Numerical approximation for an impulse control problem arising in portfolio selection under liquidity risk*

Vathana LY VATH† Mohamed MNIF‡

29th November 2006

Abstract

We investigate numerical aspects of a portfolio selection problem studied in [10], in which we suggest a model of liquidity risk and price impact and formulate the problem as an impulse control problem under state constraint. We show that our impulse control problem could be reduced to an iterative sequence of optimal stopping problems. Given the dimension of our problem and the complexity of its solvency region, we use Monte Carlo methods instead of finite difference methods to calculate the value function, the transaction and no-transaction regions. We provide a numerical approximation algorithm as well as numerical results for the optimal transaction strategy.

Keywords: Impulse control problem, Optimal transaction strategy, Monte Carlo method, Malliavin calculus.

JEL Classification : G11.

MSC Classification (2000) : 93E20, 65C05, 91B28, 60H30.

*We would like to thank Huyên Pham for useful discussions.

†Laboratoire de Probabilités et Modèles Aléatoires, CNRS, UMR 7599, Université Paris 7, France, lyvath@math.jussieu.fr

‡Laboratoire de Modélisation Mathématique et Numérique dans les Sciences de l'Ingénieur, Ecole Nationale d'Ingénieurs de Tunis, mohamed.mnif@enit.rnu.tn

1 Introduction

In this article, we investigate numerical aspects of a portfolio selection problem studied in [10], in which we suggest a model of liquidity risk and price impact. Transactions are allowed only in discrete times and incur some fixed costs. Under the impact of liquidity risk, prices are pushed up when buying stock shares and moved down when selling shares. The investor maximizes his expected utility of terminal liquidation wealth, under a solvency constraint. This problem is formulated as an impulse control problem under state constraint. In [10], we characterize the value function as the unique constrained viscosity solution to the associated Hamilton-Jacobi-Bellman Quasi-Variational Inequality (HJBQVI). Our associated HJBQVI has, in addition to time variable, three variables : x, y, and p, respectively the cash holding, the stock holding, and the stock share price.

Hamilton-Jacobi-Bellman equations are usually solved by using numerical methods based on finite difference methods. The Howard algorithm, which consists in computing two sequences: the optimal strategy and the value function, is known to be efficient for the resolution of these types of equation. From Barles and Souganidis [1], we know that a monotone, stable and consistant scheme insures the convergence of the algorithm to the unique viscosity solution of the HJBQVI. This algorithm has a complexity in $O(N^n)$ where N is the number of points of the grid in one axis and n the dimension of the equation. Chancelier, Øksendal, and Sulem [4] used the Howard algorithm to solve numerically a bi-dimensional HJBQVI related to a problem of optimal consumption and portfolio with both fixed and proportional transaction costs. They solved the problem in a bounded domain and they assumed zero Neumann boundary conditions on the localized boundary. The disadvantage of the finite difference method is its suitability to only solve HJB equations when the solvency region has a simple shape such $\mathbb{R}_+^n$ or when its boundaries are straight. In the latter paper, the solvency region presents some corners. The authors omitted the points of the domain where either the number of shares or the amount of money in the portfolio is non-positive.

Korn [8] studied the problem of portfolio optimization with strictly positive transaction costs and impulse control. He presented a sequence of optimal stopping problems where the reward function is expressed in terms of the impulse operator. He proved the convergence of the sequence of optimal stopping problems towards the value function of the initial problem. Chancelier, Øksendal and Sulem [4] suggested an iterative method to solve the impulse control problem. They considered an auxiliary value function where the transactions number is bounded by a positive number.

In this article, we prove that both iterative methods coincide. We study numerically our problem by reducing the impulse control problem to an iterative sequence of optimal stopping problems. Then, we introduce a numerical approximation algorithm for every optimal stopping problem based on ideas of Monte Carlo numerical procedure which requires the computation of many conditional expectations. Several methods can be used for the valuation of these regression functions. We choose the Malliavin Calculus based Method suggested in [7] and then developed in [2]. Our numerical approach named value-iteration algorithm could be adapted to every shape of the solvency region and we don't need to assume some artificial boundary conditions.

The paper is organized as follows. We first show that the value function could be obtained as the limit of an iterative procedure when each step is an optimal stopping problem and the reward function is related to the impulse operator. We then provide a numerical method based on Malliavin calculus and give numerical results for the optimal transaction strategy.

2 Problem formulation

This section presents the details of the model. Let $(\Omega, \mathcal{F}, I\!\!P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ supporting an one-dimensional Brownian motion W on a finite horizon $[0,T]$, $T<\infty$. We consider a continuous time financial market model consisting of a money market account yielding a constant interest rate $r \geq 0$ and a risky asset (or stock) of price process $P = (P_t)$. We denote by X_t the amount of money (or cash holdings) and by Y_t the number of shares in the stock held by the investor at time t.

