(M_1),\Omega] = [\rho(N_0),\Omega] = 0$. Then a $(\bar\rho,\Omega)$-$\tilde{\mathfrak{sv}}$-*primary field *is a $\bar\rho|_{\mathfrak{sv}_0}$-$\mathfrak{sv}$-primary field $\Phi(t,r,\zeta)$, local with N, such that*

$$\begin{aligned} N(t_1)\Phi^{(\mu),\xi}(t_2,\zeta) &\sim \frac{(\xi+2\zeta\partial_\zeta)\Phi^{(\mu),\xi}(t_2,\zeta) - \rho(N_0)^\mu_\nu \Phi^{(\nu),\xi}(t_2,\zeta)}{t_1-t_2} \\ &+ \frac{\frac{1}{2}\partial_\zeta \Phi^{(\mu),\xi-2}(t_2,\zeta) - \rho(Y_{\frac{1}{2}})^\mu_\nu \Phi^{(\nu),\xi-1}(t_2,\zeta) - 2\zeta\rho(M_1)^\mu_\nu \Phi^{(\nu),\xi}(t_2,\zeta) - \Omega^\mu_\nu \Phi^{(\nu),\xi}(t_2,\zeta)}{(t_1-t_2)^2} \\ &- \frac{\rho(M_1)^\mu_\nu \Phi^{(\nu),\xi-2}(t_2,\zeta)}{(t_1-t_2)^3} \end{aligned} \tag{2.15}$$

In the case $\Omega = 0$, we shall simply say that Φ is $\bar\rho$-$\tilde{\mathfrak{sv}}$-primary.

Remark : Mind that in these OPE and in all the following ones, ζ is considered only as a parameter, as we mentioned earlier.

The operator Ω for $\tilde{\mathfrak{sv}}$-primary fields does not follow from the coinduction method. However, it appears in all our examples, including for the superfield $\mathcal{L}$ with components L, Y, M, N with the adjoint action of $\tilde{\mathfrak{sv}}$ on itself (see §2.2 below).

Proposition 2.1.2

Suppose Φ is a (ρ,Ω)-$\tilde{\mathfrak{sv}}$-primary field. Then the adjoint action of $\tilde{\mathfrak{sv}}$ on Φ is given by the formulas of Theorem 1.1 except for the action of the N-generators which are twisted *as follows:*

$$\begin{aligned} \tilde\rho(N_f) &= \left(-f(t)(r\partial_r + 2\zeta\partial_\zeta) - \frac{1}{2}f'(t)r^2\partial_\zeta\right) \otimes \mathrm{Id}_{\mathcal{H}_\rho} + f(t)\rho(N_0) \\ &+ f'(t)r\rho(Y_{\frac{1}{2}}) + (\frac{1}{2}f''(t)r^2 + 2\zeta f'(t))\rho(M_1) + f'(t)\Omega; \end{aligned} \tag{2.16}$$

Proof. Straightforward computations. One may in particular check that the twisted representation is indeed a representation of $\tilde{\mathfrak{sv}}$. □

Note that the usual conformal fields of weight λ are a particular case of this construction: they correspond to ρ-$\mathfrak{sv}$-conformal fields Φ with only one component $\Phi = \Phi^{(0)}(t)$, commuting with N, Y, M, such that ρ is the one-dimensional character given by $\rho(L_0) = -\lambda$, $\rho(N_0) = \rho(Y_{\frac{1}{2}}) = \rho(M_1) = 0$.

2.2 A superfield interpretation

Similarly to the case of superconformal field theory (see [29], §5.9), one may consider the fields L, Y, M, N as four components of the same superfield $\mathcal{L}$. To construct $\mathcal{L}$, we first need to go over to the 'Heisenberg' point of view by setting

$$\bar{L}(t,r,\zeta) := e^{\zeta M_0} e^{rY_{-\frac{1}{2}}} L(t) e^{-rY_{-\frac{1}{2}}} e^{-\zeta M_0} \tag{2.17}$$

and similarly for $\bar{Y}, \bar{M}, \bar{N}$, the quantum generators $Y_{-\frac{1}{2}}$, resp. M_0 corresponding to the infinitesimal generators of space, resp. ζ-translations.

In the following, the sign ∂ alone always indicates a derivative with respect to time. Differences of coordinates are abbreviated as $t_{12} = t_1 - t_2$, $r_{12} = r_1 - r_2$, $\zeta_{12} = \zeta_1 - \zeta_2$.

Lemma 2.2.1

1. *The Heisenberg fields $\bar{L}, \bar{Y}, \bar{N}, \bar{M}$ read*

$$\begin{aligned} \bar{L}(t,r) &= L(t) + \frac{1}{2} r \partial Y(t) + \frac{r^2}{4} \partial^2 M(t); \\ \bar{Y}(t,r) &= Y(t) + r \partial M(t); \\ \bar{M}(t) &= M(t); \\ \bar{N}(t,r,\zeta) &= N(t) - rY(t) - \frac{r^2}{2} \partial M(t) - 2\zeta M(t). \end{aligned} \tag{2.18}$$

2. *Operator product expansions are given by the following formulas:*

$$\bar{L}(t_1,r_1)\bar{L}(t_2,r_2) \sim \frac{\partial_{t_2}\bar{L}(t_2,r_2)}{t_{12}} + \frac{2\bar{L}(t_2,r_2) - \frac{1}{4} r_{12} \partial_{t_2} \bar{Y}(t_2,r_2)}{t_{12}^2} - \frac{3}{2}\frac{r_{12}}{t_{12}^3} \bar{Y}(t_2,r_2) + \frac{\frac{c}{2} + \frac{3}{2} r_{12}^2 \bar{M}(t_2)}{t_{12}^4}; \tag{2.19}$$

$$\bar{L}(t_1,r_1)\bar{Y}(t_2,r_2) \sim \frac{\partial \bar{Y}(t_2,r_2)}{t_{12}} + \frac{\frac{3}{2}\bar{Y}(t_2,r_2) - \frac{1}{2} r_{12} \partial_{r_2} \bar{Y}(t_2,r_2)}{t_{12}^2} - 2\frac{r_{12}}{t_{12}^3} \bar{M}(t_2);$$

$$\bar{Y}(t_1,r_1)\bar{L}(t_2,r_2) \sim \frac{\partial_{r_2}\bar{L}(t_2,r_2)}{t_{12}} + \frac{3}{2}\frac{\bar{Y}(t_2,r_2)}{t_{12}^2} - \frac{2r_{12}\bar{M}(t_2)}{t_{12}^3};$$

$$\bar{L}(t_1,r_1)\bar{M}(t_2) \sim \frac{\partial \bar{M}(t_2)}{t_{12}} + \frac{\bar{M}(t_2)}{t_{12}^2}, \quad \bar{M}(t_1)\bar{L}(t_2,r_2) \sim \frac{\bar{M}(t_2)}{t_{12}^2};$$

$$\bar{L}(t_1,r_1)\bar{N}(t_2,r_2,\zeta_2) \sim \frac{\partial \bar{N}(t_2,r_2,\zeta_2)}{t_{12}} + \frac{\bar{N}(t_2,r_2,\zeta_2) - \frac{1}{2} r_{12} \partial_{r_2} \bar{N}(t_2,r_2,\zeta_2)}{t_{12}^2}$$

$$+\frac{1}{2}\frac{r_{12}^2\partial_\zeta\bar{N}(t_2,r_2,\zeta_2)}{t_{12}^3}+\frac{\alpha}{t_{12}^3};$$

$$\bar{N}(t_1,r_1,\zeta_1)\bar{L}(t_2,r_2)\sim\left(-\frac{r_{12}\partial_{r_2}\bar{L}(t_2,r_2)}{t_{12}}+\frac{-\frac{3}{2}r_{12}\bar{Y}(t_2,r_2)-2\zeta_{12}\bar{M}(t_2)}{t_{12}^2}+\frac{r_{12}^2\bar{M}(t_2)}{t_{12}^3}\right)$$

$$+\frac{\bar{N}(t_2,r_2,\zeta_2)}{t_{12}^2};$$

$$\bar{Y}(t_1,r_1)\bar{Y}(t_2,r_2)\sim\frac{\partial\bar{M}(t_2)}{t_{12}}+\frac{2\bar{M}(t_2)}{t_{12}^2},\quad \bar{Y}(t_1,r_1)\bar{M}(t_2)\sim\bar{M}(t_1)\bar{M}(t_2)\sim 0;$$

$$\bar{N}(t_1,r_1,\zeta_1)\bar{Y}(t_2,r_2)\sim\frac{-r_{12}\partial_{r_2}\bar{Y}(t_2,r_2)+\bar{Y}(t_2,r_2)}{t_{12}}-\frac{2r_{12}\bar{M}(t_2)}{t_{12}^2};$$

$$\bar{Y}(t_1,r_1)\bar{N}(t_2,r_2,\zeta_2)\sim\frac{-\bar{Y}(t_2,r_2)}{t_{12}}+\frac{2r_{12}\bar{M}(t_2)}{t_{12}^2};$$

$$\bar{N}(t_1,r_1,\zeta_1)\bar{M}(t_2)\sim\frac{2\bar{M}(t_2)}{t_{12}},\quad \bar{M}(t_1)\bar{N}(t_2,r_2,\zeta_2)\sim\frac{\partial_{\zeta_2}\bar{N}(t_2,r_2,\zeta_2)}{t_{12}};$$

$$\bar{N}(t_1,r_1,\zeta_1)\bar{N}(t_2,r_2,\zeta_2)\sim\left(\frac{-(2\zeta_{12}\partial_{\zeta_2}+r_{12}\partial_{r_2})\bar{N}(t_2,r_2,\zeta_2)}{t_{12}}+\frac{1}{2}\frac{r_{12}^2\partial_{\zeta_2}\bar{N}(t_2,r_2,\zeta_2)}{t_{12}^2}\right)+\frac{\kappa}{t_{12}^2}. \tag{2.20}$$

3. *A field $\Phi=\sum_\mu\Phi^{(\mu)}e_\mu$ is a (ρ,Ω)-$\tilde{\mathfrak{sv}}$-primary field if and only if the following relations hold (we omit the argument (t_2,r_2,ζ_2) of the field Φ in the right-hand side of the equations):*

$$\bar{L}(t_1,r_1)\Phi^{(\mu)}(t_2,r_2,\zeta_2)\sim\frac{\partial_{t_2}\Phi^{(\mu)}}{t_{12}}-\frac{1}{2}\frac{r_{12}\partial_{r_2}\Phi^{(\mu)}}{t_{12}^2}-\frac{\rho(L_0)^\mu_\nu\Phi^{(\nu)}}{t_{12}^2}$$
$$+\frac{\frac{1}{2}r_{12}^2\partial_\zeta\Phi^{(\mu)}+r_{12}\rho(Y_{\frac{1}{2}})^\mu_\nu\Phi^{(\nu)}}{t_{12}^3}-\frac{3}{2}\frac{r_{12}^2\rho(M_1)^\mu_\nu\Phi^{(\nu)}}{t_{12}^4}; \tag{2.21}$$

$$\bar{Y}(t_1,r_1)\Phi^{(\mu)}(t_2,r_2,\zeta_2)\sim\frac{\partial_{r_2}\Phi^{(\mu)}}{t_{12}}-\frac{r_{12}\partial_\zeta\Phi^{(\mu)}}{t_{12}^2}-\frac{\rho(Y_{\frac{1}{2}})^\mu_\nu\Phi^{(\nu)}}{t_{12}^2}+\frac{2r_{12}\rho(M_1)^\mu_\nu\Phi^{(\nu)}}{t_{12}^3}; \tag{2.22}$$

$$\bar{M}(t_1)\Phi^{(\mu)}(t_2,r_2,\zeta_2)\sim\frac{\partial_{\zeta_2}\Phi^{(\mu)}}{t_{12}}-\frac{\rho(M_1)^\mu_\nu\Phi^{(\nu)}}{t_{12}^2}; \tag{2.23}$$

$$\bar{N}(t_1,r_1,\zeta_1)\Phi^{(\mu)}(t_2,r_2,\zeta_2)\sim\frac{-(r_{12}\partial_{r_2}+2\zeta_{12}\partial_{\zeta_2})\Phi^{(\mu)}-\rho(N_0)^\mu_\nu\Phi^{(\nu)}}{t_{12}}$$
$$+\frac{\frac{1}{2}r_{12}^2\partial_{\zeta_2}\Phi^{(\mu)}+r_{12}\rho(Y_{\frac{1}{2}})^\mu_\nu\Phi^{(\nu)}+2\zeta_{12}\rho(M_1)^\mu_\nu\Phi^{(\nu)}+\Omega^\mu_\nu\Phi^{(\nu)}}{t_{12}^2}-\frac{r_{12}^2\rho(M_1)^\mu_\nu\Phi^{(\nu)}}{t_{12}^3} \tag{2.24}$$

Putting all this together, one gets:

Theorem 2.2.2

Set $c=\kappa=\alpha=0$. Then:

(i) The four-dimensional field

$$\mathcal{L}(t,r,\zeta) = \begin{pmatrix} \bar{L} \\ \bar{Y} \\ \bar{M} \\ \bar{N} \end{pmatrix} (t,r,\zeta) \tag{2.25}$$

is ρ-$\mathfrak{sv}$-primary for the representation ρ defined by:

$$\rho(L_0) = \begin{pmatrix} -2 & & & \\ & -\frac{3}{2} & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \ \rho(Y_{\frac{1}{2}}) = \begin{pmatrix} 0 & -\frac{3}{2} & & \\ & 0 & -2 & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \ \rho(M_1) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}. \tag{2.26}$$

(ii) It is not *$\rho - \tilde{\mathfrak{sv}}$-primary.*

Proof. Straightforward computations. Note that $\rho(M_1)$ is proportional to $\rho(Y_{\frac{1}{2}})^2$, see the remarks preceding Definition 1.4.

So what happened ? Setting $\rho(N_0) = \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & -2 & \\ & & & 0 \end{pmatrix}$, one gets a representation of $\tilde{\mathfrak{sv}}_0 = (\langle L_0\rangle \oplus \langle N_0\rangle) \ltimes \langle Y_{\frac{1}{2}}, M_1\rangle$ and $\mathcal{L}$ looks $\rho - \tilde{\mathfrak{sv}}$-primary, except for the last term $\frac{\bar{N}}{t_{12}^2}$ in the above OPE $\bar{N}.\bar{L}$. Fortunately, a supplementary matrix Ω as in Definition 2.1.1. (3) allows to take into account this term:

Theorem 2.2.3

Set $c = \kappa = \alpha = 0$. Then $\mathcal{L}$ is (ρ, Ω)-$\tilde{\mathfrak{sv}}$-primary if one sets $\Omega = \begin{pmatrix} 0 & 0 & 0 & -1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$.

Proof. Straightforward computations.

3 Construction by $U(1)$-currents or $a\bar{b}$-theory

Now that the definition of what is intended by $\tilde{\mathfrak{sv}}$*-primary* has been completed, we proceed to give explicit examples. The rest of the article is devoted to the detailed analysis of a vertex algebra constructed out of two bosons (called : *$a\bar{b}$-model*) containing a representation of $\tilde{\mathfrak{sv}}$ and $\tilde{\mathfrak{sv}}$-primary fields of any L_0-weight.

3.1 Definition of the $\widetilde{\mathfrak{sv}}$-fields

We shall use here a classical construction of current algebras given in all generality in [29]. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a (finite-dimensional) super-vector space, with even generators a^i, $i = 1, \ldots, N$ for $V_{\bar{0}}$ and odd generators $b^{+,i}$, $b^{-,i}$, $i = 1, \ldots, M$ for $V_{\bar{1}}$ (supposed to be even-dimensional).

Definition 3.1.1

1. *The* bosonic supercurrents *associated with V (see [29], section 3.5) are the mutually local N bosonic fields* $a^i(z) = \sum_{n\in\mathbb{Z}} a^i_n z^{-n-1}$ *and the 2M fermionic fields* $b^{\pm,i}(z) = \sum_{n\in\mathbb{Z}} b^{\pm,i}_n z^{-n-1}$ *with the following non-trivial OPE's:*

$$a^i(z)a^j(w) \sim \frac{\delta^{i,j}}{(z-w)^2} \tag{3.1}$$

$$b^{\pm,i}(z)b^{\mp,j}(w) \sim \pm\frac{\delta^{i,j}}{(z-w)^2} \tag{3.2}$$

or, equivalently, with the following non-trivial Lie brackets in mode decomposition

$$[a^i_n, a^j_m]_- = n\delta^{i,j}\delta_{n+m,0} \tag{3.3}$$

$$[b^{+,i}_n, b^{-,j}_m]_+ = n\delta^{i,j}\delta_{n+m,0}. \tag{3.4}$$

2. *The* fermionic supercurrents *associated with V (see [29], sections 2.5 and 3.6) are the mutually local N fermionic fields* $\bar{a}^i(z) = \sum_{n\in\mathbb{Z}} \bar{a}^i_n z^{-n-\frac{1}{2}}$ *and the (2M) bosonic fields* $\bar{b}^{\pm,i}(z) = \sum_{n\in\mathbb{Z}} \bar{b}^{\pm,i}_n z^{-n-\frac{1}{2}}$ *with the following non-trivial OPE's:*

$$\bar{a}^i(z)\bar{a}^j(w) \sim \frac{\delta^{i,j}}{z-w} \tag{3.5}$$

$$\bar{b}^{\pm,i}(z)\bar{b}^{\mp,j}(w) \sim \pm\frac{\delta^{i,j}}{z-w} \tag{3.6}$$

or, equivalently, with the following non-trivial Lie brackets in mode decomposition

$$[\bar{a}^i_n, \bar{a}^j_m]_+ = \delta^{i,j}\delta_{n+m,0} \tag{3.7}$$

$$[\bar{b}^{+,i}_n, \bar{b}^{-,j}_m]_- = \delta^{i,j}\delta_{n+m,0}. \tag{3.8}$$

Remark: The bosonic supercurrents $\bar{b}^{\pm,i}$ (with unusual parity considering their half-integer weight) are sometimes called *symplectic bosons* in the physical literature, see for instance [15, 8].

Proposition 3.1.2 (see [29], sections 3.5 and 3.6)

Consider the canonical Fock realization of the superalgebra generated by $a^i, b^{i,\pm}$ *(obtained by requiring that* $a^i, b^{i,\pm}$, $i \geq 0$, *vanish on the vacuum vector* $|0\rangle$*). Then*

$$\langle 0 \mid a^i(z)a^j(w) \mid 0\rangle = \delta^{i,j}(z-w)^{-2}, \quad \langle 0 \mid b^{\pm,i}(z)b^{\mp,j}(w) \mid 0\rangle = \pm\delta^{i,j}(z-w)^{-2} \tag{3.9}$$

and

$$\langle 0 \mid \bar{a}^i(z)\bar{a}^j(w) \mid 0\rangle = \delta^{i,j}(z-w)^{-1}, \quad \langle 0 \mid \bar{b}^{\pm,i}(z)\bar{b}^{\mp,j}(w) \mid 0\rangle = \pm\delta^{i,j}(z-w)^{-1}. \tag{3.10}$$

One may build Virasoro fields out of these supercurrents, one for each type of currents:

$$L_a(t) = \frac{1}{2} : a^2 : (t), \quad L_b(t) =: b^+b^- : (t) \tag{3.11}$$

with central charge 1, viz. -2;

$$L_{\bar{a}}(t) = -\frac{1}{2} : \bar{a}(\partial\bar{a}) : (t), \quad L_{\bar{b}}(t) = \frac{1}{2}\left(: \bar{b}^+\partial\bar{b}^- : (t) - : \bar{b}^-\partial\bar{b}^+ : (t)\right) \tag{3.12}$$

with central charge $\frac{1}{2}$, viz. -1.