Liquidity constraints. We assume that the investor can only trade discretely on $[0,T)$. This is modelled through an impulse control strategy $\alpha = (\tau_n, \zeta_n)_{n\geq 1}$: $\tau_1 \leq \ldots \tau_n \leq \ldots < T$ are stopping times representing the intervention times of the investor and ζ_n, $n \geq 1$, are $\mathcal{F}_{\tau_n}$-measurable random variables valued in $I\!\!R$ and giving the number of stock purchased if $\zeta_n \geq 0$ or sold if $\zeta_n < 0$ at these times. The sequence (τ_n, ζ_n) may be a priori finite or infinite. The dynamics of Y is then given by :

$$Y_s = Y_{\tau_n}, \qquad \tau_n \leq s < \tau_{n+1} \tag{2.1}$$

$$Y_{\tau_{n+1}} = Y_{\tau_n} + \zeta_{n+1}. \tag{2.2}$$

Notice that we do not allow trade at the terminal date T, which is the liquidation date.

Price impact. The large investor affects the price of the risky stock P by his purchases and sales : the stock price goes up when the trader buys and goes down when he sells and the impact is increasing with the size of the order. We then introduce a price impact positive function $Q(\zeta, p)$ which indicates the post-trade price when the large investor trades a position of ζ shares of stock at a pre-trade price p. In absence of price impact, we have $Q(\zeta,p) = p$. Here, we have $Q(0,p) = p$ meaning that no trading incurs no impact and Q is nondecreasing in ζ with $Q(\zeta,p) \geq$ (resp. $\leq$) p for $\zeta \geq$ (resp. $\leq$) 0. Actually, in the rest of the paper, we consider a price impact function in the form

$$Q(\zeta,p) = pe^{\lambda\zeta}, \qquad \text{where } \lambda > 0. \tag{2.3}$$

The proportionality factor $e^{\lambda\zeta}$ represents the price increase (resp. discount) due to the ζ shares bought (resp. sold). The positive constant λ measures the fact that larger trades generate larger quantity impact, everything else constant. This form of price impact function is consistent with both the asymmetric information and inventory motives in the market microstructure literature (see [9]).

We then model the dynamics of the price impact as follows. In the absence of trading, the price process is governed by

$$dP_s = P_s(bds + \sigma dW_s), \qquad \tau_n \leq s < \tau_{n+1}, \tag{2.4}$$

where b, σ are constants with $\sigma > 0$. When a discrete trading $\Delta Y_s := Y_s - Y_{s^-} = \zeta_{n+1}$ occurs at time $s = \tau_{n+1}$, the price jumps to $P_s = Q(\Delta Y_s, P_{s^-})$, i.e.

$$P_{\tau_{n+1}} \;=\; Q(\zeta_{n+1}, P_{\tau_{n+1}^-}). \tag{2.5}$$

Notice that with this modelling of price impact, the price process P is always strictly positive, i.e. valued in $\mathbb{R}_+^* = (0, \infty)$.

Cash holdings. We denote by $\theta(\zeta, p)$ the cost function, which indicates the amount for a (large) investor to buy or sell ζ shares of stock when the pre-trade price is p :

$$\theta(\zeta, p) \;=\; \zeta Q(\zeta, p).$$

In absence of transaction, the process X grows deterministically at exponential rate r :

$$dX_s \;=\; rX_s ds, \qquad \tau_n \leq s < \tau_{n+1}. \tag{2.6}$$

When a discrete trading $\Delta Y_s = \zeta_{n+1}$ occurs at time $s = \tau_{n+1}$ with pretrade price $P_{s^-} = P_{\tau_{n+1}^-}$, we assume that in addition to the amount of stocks $\theta(\Delta Y_s, P_{s^-}) = \theta(\zeta_{n+1}, P_{\tau_{n+1}^-})$, there is a fixed cost $k > 0$ to be paid. This results in a variation of cash holdings by $\Delta X_s := X_s - X_{s^-} = -\theta(\Delta Y_s, P_{s^-}) - k$, i.e.

$$X_{\tau_{n+1}} \;=\; X_{\tau_{n+1}^-} - \theta(\zeta_{n+1}, P_{\tau_{n+1}^-}) - k. \tag{2.7}$$

The assumption that any trading incurs a fixed cost of money to be paid will rule out continuous trading, i.e. optimally, the sequence (τ_n, ζ_n) is not degenerate in the sense that for all n, $\tau_n < \tau_{n+1}$ and $\zeta_n \neq 0$ a.s. A similar modelling of fixed transaction costs is considered in [12] and [8].