For the appropriate Virasoro field, the bosonic supercurrents a^i, b^i are primary with conformal weight 1, while the fermionic supercurrents $\bar{a}^i, \bar{b}^{\pm,i}$ are primary with conformal weight $\frac{1}{2}$. The simplest way to construct a Lie algebra isomorphic to an appropriately centrally extended $\tilde{\mathfrak{sv}}$ with these generating fields is the following:

Definition 3.1.3

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with $V_{\bar{0}} = \mathbb{R}a$ and $V_{\bar{1}} = \mathbb{R}b^+ \oplus \mathbb{R}b^-$. Then $\widetilde{\mathfrak{sv}}_{(0,-1,0)}$-fields L, N, Y, M may be defined as follows:

$$L = L_a + L_{\bar{b}} \quad \text{with zero central charge;} \tag{3.13}$$

$$N = - : \bar{b}^+\bar{b}^- : \quad \text{with central charge } -1; \tag{3.14}$$

$$Y =: a\bar{b}^+ : ; \tag{3.15}$$

$$M = \frac{1}{2} : (\bar{b}^+)^2 : \tag{3.16}$$

Let us first check explicitly that one retrieves the OPE (2.6,2.7,2.8, 2.9,2.10) with this definition:

$$: a^2 : (t_1) : a\bar{b}^+ : (t_2) \sim \frac{2 : a(t_1)\bar{b}^+(t_2) :}{(t_1-t_2)^2} \sim \frac{2 : a\bar{b}^+ : (t_2)}{(t_1-t_2)^2} + \frac{2 : \partial a\bar{b}^+ : (t_2)}{t_1-t_2};$$

$$: \bar{b}^+\partial\bar{b}^- : (t_1) \ : a\bar{b}^+ : (t_2) \sim \frac{: \bar{b}^+(t_1)a(t_2) :}{(t_1-t_2)^2} \sim \frac{: a\bar{b}^+ : (t_2)}{(t_1-t_2)^2} + \frac{: a\partial\bar{b}^+ : (t_2)}{t_1-t_2};$$

$$: \bar{b}^-\partial\bar{b}^+ : (t_1) \ : a\bar{b}^+ : (t_2) \sim -\frac{: \partial\bar{b}^+(t_1)a(t_2) :}{t_1-t_2} \sim -\frac{: \partial\bar{b}^+a : (t_2)}{t_1-t_2}$$

so $L(t_1)Y(t_2) \sim \frac{\partial Y(t_2)}{t_1-t_2} + \frac{\frac{3}{2}Y(t_2)}{(t_1-t_2)^2}$.

Similarly,

$$: a^2 : (t_1) \ : (\bar{b}^+)^2 : (t_2) \sim 0;$$

$$: \bar{b}^+\partial\bar{b}^- : (t_1) \ : (\bar{b}^+)^2 : (t_2) \sim \frac{2 : \bar{b}^+(t_1)\bar{b}^+(t_2) :}{(t_1-t_2)^2} \sim \frac{2 : (\bar{b}^+)^2(t_2)}{(t_1-t_2)^2} + 2\frac{: \partial\bar{b}^+\bar{b}^+ : (t_2)}{t_1-t_2};$$

$$: \bar{b}^-\partial\bar{b}^+ : (t_1) \ : (\bar{b}^+)^2 : (t_2) \sim -2\frac{: \partial\bar{b}^+\bar{b}^+ : (t_2)}{t_1-t_2}$$

so $L(t_1)M(t_2) \sim \frac{\partial M(t_2)}{t_1-t_2} + \frac{M(t_2)}{(t_1-t_2)^2}$.

Finally,

$$Y(t_1)Y(t_2) = :a\bar{b}^+:(t_1):a\bar{b}^+:(t_2) \sim \frac{:\bar{b}^+(t_1)\bar{b}^+(t_2):}{(t_1-t_2)^2}$$
$$\sim \frac{:(\bar{b}^+)^2:(t_2)}{(t_1-t_2)^2} + \frac{:\bar{b}^+\partial\bar{b}^+:(t_2)}{t_1-t_2} = \frac{2M(t_2)}{(t_1-t_2)^2} + \frac{\partial M(t_2)}{t_1-t_2} \qquad (3.17)$$

and $Y(t_1)M(t_2) \sim 0$, $M(t_1)M(t_2) \sim 0$, so one is done for the $\mathfrak{sv}$-fields L, Y, M. Then

$$N(t_1)N(t_2) =: \bar{b}^+\bar{b}^-:(t_1):\bar{b}^+\bar{b}^-:(t_2) \sim -\frac{1}{(t_1-t_2)^2}$$

(the terms of order one cancel each other);

$$:\bar{b}^+\partial\bar{b}^-:(t_1):\bar{b}^+\bar{b}^-:(t_2) \sim \frac{:\partial(\bar{b}^+\bar{b}^-):(t_2)}{t_1-t_2} + \frac{:\bar{b}^+\bar{b}^-:(t_2)}{(t_1-t_2)^2}$$
$$:\bar{b}^-\partial\bar{b}^+:(t_1):\bar{b}^+\bar{b}^-:(t_2) \sim -\frac{:\partial(\bar{b}^+\bar{b}^-):(t_2)}{t_1-t_2} - \frac{:\bar{b}^+\bar{b}^-:(t_2)}{(t_1-t_2)^2}$$

hence $L(t_1)N(t_2) \sim \frac{\partial N(t_2)}{t_1-t_2} + \frac{N(t_2)}{t_1-t_2}$; finally,

$$N(t_1)Y(t_2) = -:\bar{b}^+\bar{b}^-:(t_1):a\bar{b}^+:(t_2) \sim \frac{Y(t_2)}{t_1-t_2}$$

and

$$N(t_1)M(t_2) = -\frac{1}{2}:\bar{b}^+\bar{b}^-:(t_1):(\bar{b}^+)^2:(t_2) \sim \frac{2M(t_2)}{t_1-t_2}.$$

Definition 3.1.4

The constrained 3D-Dirac equation *(or: constrained Dirac equation for short) is the set of following equations for a spinor field $(\phi_1,\phi_2) = (\phi_1(t,r,\zeta),\phi_2(t,r,\zeta))$ on $\mathbb{R}^3$:*

$$\partial_r\phi_0 = \partial_t\phi_1 \qquad (3.18)$$
$$\partial_r\phi_1 = \partial_\zeta\phi_0 \qquad (3.19)$$
$$\partial_\zeta\phi_1 = 0. \qquad (3.20)$$

Theorem 3.1.5

1. *The space of solutions of the constrained 3D-Dirac equation is in one-to-one correspondence with the space of triples (h_0^-, h_0^+, h_1) of functions of t only: a natural bijection may be obtained by setting*

$$\phi_0(t,r,\zeta) = (h_0^-(t) + \zeta h_0^+(t)) + rh_1(t) + \frac{r^2}{2}\partial h_0^+(t) \qquad (3.21)$$
$$\phi_1(t,r,\zeta) = \int_0^t h_1(u)\ du + rh_0^+(t) \qquad (3.22)$$

2. Put

$$\Phi^{(0)}(t,r,\zeta) = (\bar{b}^-(t) + \zeta\bar{b}^+(t)) + ra(t) + \frac{r^2}{2}\partial\bar{b}^+(t) \tag{3.23}$$

and

$$\Phi^{(1)}(t,r,\zeta) = (\int a)(t) + r\bar{b}^+(t) \tag{3.24}$$

where

$$\int a = -\sum_{n\neq 0} a_n \frac{t^{-n}}{n} + a_0 \log t + \pi_0, \quad [a_0,\pi_0] = 1 \tag{3.25}$$

is the logarithmic bosonic field defined for instance in [13]. Then $\Phi := \left(\begin{array}{c}\Phi^{(0)}\\ \Phi^{(1)}\end{array}\right)$ *is a* ρ*-*$\tilde{\mathfrak{sv}}$*-primary field, where* ρ *is the two-dimensional character defined by*

$$\rho(L_0) = \left(\begin{array}{cc} -\frac{1}{2} & \\ & 0 \end{array}\right), \quad \rho(N_0) = \left(\begin{array}{cc} 1 & \\ & 0 \end{array}\right), \quad \rho(Y_{\frac{1}{2}}) = \rho(M_1) = 0. \tag{3.26}$$

3. The two-point functions $\mathcal{C}^{\mu,\nu}(t_1,r_1,\zeta_1;t_2,r_2,\zeta_2) := \langle 0|\ \Phi^{(\mu)}(t_1,r_1,\zeta_1)\Phi^{(\nu)}(t_2,r_2,\zeta_2)\ |0\rangle$, $\mu,\nu = 0,1$, *are given by*

$$\mathcal{C}^{0,0} = \frac{1}{t}(\zeta - \frac{r^2}{2t}),\ \mathcal{C}^{0,1} = \mathcal{C}^{1,0} = r,\ \mathcal{C}^{1,1} = \ln t \tag{3.27}$$

where $t = t_1 - t_2, r = r_1 - r_2, \zeta = \zeta_1 - \zeta_2$.

Remark.

The free boson $\int a$ is not conformal in the usual sense since it contains a logarithmic term, contrary to the vertex operators built as exponentials of $\int a$ that we shall use in the following sections. In this very particular case, one needs to consider a_0, π_0 as a couple of usual annihilation/creation operators in order for the scalar product $\langle 0|\ (\int a)(t_1)(\int a)(t_2)\ |0\rangle$ to make sense, so that a_0 and π_0 are adjoint to each other. The usual normalization is quite different.

Proof.

1. Let (ϕ,ψ) be a solution of the constrained Dirac equation. Then $\partial_r^2\psi = \partial_t\partial_\zeta\psi = 0$ so

$$\psi(t,r,\zeta) = \psi_0(t) + r\psi_1(t). \tag{3.28}$$

On the other hand, $\partial_\zeta^2\phi = \partial_\zeta\partial_r\psi = 0$, $\partial_\zeta\phi = \psi_1$ and $\partial_r\phi = \partial_t\psi_0 + r\partial_t\psi_1$, hence, by putting together everything,

$$\phi(t,r,\zeta) = \phi^{00}(t) + \psi_1(t)\zeta + \psi_0'(t)r + \psi_1'(t)\frac{r^2}{2}. \tag{3.29}$$

Now one just needs to set $h_0^- := \phi^{00}, h_0^+ = \psi_1$ and $h_1 = \psi_0'$.

2. This follows directly from Definition 2.1 once one has established the following easy relations

$$L(t_1)\partial\bar{b}^+(t_2) \sim \frac{\partial^2\bar{b}^+(t_2)}{t_1-t_2} + \frac{3}{2}\frac{\partial\bar{b}^+(t_2)}{(t_1-t_2)^2} + \frac{\bar{b}^+(t_2)}{(t_1-t_2)^3} \quad (3.30)$$

$$N(t_1)(\bar{b}^-(t_2)+\zeta\bar{b}^+(t_2)) \sim \frac{-\bar{b}^-(t_2)+\zeta\bar{b}^+(t_2)}{t_1-t_2} \quad (3.31)$$

$$N(t_1)a(t_2) \sim 0 \quad (3.32)$$

$$N(t_1)\partial\bar{b}^+(t_2) \sim \partial_{t_2}\left(\frac{\bar{b}^+(t_2)}{t_1-t_2}\right) = \frac{\partial\bar{b}^+(t_2)}{t_1-t_2} + \frac{\bar{b}^+(t_2)}{(t_1-t_2)^2} \quad (3.33)$$

$$Y(t_1)(\bar{b}^-(t_2)+\zeta\bar{b}^+(t_2)) \sim \frac{a(t_2)}{t_1-t_2} \quad (3.34)$$

$$Y(t_1)a(t_2) \sim \frac{\partial\bar{b}^+(t_2)}{t_1-t_2} + \frac{\bar{b}^+(t_2)}{(t_1-t_2)^2} \quad (3.35)$$

$$Y(t_1)\bar{b}^+(t_2) \sim 0 \quad (3.36)$$

together with the fact that $\bar{b}^\pm$, resp. a, are L-conformal with conformal weight $\frac{1}{2}$ (resp. 1).

3. Straightforward.

In particular, one retrieves the fact that the classical constrained Dirac equation is $\mathfrak{sv}$-invariant, see [39] for a discussion and generalizations. Unfortunately, one can hardly say that this is an interesting physical equation.

We give thereafter two other examples. They exhaust all possibilities of $\tilde{\mathfrak{sv}}$-primary linear fields of this model and are only given for the sake of completeness.

Lemma 3.1.6

1. *The* trivial *field $\bar{b}^+(t)$ is a ρ-Schrödinger-conformal field, where ρ is the one-dimensional character defined by*

$$\rho(L_0) = -\frac{1}{2}, \quad \rho(N_0) = -1, \quad \rho(Y_{\frac{1}{2}}) = \rho(M_1) = 0. \quad (3.37)$$

The associated two-point function vanishes.

2. *Put $\Phi^{(0)}(t,r,\zeta) = a(t) + r\partial\bar{b}^+$ and $\Phi^{(1)}(t,r,\zeta) = -\bar{b}^+$. Then $\Phi = \begin{pmatrix} \Phi^{(0)} \\ \Phi^{(1)} \end{pmatrix}$ is a ρ-$\tilde{\mathfrak{sv}}$-primary field, where ρ is the two-dimensional representation defined by*

$$\rho(L_0) = \begin{pmatrix} -\frac{1}{2} & \\ & -1 \end{pmatrix}, \quad \rho(N_0) = \begin{pmatrix} -1 & \\ & 0 \end{pmatrix}, \quad \rho(Y_{\frac{1}{2}}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(M_1) = 0. \quad (3.38)$$

The two-point functions of the field Φ are given by

$$\mathcal{C}^{0,0} = t^{-2}, \ \mathcal{C}^{0,1} = \mathcal{C}^{1,0} = \mathcal{C}^{1,1} = 0. \quad (3.39)$$

Proof.

1. Straightforward.
2. Follows from preceding computations.

□

3.2 Construction of the generalized polynomial fields ${}_\alpha\Phi_{j,k}$

We shall introduce in this paragraph more general fields. Take any polynomial $P = P(\bar{b}^-, \bar{b}^+, \partial\bar{b}^+, a, \int a)$ where

$$(\int a)(t) := -\sum_{n\neq 0} a_n \frac{t^{-n}}{n} + a_0 \log t + \pi_0, \quad [a_0, \pi_0] = 1 \tag{3.40}$$

is the usual logarithmic bosonic field of conformal field theory from which vertex operators are built. Since $[\bar{b}^+_n, \bar{b}^-_m] = 0$ if $nm > 0$ and similarly for the commutators of any of the fields $\bar{b}^-, \bar{b}^+, \partial\bar{b}^+, a, \int a$, the normal ordering is *commutative* and the field $:P:$ is well defined.

Let us introduce first for convenience the following notation for the coefficients of OPE of two mutually local fields.

Definition 3.2.1.

Let A, B be two mutually local fields: their OPE is given as

$$A(t_1)B(t_2) \sim \sum_{k=0}^{\infty} \frac{C_k(t_2)}{t_{12}^k} \tag{3.41}$$

for some fields $C_0(t), C_1(t), \ldots, C_p(t), \ldots$ which vanish for p large enough.

We shall denote by $A_{(k)}B$, $k = 0, 1, \ldots$ the field C_k.

Theorem 3.2.2.

Let P be any polynomial in the fields $\bar{b}^\pm$, $\partial\bar{b}^+$, a and $\int a$. Then

$$\begin{aligned} L(t_1):P:(t_2) \quad &\sim \quad \frac{:\partial P:(t_2)}{t_{12}} + \frac{\left(\frac{1}{2}(\bar{b}^-\partial_{\bar{b}^-} + \bar{b}^+\partial_{\bar{b}^+}) + \frac{3}{2}\partial\bar{b}^+\partial_{\partial\bar{b}^+} + a\partial_a + \frac{1}{2}\partial^2_{\int a}\right)P:(t_2)}{t_{12}^2} \\ &+ \quad \frac{:\left(\bar{b}^+\partial_{\partial\bar{b}^+} + \partial_{\int a}\partial_a\right)P:(t_2)}{t_{12}^3} + \frac{1}{2}\frac{:(\partial_a^2 + \partial_{\bar{b}^-}\partial_{\partial\bar{b}^+})P:(t_2)}{t_{12}^4} \end{aligned} \tag{3.42}$$

$$\begin{aligned} N(t_1)P(t_2) \quad &\sim \quad \frac{\left(\bar{b}^+\partial_{\bar{b}^+} + \partial\bar{b}^+\partial_{\partial\bar{b}^+} - \bar{b}^-\partial_{\bar{b}^-}\right)P:(t_2)}{t_{12}} + \frac{\left(\bar{b}^+\partial_{\partial\bar{b}^+} + \partial_{\bar{b}^-}\partial_{\bar{b}^+}\right)P:(t_2)}{t_{12}^2} \\ &+ \quad \frac{(\partial_{\bar{b}^-}\partial_{\partial\bar{b}^+})P:(t_2)}{t_{12}^3} \end{aligned} \tag{3.43}$$

$$
\begin{aligned}
Y(t_1):P:(t_2) \quad &\sim \quad \frac{:\left(a\partial_{\bar{b}^-}+\partial\bar{b}^+\partial_a+\bar{b}^+\partial_{\int a}\right)P:(t_2)}{t_{12}}+\frac{:(\bar{b}^+\partial_a+\partial_{\bar{b}^-}\partial_{\int a})P:(t_2)}{t_{12}^2} \\
&+ \quad \frac{:\partial_a\partial_{\bar{b}^-}P:(t_2)}{t_{12}^3}
\end{aligned}
\tag{3.44}
$$

$$
M(t_1):P:(t_2)\sim\frac{:\bar{b}^+\partial_{\bar{b}^-}P:(t_2)}{t_{12}}+\frac{1}{2}\frac{:\partial_{\bar{b}^-}^2P:(t_2)}{t_{12}^2}
\tag{3.45}
$$

Proof.

Consider the monomial $P=P_{jklmn}=(\bar{b}^-)^j(\bar{b}^+)^k(\partial\bar{b}^+)^l a^m(\int a)^n$. Let us compute $L_{(n)}:P:$, $n\geq 0$ first. Apart from the contribution of the logarithmic field $\int a$ which has special properties, one may deduce the coefficient of the terms of order t_{12}^{-1} and t_{12}^{-2} directly from general considerations (see [29]): the field $(\bar{b}^-)^j(\bar{b}^+)^k(\partial\bar{b}^+)^l a^m$ is quasiprimary with conformal weight $\frac{j+k}{2}+\frac{3l}{2}+m$. The contribution from the field $\int a$ reads

$$
\begin{aligned}
L(t_1):P:(t_2) \quad &\sim \quad \frac{1}{2}:a^2:(t_1):P_{jkl00}a^m(\int a)^n:+\dots \\
&\sim \quad n\frac{:P_{jkl00}a^{m+1}(\int a)^{n-1}:(t_2)}{t_{12}}+\frac{n(n-1)}{2}\frac{:P_{jkl00}a^m(\int a)^{n-1}:(t_2)}{t_{12}^2} \\
&+ \quad mn\frac{:P_{jkl00}a^{m-1}(\int a)^{n-1}:(t_2)}{t_{12}^3}+\dots
\end{aligned}
\tag{3.46}
$$

in accordance with the Theorem. So, if we prove that $L_{(n)}:P_{jklm0}:$ agree with (3.42) for $n\geq 2$, we are done. One gets (leaving aside the poles of order 1 or 2)

$$\frac{1}{2}:a^2:(t_1)a^m(t_2)\sim\frac{1}{2}m(m-1)\frac{(\bar{b}^-)^j(\bar{b}^+)^k(\partial\bar{b}^+)^l a^{m-2}:(t_2)}{t_{12}^4}+\dots \text{ (double contraction);}$$

$$\frac{1}{2}:\bar{b}^+\partial\bar{b}^-:(t_1):(\partial\bar{b}^+)^l:(t_2)\sim l\frac{:(\bar{b}^-)^j(\bar{b}^+)^k(\partial\bar{b}^+)^{l-1}a^m:(t_2)}{t_{12}^3}+\dots \text{ (simple contraction);}$$

$$\frac{1}{2}:\bar{b}^+\partial\bar{b}^-:(t_1):(\bar{b}^-)^j(\bar{b}^+)^k:(t_2)\sim\frac{jk}{2}\frac{(\bar{b}^-)^{j-1}(\bar{b}^+)^{k-1}(\partial\bar{b}^+)^l a^m:(t_2)}{t_{12}^3} \text{ (double contraction);}$$

$$\frac{1}{2}:\bar{b}^+\partial\bar{b}^-:(t_1):(\bar{b}^-)^j(\partial\bar{b}^+)^l:(t_2)\sim jl\frac{:(\bar{b}^-)^{j-1}(\bar{b}^+)^k(\partial\bar{b}^+)^{l-1}a^m:(t_2)}{t_{12}^4} \text{ (double contraction);}$$

$$-\frac{1}{2}:\bar{b}^-\partial\bar{b}^+:(t_1):(\bar{b}^-)^j(\bar{b}^+)^k:(t_2)\sim-\frac{jk}{2}\frac{(\bar{b}^-)^{j-1}(\bar{b}^+)^{k-1}(\partial\bar{b}^+)^l a^m:(t_2)}{t_{12}^3} \text{ (double contraction);}$$

$$-\frac{1}{2}:\bar{b}^-\partial\bar{b}^+:(t_1):(\bar{b}^-)^j(\partial\bar{b}^+)^l:(t_2)\sim-\frac{jl}{2}\frac{:(\bar{b}^-)^{j-1}(\bar{b}^+)^k(\partial\bar{b}^+)^{l-1}a^m:(t_2)}{t_{12}^4} \text{ (double contraction)}$$

hence the result.