Liquidation value and solvency constraint. The solvency constraint is a key issue in portfolio/consumption choice problem. The point is to define in an economically meaningful way what is the portfolio value of a position in cash and stocks. In our context, we introduce the liquidation function $\ell(y, p)$ representing the value that an investor would obtained by liquidating immediately his stock position y by a single block trade, when the pre-trade price is p. It is given by :

$$\ell(y, p) \;=\; -\theta(-y, p).$$

If the agent has the amount x in the bank account, the number of shares y of stocks at the pre-trade price p, i.e. a state value $z = (x, y, p)$, his net wealth or liquidation value is given by :

$$L(z) \;=\; \max[L_0(z), L_1(z)] 1_{y \geq 0} + L_0(z) 1_{y<0}, \tag{2.8}$$

where

$$L_0(z) \;=\; x + \ell(y, p) - k, \qquad L_1(z) \;=\; x.$$

The interpretation is the following. $L_0(z)$ corresponds to the net wealth of the agent when he liquidates his position in stock. Moreover, if he has a long position in stock, i.e. $y \geq 0$, he can also choose to bin his stock shares, by keeping only his cash amount, which leads to a net wealth $L_1(z)$. This last possibility may be advantageous, i.e. $L_1(z) \geq L_0(z)$, due to the fixed cost k. Hence, globally, his net wealth is given by (2.8). In the absence of liquidity risk, i.e $\lambda = 0$, and fixed transaction cost, i.e. $k = 0$, we recover the usual definition of wealth $L(z) = x + py$. Our definition (2.8) of liquidation value is also consistent with the one in transaction cost models where portfolio value is measured after stock position is liquidated and rebalanced in cash, see e.g. [5] and [13]. Another alternative would be to measure the portfolio value separately in cash and stock as in [6] for transaction cost models. This study would lead to multidimensional utility functions and is left for future research.

We then naturally introduce the liquidation solvency region (see Figure 1) :

$$\mathcal{S} \;=\; \big\{z = (x,y,p) \in I\!R \times I\!R \times I\!R_+^* \;: L(z) > 0\big\},$$

and we denote its boundary and its closure by

$$\partial\mathcal{S} \;=\; \big\{z = (x,y,p) \in I\!R \times I\!R \times I\!R_+^* \;: L(z) = 0\big\} \quad \text{and} \quad \bar{\mathcal{S}} \;=\; \mathcal{S} \cup \partial\mathcal{S}.$$

The boundary of the solvency region may then be explicited as follows (see Figures 2 and 3) :

$$\partial\mathcal{S} \;=\; \partial_\ell^-\mathcal{S} \cup \partial^y\mathcal{S} \cup \partial_0^x\mathcal{S} \cup \partial_1^x\mathcal{S} \cup \partial_2^x\mathcal{S} \cup \partial_\ell^+\mathcal{S},$$

where

$$\begin{aligned}
\partial_\ell^-\mathcal{S} &= \big\{z = (x,y,p) \in I\!R \times I\!R \times I\!R_+^* \;: x + \ell(y,p) = k,\; y \leq 0\big\}\\
\partial^y\mathcal{S} &= \big\{z = (x,y,p) \in I\!R \times I\!R \times I\!R_+^* \;: 0 \leq x < k,\; y = 0\big\}\\
\partial_0^x\mathcal{S} &= \big\{z = (x,y,p) \in I\!R \times I\!R \times I\!R_+^* \;: x = 0,\; y > 0,\; p < k\lambda e\big\}\\
\partial_1^x\mathcal{S} &= \big\{z = (x,y,p) \in I\!R \times I\!R \times I\!R_+^* \;: x = 0,\; 0 < y < y_1(p)),\; p \geq k\lambda e\big\}\\
\partial_2^x\mathcal{S} &= \big\{z = (x,y,p) \in I\!R \times I\!R \times I\!R_+^* \;: x = 0,\; y > y_2(p),\; p \geq k\lambda e\big\}\\
\partial_\ell^+\mathcal{S} &= \big\{z = (x,y,p) \in I\!R \times I\!R \times I\!R_+^* \;: x + \ell(y,p) = k,\; y_1(p) \leq y \leq y_2(p),\; p \geq k\lambda e\big\}.
\end{aligned}$$

In the sequel, we also introduce the corner lines in $\partial\mathcal{S}$:

$$\begin{aligned}
D_0 \;&=\; \{(0,0)\} \times I\!R_+^* \;\subset\; \partial^y\mathcal{S}, \qquad & D_k \;&=\; \{(k,0)\} \times I\!R_+^* \;\subset\; \partial_\ell^-\mathcal{S}\\
C_1 \;&=\; \{(0,y_1(p),p) \;: p \in I\!R_+^*\} \;\subset\; \partial_\ell^+\mathcal{S}, \qquad & C_2 \;&=\; \{(0,y_2(p),p) \;: p \in I\!R_+^*\} \;\subset\; \partial_\ell^+\mathcal{S}.
\end{aligned}$$

Admissible controls. Given $t \in [0,T]$, $z = (x,y,p) \in \bar{\mathcal{S}}$ and an initial state $Z_{t^-} = z$, we say that the impulse control strategy $\alpha = (\tau_n, \zeta_n)_{n\geq 1}$ is admissible if the process $Z_s = (X_s, Y_s, P_s)$ given by (2.1)-(2.2)-(2.4)-(2.5)-(2.6)-(2.7) (with the convention $\tau_0 = t$) lies in $\bar{\mathcal{S}}$ for all $s \in [t,T]$. We denote by $\mathcal{A}(t,z)$ the set of all such policies. We shall see later that this set of admissible controls is nonempty for all $(t,z) \in [0,T] \times \bar{\mathcal{S}}$. In the sequel, for $t \in [0,T]$, $z = (x,y,p) \in \bar{\mathcal{S}}$, we also denote $Z_s^{0,t,z} = (X_s^{0,t,x}, y, P_s^{0,t,p})$, $t \leq s \leq T$, the state process when no transaction (i.e. no impulse control) is applied between t and T, i.e. the solution to :

$$dZ_s^0 \;=\; \begin{pmatrix} rX_s^0 \\ 0 \\ bP_s^0 \end{pmatrix} ds + \begin{pmatrix} 0 \\ 0 \\ \sigma P_s^0 \end{pmatrix} dW_s, \qquad (2.9)$$

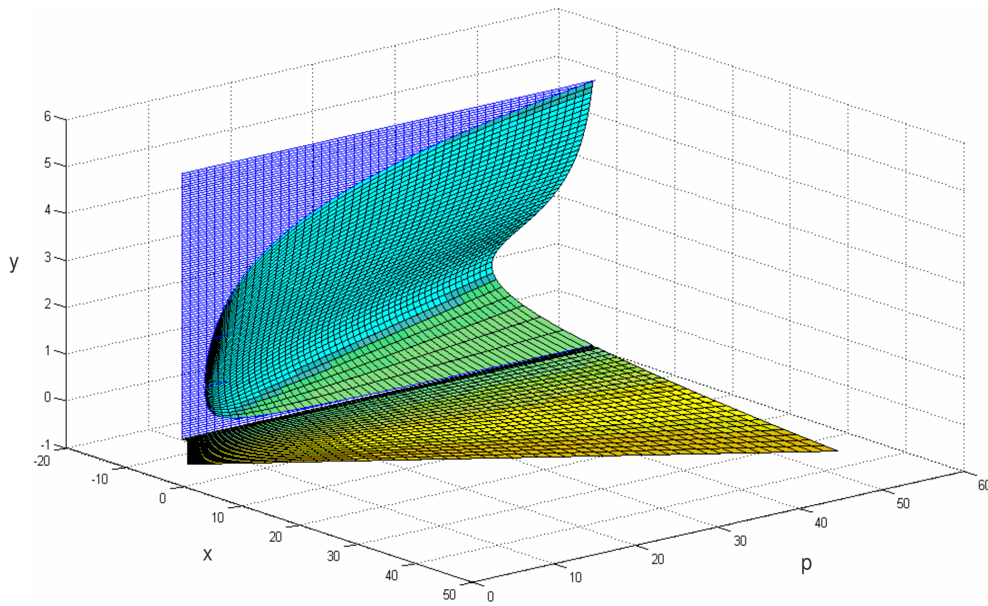

Figure 1: The solvency region when k = 1, λ = 1

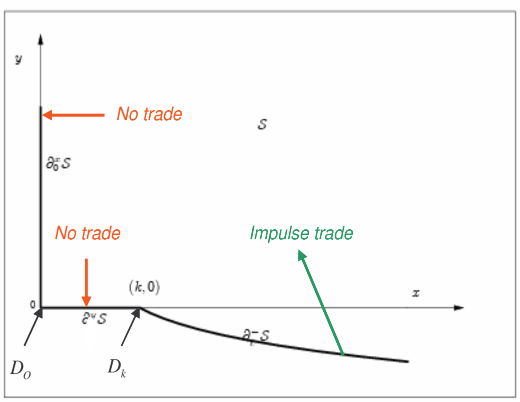

Figure 2: The solvency region when p < kλe

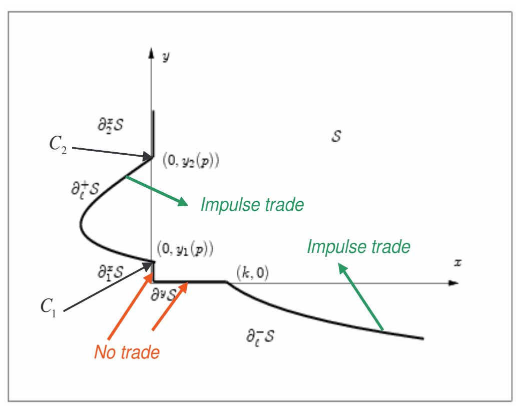

Figure 3: The solvency region when p > kλe

starting from z at time t.