Let us consider now the OPE of N with P. The fields a and $\int a$ giving no contribution, one may just as well assume that $m=n=0$. Then

$$-:\bar{b}^+\bar{b}^-:(t_1)\quad :(\bar{b}^-)^j(\bar{b}^+)^k(\partial\bar{b}^+)^l:(t_2)\sim\left(\frac{(k-j)(\bar{b}^-)^j(\bar{b}^+)^k(\partial\bar{b}^+)^l}{t_{12}}+l\frac{:\bar{b}^+(t_1)(:(\bar{b}^-)^j(\bar{b}^+)^k(\partial\bar{b}^+)^{l-1}:)(t_2):}{t_{12}^2}\right)$$
$$+\left(\frac{jk:(\bar{b}^-)^{j-1}(\bar{b}^+)^{k-1}(\partial\bar{b}^+)^l:(t_2)}{t_{12}^2}+jl\frac{:(\bar{b}^-)^{j-1}(\bar{b}^+)^{k-1}(\partial\bar{b}^+)^{l-1}:(t_2)}{t_{12}^3}\right)$$

adding the terms coming from a single contraction to the terms coming from a double contraction. Hence the result. The OPE of M with P follows easily from the same rules. Finally,

$$Y(t_1):P:(t_2) = :a\bar{b}^+:(t_1):(\bar{b}^-)^j(\bar{b}^+)^k(\partial\bar{b}^+)^l a^m(\int a)^n:(t_2) \tag{3.47}$$
$$\sim \left(j\frac{:(\bar{b}^-)^{j-1}(\bar{b}^+)^k(\partial\bar{b}^+)^l a^{m+1}(\int a)^n:(t_2)}{t_{12}}+n\frac{:(\bar{b}^-)^j(\bar{b}^+)^{k+1}(\partial\bar{b}^+)^l a^m(\int a)^{n-1}:(t_2)}{t_{12}}\right.$$
$$+\left. m\frac{:\bar{b}^+(t_1)(:(\bar{b}^-)^j(\bar{b}^+)^k(\partial\bar{b}^+)^l a^{m-1}(\int a)^n:(t_2):}{t_{12}^2}\right)$$
$$+\left(nj\frac{:(\bar{b}^-)^{j-1}(\bar{b}^+)^k(\partial\bar{b}^+)^l a^m(\int a)^{n-1}:(t_2)}{t_{12}^2}+mj\frac{:(\bar{b}^-)^{j-1}(\bar{b}^+)^k(\partial\bar{b}^+)^l a^{m-1}(\int a)^n:(t_2)}{t_{12}^3}\right) \tag{3.48}$$

(separating once more the terms coming from a single contraction from the terms with a double contraction) hence (3.44). □

In any case, the $\widetilde{\mathfrak{sv}}$-fields preserve this space of polynomial fields. The reason why we chose not to include powers of $\partial\bar{b}^-$ or ∂a for instance, or higher derivatives of the field $\bar{b}^+$, will appear clearly in a moment. Take a ρ-Schrödinger-conformal field $\Phi=(\Phi^{(\mu)})_\mu$ and suppose it has a formal expansion of the type $\sum_{\xi,\nu}\Phi^{(\mu),\xi,\sigma}(t)r^\xi\zeta^\sigma$ as in subsection 2.1, with σ varying in a set of real values of the same type as for ξ, while the $\Phi^{(\mu),\xi,\sigma}$ are polynomials in the variable $\int a$, $\bar{b}^\pm$ and their derivatives of any order. Suppose $\Phi^{(\mu),\xi,\sigma}\neq 0$ for a negative value of ξ. Then

$$\Phi^{(\mu),\xi,\sigma}=Y_{(0)}\frac{\Phi^{(\mu),\xi-1,\sigma}}{\xi}=\frac{1}{\xi}(a\partial_{\bar{b}^-}+\ldots)\Phi^{(\mu),\xi-1,\sigma} \tag{3.49}$$

hence $\Phi^{(\mu),\xi-1,\sigma}$ must include a monomial P_{jklmn} with m strictly less than for all the monomials in $\Phi^{(\mu),\xi,\sigma}$. But this argument can be repeated indefinitely, going down one step $\xi\to\xi-1$ at a time, and one ends with a contradiction if negative powers of a are not allowed. The same goes for σ since

$$\Phi^{(\mu),\xi,\sigma}=M_{(0)}\frac{\Phi^{(\mu),\xi,\sigma-1}}{\sigma}=\frac{1}{\sigma}\bar{b}^+\partial\bar{b}^-\Phi^{(\mu),\xi,\sigma-1}. \tag{3.50}$$

A moment's thought proves then that if the $\Phi^{(\mu),\xi,\sigma}$ are to be polynomials, then the indices ξ and σ should be positive integers and all the terms $\Phi^{(\mu),\xi,\sigma}$ may be obtained from the *lowest degree component fields* $\Phi^{(\mu),0,0}$ by using Definition 2.1.1; in particular, $Y_{(0)}\Phi^{(\mu)}=\partial_r\Phi^{(\mu)}$ and

$M_{(0)}\Phi^{(\mu)} = \partial_\zeta \Phi^{(\mu)}$: by applying the operators $Y_{(0)}$ and $M_{(0)}$ to $\Phi^{(\mu),0,0}$, one retrieves the whole series $\Phi^{(\mu)} = \sum\sum_{\xi,\sigma=0}^{\infty} \Phi^{(\mu),\xi,\sigma}(t) r^\xi \zeta^\sigma$.

Now $\Phi^{(\mu),0,0}$ may contain neither powers of $\partial\bar{b}^{\pm}$ (otherwise Theorem 3.2.1 gives $L_{(2)}\Phi^{(\mu),0,0} \neq 0$ and formula (2.11) proves that this is impossible) nor powers of a, except, possibly, for fields of the type $(\bar{b}^+)^k a$ (otherwise Theorem 3.2.1 shows that $Y_{(2)}\Phi^{(\mu),0,0} \neq 0$ or $L_{(2)}\Phi^{(\mu),0,0} \neq 0$ or $L_{(3)}\Phi^{(\mu),0,0} \neq 0$, and this is contradictory with formula (2.11) or (2.12)). Higher derivatives of the previous fields would yield higher order singularities in the OPE with L for instance. Note also that powers of $\int a$ may be freely included under the previous conditions and entail no supplementary constraint.

Hence (discarding fields such that $\Phi^{(\mu),0,0}$ is linear in a, which are not very interesting, as one sees by considering the rather trivial action of the $\widetilde{\mathfrak{sv}}$-fields on them and their disappointingly simple n-point functions), one is led to consider the following family of fields, where we make use of the *vertex operator* $V_\alpha := \exp \alpha \int a$ ($\alpha \in \mathbb{R}$), see [13] for instance. Vertex operators are known to be primary; with our normalization, V_α is L-primary with conformal weight $\frac{\alpha^2}{2}$.

Definition 3.2.3

Set for $\alpha \in \mathbb{C}, j, k = 0, 1, \ldots$

$$ {}_\alpha\phi_{j,k}(t) =: (\bar{b}^-)^j (\bar{b}^+)^k V_\alpha : (t) \tag{3.51}$$

and

$$\phi_{j,k}(t) = {}_0\phi_{j,k}(t) =: (\bar{b}^-)^j (\bar{b}^+)^k : (t). \tag{3.52}$$

All these fields appear to be the lowest-degree component fields of ρ-$\widetilde{\mathfrak{sv}}$-primary fields. The operator $\rho(Y_{\frac{1}{2}})$ is trivial if $\alpha = 0$; in the contrary case, $\rho(M_1)$ may be expressed as a coefficient times $(\rho(Y_{\frac{1}{2}}))^2$, in accordance with the discussion preceding Definition 1.4. Since ρ is quite different according to whether $\alpha \neq 0$ or $\alpha = 0$, and also for the sake of clarity, we will state two different theorems.

Theorem 3.2.4 (construction of the polynomial fields $\Phi_{j,k}$)

1. *Set*

$$\Phi_{j,k}^{(0),0}(t,\zeta) = \sum_{m=0}^{j} \left(\begin{array}{c} m \\ j \end{array} \right) \zeta^m : (\bar{b}^-)^{j-m} (\bar{b}^+)^{k+m} : (t) \tag{3.53}$$

and define inductively a series of fields $\Phi_{j,k}^{(\mu),\xi}$ *(*$\mu, \xi = 0, 1, 2, \ldots$*) by setting*

$$\Phi_{j,k}^{(\mu+1),\xi}(t,\zeta) = -\frac{1}{2} : \partial^2_{\bar{b}^-} \Phi_{j,k}^{(\mu),\xi} : (t,\zeta) \tag{3.54}$$

and

$$\Phi_{j,k}^{(\mu),\xi+1}(t,\zeta) = \frac{1}{1+\xi} : (a\partial_{\bar{b}}^{-} + \partial\bar{b}^+ \partial_a) \Phi_{j,k}^{(\mu),\xi} : (t,\zeta) \tag{3.55}$$

Then $\Phi^{(\mu)}_{j,k} = 0$ *for* $\mu > [j/2]$ *and*

$$\mathbf{\Phi}_{j,k} := (\Phi^{(\mu)}_{j,k})_{0\le\mu\le[j/2]}, \quad \Phi^{(\mu)}_{j,k}(t,r,\zeta) := \sum_{\xi\ge 0} \Phi^{(\mu),\xi}_{j,k}(t,\zeta) r^{\xi} \tag{3.56}$$

defines a ρ-$\langle N_0\rangle \ltimes \mathfrak{sv}$*-primary field,* ρ *being the representation of* $\tilde{\mathfrak{sv}}_0$ *defined by*

$$\rho(L_0) = -\left[\frac{j+k}{2}\mathrm{Id} - \sum_{\mu=0}^{[j/2]} \mu E^{\mu}_{\mu}\right] \tag{3.57}$$

$$\rho(N_0) = -\left[(k-j)\mathrm{Id} + 2\sum_{\mu=0}^{[j/2]} \mu E^{\mu}_{\mu}\right] \tag{3.58}$$

$$\rho(Y_{\frac{1}{2}}) = 0 \tag{3.59}$$

$$\rho(M_1) = \sum_{\mu=0}^{[j/2]-1} E^{\mu}_{\mu+1} \tag{3.60}$$

where E^{μ}_{ν} *is the* $([j/2]+1)\times([j/2]+1)$ *elementary matrix, with a single coefficient* 1 *at the intersection of the* μ*-th line and the* ν*-th row.*

2. *Set* $\mathbf{\Phi} = (\Phi^{(0)}_{j,k})_{j,k=0,1,\ldots}$*. Then* $\mathbf{\Phi}$ *is a* (ρ,Ω)-$\tilde{\mathfrak{sv}}$*-primary field if* ρ,Ω *are defined as follows:*

$$\begin{aligned} \rho(L_0)\Phi^{(0)}_{j,k} &= -\frac{j+k}{2}\Phi^{(0)}_{j,k}; \\ \rho(Y_{\frac{1}{2}})\Phi^{(0)}_{j,k} = 0; \quad \rho(M_1)\Phi^{(0)}_{j,k} &= -\frac{1}{2}j(j-1)\Phi^{(0)}_{j-2,k}; \\ \rho(N_0)\Phi^{(0)}_{j,k} &= (j-k)\Phi^{(0)}_{j,k}; \\ \Omega\Phi^{(0)}_{j,k} &= jk\Phi^{(0)}_{j-1,k-1}. \end{aligned} \tag{3.61}$$

Remarks.

1. Both representations ρ are of course the same; the passage from the first action on the $\Phi^{(\mu)}_{j,k}$ to the action on $\mathbf{\Phi}$ is given by the relation

$$\Phi^{(\mu)}_{j,k} = (-\frac{1}{2})^{\mu} j(j-1)\ldots(j-2\mu+1)\Phi^{(0)}_{j-2\mu,k}. \tag{3.62}$$

The second case in the Theorem is an extension of the first one when one wants to consider covariance under all N-generators (not only under N_0), which makes things more complicated.

2. Formally, one has

$$\Phi^{(\mu)}_{j,k} =: \exp r(a\partial_{\bar{b}^-} + \partial\bar{b}^+\partial_a).\Phi^{(\mu),0}_{j,k} : \tag{3.63}$$

since $Y_{(0)} \equiv \partial_r \equiv a\partial_{\bar{b}^-} + \partial\bar{b}^+\partial_a$ when applied to a polynomial $\mathfrak{sv}$-primary field of the form $P(\bar{b}^\pm, \partial\bar{b}^+, a)$. Hence, by the Campbell-Hausdorff formula

$$\exp(A+B) = \exp\frac{1}{2}[B,A]\exp A\exp B, \tag{3.64}$$

valid if $[A,[A,B]] = [B,[A,B]] = 0$, one may also write

$$\Phi^{(\mu)}_{j,k} =: \exp ra\partial_{\bar{b}^-}\exp\frac{r^2}{2}\partial\bar{b}^+\partial_{\bar{b}^-}\ .\ \Phi^{(\mu),0}_{j,k}. \tag{3.65}$$

Proof.

1. First of all, $\Phi^{(\mu),\xi}$ is well-defined only because the operators $\partial^2_{\bar{b}^-}$ and $a\partial_{\bar{b}^-} + \partial\bar{b}^+\partial_a$ (giving the shifts $\mu \to \mu+1$ and $\xi \to \xi+1$) commute. Let us check successively the covariance under the action of M, Y, N_0, L.

   - One finds from (3.45)

$$\begin{aligned} M_{(0)}\Phi^{(0),0}(t,\zeta) &= \bar{b}^+\partial_{\bar{b}^-}\Phi^{(0),0}(t,\zeta) = \sum_{m=0}^{j}\binom{j}{m}(j-m)\ :(\bar{b}^-)^{j-m-1}(\bar{b}^+)^{k+m+1}:\ (t)\zeta^m \\ &= \partial_\zeta\Phi^{(0),0}(t,\zeta); \end{aligned} \tag{3.66}$$

$$M_{(1)}\Phi^{(0),0}(t,\zeta) = \frac{1}{2}\partial^2_{\bar{b}^-}\Phi^{(0),0}(t,\zeta) = -\Phi^{(1),0}(t,\zeta) \tag{3.67}$$

   which is coherent with formula (2.13) and the definition (3.60) of $\rho(M_1)$. The field $\Phi_{j,k}$ is M-covariant if $\bar{b}^+\partial_{\bar{b}^-}\Phi^{(\mu),\xi}(t,\zeta) = \partial_\zeta\Phi^{(\mu),\xi}(t,\zeta)$ for every $\mu,\xi \geq 0$. But this is true for $\mu,\xi = 0$ and $[\bar{b}^+\partial_{\bar{b}^-}, \partial_{\bar{b}^-}] = [\bar{b}^+\partial_{\bar{b}^-}, a\partial_{\bar{b}^-} + \partial\bar{b}^+\partial_a] = 0$. Hence this is true for all values of μ,ξ by induction.

   - The action of $Y_{(0)}$ on $\Phi^{(\mu),\xi}$ is correct by definition - compare with formulas (3.44) and (3.55). One has $Y_{(1)}\Phi^{(\mu),0} = 0$ because $\partial_a\Phi^{(\mu),0} = 0$, which is coherent with (2.12) if one sets $\rho(Y_{\frac{1}{2}}) = 0$. To prove that $Y_{(1)}\Phi^{(\mu),\xi} = \bar{b}^+\partial_a\Phi^{(\mu),\xi}$ coincides with $\partial_\zeta\Phi^{(\mu),\xi-1} = \bar{b}^+\partial_{\bar{b}^-}\Phi^{(\mu),\xi-1}$, one uses induction on ξ and the commutator relation $[\bar{b}^+\partial_a, a\partial_{\bar{b}^-} + (\partial\bar{b}^+)\partial_a] = \bar{b}^+\partial_{\bar{b}^-}$. If this holds for some $\xi \geq 0$, then

$$\begin{aligned} \bar{b}^+\partial_a\Phi^{(\mu),\xi+1} &= \frac{1}{1+\xi}(\bar{b}^+\partial_a)(a\partial_{\bar{b}^-} + (\partial\bar{b}^+)\partial_a)\Phi^{(\mu),\xi} \\ &= \frac{1}{1+\xi}\left[(a\partial_{\bar{b}^-} + (\partial\bar{b}^+)\partial_a)(\bar{b}^+\partial_{\bar{b}^-})\Phi^{(\mu),\xi-1} + \bar{b}^+\partial_{\bar{b}^-}\Phi^{(\mu),\xi}\right] \end{aligned}$$

$$= \frac{1}{1+\xi}\left[(\bar{b}^+\partial_{\bar{b}^-})(a\partial_{\bar{b}^-}+(\partial\bar{b}^+)\partial_a)\Phi^{(\mu),\xi-1}+\bar{b}^+\partial_{\bar{b}^-}\Phi^{(\mu),\xi}\right]$$
$$= \frac{1}{1+\xi}\left[\xi\bar{b}^+\partial_{\bar{b}^-}\Phi^{(\mu),\xi}+\bar{b}^+\partial_{\bar{b}^-}\Phi^{(\mu),\xi}\right]$$
$$= \bar{b}^+\partial_{\bar{b}^-}\Phi^{(\mu),\xi} \tag{3.68}$$

by (3.55).

- One has $Y_{(2)}\Phi^{(\mu),0}=0$ by (3.44) and, supposing that $Y_{(2)}\Phi^{(\mu),\xi}=\partial_{\bar{b}^-}\partial_a\Phi^{(\mu),\xi}$ coincides with $-2\Phi^{(\mu+1),\xi-1}=-2\partial_\zeta\Phi^{(\mu),\xi-1}=\partial^2_{\bar{b}^-}\Phi^{(\mu),\xi-1}$ for some $\xi\geq 0$, then

$$\partial_{\bar{b}^-}\partial_a\Phi^{(\mu),\xi+1}=\frac{1}{1+\xi}(\partial_{\bar{b}^-}\partial_a)(a\partial_{\bar{b}^-}+\partial\bar{b}^+\partial_a)\Phi^{(\mu),\xi}=\partial^2_{\bar{b}^-}\Phi^{(\mu),\xi} \tag{3.69}$$

by a proof along the same lines, since $[\partial_{\bar{b}^-}\partial_a, a\partial_{\bar{b}^-}+\partial_{\bar{b}^+}\partial_a]=\partial^2_{\bar{b}^-}$.

- Since $N_{(0)}$ acts as $\bar{b}^+\partial_{\bar{b}^+}-\bar{b}^-\partial_{\bar{b}^-}$ on $\Phi^{(\mu),0}$, it simply measures the difference of degrees in $\bar{b}^+$ and $\bar{b}^-$ (for polynomial fields which depends only on $\bar{b}^\pm$ and not on their derivatives). Hence one sees easily that $N_{(0)}\Phi^{(\mu),0}=(2\zeta\partial_\zeta-j+k+2\mu)\Phi^{(\mu),0}$, which is formula (2.15). Then $Y_{(0)}\equiv a\partial_{\bar{b}^-}+(\partial\bar{b}^+)\partial_a$ increases by 1 the eigenvalue of $N_{(0)}$, see (3.43), which is also coherent with (2.15).

- There remains to check for the action of $L_{(i)}$, $i=2,3$. Supposing that $L_{(2)}\Phi^{(\mu),\xi}=\bar{b}^+\partial_{\partial\bar{b}^+}\Phi^{(\mu),\xi}$ coincides with $\frac{1}{2}\partial_\zeta\Phi^{(\mu),\xi-2}=\frac{1}{2}\bar{b}^+\partial_{\bar{b}^-}\Phi^{(\mu),\xi-2}$ for some $\xi\geq 0$, then

$$\bar{b}^+\partial_{\partial\bar{b}^+}\Phi^{(\mu),\xi+1}=\frac{1}{1+\xi}\left[\frac{1}{2}(a\partial_{\bar{b}^-}+\partial\bar{b}^+\partial_a)(\bar{b}^+\partial_{\bar{b}^-})\Phi^{(\mu),\xi-2}+\bar{b}^+\partial_a\Phi^{(\mu),\xi}\right]; \tag{3.70}$$

since (as we just proved) $Y_{(1)}\Phi^{(\mu),\xi}=\bar{b}^+\partial_a\Phi^{(\mu),\xi}=\bar{b}^+\partial_{\bar{b}^-}\Phi^{(\mu),\xi-1}$, one gets $L_{(2)}\Phi^{(\mu),\xi+1}=\frac{1}{2}\partial_\zeta\Phi^{(\mu),\xi-1}$.

Finally, supposing that $2L_{(3)}\Phi^{(\mu),\xi}=(\partial_a^2+\partial_{\bar{b}^-}\partial_{\partial\bar{b}^+})\Phi^{(\mu),\xi}$ coincides with $-3\Phi^{(\mu)+1,\xi-2}=\frac{3}{2}\partial^2_{\bar{b}^-}\Phi^{(\mu),\xi-2}$, then, using the commutator relation $[\partial_a^2+\partial_{\bar{b}^-}\partial_{\partial\bar{b}^+}, a\partial_{\bar{b}^-}+(\partial\bar{b}^+)\partial_a]=3\partial_{\bar{b}^-}\partial_a$ and the above equality $Y_{(2)}\Phi^{(\mu),\xi}=\partial_{\bar{b}^-}\partial_a\Phi^{(\mu),\xi}=\partial^2_{\bar{b}^-}\Phi^{(\mu),\xi-1}$, one finds

$$(\partial_a^2+\partial_{\bar{b}^-}\partial_{\partial\bar{b}^+})\Phi^{(\mu),\xi+1}=\frac{1}{\xi+1}\left[\frac{3}{2}(a\partial_{\bar{b}^+}+\partial\bar{b}^+\partial_a)\partial^2_{\bar{b}^-}\Phi^{(\mu),\xi-2}+3\partial_{\bar{b}^-}\partial_a\Phi^{(\mu),\xi}\right]=\frac{3}{2}\partial^2_{\bar{b}^-}\Phi^{(\mu),\xi-1}. \tag{3.71}$$

2. First note that

$$\partial_{\bar{b}^+}\partial_{\bar{b}^-}\Phi^{(0)}_{j,k} = \sum_{m\geq 0}(j-m)(k+m)\begin{pmatrix} j \\ m\end{pmatrix}(\bar{b}^-)^{j-m-1}(\bar{b}^+)^{k+m-1}$$
$$= jk\sum_{m\geq 0}\begin{pmatrix} j-1 \\ m\end{pmatrix}(\bar{b}^-)^{j-m-1}(\bar{b}^+)^{k+m-1}$$
$$+ j(j-1)\sum_{m\geq 1}\begin{pmatrix} j-2 \\ m-1\end{pmatrix}(\bar{b}^-)^{j-m-1}(\bar{b}^+)^{k+m-1}$$

$$\begin{aligned} &= jk\Phi^{(0)}_{j-1,k-1} + \zeta j(j-1)\Phi^{(0)}_{j-2,k} \\ &= -\Omega\Phi^{(0)}_{j,k} - 2\zeta\rho(M_1)\Phi^{(0)}_{j,k} \end{aligned} \tag{3.72}$$

by Remark 1. following Theorem 3.2.4.

Hence one has identified the action of $N_{(1)}$ on $\Phi^{(0),0}_{j,k}$ as the correct one. Suppose now that $N_{(1)}\Phi^{(0),\xi} = (\bar{b}^+\partial_{\partial\bar{b}^+} + \partial_{\bar{b}^+}\partial_{\bar{b}^-})\Phi^{(0),\xi}$ coincides with $\frac{1}{2}\partial_\zeta\Phi^{(0),\xi-2} + \zeta j(j-1)\Phi^{(0),\xi}_{j-2,k} + jk\Phi^{(0),\xi}_{j-1,k-1}$ for some ξ. Then, by commuting $\bar{b}^+\partial_{\partial\bar{b}^+} + \partial_{\bar{b}^+}\partial_{\bar{b}^-}$ through $Y_{(0)} = a\partial_{\bar{b}^-} + \partial\bar{b}^+\partial_a$, one gets

$$(\bar{b}^+\partial_{\partial_{\bar{b}^+}} + \partial_{\bar{b}^+}\partial_{\bar{b}^-})\Phi^{(0),\xi+1}_{j,k} =$$
$$\frac{1}{1+\xi}\left[(a\partial_{\bar{b}^-} + \partial\bar{b}^+\partial_a)(\frac{1}{2}\partial_\zeta\Phi^{(0),\xi-2} + \zeta j(j-1)\Phi^{(0),\xi}_{j-2,k} + jk\Phi^{(0),\xi}_{j-1,k-1}) + \bar{b}^+\partial_a\Phi^{(0),\xi}\right] \tag{3.73}$$

and $Y_{(1)}\Phi^{(0),\xi} = \bar{b}^+\partial_a\Phi^{(0),\xi}_{j,k} = \partial_\zeta\Phi^{(0),\xi-1}_{j,k}$ as we have just proved, hence $N_{(1)}\Phi^{(0),\xi+1}$ is given by the correct formula.

Finally, $N_{(2)}\Phi^{(0),\xi}_{j,k} = \partial_{\bar{b}^-}\partial_{\partial\bar{b}^+}\Phi^{(0),\xi}$ must be identified with $-\partial_\zeta\Phi^{(0),\xi-2}_{j,k}$ (which is certainly true for $\xi = 0$). Supposing this holds for some ξ,

$$\begin{aligned} \partial_{\bar{b}^-}\partial_{\partial\bar{b}^+}\Phi^{(0),\xi+1} &= \frac{1}{1+\xi}(\partial_{\bar{b}^-}\partial_{\partial\bar{b}^+})(a\partial_{\bar{b}^-} + \partial\bar{b}^+\partial_a)\Phi^{(0),\xi} \\ &= \frac{1}{1+\xi}\left[-(a\partial_{\bar{b}^-} + \partial\bar{b}^+\partial_a)\partial_\zeta\Phi^{(0),\xi-2} + \partial_{\bar{b}^-}\partial_a\Phi^{(0),\xi}\right] \end{aligned} \tag{3.74}$$

since $[\partial_{\bar{b}^-}\partial_{\partial\bar{b}^+}, a\partial_{\bar{b}^-} + \partial\bar{b}^+\partial_a] = \partial_{\bar{b}^-}\partial_a$; we now use the previous result $Y_{(2)}\Phi^{(\mu),\xi} = \partial_{\bar{b}^-}\partial_a\Phi^{(\mu),\xi} = -2\partial_\zeta\Phi^{(0),\xi-1}$ and conclude by induction.

Theorem 3.2.5 (construction of the generalized polynomial fields ${}_\alpha\Phi_{j,k}$)

1. *Set*

$${}_\alpha\Phi^{(0),0}_{j,k}(w,\zeta) = \sum_{m=0}^{j}\binom{j}{m}\zeta^m : (\bar{b}^-)^{j-m}(\bar{b}^+)^{k+m}V_\alpha : \tag{3.75}$$

and define inductively a series of fields ${}_\alpha\Phi^{(\mu),\xi} = {}_\alpha\Phi^{(\mu),\xi}_{j,k}$ $(\mu,\xi = 0,1,2,\ldots)$ *by setting*

$${}_\alpha\Phi^{(\mu+1),\xi}(w,\zeta) = \frac{i}{\sqrt{2}} : \partial_{\bar{b}^-}\,{}_\alpha\Phi^{(\mu),\xi} : (w,\zeta) \tag{3.76}$$

and

$${}_\alpha\Phi^{(\mu),\xi+1}(w,\zeta) = \frac{1}{1+\xi} : (a\partial^-_{\bar{b}} + \partial\bar{b}^+\partial_a + \alpha\bar{b}^+)\,{}_\alpha\Phi^{(\mu),\xi} : (w,\zeta) \tag{3.77}$$

Then

$${}_\alpha\Phi_{j,k} := ({}_\alpha\Phi^{(\mu)})_{0\le i\le j}, \quad {}_\alpha\Phi^{(\mu)}(t,r,\zeta) = \sum_{\xi\ge 0}{}_\alpha\Phi^{(\mu),\xi}(w,\zeta)r^\xi \tag{3.78}$$

defines a ρ-$\langle N_0\rangle \ltimes \mathfrak{sv}$-primary field, ρ being the representation of $\widetilde{\mathfrak{sv}}_0$ defined by

$$\rho(L_0) = -\left[\frac{j+k+\alpha^2}{2}\mathrm{Id} - \frac{1}{2}\sum_{\mu=0}^{j}\mu E_{\mu,\mu}\right] \tag{3.79}$$

$$\rho(N_0) = -\left[(k-j)\mathrm{Id} + \sum_{\mu=0}^{j}\mu E_{\mu,\mu}\right] \tag{3.80}$$

$$\rho(Y_{\frac{1}{2}}) = \mathrm{i}\alpha\sqrt{2}\sum_{\mu=0}^{j-1} E_{\mu,\mu+1} \tag{3.81}$$

$$\rho(M_1) = -\frac{1}{2}(\frac{1}{\alpha}\rho(Y_{\frac{1}{2}}))^2 = \sum_{\mu=0}^{j-2} E_{\mu,\mu+2} \tag{3.82}$$

2. *Set ${}_{\boldsymbol\alpha}\boldsymbol\Phi = ({}_\alpha\Phi^{(0)}_{j,k})_{j,k=0,1,\ldots}$. Then ${}_{\boldsymbol\alpha}\boldsymbol\Phi$ is a (ρ,Ω)-$\tilde{\mathfrak{sv}}$-primary field if ρ,Ω are defined as follows:*

$$\begin{aligned}
\rho(L_0)\,{}_\alpha\Phi^{(0)}_{j,k} &= -\frac{j+k}{2}\,{}_\alpha\Phi^{(0)}_{j-2,k};\\
\rho(Y_{\frac{1}{2}})\Phi^{(0)}_{j,k} = -\alpha j\Phi^{(0)}_{j-1,k};\quad \rho(M_1)\,{}_\alpha\Phi^{(0)}_{j,k} &= -\frac{1}{2}j(j-1)\,{}_\alpha\Phi^{(0)}_{j-2,k};\\
\rho(N_0)\,{}_\alpha\Phi^{(0)}_{j,k} &= (j-k)\,{}_\alpha\Phi^{(0)}_{j,k};\\
\Omega\,{}_\alpha\Phi^{(0)}_{j,k} &= jk\,{}_\alpha\Phi^{(0)}_{j-1,k-1}.
\end{aligned} \tag{3.83}$$

Remark.

- The coherence between the two representations is given this time by:

$$ {}_\alpha\Phi^{(\mu)}_{j,k} = \left(\frac{\mathrm{i}}{\sqrt{2}}\right)^k j(j-1)\ldots(j-k+1)\Phi^{(0)}_{j-\mu,k}. \tag{3.84}$$

- One may write formally

$$\begin{aligned}
{}_\alpha\Phi^{(\mu)}_{j,k} &= \ :\exp r\left(a\partial_{\bar b^-} + \partial\bar b^+\partial_a + \alpha\bar b^+\right)\ .\Phi^{(\mu),0}_{j,k}\ :\\
&= \ :\exp \alpha r\bar b^+\ .\ \exp ra\partial_{\bar b^-}\ .\ \exp\frac{r^2}{2}\partial\bar b^+\partial_{\bar b^-}\Phi^{(\mu),0}_{j,k}\ :
\end{aligned} \tag{3.85}$$

Proof.

The proof is almost the same, with just a few modifications. We shall follow the proof of Theorem 3.2.4 line by line and rewrite only what has to be changed.

- One has $Y_{(1)}\,{}_\alpha\Phi^{(\mu),0}(\zeta) = \alpha\partial_{\bar b^-}\,{}_\alpha\Phi^{(\mu),0}(\zeta)$, to be identified with $-\rho(Y_{\frac12})^\mu_\nu\,{}_\alpha\Phi^{(\nu),0}$. Hence one must set, in accordance with (3.81)

$$ {}_\alpha\Phi^{(\mu)+1,0} = \frac{i}{\sqrt{2}}\partial_{\bar b^-}\,{}_\alpha\Phi^{(\mu),0} \tag{3.86}$$

so ${}_\alpha\Phi^{(\mu)+2,0} = -\frac12\partial^2_{\bar b^-}\,{}_\alpha\Phi^{(\mu),0}$ as in Theorem 3.2.4, with a double shift instead in the indices i.

Suppose now $Y_{(1)}\,{}_\alpha\Phi^{(\mu),\xi} = (\bar b^+\partial_a+\alpha\partial_{\bar b^-})\,{}_\alpha\Phi^{(\mu),\xi}$ coincides with $\partial_\zeta\,{}_\alpha\Phi^{(\mu),\xi-1} - i\alpha\sqrt2\,{}_\alpha\Phi^{(\mu)+1,\xi} = \bar b^+\partial_{\bar b^-}\,{}_\alpha\Phi^{(\mu),\xi-1} - i\alpha\sqrt2\,{}_\alpha\Phi^{(\mu)+1,\xi}$: then the commutator relation $[\bar b^+\partial_a + \alpha\partial_{\bar b^-}, a\partial_{\bar b^-} + \partial\bar b^+\partial_a + \alpha\bar b^+] = \bar b^+\partial_{\bar b^-}$ yields

$$Y_{(1)}\,{}_\alpha\Phi^{(\mu),\xi+1} = \tag{3.87}$$
$$\frac{1}{\xi+1}\left\{(a\partial_{\bar b^-} + (\partial\bar b^+)\partial_a + \alpha\bar b^+)(\bar b^+\partial_{\bar b^-}\,{}_\alpha\Phi^{(\mu),\xi-1} - i\alpha\sqrt2\,{}_\alpha\Phi^{(\mu)+1,\xi}) + \bar b^+\partial_{\bar b^-}\,{}_\alpha\Phi^{(\mu),\xi}\right\}$$
$$= \bar b^+\partial_{\bar b^-}\,{}_\alpha\Phi^{(\mu),\xi} - i\alpha\sqrt2\Phi^{(\mu)+1,\xi+1} \tag{3.88}$$

Then $Y_{(2)} = \partial_{\bar b^-}\partial_a$ and $[\partial_{\bar b^-}\partial_a, Y_{(0)}] = \partial^2_{\bar b^-}$ as in Theorem 3.2.4, so covariance under $Y_{(2)}$ holds true.

- The action of $N_{(0)}, N_{(1)}, N_{(2)}$ on ${}_\alpha\Phi^{(\mu)}$ or ${}_\alpha\Phi^{(0)}_{j,k}$ is exactly as in Theorem 3.2.4 since $N = -:\bar b^+\bar b^-:$ does not involve neither the free boson a nor its integral.

- One must still check for $L_{(2)}$ (nothing changes for $L_{(3)}$). Suppose that $L_{(2)}\,{}_\alpha\Phi^{(\mu),\xi} = \bar b^+\partial_{\partial\bar b^+} + \alpha\partial_a$ coincides with $\frac12\partial_\zeta\,{}_\alpha\Phi^{(\mu),\xi-2} - \mathrm{i}\alpha\sqrt2\,{}_\alpha\Phi^{(\mu+1),\xi-1} = \frac12\bar b^+\partial_{\bar b^-}\,{}_\alpha\Phi^{(\mu),\xi-2} - \mathrm{i}\alpha\sqrt2\,{}_\alpha\Phi^{(\mu)+1,\xi-1}$. Then, using

$$[\bar b^+\partial_{\partial\bar b^+} + \alpha\partial_a, a\partial_{\bar b^-} + (\partial\bar b^+)\partial_a + \alpha\bar b^+]\,{}_\alpha\Phi^{(\mu),\xi} = (\bar b^+\partial_a + \alpha\partial_{\bar b^-})\,{}_\alpha\Phi^{(\mu),\xi} = \partial_\zeta\,{}_\alpha\Phi^{(\mu),\xi-1} - \mathrm{i}\alpha\sqrt2\Phi^{(\mu+1),\xi} \tag{3.89}$$

(see computation of $Y_{(1)}\,{}_\alpha\Phi^{(\mu),\xi}$ above) one gets

$$\begin{aligned}(\bar b^+\partial_{\partial\bar b^+} + \alpha\partial_a)\,{}_\alpha\Phi^{(\mu),\xi+1} &= \frac{1}{1+\xi}[(a\partial_{\bar b^-} + (\partial\bar b^+)\partial_a + \alpha\bar b^+)(\frac12\bar b^+\partial_{\bar b^-}\,{}_\alpha\Phi^{(\mu),\xi-2} - \mathrm{i}\alpha\sqrt2\,{}_\alpha\Phi^{(\mu)+1,\xi-1}) \\ &+ \partial_\zeta\Phi^{(\mu),\xi-1} - \mathrm{i}\alpha\sqrt2\,{}_\alpha\Phi^{(\mu)+1,\xi}] \\ &= \frac12\bar b^+\partial_{\bar b^-}\,{}_\alpha\Phi^{(\mu),\xi-1} - \mathrm{i}\alpha\sqrt2\,{}_\alpha\Phi^{(\mu)+1,\xi}.\end{aligned} \tag{3.90}$$

□

We shall now start computing explicitly the simplest n-point functions of the $\tilde{\mathfrak{sv}}$-primary fields we have just defined.

4 Correlators of the polynomial and generalized polynomial fields

We obtain below the two-point functions of the generalized polynomial fields ${}_\alpha\Phi_{j,k}$ (see Propositions 4.1 and 4.2) and the three-point functions in the case $\alpha = 0$, see Proposition 4.3 (computations are much more involved in the case $\alpha \neq 0$).

Proposition 4.1 (computation of the two-point functions when $\alpha = 0$)

Set $t = t_1 - t_2, r = r_1 - r_2, \zeta = \zeta_1 - \zeta_2$ *for the differences of coordinates. Then the two-point function*

$$\mathcal{C}(t_1, r_1, \zeta_1; t_2, r_2, \zeta_2) := \langle 0 \mid \Phi^{(0)}_{j_1,k_1}(t_1, r_1, \zeta_1)\Phi^{(0)}_{j_2,k_2}(t_2, r_2, \zeta_2) \mid 0\rangle$$

is equal to

$$\mathcal{C}(t, r, \zeta) = \delta_{j_1+k_1,j_2+k_2}(-1)^{k_2} j_1!\ k_1! \begin{pmatrix} j_2 \\ k_1 \end{pmatrix} t^{-(j_1+k_1)}(\zeta - \frac{r^2}{2t})^{j_1-k_2} \tag{4.1}$$

if $j_1 \geq k_2$ *and* 0 *else.*

Proof.

We use the covariance of $\mathcal{C}$ under the finite subalgebra $\rho_{j,k}(L_{\pm 1,0})$, $\rho_{j,k}(Y_{\pm\frac{1}{2}})$, $\rho_{j,k}(N_0)$. In particular, $\mathcal{C}$ is a function of the differences of coordinates t, r, ζ only. Covariance under

$$\rho(L_0) = -\sum_{i=1}^{2}(t_i\partial_{t_i} + \frac{1}{2}r_i\partial_{r_i}) - \frac{1}{2}\sum_{i=1}^{2}(j_i + k_i), \quad \rho(Y_{\frac{1}{2}}) = -\sum_{i=1}^{2} t_i\partial_{r_i} + r_i\partial_{\zeta_i}, \tag{4.2}$$

$$\rho(L_1) = -\sum_{i=1}^{2} t_i^2\partial_{t_i} + t_i r_i\partial_{r_i} + \frac{r_i^2}{2}\partial_{\zeta_i} - \sum_{i=1}^{2} t_i(j_i + k_i) \tag{4.3}$$

yields quite generally (see [21], [23])

$$\mathcal{C} = C.\delta_{j_1+k_1,j_2+k_2} t^{-(j_1+k_1)} f(\zeta - \frac{r^2}{2t}) \tag{4.4}$$

for some function f.