Investment problem. We consider an utility function U from $\mathbb{R}_+$ into $\mathbb{R}$, strictly increasing, concave and w.l.o.g. $U(0) = 0$, and s.t. there exist $K \geq 0$, $\gamma \in [0,1)$:

$$U(w) \quad \leq \quad K w^{\gamma}, \quad \forall w \geq 0, \tag{2.10}$$

We denote U_L the function defined on $\bar{\mathcal{S}}$ by :

$$U_L(z) \quad = \quad U(L(z)).$$

We study the problem of maximizing the expected utility from terminal liquidation wealth and we then consider the value function :

$$v(t,z) \quad = \quad \sup_{\alpha \in \mathcal{A}(t,z)} \mathbb{E}\left[e^{-r(T-t)} U_L(Z_T)\right], \qquad (t,z) \in [0,T] \times \bar{\mathcal{S}}. \tag{2.11}$$

3 Viscosity solution of the associated Quasi-variational Hamilton-Jacobi-Bellman inequality

In Ly Vath, Mnif and Pham [10], we derived the HJB quasi-variational inequality satisfied by the value function (2.11)

$$\min\left[-\frac{\partial v}{\partial t} - \mathcal{L}v\ ,\ v - \mathcal{H}v\right] \quad = \quad 0, \quad \text{ on } \quad [0,T) \times \mathcal{S}, \tag{3.1}$$

where $\mathcal{L}$ as the infinitesimal generator associated to the system (2.9) corresponding to a no-trading period :

$$\mathcal{L}\varphi \quad = \quad rx\frac{\partial \varphi}{\partial x} + bp\frac{\partial \varphi}{\partial p} + \frac{1}{2}\sigma^2 p^2 \frac{\partial^2 \varphi}{\partial p^2} - r\varphi,$$

$\mathcal{H}$ is the impulse operator defined by

$$\mathcal{H}\varphi(t,z) \quad = \quad \sup_{\zeta \in \mathcal{C}(z)} \varphi(t, \Gamma(z,\zeta)), \quad (t,z) \in [0,T] \times \bar{\mathcal{S}},$$

Γ is the impulse transaction function defined from $\bar{\mathcal{S}} \times \mathbb{R}$ into $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$:

$$\Gamma(z,\zeta) \quad = \quad (x - \theta(\zeta,p) - k, y + \zeta, Q(\zeta,p)), \quad z = (x,y,p) \in \bar{\mathcal{S}},\ \zeta \in \mathbb{R},$$

and $\mathcal{C}(z)$ the set of admissible transactions :

$$\mathcal{C}(z) \quad = \quad \left\{\zeta \in \mathbb{R}\ : \Gamma(z,\zeta)\ \in\ \bar{\mathcal{S}}\right\} \ = \ \left\{\zeta \in \mathbb{R}\ : L(\Gamma(z,\zeta))\ \geq\ 0\right\}.$$

We related the value function (2.11) and the associated HJB quasi-variational inequality (3.1) by means of constrained viscosity solutions. The definition of viscosity solutions is given as follows:

Definition 3.1 *(i) Let $\mathcal{O} \subset \bar{\mathcal{S}}$. A locally bounded function u on $[0,T) \times \bar{\mathcal{S}}$ is a viscosity subsolution (resp. supersolution) of (3.1) in $[0,T) \times \mathcal{O}$ if for all $(\bar{t},\bar{z}) \in [0,T) \times \mathcal{O}$ and $\varphi \in C^{1,2}([0,T) \times \bar{\mathcal{S}})$ s.t. $(u^* - \varphi)(\bar{t},\bar{z}) = 0$ (resp. $(u_* - \varphi)(\bar{t},\bar{z}) = 0$) and $(\bar{t},\bar{z})$ is a maximum of $u^* - \varphi$ (resp. minimum of $u_* - \varphi$) on $[0,T) \times \mathcal{O}$, we have*

$$\min\left[-\frac{\partial\varphi}{\partial t}(\bar{t},\bar{z}) - \mathcal{L}\varphi(\bar{t},\bar{z}), u^*(\bar{t},\bar{z}) - \mathcal{H}u^*(\bar{t},\bar{z})\right] \leq 0 \tag{3.2}$$

$$(\ \textit{resp.} \quad \geq \ 0). \tag{3.3}$$

(ii) A locally bounded function u on $[0,T) \times \bar{\mathcal{S}}$ is a constrained viscosity solution of (3.1) in $[0,T) \times \mathcal{S}$ if u is a viscosity subsolution of (3.1) in $[0,T) \times \bar{\mathcal{S}}$ and a viscosity supersolution of (3.1) in $[0,T) \times \mathcal{S}$.

In Ly Vath, Mnif and Pham [10], we obtained the following characterization

Theorem 3.1 *The value function v is continuous on $[0,T) \times \mathcal{S}$ and is the unique (in $[0,T) \times \mathcal{S}$) constrained viscosity solution to (3.1) satisfying the boundary and terminal condition :*

$$\lim_{\substack{(t',z') \to (t,z) \\ z' \in \mathcal{S}}} v(t',z') = 0, \qquad \forall (t,z) \in [0,T) \times D_0 \tag{3.4}$$

$$\lim_{\substack{(t,z') \to (T,z) \\ t < T,\, z' \in \mathcal{S}}} v(t,z') = \max[U_L(z), \mathcal{H}U_L(z)], \qquad \forall z \in \bar{\mathcal{S}}, \tag{3.5}$$

and the growth condition :

$$|v(t,z)| \leq K\left(1 + \left(x + \frac{p}{\lambda}\right)\right)^{\gamma}, \quad \forall (t,z) \in [0,T) \times \mathcal{S} \tag{3.6}$$

for some positive constant $K < \infty$.

4 Convergence of the iterative scheme

We first introduce the following subsets of $\mathcal{A}(t,z)$, the set of the admissible impulse control strategies :

$$\mathcal{A}_n(t,z) := \{\alpha = (\tau_k, \xi_k)_{k=0,\ldots,n} \in \mathcal{A}(t,z)\}$$

and the corresponding value function v_n, which describes the value function when the investor is required to trade at most n times:

$$v_n(t,z) := \sup_{\alpha \in \mathcal{A}_n(t,z)} I\!E[e^{-r(T-t)} U_L(Z_T)] \quad (t,z) \in [0,T] \times \overline{\mathcal{S}}. \tag{4.1}$$

For $t \in [0,T]$ and $z = (x,y,p) \in \overline{\mathcal{S}}$, if x, y are both nonnegative, we clearly have $L(Z_s^{0,t,z}) \geq 0$, and so $\mathcal{A}_0(t,z)$ is nonempty. Otherwise, if $x < 0$, $y \geq 0$ or $x \geq 0$, $y < 0$, due to the diffusion term $P^{0,t,z}$, it is clear that the probability for $L(Z_s^{0,t,z})$ to be negative before time T, is strictly positive, so that $\mathcal{A}_0(t,z)$ is empty. Hence, the value function for $n = 0$ is initialized to:

$$v_0(t,z) = \begin{cases} \mathbb{E}\left[e^{-r(T-t)}U_L(Z_T^{0,t,z})\right] & \text{if } x \geq 0,\ y \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

We now show the convergence of the sequence of the value functions v_n towards our initial value function v.

Lemma 4.1 *For all* $(t,z) \in \mathcal{S}$

$$\lim_{n\to\infty} v_n(t,z) \ = \ v(t,z).$$

Proof. From the definition of $\mathcal{A}_n(t,z)$, we have:

$$\mathcal{A}_n(t,z) \ \subset \ \mathcal{A}_{n+1}(t,z) \ \subset \ \mathcal{A}(t,z).$$

As such,

$$v_n(t,z) \ \leq \ v_{n+1}(t,z) \ \leq \ v(t,z),$$

which gives the existence of the limit and the first inequality:

$$\lim_{n\to\infty} v_n(t,z) \ \leq \ v(t,z). \tag{4.2}$$

Given $\varepsilon > 0$, from the definition of v, there exists an impulse control $\alpha \ = \ (\tau_1, \tau_2, ...; \xi_1, \xi_2, ...) \in \mathcal{A}(t,z)$ such that

$$\mathbb{E}[e^{-r(T-t)}U_L(Z_T^\alpha)] \ \geq \ v(t,z) - \varepsilon, \tag{4.3}$$

with Z^α diffusing under the impulse control α.
We now set the control

$$\alpha_n \ := \ (\tau_1, \tau_2, ..., \tau_{n-1}, \underline{\tau}; \xi_1, \xi_2, ..., \xi_{n-1}, y_{\tau_{n-1}}),$$

where $\tau_{n-1} \ < \ \underline{\tau} \ < \ \min\{\tau_n, T\}$. We see that $\alpha_n \in \mathcal{A}_n(t,z)$ and consider the corresponding process $Z^{(\alpha_n)}$. Using Fatou lemma, we obtain:

$$\liminf_{n\to\infty} \mathbb{E}[e^{-r(T-t)}U_L(Z_T^{(\alpha_n)})] \ \geq \ \mathbb{E}[\liminf_{n\to\infty} e^{-r(T-t)}U_L(Z_T^{(\alpha_n)})] \ = \ \mathbb{E}[e^{-r(T-t)}U_L(Z_T^\alpha)] \tag{4.4}$$

Using (4.3) and (4.4), we obtain

$$\liminf_{n\to\infty} v_n(t,z) \ \geq \ \liminf_{n\to\infty} \mathbb{E}[e^{-r(T-t)}U_L(Z_T^{(\alpha_n)})] \ \geq \ v(t,z) - \varepsilon.$$

As we obtain the latter inequality with an arbitrary $\varepsilon > 0$, and combining with the relation (4.2), we obtain the desired result:

$$\lim_{n\to\infty} v_n(t,z) \ = \ v(t,z).$$

□

Theorem 4.1 *We define $\varphi_n(t,z)$ iteratively as a sequence of optimal stopping problems:*

$$\begin{aligned}\varphi_{n+1}(t,z) &= \sup_{\tau\in\mathcal{S}_{t,T}} \mathbb{E}\left[e^{-r(\tau-t)}\mathcal{H}\varphi_n(\tau, Z_\tau^{0,t,z})\right],\\ \varphi_0(t,z) &= v_0(t,z),\end{aligned}$$

where $\mathcal{S}_{t,T}$ is the set of stopping times in $[t,T]$. Then

$$\varphi_n(t,z) \;=\; v_n(t,z).$$

Remark 4.1 Theorem 4.1 together with Lemma 4.1 show that

$$\lim_{n\to\infty}\varphi_n(t,z) \;=\; v(t,z), \qquad (t,z)\in[0,T]\times\mathcal{S}.$$

so that the iteration scheme for φ_n provides an approximation for v.