Suppose $j_1 + k_1 = j_2 + k_2$. Assuming the extra covariance under $\tilde{\rho}_k(N_0) = -\sum_{i=1}^{2}(r_i\partial_{r_i} + 2\zeta_i\partial_{\zeta_i}) + (j_1 - k_1) + (j_2 - k_2) \equiv -(r\partial_r + 2\zeta\partial_\zeta) + 2(j_1 - k_2)$, one gets $vf'(v) = (j_1 - k_2)f$, hence $f(v) = v^{j_1-k_2}$ up to a constant (see also Proposition A.1 in Appendix A). The coefficient $(-1)^{k_2} j_1! k_1! \begin{pmatrix} j_2 \\ k_1 \end{pmatrix}$ may be obtained from the coefficient C of the term of highest degree in t (i.e. the least singular term in t) in the formal series in $r_{1,2}, \zeta_{1,2}$. Since $Y_{(0)} \equiv a\partial_{\bar{b}^-} + \partial\bar{b}^+\partial_a$ maps an L-quasiprimary field of weight, say, λ into an L-quasiprimary field of weight $\lambda + \frac{1}{2}$, it is clear that C can be read from

$$\begin{aligned} \mathcal{C}_0 &= \langle 0 \mid \Phi^{(0),0}(t_1, r_1, \zeta_1)\Phi^{(0),0}(t_2, r_2, \zeta_2) \mid 0\rangle \\ &= \sum_{m_1,m_2} \zeta^{m_1+m_2} \begin{pmatrix} j_1 \\ m_1 \end{pmatrix}\begin{pmatrix} j_2 \\ m_2 \end{pmatrix} \langle 0 \mid\ :(\bar{b}^-)^{j_1-m_1}(\bar{b}^+)^{k_1+m_1}:(t_1)\ \ :(\bar{b}^-)^{j_2-m_2}(\bar{b}^+)^{k_2+m_2}:(t_2) \mid 0\rangle \end{aligned} \tag{4.5}$$

For the same reason, $\mathcal{C}_0$ must be equal to $Ct^{-(j_1+k_1)}(\zeta_1 - \zeta_2)^{j_1-k_2}$. One gets immediately $C = 0$ for $j_1 < k_2$. In the contrary case, one gets C by looking for the coefficient of $\zeta_2^{j_1-k_2}$, which is given by

$$(-1)^{j_1-k_2}\langle 0 \mid \Big(: (\bar{b}^-)^{j_1}(\bar{b}^+)^{k_1} : (t_1)\Big) \left(\left(\begin{array}{c} j_2 \\ j_1-k_2 \end{array} \right) : (\bar{b}^-)^{j_2-(j_1-k_2)}(\bar{b}^+)^{k_2+(j_1-k_2)} : (t_2)\right) \mid 0\rangle$$
$$= (-1)^{k_2} t^{-(j_1+k_1)} \left(\begin{array}{c} j_2 \\ k_1 \end{array} \right) j_1! k_1!. \tag{4.6}$$

□

Proposition 4.2 (computation of the two-point functions when $\alpha \neq 0$)
Set $t = t_1 - t_2, r = r_1 - r_2, \zeta = \zeta_1 - \zeta_2$ for the differences of coordinates. Write

$$\mathcal{C}^{\mu_1,\mu_2}_{(\alpha_1,j_1,k_1),(\alpha_2,j_2,k_2)} := \langle 0 \mid {}_{\alpha_1}\Phi^{(\mu_1)}_{j_1,k_1}(t_1,r_1,\zeta_1)\ {}_{\alpha_2}\Phi^{(\mu_2)}_{j_2,k_2}(t_2,r_2,\zeta_2) \mid 0\rangle. \tag{4.7}$$

Then:

(i) the two-point functions vanish unless $\alpha_1 = -\alpha_2$ and $j_1 \geq k_2$ and $j_2 \geq k_1$;

(ii) suppose that $j_1 = j_2 := j$, $k_1 = k_2 = 0$ and $\alpha := \alpha_1 = -\alpha_2$. Then

$$\mathcal{C}^{\mu_1,\mu_2}_{(\alpha,j,0),(-\alpha,j,0)} = t^{-j-\alpha^2+\frac{\mu_1+\mu_2}{2}} \sum_{\delta=\max(\mu_1,\mu_2)}^{j} c^\delta_j \frac{(i\alpha\sqrt{2})^{\delta-\mu_1}(-i\alpha\sqrt{2})^{\delta-\mu_2}}{(\delta-\mu_1)!(\delta-\mu_2)!} (\frac{r^2}{t})^{\delta-\frac{\mu_1+\mu_2}{2}} (\zeta - \frac{r^2}{2t})^{j-\delta} \tag{4.8}$$

where $c^\delta_j = (-1)^\delta \frac{(j!)^2}{2^\delta (j-\delta)!}$.

Remark. All the other cases may be deduced easily from formula (4.8) since, if $j_1 \geq k_2$ and $j_2 \geq k_1$ and (without loss of generality) $(j_2+k_2)-(j_1+k_1) = \Delta \geq 0$,

$$\mathcal{C}^{\mu_1,\mu_2}_{(\alpha,j_1,k_1),(-\alpha,j_2,k_2)}$$
$$= \left(k_1! \left(\begin{array}{c} j_1+k_1 \\ k_1 \end{array} \right) . k_2! \left(\begin{array}{c} j_2+k_2 \\ k_2 \end{array} \right) \right)^{-1} \langle 0 \mid \left[(\bar{b}^+\partial_{\bar{b}^-})^{k_1}\ {}_{\alpha}\Phi^{(\mu_1)}_{j_1+k_1,0} \right] \left[(\bar{b}^+\partial_{\bar{b}^-})^{k_2}\ {}_{-\alpha}\Phi^{(\mu_2)}_{j_2+k_2,0} \right] \mid 0\rangle$$
$$= \left(k_1! \left(\begin{array}{c} j_1+k_1 \\ k_1 \end{array} \right) . k_2! \left(\begin{array}{c} j_2+k_2 \\ k_2 \end{array} \right) . \Delta! \left(\begin{array}{c} j_1+k_1+\Delta \\ \Delta \end{array} \right) \right)^{-1}$$
$$\langle 0 \mid \left[\partial^\Delta_{\bar{b}^-} (\bar{b}^+\partial_{\bar{b}^-})^{k_1}\ {}_{\alpha}\Phi^{(\mu_1)}_{j_2+k_2,0} \right] \left[(\bar{b}^+\partial_{\bar{b}^-})^{k_2}\ {}_{-\alpha}\Phi^{(\mu_2)}_{j_2+k_2,0} \right] \mid 0\rangle$$
$$= \frac{(i\sqrt{2})^\Delta}{k_1! \left(\begin{array}{c} j_1+k_1 \\ k_1 \end{array} \right) . k_2! \left(\begin{array}{c} j_2+k_2 \\ k_2 \end{array} \right) . \Delta! \left(\begin{array}{c} j_1+k_1+\Delta \\ \Delta \end{array} \right)} \partial^{k_1}_{\zeta_1} \partial^{k_2}_{\zeta_2} \mathcal{C}^{\mu_1+\Delta,\mu_2}_{(\alpha,j_2+k_2,0),(-\alpha,j_2+k_2,0)} \tag{4.9}$$

thanks to the fact that $\partial_\zeta\ {}_{\alpha}\Phi^{(\mu)}_{j,k} = \bar{b}^+\partial_{\bar{b}^-}\ {}_{\alpha}\Phi^{(\mu)}_{j,k}$ and ${}_{\alpha}\Phi^{(\mu+1)}_{j,k} = -\frac{i}{\sqrt{2}}\partial_{\bar{b}^-}\ {}_{\alpha}\Phi^{(\mu)}_{j,k}$.

Proof.

We only prove (ii) since (i) is clear from the preceding computations. Applying Proposition B.1 from Appendix B, with $d = j+1$, $\lambda_{1,2} = \frac{\alpha^2+j}{2}$, $\alpha_{1,2} = \pm i\alpha\sqrt{2}$ and $\lambda'_{1,2} = -j$, one gets (4.8). There remains to find the coefficients c_δ^j. Let us first explain how to find c_j^j. One has

$$\begin{aligned}
\mathcal{C}^{j,j} &= \langle 0 \mid {}_\alpha\Phi_{j,0}^{(j)} \, {}_{-\alpha}\Phi_{j,0}^{(j)} \mid 0\rangle = c_j^j t^{-\alpha^2} \\
&= (\frac{i}{\sqrt{2}})^{2j} \langle 0 \mid \left((\partial_{\bar{b}^-})^j \, {}_\alpha\Phi_{j,0}^{(0)} \right) \left((\partial_{\bar{b}^-})^j \, {}_{-\alpha}\Phi_{j,0}^{(0)} \right) \mid 0\rangle \text{ by Theorem 3.2.5} \\
&= (-1)^j 2^{-j} (j!)^2 \langle 0 \mid \, {}_\alpha\Phi_{0,0}^{(0)} \, {}_{-\alpha}\Phi_{0,0}^{(0)} \mid 0\rangle \\
&= (-1)^j 2^{-j} (j!)^2 \langle 0 \mid \; : \exp \alpha r_1 \bar{b}^+ V_\alpha(t_1) : \; : \exp -\alpha r_2 \bar{b}^+ V_{-\alpha}(t_2) : \; \mid 0\rangle \\
&= (-1)^j 2^{-j} (j!)^2 t^{-\alpha^2}.
\end{aligned} \tag{4.10}$$

By the same trick, one gets (by deriving $j - \varepsilon$ times with respect to $\bar{b}^-$)

$$\begin{aligned}
\mathcal{C}^{j-\varepsilon, j-\varepsilon} &= (c_j^{j-\varepsilon} (\zeta - \frac{r^2}{2t})^\varepsilon + O(r)) t^{-\varepsilon - \alpha^2} \\
&= (-1)^{j-\varepsilon} 2^{-(j-\varepsilon)} (j(j-1)\ldots(\varepsilon+1))^2 \langle 0 \mid \, {}_\alpha\Phi_{\varepsilon,0}^{(0)} \, {}_{-\alpha}\Phi_{\varepsilon,0}^{(0)} \mid 0\rangle
\end{aligned} \tag{4.11}$$

and one may identify the lowest degree component in r – which does *not* depend on α, up to a multiplication by the factor $t^{-\alpha^2}$ – by setting $r_1 = r_2$,

$$\begin{aligned}
\mathcal{C}^{j-\varepsilon, j-\varepsilon}(r_1 = r_2) &= c_j^{j-\varepsilon} \zeta^\varepsilon t^{-\varepsilon-\alpha^2} \\
&= (-1)^{j-\varepsilon} 2^{-(j-\varepsilon)} \left(\frac{j!}{\varepsilon!} \right)^2 t^{-\alpha^2} \langle 0 \mid \Phi_{\varepsilon,0}^{(0)} \Phi_{\varepsilon,0}^{(0)} \mid 0\rangle \\
&= (-1)^{j-\varepsilon} 2^{-(j-\varepsilon)} \left(\frac{(j!)^2}{\varepsilon!} \right) \zeta^\varepsilon t^{-\varepsilon-\alpha^2}
\end{aligned} \tag{4.12}$$

by Proposition 4.1. □

Proposition 4.3 (computation of the three-point functions when $\alpha = 0$)

The following formula holds:

$$\langle \Phi_{j_1,0}^{(0)}(t_1, r_1, \zeta_1) \Phi_{j_2,0}^{(0)}(t_2, r_2, \zeta_2) \Phi_{j_3,0}^{(0)}(t_3, r_3, \zeta_3) \rangle = \frac{j_1! j_2! j_3!}{(\frac{1}{2}(j_1+j_3-j_2))!(\frac{1}{2}(j_2+j_3-j_1))!(\frac{1}{2}(j_1+j_2-j_3))!}$$
$$\left(\frac{\xi_{12}}{t_{12}} \right)^{\frac{1}{2}(j_1+j_2-j_3)} \left(\frac{\xi_{13}}{t_{13}} \right)^{\frac{1}{2}(j_1+j_3-j_2)} \left(\frac{\xi_{23}}{t_{23}} \right)^{\frac{1}{2}(j_2+j_3-j_1)} \tag{4.13}$$

where $\xi_{ij} := \zeta_{ij} - \frac{r_{ij}^2}{2t_{ij}}$.

Remark.

All three-point correlators for the case $\alpha = 0$ can be obtained easily from these results by applying a number of times the operator $\bar{b}^+ \partial_{\bar{b}^-}$ or equivalently ∂_ζ.

Proof.

Denote by $\mathcal{C}(t_i, r_i, \zeta_i) = \langle \Phi_{j_1}(t_1, r_1, \zeta_1)\Phi_{j_2}(t_2, r_2, \zeta_2)\Phi_{j_3}(t_3, r_3, \zeta_3)\rangle$ the three-point function. By Theorem A.3,

$$\mathcal{C} = C t_{12}^{-\alpha} t_{23}^{-\beta} t_{13}^{-\gamma} (\xi_{12}^{\alpha}\xi_{23}^{\beta}\xi_{31}^{\gamma} + \Gamma(\xi_{12}, \xi_{13}, \xi_{23})) \tag{4.14}$$

where C is a constant, $\alpha = \frac{j_1+j_2-j_3}{2}, \beta = \frac{j_1+j_3-j_2}{2}, \gamma = \frac{j_2+j_3-j_1}{2}$, and $\Gamma = \Gamma(\xi_{12}, \xi_{23}, \xi_{13})$ is any linear combination (with constant coefficients) of monomials $\xi_{12}^{\alpha'}\xi_{23}^{\beta'}\xi_{31}^{\gamma'}$ with $(\alpha', \beta', \gamma') \neq (\alpha, \beta, \gamma)$ and $\alpha' + \beta' + \gamma' = \frac{J}{2}$. Suppose $t_3 \neq t_1, t_2$ and look at the degree of the pole in $\frac{1}{t_{12}}$ in $\mathcal{C}$ considered as a function of t_1, t_2, t_3 and $\zeta_1, \zeta_2, \zeta_3$. Each term in the asymptotic expansion of Φ_{j_i} in powers of r_i, ζ_i is a polynomial of degree j_i in the fields $a, \bar{b}^-, b^+, \partial_{\bar{b}^+}$. The covariance $\mathcal{C} = \langle \Phi_1\Phi_2\Phi_3\rangle$ may be computed as any polynomial of Gaussian variables by using Wick's theorem; calling a_{ij} the number of couplings of Φ_i with Φ_j, an easy argument yields $j_1 = a_{12} + a_{13}, j_2 = a_{12} + a_{23}, j_3 = a_{13} + a_{23}$, hence in particular $a_{12} = \alpha$. Hence $\mathcal{C}$ has a pole in $\frac{1}{t_{12}}$ of degree at most 2α and Γ may not contain any term of the type $\xi_{12}^{\alpha'}\xi_{23}^{\beta}\xi_{31}^{\gamma'}$ with $\alpha' > \alpha$. By taking into account the poles in $\frac{1}{t_{23}}$ and $\frac{1}{t_{13}}$, one sees that $\Gamma = 0$.

There remains to compute the coefficient C. By rewriting $\mathcal{C}$ as

$$\mathcal{C} = \sum_{\alpha'+\beta'+\gamma'=J/2} C_{\alpha',\beta',\gamma'} \xi_{12}^{\alpha'}\xi_{13}^{\beta'}(\xi_{12} + \xi_{23} + \xi_{31})^{\gamma'}, \tag{4.15}$$

and using $\xi_{31} = (\xi_{12} + \xi_{23} + \xi_{31}) - \xi_{12} - \xi_{23}$, one sees that $C = C_{\alpha,\beta,\gamma}$. Now a minute's thought shows that the coefficient of

$$(\zeta_1^0\zeta_2^{\beta}\zeta_3^{\gamma})(r_1^0 r_2^0 r_2^{2\gamma}) t_{23}^{-2\gamma} t_{12}^{-\alpha} t_{13}^{-\beta} \tag{4.16}$$

in $\mathcal{C}$ is equal to $(-1)^{J/2} 2^{-\gamma} C_{\alpha,\beta,\gamma}$. hence (using the asymptotic expansion of Φ_1, Φ_2 and Φ_3 in powers of ζ_i, r_i) $C_{\alpha,\beta,\gamma}$ may be computing by extracting the coefficient of $t_{23}^{-2\gamma} t_{12}^{-\alpha} t_{13}^{-\beta}$ in

$$\langle 0 \mid \,:(\bar{b}^-)^{j_1}: (t_1) : \begin{pmatrix} j_2 \\ \alpha \end{pmatrix} (\bar{b}^-)^{j_2-\alpha}(\bar{b}^+)^{\alpha} : (t_2) : \frac{(a\partial_{\bar{b}^-} + \partial\bar{b}^+\partial_a)^{2\gamma}}{(2\gamma)!} \begin{pmatrix} j_3 \\ \beta \end{pmatrix} (\bar{b}^-)^{j_3-\beta}(\bar{b}^+)^{\beta} : (t_3) \,|0\rangle,$$

and multiplying by $(-1)^{J/2} 2^{-\gamma}$. Now the coefficient of $r^{2\gamma}$ in $\exp rY_{(0)} \,.\, \left((\bar{b}^-)^{j_3-\beta}(\bar{b}^+)^{\beta}\right)$ is equal to

$$\begin{aligned} &\sum_{i+2j=2\gamma} r^{2\gamma} \frac{(a\partial_{\bar{b}^-})^i}{i!} \left[2^{-j} \begin{pmatrix} j_3 - \beta \\ j \end{pmatrix} (\bar{b}^-)^{j_3-\beta-j} (\partial\bar{b}^+)^j (\bar{b}^+)^{\beta} \right] \\ &= \sum_{i+2j=2\gamma} r^{2\gamma} 2^{-j} \frac{(j_3-\beta)!}{j!i!(j_3-\beta-i-j)!} (\partial\bar{b}^+)^j a^i (\bar{b}^+)^{\beta}. \end{aligned} \tag{4.17}$$

The terms with $i > 0$ do not contribute to $C_{\alpha,\beta,\gamma}$ since a can only be found in the field with the variable t_3 and does not couple to the other fields. Hence

$$\begin{aligned} C_{\alpha,\beta,\gamma} t_{23}^{-2\gamma} t_{12}^{-\alpha} t_{13}^{-\beta} \;=\; & (-1)^{J/2} \begin{pmatrix} j_3 \\ \beta \end{pmatrix} \frac{(j_3-\beta)!}{\gamma!(j_3-\beta-\gamma)!} \begin{pmatrix} j_2 \\ j_2 - \alpha \end{pmatrix} \\ & \langle 0 \mid \,:(\bar{b}^-)^{j_1}: (t_1) : (\bar{b}^-)^{j_2-\alpha}(\bar{b}^+)^{\alpha} : (t_2) : (\partial\bar{b}^+)^{J/2-\alpha-\beta}(\bar{b}^+)^{\beta} : (t_3) \mid 0\rangle. \end{aligned} \tag{4.18}$$

The first field $(\bar{b}^-)^{j_1}$ couples α times (resp. β times) with the second (resp. third) fields, yielding $(t_{12})^{-\alpha}(t_{13})^{-\beta}$ times

$$\begin{pmatrix} j_1 \\ \alpha \end{pmatrix} \alpha!(-1)^\alpha \beta!(-1)^\beta. \tag{4.19}$$

There remains the coupling of the second and third fields, namely,

$$\langle 0 \mid \ :(\bar{b}^-)^{j_2-\alpha}(t_2):(\partial\bar{b}^+)^{J/2-\alpha-\beta}:(t_3) \tag{4.20}$$

which yields $t_{23}^{-2\gamma}$ times $\begin{pmatrix} j_2-\alpha \\ J/2-\alpha-\beta \end{pmatrix} (J/2-\alpha-\beta)!(-1)^{j_2-\alpha}$.

All together one gets

$$C_{\alpha,\beta,\gamma} = \frac{j_1!j_2!j_3!}{(\frac{1}{2}(j_1+j_3-j_2))!(\frac{1}{2}(j_2+j_3-j_1))!(\frac{1}{2}(j_1+j_2-j_3))!}. \tag{4.21}$$

Hence the result.

□

5 Construction of the massive fields

All the fields constructed until now involve only *polynomials* in the unphysical variable ζ. Inverting the Laplace transform $\mathcal{L}: f_{\mathcal{M}} \to \mathcal{L}f(\zeta) = \int_0^\infty f_{\mathcal{M}} e^{\mathcal{M}\zeta}\, d\mathcal{M}$ is a priori impossible since polynomials in ζ only give derivatives of the delta-function $\delta_{\mathcal{M}}$; one may say that these fields represent singular zero-mass fields, which are a priori irrelevant from a physical point of view.

However, we believe it is possible to construct *massive fields* by combining the above polynomial fields into a formal series depending on a parameter Ξ and taking an analytic continuation, whose status is yet unclear. Let us formalize this as a conjecture:

Conjecture:

Massive fields may be obtained as an analytic continuation for $\Xi \to 0$ of series in $\Phi_{j,k}$, ${}_\alpha\Phi_{j,k}$ of the form

$$\Xi^\lambda \sum_{j,k\geq 0} a_{j,k} \Xi^{-\frac{j+k}{2}} \Phi_{j,k}(t,r,\zeta) \quad \text{or} \ \Xi^\lambda \sum_{j,k\geq 0} a_{j,k} \Xi^{-\frac{j+k}{2}}\, {}_\alpha\Phi_{j,k}(t,r,\zeta) \tag{5.1}$$

for some λ, with a non-zero radius of convergence in Ξ^{-1}.

The idea lying behind this is that the discrepancy in the scaling behaviours in t of the fields $\Phi_{j,k}$ (namely, $\Xi^{-\frac{j+k}{2}}\Phi_{j,k}(t)$ behaves as $(\Xi t)^{-\frac{j+k}{2}}$ when $t \to \infty$ since $\Phi_{j,k}$ has L_0-weight $\frac{j+k}{2}$) disappears in the above sums in the limit $\Xi \to 0$. As for the appearance of a *massive* behaviour in the limit $\Xi \to 0$, it is reminiscent of the construction of the coherent state $e^{\mathcal{M}a^\dagger}|0\rangle$, an eigenvector of the annihilation operator a in the theory of the harmonic oscillator. We hope to make this analogy more precise in the future.

We introduce in Theorem 5.1 and Theorem 5.2 below good potential candidates for massive fields. Theorems 5.1, 5.2 and 5.3 show that all two-point functions and (at least) some three-point functions may indeed be analytically extended, and give explicit expressions for the corresponding n-point functions of the *would-be* massive field. The missing part in the picture is a formal proof that all n-point functions have an analytic extension to $\Xi \to 0$. An encouraging fact is that the limit for $\Xi \to 0$ does not seem to depend (up to a physically irrelevant overall coefficient depending only on the mass) on the precise asymptotic series.

We made some attempts to prove the existence of the desired analytic extension by constructing the n-point functions as solutions of differential equations coming from the symmetries (for instance, the two-point function $\langle \psi^{\Xi}_{-1} \psi^{\Xi}_{-1} \rangle$, see below, may be computed – up to a constant – by using the covariance under $\mathfrak{sch}_1$ and under N_1, and it should be possible to compute more generally $\langle \psi^{\Xi}_{d_1} \psi^{\Xi}_{d_2} \rangle$ in the same way by induction in d_1, d_2). This scheme may work, at least for the three-point functions, but it looks like a difficult task in general, involving a precise analysis of the singularities at $\Xi = 0$ of differential operators with regular singularities.

In the case of the polynomial fields $\Phi_{j,k}$, one obtains (up to an irrelevant function of $\mathcal{M}$) the heat kernel in any even dimension (this is impossible for odd dimensions because the heat kernel then involves a square root of $t_1 - t_2$ and one should use instead non-local conformal fields in the first place instead of the bosons). In the case of the generalized polynomial fields ${}_\alpha\Phi_{j,k}$, the two-point function is non-standard, which is not surprising since the ${}_\alpha\Phi_{j,k}$ are themselves non-scalar. The exact form is new and involves a Bessel function. There are (to the best of our knowledge) no known examples at the moment of a physical model with a two-point function of this form.

Theorem 5.1 (polynomial fields $\Phi_{j,k}$)

1. *Let $d = -1, 0, 1, \ldots$ and $\Xi > 0$. Set*

$$\phi^{\Xi}_{d} := \sum_{j=0}^{\infty} \frac{\mathrm{i}^{j+d}}{\Xi^{\frac{j+1}{2}}} \frac{\sqrt{j!}}{(j+d+1)!} \Phi^{(0)}_{j+d+1,d+1}. \tag{5.2}$$

Then the inverse Laplace transform of the two-point function

$$\mathcal{C}^{\Xi}(t, r, \zeta) = \langle 0 \mid \phi^{\Xi}_{d}(t_1, r_1, \zeta_1) \phi^{\Xi}_{d}(t_2, r_2, \zeta_2) \mid 0 \rangle,$$

defined a priori for $\Xi \gg 1$, may be analytically extended to the following function:

$$(\mathcal{L}^{-1}\mathcal{C}^{\Xi})(\mathcal{M}; t, r) = e^{\mathcal{M}\Xi t} t^{-2d-1} e^{-\mathcal{M}\frac{r^2}{2t}}. \tag{5.3}$$

When $\Xi \to 0$, this goes to the standard heat kernel $K_{4d+2}(t, r) = t^{-2d-1} e^{-\mathcal{M}\frac{r^2}{2t}}$.

2. *Let $d = 0, 1, \ldots$ and $\Xi > 0$. Set*

$$\tilde{\phi}^{\Xi}_{d} := \sum_{j=1}^{\infty} \frac{\mathrm{i}^{j+d}}{\Xi^{\frac{j+1}{2}}} \frac{\sqrt{j!}}{(j+d)!} \Phi^{(0)}_{j+d,d+1}. \tag{5.4}$$

Then the inverse Laplace transform of the two-point function $\langle \tilde{\phi}_d^{\Xi} \tilde{\phi}_d^{\Xi} \rangle$ *may be analytically extended into the function* $\mathcal{M} e^{\mathcal{M}\Xi t} t^{-2d} e^{-\mathcal{M}\frac{r^2}{2t}}$. *When* $\Xi \to 0$, *this goes to* $\mathcal{M}$ *times the standard heat kernel* $K_{4d}(t,r) = t^{-2d} e^{-\mathcal{M}\frac{r^2}{2t}}$.

3. *(same hypotheses) Set*

$$\psi_{2d}^{\Xi} := \sum_{j=0}^{\infty} \mathrm{i}^j \frac{\Xi^{-j-d-\frac{3}{2}}}{j!} \Phi^{(0)}_{2j+2d+1,2d+1}. \tag{5.5}$$

Then the two-point function $\langle \psi_d^{\Xi} \psi_d^{\Xi} \rangle$ *has an analytic continuation to small* Ξ. *The inverse Laplace transform of its value for* $\Xi = 0$ *is equal (up to a constant) to* $\mathcal{M}^{2d+2} K_{4d-2}(t,r)$.

4. *(same hypotheses) Set*

$$\tilde{\psi}_{2d}^{\Xi} := \sum_{j=0}^{\infty} \mathrm{i}^j \frac{\Xi^{-j-d-\frac{1}{2}}}{j!} \Phi^{(0)}_{2j+2d,2d+1}. \tag{5.6}$$

Then the two-point function $\langle \tilde{\psi}_d^{\Xi} \tilde{\psi}_d^{\Xi} \rangle$ *has an analytic continuation to small* Ξ. *The inverse Laplace transform of its value for* $\Xi = 0$ *is equal (up to a constant) to* $\mathcal{M}^{2d+1} K_{4d}(t,r)$.

Remark. One may also define

$$\psi_d^{\Xi} := \sum_{j=0}^{\infty} \mathrm{i}^j \frac{\Xi^{-j-\frac{d}{2}-1}}{j!} \Phi^{(0)}_{2j+d+1,d+1} \tag{5.7}$$

for d odd, but similar computations (using a different connection formula for the hypergeometric function though, see proof below) show that its two-point function is equal (up to a constant) to that of ψ_{d+1}^{Ξ}, i.e. (up to a polynomial in $\mathcal{M}$) to K_{2d}. (Note however the strange-looking but necessary shift by $\frac{1}{2}$ in the powers of Ξ in the expression of the ψ_d^{Ξ} with odd index d with respect to those with an even index). Hence the need for $\tilde{\psi}_{2d}^{\Xi}$.

Proof.

Note first quite generally that, if $K_d(\mathcal{M};t,r) := \frac{e^{-\mathcal{M}\frac{r^2}{2t}}}{t^{d/2}}$ is the standard heat kernel in d dimensions, then

$$\mathcal{L}(\mathcal{M}^n K_d(\mathcal{M};t,r)) = \partial_\zeta^n \left(t^{-d/2} \left(\frac{r^2}{2t} - \zeta \right)^{-1} \right) = (-1)^{n+1} n! t^{-d/2} \left(\zeta - \frac{r^2}{2t} \right)^{-n-1}. \tag{5.8}$$

We shall use the notation $\xi := \zeta - r^2/2t$ in the proof.

1. The Laplace transform of the function $g_{\Xi}^{(d)}(\mathcal{M};t,r) := e^{\mathcal{M}\Xi t} t^{-2d-1} e^{-\mathcal{M}\frac{r^2}{2t}}$ is equal to

$$(\mathcal{L}g^{(d)})(t,r,\zeta) = -t^{-2d-1} \frac{1}{\Xi t + (\zeta - \frac{r^2}{2t})} = -\sum_{j=0}^{\infty} (-1)^j \Xi^{-j-1} t^{-2(d+1)-j} (\zeta - \frac{r^2}{2t})^j$$

(provided that the series converges, or taken in a formal sense). Then Proposition 4.1 shows that the two-point function of the field ϕ_d^{Ξ} defined above is equal to this series.

2. Set $\tilde{g}_{\Xi}^{(d)}(\mathcal{M};t,r) := \mathcal{M}e^{\mathcal{M}\Xi t}t^{-2d}e^{-\mathcal{M}\frac{r^2}{2t}}$: then

$$(\mathcal{L}\tilde{g}_{\Xi}^{(d)})(t,r,\zeta) = \partial_\zeta(\mathcal{L}g_{\Xi}^{(d-\frac{1}{2})})(t,r,\zeta) = -\sum_{j=1}^{\infty} j(-1)^j t^{-2d-j-1}(\zeta - \frac{r^2}{2t})^{j-1}$$

is easily checked to be equal to the two-point function $\langle \tilde{\phi}_d^{\Xi} \tilde{\phi}_d^{\Xi} \rangle$.

3. Set

$$I^{\Xi} := \Xi^{2d+3}\partial_t^{-(2d+1)}(t^{2d}\langle \psi_d^{\Xi}\psi_d^{\Xi}\rangle)$$

where $\partial_t^{-1} = \int_0^t dt$ is the integration operator from 0 to t. Then Proposition 4.1, together with the duplication formula for the Gamma function, yield

$$\begin{aligned} I^{\Xi} &= \sum_{j\geq 0}(2j+2d+1)!\ t^{-2j-1}\frac{(-\xi^2/\Xi^2)^j}{(j!)^2} \\ &= \frac{1}{t}\cdot\frac{2^{2j+2d+1}}{\sqrt{\pi}}\sum_{j\geq 0}\frac{\Gamma(j+d+1)\Gamma(j+d+\frac{3}{2})}{\Gamma(j+1)}\frac{\left(-\left(\frac{\xi}{\Xi t}\right)^2\right)^j}{j!} \\ &= \frac{1}{t}\frac{2^{2d+1}}{\sqrt{\pi}}\Gamma(d+1)\Gamma(d+\frac{3}{2})\ {}_2F_1(d+1,d+\frac{3}{2};1;-\left(\frac{2\xi}{\Xi t}\right)^2) \end{aligned} \tag{5.9}$$

which is defined for $\Xi \gg 1$. The connection formula (see [2], 15.3.3) for the Gauss hypergeometric function ${}_2F_1$

$${}_2F_1(a,b,c;z) = (1-z)^{c-a-b}\ {}_2F_1(c-a,c-b;c;z) \tag{5.10}$$

yields

$${}_2F_1(d+1,d+\frac{3}{2};1;-(\frac{2\xi}{\Xi t})^2) = \left[1+\left(\frac{2\xi}{\Xi t}\right)^2\right]^{-2d-\frac{3}{2}}\ {}_2F_1(-d,-d-\frac{1}{2};1;-\left(\frac{2\xi}{\Xi t}\right)^2). \tag{5.11}$$

The hypergeometric function on the preceding line is simply a polynomial in Ξ^{-1} since $-d$ is a negative integer. By extracting the most singular term in Ξ^{-1}, one sees that

$$I^{\Xi} \sim_{\Xi\to 0} (-1)^d\frac{d!}{4\pi}\Gamma(d+\frac{3}{2})^2\Xi^{2d+3}\xi^{-2d-3}t^{2d+2}.$$

Hence

$$\langle \psi_d^{\Xi}\psi_d^{\Xi}\rangle \to_{\Xi\to 0} (-1)^d\frac{(2d+2)!d!}{4\pi}\Gamma(d+\frac{3}{2})^2\xi^{-2d-3}t^{1-2d} = (-1)^{d+1}\frac{d!}{4\pi}\Gamma(d+\frac{3}{2})^2\mathcal{L}(\mathcal{M}^{2d+2}K_{4d-2}(\mathcal{M};t,r)).$$

4. Same method.

$\square$

Theorem 5.2 (generalized polynomial fields ${}_\alpha\Phi_{j,k}$)

Let $\alpha \in \mathbb{R}$ and $\Xi > 0$.

1. *Set*

$$ {}_{\alpha}\phi^{\Xi} := \sum_{j=0}^{\infty} \frac{\mathrm{i}^j}{\Xi^{\frac{j+1}{2}}} \frac{1}{\sqrt{j!}}\, {}_{\alpha}\Phi^{(0)}_{j,0}. \tag{5.12} $$

Then the two-point function

$$ \mathcal{C}^{\Xi}(t,r,\zeta) = \langle 0 \mid {}_{\alpha}\phi^{\Xi}(t_1,r_1,\zeta_1)\, {}_{-\alpha}\phi^{\Xi}(t_2,r_2,\zeta_2) \mid 0\rangle $$

has an analytic continuation to small Ξ*, and its inverse Laplace transform at* $\Xi = 0$ *is equal to*

$$ \widetilde{\mathcal{C}}_{\mathcal{M}}(t,r) = -t^{1-\alpha^2} e^{-\mathcal{M}r^2/2t} I_0(2|\alpha|\sqrt{\mathcal{M}r^2/t}) \tag{5.13} $$

where I_0 *is the modified Bessel function of order 0.*

2. *Set*

$$ {}_{\alpha}\psi^{\Xi} := \sum_{j=0}^{\infty} (-1)^j \frac{\Xi^{-j-\frac{1}{2}}}{j!}\, {}_{\alpha}\Phi^{(0)}_{2j,0}. \tag{5.14} $$

Then the same results hold for the two-point function $\langle\, {}_{\alpha}\psi^{\Xi}\, {}_{-\alpha}\psi^{\Xi}\rangle$ *(up to an overall multiplicative constant).*

Remark: if one replaces α with $\mathrm{i}\alpha$, then the two-point function involves this time the Bessel function J_0.

Proof.

1. By applying Proposition 4.2, one gets

$$ \begin{aligned} \mathcal{C}(t,r,\zeta) &= \sum_{j\geq 0} \frac{(-1)^j}{\Xi^{j+1}} t^{-j-\alpha^2} \sum_{\delta=0}^{j} (-1)^{\delta} \frac{j!}{2^{\delta}(j-\delta)!} \frac{(2\alpha^2)^{\delta}}{(\delta!)^2} \left(\frac{r^2}{t}\right)^{\delta} (\zeta - \frac{r^2}{2t})^{j-\delta} \\ &= \frac{t^{-\alpha^2}}{\Xi} \sum_{n=0}^{\infty} \left(-\frac{\zeta - r^2/2t}{\Xi t}\right)^n \sum_{\delta=0}^{\infty} \frac{(n+\delta)!}{n!(\delta!)^2} \left(\frac{\alpha^2 r^2}{\Xi t^2}\right)^{\delta}. \end{aligned} \tag{5.15} $$

The function

$$ f(y) = \sum_{\delta=0}^{\infty} \frac{\begin{pmatrix} n+\delta \\ n \end{pmatrix}}{\delta!} y^{\delta} = \sum_{\delta=0}^{\infty} \frac{(n+\delta)!}{n!(\delta!)^2} y^{\delta} $$

is entire and admits a Laplace transform

$$ h(\lambda) = \mathcal{L}f(\lambda) = \int_0^{\infty} f(y) e^{-\lambda y}\, dy = \sum_{\delta=0}^{\infty} \begin{pmatrix} n+\delta \\ n \end{pmatrix} \lambda^{-\delta-1} = \frac{1}{\lambda}(1-1/\lambda)^{-n-1} = \lambda^n(\lambda-1)^{-n-1} $$

which is given by a converging series for $\lambda > 1$; by inverting the Laplace transform, one gets

$$f(y) = \partial_y^n \left(\frac{(-1)^n y^n}{n!} e^y \right).$$

An application of Leibniz formula gives

$$\partial_y^n \left(\frac{y^n}{n!} e^y\right) = \left(\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{y^k}{k!}\right) e^y.$$

By putting everything together and setting $y = \frac{\alpha^2 r^2}{\Xi t^2}$, one gets

$$\begin{aligned} \mathcal{C}(t,r,\zeta) &= \frac{t^{-\alpha^2}}{\Xi} \sum_{n=0}^{\infty} \left(\frac{\zeta - r^2/2t}{\Xi t} \right)^n \left(\sum_{k=0}^{n} \frac{\begin{pmatrix} n \\ k \end{pmatrix}}{k!} \left(\frac{\alpha^2 r^2}{\Xi t^2} \right)^k \right) e^{\frac{\alpha^2 r^2}{\Xi t^2}} \\ &= \frac{t^{-\alpha^2}}{\Xi} \sum_{k=0}^{\infty} \frac{1}{k!} \left[\sum_{n=k}^{\infty} \begin{pmatrix} n \\ k \end{pmatrix} \left(\frac{\zeta - r^2/2t}{\Xi t} \right)^n \right] \left(\frac{\alpha^2 r^2}{\Xi t^2} \right)^k e^{\frac{\alpha^2 r^2}{\Xi t^2}}. \qquad (5.16) \end{aligned}$$

By comparing with the generating series

$$\sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \begin{pmatrix} n \\ k \end{pmatrix} a^n \right) x^k = \sum_{n=0}^{\infty} a^n \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k = \sum_{n=0}^{\infty} [a(1+x)]^n = \frac{1}{1-a} \sum_{k=0}^{\infty} \left(\frac{a}{1-a} \right)^k x^k,$$

one gets

$$\mathcal{C}(t,r,\zeta) = -\frac{t^{1-\alpha^2}}{\zeta - r^2/2t - \Xi t} \exp\left[-\alpha^2 \frac{r^2}{t} \left(\frac{1}{\zeta - r^2/2t - \Xi t} \right) \right]. \qquad (5.17)$$

One finds in [14] $\mathcal{L}^{-1}(\lambda^{-1} e^{a/\lambda})(t) = I_0(2\sqrt{at})$, $a > 0$ (mind our unusual convention for the Laplace transform with respect to the mass $\mathcal{M}$!), where I_0 is the modified Bessel function of order 0. Hence

$$\mathcal{C}_{\mathcal{M}}(t,r) = -t^{1-\alpha^2} e^{\mathcal{M}\Xi t} e^{-\mathcal{M}r^2/2t} I_0(2|\alpha|\sqrt{\mathcal{M}r^2/t}).$$

2. The method is the same but computations are considerably more involved. Set $y := \frac{\alpha^2 r^2}{\Xi t^2}$ and $x = -\frac{4\xi}{\Xi t}$. Applying Proposition 4.2 yields this time

$$\mathcal{C}(t,r,\zeta) = \frac{t^{-\alpha^2}}{\Xi} \sum_{n=0}^{\infty} \left(-\frac{\zeta - r^2/2t}{\Xi t} \right)^n \sum_{\delta \geq 0, \delta + n \equiv 0[2]} (-1)^{\frac{n+\delta}{2}} \frac{((n+\delta)!)^2}{n! [(\frac{1}{2}(n+\delta))!]^2 (\delta!)^2} \left(\frac{\alpha^2 r^2}{\Xi t^2} \right)^\delta.$$

Let $h(\lambda)$ be the Laplace transform of $\mathcal{C}$ with respect to y. Formally, this is equivalent to replacing $y^\delta/\delta!$ by $\lambda^{-\delta-1}$. Separating the cases n, δ even, resp. odd, and using the duplication formula for the Gamma function, one gets

$$\begin{aligned} h(\lambda) &= \frac{1}{\Xi\sqrt{\pi}\lambda t^{\alpha^2}}\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(n+m+\frac{1}{2})^2}{\Gamma(m+\frac{1}{2})m!}(\frac{2}{\lambda})^{2m}\right. \\ &- \left.\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(n+m+\frac{3}{2})^2}{\Gamma(m+1)\Gamma(m+\frac{3}{2})}\left(\frac{2}{\lambda}\right)^{2m+1}\right] \\ &= \frac{1}{\Xi\sqrt{\pi}\lambda t^{\alpha^2}}\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\ {}_2F_1(n+\frac{1}{2},n+\frac{1}{2};\frac{1}{2};-\frac{4}{\lambda^2})\frac{\Gamma(n+\frac{1}{2})^2}{\Gamma(\frac{1}{2})}\right. \\ &- \left.\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\ {}_2F_1(n+\frac{3}{2},n+\frac{3}{2};\frac{3}{2};-\frac{4}{\lambda^2})\frac{2}{\lambda}\frac{\Gamma(n+\frac{3}{2})^2}{\Gamma(\frac{3}{2})}\right]. \end{aligned} \tag{5.18}$$

Hence $h(\lambda) = \frac{1}{\Xi\sqrt{\pi}\lambda t^{\alpha^2}}(T_1(\lambda)+T_2(\lambda))$ where (using once more the duplication formula and connection formulas for the hypergeometric function)

$$T_1(\lambda) = \sqrt{\pi}(1+\frac{4}{\lambda^2})^{-\frac{1}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{2})_n}{n!}\left(\frac{x}{2}\right)^{2n}(1+\frac{4}{\lambda^2})^{-2n}\ {}_2F_1(-n,-n;\frac{1}{2};-\frac{4}{\lambda^2})$$

and

$$T_2(\lambda) = -\sqrt{\pi}\frac{x}{\lambda}(1+\frac{4}{\lambda^2})^{-\frac{3}{2}}\ .\ \sum_{n=0}^{\infty}(-1)^n\frac{(\frac{3}{2})_n}{n!}(\frac{x}{2})^{2n}(1+\frac{4}{\lambda^2})^{-2n}\ {}_2F_1(-n,-n;\frac{3}{2};-\frac{4}{\lambda^2}).$$

These hypergeometric functions are simple polynomials since they have negative integer arguments; however, the sum obtained by expanding these polynomials looks hopelessly intricate. We use instead the following formula

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{(c)_{2n}}{n!(a)_n}(-v^2)^n\ {}_2F_1(-n,1-a-n;b;u^2) \\ &= 2^{a+b-c-2}u^{1-b}v^{-c}\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\int_0^{\infty}J_{a-1}(w)J_{b-1}(uw)\exp(-\frac{w}{2v})w^{c-a-b+1}dw, \end{aligned} \tag{5.19}$$

(see [19], formula (65.3.11)), valid if Re $(c) > 0$, Re $(v) > 0$, Re $(\frac{1}{2v} \pm iu) > 0$, Re $(a+b+c) > 0$.