Remark 4.2 The value function φ_n satisfies the system of variational inequalities, which can be solved by induction starting from φ_0:

$$\min\left[-\frac{\partial\varphi_{n+1}}{\partial t} - \mathcal{L}\varphi_{n+1}\ ,\ \varphi_{n+1} - \mathcal{H}\varphi_n\right] \;=\; 0, \quad (t,z)\in[0,T)\times\mathcal{S},$$

together with the terminal condition:

$$\varphi_{n+1}(T,z) \;=\; \mathcal{H}\varphi_n(T,z).$$

Proof of Theorem 4.1. We show by induction that $v_n(t,z) = \varphi_n(t,z)$, for all n. First, we have $v_0 = \varphi_0$. Considering an impulse control strategy $\alpha_1 = (\tau,\xi) \in \mathcal{A}_1(t,z)$, we clearly have

$$\begin{aligned}\varphi_1(t,z) &\geq \mathbb{E}[e^{-r(\tau-t)}\mathcal{H}\varphi_0(\tau, Z_\tau^{0,t,z})],\\ &\geq \mathbb{E}[e^{-r(\tau-t)}\mathcal{H}v_0(\tau, Z_\tau^{0,t,z})].\end{aligned}$$

From the definition of the operator $\mathcal{H}$, we obtain

$$\varphi_1(t,z) \;\geq\; \mathbb{E}[e^{-r(\tau-t)}v_0(\tau,\Gamma(Z_\tau^{0,t,z},\xi))], \quad \forall\alpha_1 = (\tau,\xi)\in\mathcal{A}_1(t,z). \tag{4.5}$$

Let $Z^{(\alpha_1)}$ be the diffusion of Z, starting at time t, with $Z_t^{(\alpha_1)} = z$, and evolving under the impulse control α_1. Relation (4.5) becomes:

$$\varphi_1(t,z) \;\geq\; E[e^{-r(\tau-t)}v_0(\tau, Z_\tau^{(\alpha_1)})], \quad \forall\ \alpha_1 = (\tau,\xi)\ \in\ \mathcal{A}_1(t,z). \tag{4.6}$$

Given the arbitrariness of α_1 and by using the dynamic programming principle applied to $v_1(t,z)$, we obtain

$$\varphi_1(t,z) \;\geq\; v_1(t,z).$$

From the definition of φ_1, for a given $\varepsilon > 0$, there exists τ^* such that

$$\varphi_1(t,z) - \varepsilon \;\leq\; \mathbb{E}[e^{-r(\tau^*-t)}\mathcal{H}\varphi_0(\tau^*, Z_{\tau^*}^{0,t,z})]. \tag{4.7}$$

From the compactness of the set of admissible transactions, there exists ξ^* such that

$$\begin{aligned}\varphi_1(t,z)-\varepsilon &\leq \mathbb{E}[e^{-r(\tau^*-t)}v_0(\tau^*,\Gamma(Z^{0,t,z}_{\tau^*},\xi^*))],\\ &\leq E[e^{-r(\tau^*-t)}v_0(\tau^*,Z^{(*)}_{\tau^*})],\end{aligned}$$

where $Z^{(*)}$ is the processus starting at time t, with $Z^{(*)}_t = z$, and evolving under the impulse control $\alpha^* := (\tau^*, \xi^*)$.
Using the dynamic programming principle applied on $v_1(t,z)$, we obtain

$$\varphi_1(t,z)-\varepsilon \quad\leq\quad v_1(t,z).$$

The latter inequality is satisfied for any value of $\varepsilon > 0$, as such, we have

$$\varphi_1(t,z) \;\leq\; v_1(t,z),$$

which leads to $\varphi_1(t,z) = v_1(t,z)$, for all $(t,z) \in [0,T)\times\mathcal{S}$.
By induction, assuming that for a given n, we have $\varphi_n(t,z) = v_n(t,z)$, we will prove that $\varphi_{n+1}(t,z) = v_{n+1}(t,z)$. By definition, we have for any $\alpha_{n+1} = (\tau_1, ..., \tau_{n+1}, \xi_1, ..., \xi_{n+1}) \in \mathcal{A}_{n+1}(t,z)$,

$$\begin{aligned}\varphi_{n+1}(t,z) &\geq \mathbb{E}[e^{-r(\tau_1-t)}\mathcal{H}\varphi_n(\tau_1,Z^{0,t,z}_{\tau_1})],\\ &\geq \mathbb{E}[e^{-r(\tau_1-t)}v_n(\tau_1,\Gamma(Z^{0,t,z}_{\tau_1},\xi_1))],\\ &\geq \mathbb{E}[e^{-r(\tau_1-t)}v_n(\tau_1,Z^{(n+1)}_{\tau_1})],\end{aligned} \tag{4.8}$$