Hence

$$T_1(\lambda) = -2x^{-1}\sqrt{\pi}(1+\frac{4}{\lambda^2})^{\frac{1}{2}}\int_0^{\infty}dwJ_0(w)\cosh(\frac{2w}{\lambda})\exp\left(-2wx^{-1}(1+\frac{4}{\lambda^2})\right) \tag{5.20}$$

and

$$T_2(\lambda) = -2x^{-1}\sqrt{\pi}(1+\frac{4}{\lambda^2})^{\frac{1}{2}}\int_0^{\infty}dwJ_0(w)\sinh(\frac{2w}{\lambda})\exp\left(-2wx^{-1}(1-\frac{4}{\lambda^2})\right). \tag{5.21}$$

By applying the following formula [16]

$$\int_0^\infty dw\ e^{-\beta w} J_0(w) = \frac{1}{\sqrt{1+\beta^2}}$$

(Laplace transform of the Bessel function) and expanding the cosinh and sinh functions into exponentials, one gets

$$T_1 = -\sqrt{\pi} x^{-1} (1+\frac{4}{\lambda^2})^{\frac12} \left[\left(1 + \left(-\frac{2}{\lambda} + \frac{2}{x}(1+\frac{4}{\lambda^2})\right)^2\right)^{-\frac12} + \left(1 + \left(\frac{2}{\lambda} + \frac{2}{x}(1+\frac{4}{\lambda^2})\right)^2\right)^{-\frac12} \right] \tag{5.22}$$

and

$$T_2 = -\sqrt{\pi} x^{-1} (1+\frac{4}{\lambda^2})^{\frac12} \left[\left(1 + \left(-\frac{2}{\lambda} + \frac{2}{x}(1+\frac{4}{\lambda^2})\right)^2\right)^{-\frac12} - \left(1 + \left(\frac{2}{\lambda} + \frac{2}{x}(1+\frac{4}{\lambda^2})\right)^2\right)^{-\frac12} \right] \tag{5.23}$$

Using

$$(1+\frac{4}{\lambda^2})^{\frac12} \left[1 + \left(-\frac{2}{\lambda} + \frac{2}{x}(1+\frac{4}{\lambda^2})\right)^2 \right]^{-\frac12} = \frac{\lambda}{\sqrt{(1+\frac{4}{x^2})\lambda^2 - \frac{8}{x}\lambda + \frac{16}{x^2}}}) \tag{5.24}$$

and the inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{1}{\sqrt{a\lambda^2 + 2b\lambda + c}}\right)(y) = \frac{1}{\sqrt{a}} e^{-\frac{b}{a}y} J_0(y\sqrt{\frac{c}{a} - \frac{b^2}{a^2}}) \tag{5.25}$$

one gets

$$h(\lambda) = \frac{-2}{\Xi x t^{\alpha^2}} \frac{1}{\sqrt{(1+\frac{4}{x^2})\lambda^2 - \frac{8}{x}\lambda + \frac{16}{x^2}}}$$

hence

$$\begin{aligned} C(t,r,\zeta) &= \frac{-2}{\Xi t^{\alpha^2} \sqrt{\frac{16\xi^2}{\Xi^2 t^2} + 4}} \exp\left(-\frac{16\xi/\Xi t}{16\xi^2/\Xi^2 t^2 + 4} \frac{\alpha^2 r^2}{\Xi t^2}\right) J_0\left(\frac{8\alpha^2 r^2/\Xi t^2}{16\xi^2/\Xi^2 t^2 + 4}\right) \\ &\sim_{\Xi\to 0} -\frac{1}{2\xi} t^{1-\alpha^2} \exp(-\frac{\alpha^2 r^2}{\xi t}) \end{aligned} \tag{5.26}$$

which is (up to a constant) exactly the same expression we got for the two-point function $\langle {}_{\alpha}\phi^{\Xi}\ {}_{-\alpha}\phi^{\Xi} \rangle$.

□

We did not manage to compute explicitly the three-point functions $\langle \psi^{\Xi}_{d_1} \psi^{\Xi}_{d_2} \psi^{\Xi}_{d_3} \rangle$ except in the simplest case $d_1 = d_2 = d_3 = -1$ (see the remark after Theorem 4.1 for the definition of ψ^{Ξ}_{-1}). One obtains:

Theorem 5.3

Let

$$\mathcal{C}^{\Xi} := \langle \psi^{\Xi}_{-1}(t_1,r_1,\zeta_1)\psi^{\Xi}_{-1}(t_2,r_2,\zeta_2)\psi^{\Xi}_{-1}(t_3,r_3,\zeta_3)\rangle \tag{5.27}$$

be the three point-function of the massive field

$$\psi^{\Xi}_{-1} := \sum_{j=0}^{\infty} \mathrm{i}^j \frac{\Xi^{-j-\frac{1}{2}}}{j!}\Phi^{(0)}_{2j,0}$$

defined in Theorem 5.1. Then

$$\mathcal{C}^{\Xi} \to_{\Xi\to 0} C.\left(\frac{t_{12}t_{23}t_{31}}{\xi_{12}\xi_{23}\xi_{31}}\right)^{\frac{1}{2}} \tag{5.28}$$

where $\xi_{ij} := \zeta_{ij} - \frac{r_{ij}^2}{2t_{ij}}$.

An inverse Laplace transform of $\mathcal{C}^{\Xi}$ *with respect to the* ζ*-parameters yields the following three-point function in terms of the masses :*

(5.29)

Proof.

Let $x_1 = \mathrm{i}\frac{\xi_{12}\xi_{13}t_{23}}{t_{12}t_{13}\xi_{23}\Xi}$, $x_2 = \mathrm{i}\frac{\xi_{23}\xi_{21}t_{31}}{t_{23}t_{21}\xi_{31}\Xi}$ and $x_3 = \mathrm{i}\frac{\xi_{31}\xi_{32}t_{12}}{t_{31}t_{32}\xi_{12}\Xi}$. Then Proposition 4.3.3 yields

$$\mathcal{C}^{\Xi} := \Xi^{-\frac{3}{2}} \sum_{j_1,j_2\geq 0} \frac{x_1^{j_1}x_2^{j_2}}{j_1!j_2!}(2j_1)!(2j_2)! \sum_{|j_1-j_2|\leq j_3\leq j_1+j_2} \frac{x_3^{j_3}}{j_3!}\frac{(2j_3)!}{(j_1+j_3-j_2)!(j_1+j_2-j_3)!(j_2+j_3-j_1)!}. \tag{5.30}$$

Write

$$\frac{1}{(j_1+j_2-j_3)!} = (-1)^{j_3-j_1-j_2}\lim_{\varepsilon\to 0}\varepsilon\Gamma(j_3-j_1-j_2+\varepsilon). \tag{5.31}$$

This form of the complement formula for the Gamma function is valid whatever the argument. Then

$$\mathcal{C}^{\Xi} = \Xi^{-\frac{3}{2}} \sum_{j_1,j_2\geq 0} \frac{x_1^{j_1}x_2^{j_2}}{j_1!j_2!}(2j_1)!(2j_2)!\ I_3(j_1,j_2;x_3) \tag{5.32}$$

where (by using also the duplication formula for the Gamma function)

$$\begin{aligned} I_3(j_1,j_2;x_3) &= (-1)^{j_1+j_2}\lim_{\varepsilon\to 0}\varepsilon \sum_{j_3=|j_1-j_2|}^{\infty}(-4x_3)^{j_3}\frac{\Gamma(j_3+\frac{1}{2})}{\sqrt{\pi}}\frac{\Gamma(j_3-j_1-j_2+\varepsilon)}{\Gamma(j_3+(j_1-j_2)+1)\Gamma(j_3+(j_2-j_1)+1)} \\ &= \frac{(-1)^{j_1+j_2}}{\sqrt{\pi}}(-4x_3)^{|j_1-j_2|}\sum_{j=0}^{\infty}\frac{(-4x_3)^j}{j!}\frac{\Gamma(j+|j_1-j_2|+\frac{1}{2})\Gamma(j-2\min(j_1,j_2)+\varepsilon)}{\Gamma(j+2|j_1-j_2|+1)} \\ &= \frac{(-1)^{j_1+j_2}}{\sqrt{\pi}}(-4x_3)^{|j_1-j_2|}\ {}_2\bar{F}_1(|j_1-j_2|+\frac{1}{2},\varepsilon-2\min(j_1,j_2);2|j_1-j_2|+1;-4x_3) \end{aligned} \tag{5.33}$$

The symbol ${}_2\bar{F}_1$ stands for Gauss' hypergeometric function apart from a different normalization, namely,

$$ {}_2\bar{F}_1(a,b,c;z) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\ {}_2F_1(a,b,c;z) = \sum_{n\geq 0} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}. \tag{5.34} $$

Now the well-known formula connecting the behaviour around 0 with the behaviour around infinity of the hypergeometric function, see [2] for instance, yields

$$ \begin{aligned} & {}_2\bar{F}_1(|j_1-j_2|+\frac{1}{2}, \varepsilon - 2\min(j_1,j_2); 2|j_1-j_2|+1; -4x_3) \\ & = \Gamma(\varepsilon - j_1 - j_2 - \frac{1}{2}) \left(\frac{1}{4x_3}\right)^{|j_1-j_2|+\frac{1}{2}}\ {}_2F_1(|j_1-j_2|+\frac{1}{2}, \frac{1}{2}-|j_1-j_2|; j_1+j_2+\frac{3}{2}; -\frac{1}{4x_3}) \\ & + \frac{\Gamma(\varepsilon - 2\min(j_1,j_2))\Gamma(j_1+j_2+\frac{1}{2})}{\Gamma(2\max(j_1,j_2)+1-\varepsilon)} \left(\frac{1}{4x_3}\right)^{-2\min(j_1,j_2)} \\ & {}_2F_1(-2\min(j_1,j_2), -2\max(j_1,j_2); \frac{1}{2}-j_1-j_2; -\frac{1}{4x_3}). \end{aligned} \tag{5.35} $$

In the limit $\varepsilon \to 0$, only the second term in the right-hand side has a pole, $\Gamma(\varepsilon - 2\min(j_1,j_2)) \sim_{\varepsilon\to 0} \frac{1}{\Gamma(1+2\min(j_1,j_2))}\frac{1}{\varepsilon}$, hence

$$ I_3 = \frac{(4x_3)^{j_1+j_2}}{\sqrt{\pi}} \frac{\Gamma(j_1+j_2+\frac{1}{2})}{\Gamma(1+2j_1)\Gamma(1+2j_2)}\ {}_2F_1(-2j_1,-2j_2;\frac{1}{2}-j_1-j_2;-\frac{1}{4x_3}) \tag{5.36} $$

and

$$ \mathcal{C}^{\Xi} = \frac{\Xi^{-\frac{3}{2}}}{\sqrt{\pi}} \sum_{j_1,j_2\geq 0} \frac{(4x_1x_3)^{j_1}(4x_2x_3)^{j_2}}{j_1!j_2!}\Gamma(j_1+j_2+\frac{1}{2})\ {}_2F_1(-2j_1,-2j_2;\frac{1}{2}-j_1-j_2;-\frac{1}{4x_3}). \tag{5.37} $$

Kummer's quadratic transformation formulas for the hypergeometric functions give (see [2], 15.3.22)

$$ {}_2F_1(-2j_1,-2j_2;\frac{1}{2}-j_1-j_2;-\frac{1}{4x_3}) = {}_2F_1(-j_1,-j_2;\frac{1}{2}-j_1-j_2;1-(1+\frac{1}{2x_3})^2). \tag{5.38} $$

Now for any β

$$ \sum_{j_1,j_2\geq 0} \frac{y_1^{j_1}}{j_1!}\frac{y_2^{j_2}}{j_2!}\Gamma(j_1+j_2+\beta) = \sum_{j_1\geq 0}\frac{y_1^{j_1}}{j_1!}\Gamma(j_1+\beta)(1-y_2)^{-j_1-\beta} = \Gamma(\beta)(1-y_1-y_2)^{-\beta}. \tag{5.39} $$

Applying this formula to each term in the series expansion of the above hypergeometric function yields

$$
\begin{aligned}
\mathcal{C}^{\Xi} &= \frac{\Xi^{-\frac{3}{2}}}{\sqrt{\pi}} \sum_{k\geq 0} \frac{\left(16x_3^2 x_1 x_2 ((1+\frac{1}{2x_3})^2-1)\right)^k}{k!} \sum_{l_1,l_2\geq 0} \frac{(4x_1x_3)^{l_1}(4x_2x_3)^{l_2}}{l_1! l_2!} \Gamma(l_1+l_2+(k+\frac{1}{2})) \\
&= \frac{\Xi^{-\frac{3}{2}}}{\sqrt{\pi}} \sum_{k\geq 0} \frac{\left(16x_3^2 x_1 x_2 ((1+\frac{1}{2x_3})^2-1)\right)^k}{k!} \Gamma(k+\frac{1}{2})(1-4x_1x_3-4x_2x_3)^{-k-\frac{1}{2}} \\
&= \Xi^{-\frac{3}{2}} \left(1-4x_3(x_1+x_2)+16x_3^2 x_1 x_2 (1-(1+\frac{1}{2x_3})^2)\right)^{-\frac{1}{2}} \\
&= \Xi^{-\frac{3}{2}} \left(1-4x_1x_2-4x_1x_3-4x_2x_3-16x_1x_2x_3\right)^{-\frac{1}{2}} \qquad (5.40)
\end{aligned}
$$

hence the limit when $\Xi \to 0$.

Casting this result into the usual coordinates $(\mathcal{M}, t, r)$ (i.e. taking an inverse Laplace transform) is a technical task, although some partial results are available through the usual results in conformal field theory (see Remark after Theorem A.3 in the Appendix).

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Appendix A. Two- and three-point functions for general coinduced fields

Let Φ_i, $i = 1, 2, \dots$ be $\tilde{\mathfrak{sch}}_1$-quasi-primary fields. The general problem we address in this Appendix is: what is the most general n-point function $\langle 0 \mid \Phi_1(t_1, r_1, \zeta_1) \dots \Phi_n(t_n, r_n, \zeta_n) \mid 0\rangle$ compatible with the constraints coming from symmetries ?

It has been solved in general (see [21, 23, 24] and [4], Appendix B) for scalar massive $\mathfrak{sch}_1$-quasi-primary fields, i.e. for fields such that the representation ρ of $\mathfrak{sv}_0 = \langle L_0\rangle \ltimes \langle Y_{\frac{1}{2}}, M_1\rangle$ is one-dimensional, namely $\rho(L_0) = -\lambda$ (where λ is the scaling exponent [3] of the field) and $\rho(Y_{\frac{1}{2}}) = 0$. Note that in the whole discussion, the value of $\rho(M_1)$ is irrelevant since M_1 does not belong to $\tilde{\mathfrak{sch}}_1$. Let us recall the results for two- and three-point functions. In the following proposition, we also consider the natural extension to scalar $\tilde{\mathfrak{sch}}_1$-quasi-primary fields:

Definition A.1

A scalar (λ, λ')-quasi-primary field *is a ρ-$\tilde{\mathfrak{sch}}_1$-quasi-primary field for which ρ is scalar, with $\rho(L_0) = -\lambda$, $\rho(N_0) = -\lambda'$ and $\rho(Y_{\frac{1}{2}}) = 0$.*

When speaking of two-point functions, we shall generally use the notation $t = t_1 - t_2$, $r = r_1 - r_2$, $\zeta = \zeta_1 - \zeta_2$. The notations $u = r^2/2t$, $\xi = \zeta - u = \zeta - r^2/2t$ will also show up frequently.

Note quite generally that the Bargmann superselection rule (due to the covariance under the phase shift M_0), as mentioned in the Introduction, forbids scalar massive fields $\Phi_1, \dots, \Phi_n$ with total mass $\mathcal{M} = \mathcal{M}_1 + \dots + \mathcal{M}_n$ different from 0 to have a non-zero n-point function.

Proposition A.1

(i) Let $\Phi_{1,2}$ be two scalar $\mathfrak{sch}_1$-quasi-primary fields with scaling exponents $\lambda_{1,2}$. Then their two-point function $\mathcal{C} = \langle \Phi_1(t_1, r_1, \zeta_1)\Phi_2(t_2, r_2, \zeta_2)\rangle$ vanishes except if $\lambda_1 = \lambda_2 =: \lambda$, in which case it is equal to

$$\mathcal{C} = t^{-2\lambda} f(\zeta - \frac{r^2}{2t}) \tag{A1}$$

where f is an arbitray scaling function. The inverse Laplace transform with respect to ζ gives (up to the multiplication by an arbitrary function of the mass) for fields with the same mass $\mathcal{M}$ a generalized heat kernel,

$$\mathcal{C} = g(\mathcal{M}) t^{-2\lambda} e^{-\mathcal{M}\frac{r^2}{2t}}. \tag{A2}$$

(ii) Suppose furthermore that $\Phi_{1,2}$ are $(\lambda_i, \lambda_i') - \tilde{\mathfrak{sch}}_1$-quasi-primary, $i = 1, 2$, with $\lambda_1 = \lambda_2 =: \lambda$ (otherwise $\mathcal{C}$ vanishes). Then the two-point function is fixed (up to a constant),

$$\mathcal{C} = t^{-2\lambda}(\zeta - \frac{r^2}{2t})^{-\frac{\lambda_1' + \lambda_2'}{2}}. \tag{A3}$$

[3]Physicists usually call 'scaling exponent' $2\lambda =: x$ instead of λ. For instance, the Schrödinger field defined in the Introduction has scaling exponent $\lambda = \frac{1}{4}$ or $x = \frac{1}{2}$ depending on the convention.

The inverse Laplace transform with respect to ζ of this function yields (up to a constant)

$$\mathcal{C} = \mathcal{M}^{\frac{\lambda_1'+\lambda_2'}{2}-1} t^{-2\lambda} e^{-\mathcal{M}\frac{r^2}{2t}}. \tag{A4}$$

(iii) Let $\Phi_{1,2,3}$ be three scalar $\mathfrak{sch}_1$-quasi-primary fields with scaling exponents $\lambda_{1,2,3}$. Then

$$\begin{aligned}\mathcal{C} &= \langle \Phi_1(t_1, r_1; \mathcal{M}_1)\Phi_2(t_2, r_2; \mathcal{M}_2)\Phi_3(t_3, r_3; \mathcal{M}_3)\rangle \\ &= \delta(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3) t_{12}^{\lambda_3-\lambda_1-\lambda_2} t_{23}^{\lambda_1-\lambda_2-\lambda_3} t_{31}^{\lambda_2-\lambda_3-\lambda_1} \\ &\quad \exp\left[-\frac{\mathcal{M}_1}{2}\frac{r_{13}^2}{t_{13}} - \frac{\mathcal{M}_2}{2}\frac{r_{23}^2}{t_{23}}\right] F\left(\frac{(r_{13}t_{23} - r_{23}t_{13})^2}{t_{12}t_{23}t_{13}}\right)\end{aligned} \tag{A5}$$

where F is an arbitray scaling function.

Note that the N_0-symmetry constraint is necessary to fix (up to a constant) even the two-point function in the variables (t, r, ζ), contrary to the more rigid case of conformal invariance which fixes two- and three-point functions. That is the reason why we consider fields that are covariant under the extended Schrödinger or Schrödinger-Virasoro algebra.

The non-scalar fields considered below are actually the most general possible for finite-dimensional representations ρ (see discussion before Definition 1.4), since one does not consider $\rho(M_1)$.

Theorem A.2. (two-point functions for non-scalar fields)

Consider two d-dimensional representations ρ_i, $i = 1, 2$ of $\widetilde{\mathfrak{sv}}_0 = (\langle L_0\rangle \oplus \langle N_0\rangle) \ltimes \langle Y_{\frac{1}{2}}, M_1\rangle$ indexed by the parameters $\lambda_{1,2}, \lambda'_{1,2}, \alpha_{1,2}$ such that

$$\rho_i(L_0) = -\lambda_i \mathrm{Id} + \frac{1}{2}\sum_{\mu=0}^{d-1} \mu E_{\mu,\mu}, \ \rho_i(N_0) = -\lambda_i' \mathrm{Id} - \sum_{\mu=0}^{d-1} \mu E_{\mu,\mu} \tag{A6}$$

$$\rho_i(Y_{\frac{1}{2}}) = \alpha_i \sum_{\mu=0}^{d-2} E_{\mu,\mu+1} \tag{A7}$$

Let $\Phi_i = (\Phi_i^{\mu}(t, r, \zeta))_{\mu=0,\dots,d-1}$, be ρ_i-quasiprimary fields, $i = 1, 2$. Then their two-point functions $\mathcal{C}^{\mu,\nu} = \langle \Phi_1^{(\mu)}(t_1, r_1, \zeta_1)\Phi_2^{\nu}(t_2, r_2, \zeta_2)\rangle$ vanish unless $2(\lambda_1 - \lambda_2)$ is an integer. Supposing that $\lambda_1 = \lambda_2$, they may be expressed in terms of d arbitrary parameters $c_0, \dots, c_{d-1}$ as follows:

$$\mathcal{C}^{\mu,\nu} = t^{-\lambda+\frac{\mu+\nu}{2}} \sum_{\delta=\max(\mu,\nu)}^{d-1} c_\delta \frac{\alpha_1^{\delta-\mu}\alpha_2^{\delta-\nu}}{(\delta-\mu)!(\delta-\nu)!} (\frac{r^2}{t})^{\delta-\frac{\mu+\nu}{2}} (\zeta - \frac{r^2}{2t})^{-(\frac{\lambda'}{2}+\delta)} \tag{A8}$$

where $\lambda = \lambda_1 + \lambda_2$ ($= 2\lambda_1$ here) and $\lambda' = \lambda_1' + \lambda_2'$.

Remark. The assumption $\lambda_1 = \lambda_2$ is no restriction of generality: supposing that $\Delta := 2(\lambda_1 - \lambda_2)$ is (say) a positive integer implies a shift in the index μ with respect to ν in formula

(A8) and restricts the number of unknown constants. By working through the proof of this Theorem, it is possible to see that the $\mathcal{C}^{\mu,\nu}$ vanish for $\max(\mu,\nu) > d-1-\Delta$ (hence all of them vanish if $\Delta \geq d$) and that the other components depend on $d-\Delta$ coefficients.

Proof.

First of all, invariance under translations $\rho(L_{-1}) = -\partial_t, \rho(Y_{-\frac{1}{2}}) = -\partial_r, \rho(M_0) = -\partial_\zeta$ implies that $\mathcal{C}^{\mu,\nu}$ is a function of the differences of coordinates t, r, ζ only. Set $\lambda = \lambda_1 + \lambda_2, \lambda' = \lambda'_1 + \lambda'_2$; we do not assume anything on $\lambda_1 - \lambda_2$ for the moment. Let us write the action of $\rho(L_0) = -t\partial_t - \frac{1}{2}r\partial_r + \rho_1(L_0) \otimes \mathrm{Id} + \mathrm{Id} \otimes \rho_2(L_0)$ on $\mathcal{C}^{\mu,\nu}$. By definition, one has

$$(t\partial_t + \frac{1}{2}r\partial_r)\mathcal{C}^{\mu,\nu} = (-\lambda + \frac{\mu+\nu}{2})\mathcal{C}^{\mu,\nu} \tag{A9}$$

hence

$$\mathcal{C}^{\mu,\nu} = f^{\mu,\nu}(\zeta, u)t^{-\lambda+\frac{\mu+\nu}{2}} \tag{A10}$$

where $u := \frac{r^2}{2t}$. Then invariance under $\rho(Y_{\frac{1}{2}}) = -t\partial_r - r\partial_\zeta + \rho_1(Y_{\frac{1}{2}}) \otimes \mathrm{Id} + \mathrm{Id} \otimes \rho_2(Y_{\frac{1}{2}})$ implies

$$\begin{aligned} (t\partial_r + r\partial_\zeta)\mathcal{C}^{\mu,\nu} &= \rho_1(Y_{\frac{1}{2}})^\mu_l \mathcal{C}^{l,\nu} + \rho_2(Y_{\frac{1}{2}})^\nu_l \mathcal{C}^{\mu,l} && \text{(A11)} \\ &= \alpha_1 \mathcal{C}^{\mu+1,\nu} + \alpha_2 \mathcal{C}^{\mu,\nu+1} && \text{(A12)} \end{aligned}$$

hence

$$\sqrt{2u}(\partial_u + \partial_\zeta)f^{\mu,\nu}(\zeta,u) = \alpha_1 f^{\mu+1,\nu}(\zeta,u) + \alpha_2 f^{\mu,\nu+1}(\zeta,u). \tag{A13}$$

The solutions of the homogeneous equation associated with (A13) are the functions of $\xi := \zeta - u$. In the new set of coordinates (ξ, u), equation (A13) reads as

$$\sqrt{2u}\partial_u f^{\mu,\nu}(\xi,u) = \alpha_1 f^{\mu+1,\nu}(\xi,u) + \alpha_2 f^{\mu,\nu+1}(\xi,u). \tag{A14}$$

These coupled equations are easily solved. First, $\partial_u f^{d-1,d-1}(\xi,u) = 0$, hence $g^{d-1,d-1} := f^{d-1,d-1}$ is a function of ξ only. It is clear by decreasing induction on μ and ν that the general solution may be expressed in terms of d^2 undetermined functions $g^{\mu,\nu}(\xi)$, $0 \leq \mu,\nu \leq d-1$, through the relations

$$f^{\mu,\nu}(\xi,u) = \alpha_1 \int \frac{f^{\mu+1,\nu}(\xi,u)}{\sqrt{2u}}\, du \;+\; \alpha_2 \int \frac{f^{\mu,\nu+1}(\xi,u)}{\sqrt{2u}}\, du \;+ g^{\mu,\nu}(\xi) \tag{A15}$$

Let us now use covariance under $\rho(N_0) \equiv -r\partial_r - 2\zeta\partial_\zeta + \rho_1(N_0) \otimes \mathrm{Id} + \mathrm{Id} \otimes \rho_2(N_0)$: one gets

$$2(u\partial_u + \xi\partial_\xi)f^{\mu,\nu}(\xi,u) = -(\lambda' + \mu + \nu)f^{\mu,\nu}(\xi,u)$$

hence

$$f^{\mu,\nu}(\xi,u) := \xi^{-\frac{\lambda'+\mu+\nu}{2}} f_0^{\mu,\nu}(\frac{\xi}{u});$$

this implies immediately $g^{d-1,d-1}(\xi) = \xi^{-(\frac{\lambda'}{2}+d-1)}$ up to a multiplicative constant. Then $\int \frac{\xi^{-\frac{\lambda'+\mu+\nu+1}{2}} f_0^{\mu+1,\nu}(\frac{\xi}{u})}{\sqrt{2u}}\, du$ is homogeneous of degree $-(\lambda'+\mu+\nu)$ with respect to $2(u\partial_u + \xi\partial_\xi)$, hence the defining relations (A15) are compatible with covariance under $\rho(N_0)$, provided that $g^{\mu,\nu}(\xi)=\xi^{-\frac{\lambda'+\mu+\nu}{2}}$ up to a constant.

Covariance under $\rho(L_1)=-\sum_{i=1}^{2}(t_i^2\partial_{t_i}+t_ir_i\partial_{r_i}+\frac12 r_i^2\partial_{\zeta_i})+(2t_1\rho_1(L_0)+r_1\rho_1(Y_{\frac12}))\otimes\mathrm{Id}+\mathrm{Id}\otimes(2t_2\rho_2(L_0)+r_2\rho_2(Y_{\frac12}))$ is seen to be equivalent (after some easy computations) to the coupled equations

$$(t^2\partial_t+tr\partial_r+\frac12 r^2\partial_\zeta)\mathcal{C}^{\mu,\nu}(t,r,\zeta)=2t\rho_1(L_0)_l^\mu\mathcal{C}^{l,\nu}+r\rho_1(Y_{\frac12})_l^\mu\mathcal{C}^{l,\nu} \tag{A16}$$

Using the above Ansatz (A10) yields

$$\left[(\frac{\nu-\mu}{2}+\lambda_1-\lambda_2)+u\partial_u\right]f^{\mu,\nu}(\xi,u)=\alpha_1\sqrt{2u}f^{\mu+1,\nu}(\xi,u). \tag{A17}$$

Applying this relation to $f^{d-1,d-1}=c_{d-1}.\xi^{-(\frac{\lambda'}{2}+d-1)}$ gives $c_{d-1}=0$ unless $\lambda_1=\lambda_2$, which we assume from now on. Let us compute $f^{d-2,d-1}$ and $f^{d-1,d-2}$ before we cope with the general case; one may set $c_{d-1}=1$ for the moment. Then $f^{d-2,d-1}(\xi,u)=(\alpha_1\sqrt{2u}+c\sqrt{\xi})\xi^{-(\frac{\lambda'}{2}+d-1)}$ must satisfy $(\frac12+u\partial_u)f^{d-2,d-1}(\xi,u)=\alpha_1\sqrt{2u}\xi^{-(\frac{\lambda'}{2}+d-1)}$. The function $\mu_1\sqrt{2u}\xi^{-(\frac{\lambda'}{2}+d-1)}$ is indeed a solution of this equation, and any other solution will be a linear combination of this with some function $u^{-\frac12}h(\xi)$, hence $c=0$ and $f^{d-2,d-1}$ is totally determined by $f^{d-1,d-1}$. On the other hand, $f^{d-1,d-2}(\xi,u)=(\alpha_2\sqrt{2u}+c\sqrt{\xi})\xi^{-(\frac{\lambda'}{2}+d-1)}$ must satisfy $(-\frac12+u\partial_u)f^{d-1,d-2}(\xi,u)=\alpha_1\sqrt{2u}\xi^{-(\frac{\lambda'}{2}+d-1)}$. The general solution of this equation is

$$f^{d-1,d-2}(\xi,u)=\sqrt{2u}h(\xi)+\alpha_1\sqrt{2u}(\ln u)\xi^{-(\frac{\lambda'}{2}+d-1)}. \tag{A18}$$

Both Ansätze are clearly compatible if and only if $c=0$.

Let us now prove the general case by decreasing induction on $\max(\mu,\nu)$. Assume formula (A8) of the Theorem has been proved for $\max(\mu,\nu)>M$. Then formula (A14) gives $f^{M,M}$ up to an undetermined function $g^{M,M}(\xi)$ which is proportional to $\xi^{-\lambda'-M}$ due to covariance with respect to $\rho(N_0)$; it is compatible with formula (A8) and formula (A17). One may now go down or left along a line or a row: if for instance all $f^{M-i,M}$, $i<I$ have been found to agree with (A8), then formula (A14) again gives $f^{M+I,M}$, in accordance with (A8), up to an undetermined function $g^{M+I,M}(\xi)$. Compatibility with covariance under $\rho(L_1)$ (formula (A17)) gives $(\frac{I}{2}+u\partial_u)g^{M+I,M}(\xi)=0$, hence $g^{M+I,M}=0$ as soon as $I>0$. □

Let us now turn to the computation of the general three-point function for *scalar* quasi-primary fields.

Theorem A.3

Let $\Phi_i,\ i=1,2,3$ be (λ_i,λ'_i)-quasi-primary fields. Then their general three-point function $\mathcal{C}(t_i,r_i,\zeta_i)=\langle\Phi_1(t_1,r_1,\zeta_1)\Phi_2(t_2,r_2,\zeta_2)\Phi_3(t_3,r_3,\zeta_3)\rangle$ may be written as

$$\mathcal{C}=t_{12}^{-\lambda_1-\lambda_2+\lambda_3}t_{23}^{-\lambda_2-\lambda_3+\lambda_1}t_{13}^{-\lambda_1-\lambda_3+\lambda_2}\xi_{12}^{\frac{1}{2}(-\lambda'_1-\lambda'_2+\lambda'_3)}\xi_{23}^{\frac{1}{2}(-\lambda'_2-\lambda'_3+\lambda'_1)}\xi_{13}^{\frac{1}{2}(-\lambda'_1-\lambda'_3+\lambda'_2)}\ .\ F(\xi_{12},\xi_{13},\xi_{23}) \tag{A19}$$

where F is any function of $\xi_{12}:=\zeta_{12}-r_{12}^2/2t_{12},\xi_{13}:=\zeta_{13}-r_{13}^2/2t_{13},\xi_{23}:=\zeta_{23}-r_{23}^2/2t_{23}$ which is homogeneous of degree zero, i.e.

$$(\xi_{12}\partial_{\xi_{12}}+\xi_{13}\partial_{\xi_{13}}+\xi_{23}\partial_{\xi_{23}})F=0. \tag{A20}$$

Remark.

In the case $\lambda'_i=2\lambda_i$, F constant, one retrieves the standard result for the three-point function in 3d conformal field theory, with a Lorentzian pseudo-distance given (in light-cone coordinates) by $d^2((t_i,r_i,\zeta_i),(t_j,r_j,\zeta_j))=t_{ij}\zeta_{ij}-r_{ij}^2/2$. The explicit connection between the n-point functions in the Schrödinger/conformal cases has been made in [23] and in [25]. In the last reference, an explicit computation of the three-point function in the dual *mass* coordinates $\mathcal{M}_i,\ i=1,2,3$ is given – assuming covariance under the whole conformal group – in the case when $\lambda_1=\lambda_2$, $\mathcal{M}_1=\mathcal{M}_2,\ r_1=r_2$. The general result is a combination of two confluent hypergeometric functions. Note that in the present case, $\lambda'_i\neq 2\lambda_i$ in general, but this leads simply to a different time-dependent pre-factor. The general conformally invariant solution in coordinates $\mathcal{M},t,r$ (after removing the restriction on $\lambda_1,\mathcal{M}_1,r_1$) might be given by a generalized hypergeometric function of two variables, see [28].

Proof.

Set $r=r_1-r_3$, $r'=r_2-r_3$ and similarly for t,t' and ζ,ζ'. The covariance under the action of $L_0,Y_{\frac{1}{2}},N_0$ and L_1 yields respectively

$$(\sum_i t_i\partial_{t_i}+\frac{1}{2}\sum_i r_i\partial_{r_i}+\lambda)\mathcal{C}=0, \tag{A21}$$

$$(\sum_i t_i\partial_{r_i}+r_i\partial_{\zeta_i})\mathcal{C}=0, \tag{A22}$$

$$(\sum_i r_i\partial_{r_i}+2\zeta_i\partial_{\zeta_i}+\lambda')\mathcal{C}=0, \tag{A23}$$

$$(\sum_i t_i^2\partial_{t_i}+t_ir_i\partial_{r_i}+\frac{1}{2}r_i^2\partial_{\zeta_i}+2\lambda_it_i)\mathcal{C}=0 \tag{A24}$$

with $\rho_i(L_0)=-\lambda_i,\rho_i(N_0)=-\lambda'_i$, and $\lambda:=\sum_i\lambda_i,\lambda'=\sum_i\lambda'_i$. The function

$$\mathcal{C}=t_{12}^{-\lambda_1-\lambda_2+\lambda_3}t_{23}^{-\lambda_2-\lambda_3+\lambda_1}t_{13}^{-\lambda_1-\lambda_3+\lambda_2}\xi_{12}^{\frac{1}{2}(-\lambda'_1-\lambda'_2+\lambda'_3)}\xi_{23}^{\frac{1}{2}(-\lambda'_2-\lambda'_3+\lambda'_1)}\xi_{13}^{\frac{1}{2}(-\lambda'_1-\lambda'_3+\lambda'_2)} \tag{A25}$$

is a particular solution of this system of equations. Hence the general solution is given by $\mathcal{C}_0(t_i,r_i,\zeta_i)\mathcal{C}(t_i,r_i,\zeta_i)$, where $\mathcal{C}_0$ is any solution of the homogeneous system obtained by setting $\lambda_i,\lambda'_i=0$. By taking an appropriate linear combination of (A21) and (A23), one gets

$$(E+E')\mathcal{C}_0=0, \tag{A26}$$

where $E = \sum_i t_i \partial_{t_i} + r_i \partial_{r_i} + \zeta_i \partial_{\zeta_i}$ is the Euler operator in the variables t_i, r_i, ζ_i and similarly for E'. An appropriate linear combination of (A21), (A22) and (A24) gives

$$(t^2 \partial_t + t'^2 \partial_{t'} + tr\partial_r + t'r'\partial_{r'} + \frac{1}{2} r^2 \partial_\zeta + \frac{1}{2} r'^2 \partial_{\zeta'}) \mathcal{C}_0 = 0 \tag{A27}$$

Equation (A22) is equivalent to saying that

$$\mathcal{C}_0 := \mathcal{C}_1(t, t', \rho, \xi, \xi') \tag{A28}$$

where

$$\rho = \frac{r}{t} - \frac{r'}{t'}, \xi = \zeta - r^2/2t, \xi' = \zeta' - r'^2/2t'. \tag{A29}$$

Equation (A21) may then be rewritten

$$(t\partial_t + t'\partial_{t'} - \frac{1}{2}\rho\partial_\rho)\mathcal{C}_1 = 0 \tag{A30}$$

hence

$$\mathcal{C}_1 := \mathcal{C}_2(\tau, \tau', \xi, \xi') \tag{A31}$$

where

$$\tau = \rho^2 t, \tau' = \rho^2 t'. \tag{A32}$$

Equation (A27) reads now simply

$$(\tau^2 \partial_\tau + \tau'^2 \partial_{\tau'})\mathcal{C}_2 = 0, \tag{A33}$$

with general solution

$$\mathcal{C}_2 := \mathcal{C}_3(\frac{1}{\tau} - \frac{1}{\tau'}, \xi, \xi'). \tag{A34}$$

Set $\xi_{ij} := \zeta_{ij} - r_{ij}^2/2t_{ij}$. Then

$$\frac{1}{2}(\frac{1}{\tau} - \frac{1}{\tau'}) = \xi_{12} + \xi_{23} + \xi_{31} \tag{A35}$$

Hence the final result by taking into account the equation $(E + E')\mathcal{C}_0 = 0$. □

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