where $Z^{(n+1)}$ is the diffusion starting at time t, with $Z^{(n+1)}_t = z$ and evolves under the control α_{n+1}. Given the arbitrariness of the control α_{n+1} and by using the dynamic programming principle applied to v_{n+1}, relation (4.8) becomes:

$$\varphi_{n+1}(t,z) \quad\geq\quad v_{n+1}(t,z).$$

To prove the opposite inequality, we use the definition of φ_{n+1}. For any $\varepsilon > 0$, there exists τ^* such that

$$\varphi_{n+1}(t,z)-\varepsilon \leq \mathbb{E}[e^{-r(\tau^*-t)}\mathcal{H}\varphi_n(\tau^*,Z^{0,t,z}_{\tau^*})], \tag{4.9}$$

$$\leq \mathbb{E}[e^{-r(\tau^*-t)}\mathcal{H}v_n(\tau^*,Z^{0,t,z}_{\tau^*})]. \tag{4.10}$$

From the compactness of the set of admissible transactions, there also exists ξ^* such that

$$\mathcal{H}v_n(\tau^*,Z^{0,t,z}_{\tau^*}) \;=\; v_n(\tau^*,Z^{(\alpha^*)}_{\tau^*}),$$

where $Z^{(\alpha^*)}$, the processus starting at time t, with $Z_t = z$, evolves under the impulse control $\alpha^* := (\tau^*, \xi^*)$. Using the dynamic programming principle applied on v_{n+1}, the relation (4.10) becomes

$$\begin{aligned}\varphi_{n+1}(t,z)-\varepsilon &\leq \mathbb{E}[e^{-r(\tau^*-t)}v_n(\tau^*,Z^{(\alpha^*)}_{\tau^*})],\\ &\leq v_{n+1}(t,z).\end{aligned}$$

The inequality is obtained for any given ε, this leads to the required inequality

$$\varphi_{n+1}(t,z) \;=\; v_{n+1}(t,z).$$

□

5 Numerical study

The objective of this section is the computation of a sequence of optimal stopping problem:

$$v_{n+1}(t,z) \quad = \quad \sup_{\tau \in \mathcal{S}_{t,T}} E\left[e^{-r(\tau-t)}\mathcal{H}v_n(\tau, X_\tau^{0,t,x}, y, P_\tau^{0,t,p})\right], \; z \in \bar{\mathcal{S}}$$

and the associated trade region and the no-trade region. We choose the Monte Carlo numerical procedure for the implementation.

5.1 The Monte Carlo method

Let $\mathbf{T}_m = \{t_l = lT/m\}_{0 \leq l \leq m}$ be the partition of the time interval $\mathbf{T} = [0,T]$. We denote by h_t the time step $\frac{T}{m}$, and by $\mathcal{S}_{m,t,T}$ the subset of $\mathcal{S}_{t,T}$ defined by

$$\mathcal{S}_{m,t,T} \quad = \quad \{\tau \in \mathcal{S}_{t,T} \; ; \; \tau \in \mathbf{T}_m\}.$$

Let $h_z := (h_x, h_y, h_p) = (1/M_1, 1/M_2, 1/M_3)$, where $(M_1, M_2, M_3) \in I\!N^{*3}$ denotes the finite difference step in the state coordinate $z = (x, y, p)$. Since the liquidation solvency region is unbounded, we localize $\bar{\mathcal{S}}$ to $D = \{z \in \bar{\mathcal{S}} \text{ s.t. } -L_1 \leq x \leq L_1, \, -L_2 \leq y \leq L_2, \, 0 \leq z \leq L_3\}$, where L_1, L_2 and L_3 are positive constants. We define the grid

$$\Omega_{h_z} \quad = \quad \{z = (ih_x, jh_y, kh_p) \in D, -M_1L_1 \leq i \leq M_1L_1, -M_2L_2 \leq j \leq M_2L_2, 0 \leq k \leq M_3L_3\}.$$

For the implementation, we simulate N independent Brownian motions as follows :

$$W_{t_{l+1}} - W_{t_l} \sim N(0, h_t).$$

Then, the price path is given by

$$P^0_{t_{l+1}} \quad = \quad P^0_{t_l} e^{(b-\frac{\sigma^2}{2})h_t + \sigma(W_{t_{l+1}} - W_{t_l})}$$

For the approximation of the value function v_n at the point (t, Z_t^0) where Z_t^0 is the random vector (X_t^0, y, Z_t^0) (the randomness is only in the third component of this vector), $t \in \mathbf{T}_m$, two cases are possible :

Case 1: If $Z_t \in [-M_1, M_1] \times [-M_2, M_2] \times [-M_3, M_3]$, then

$$\hat{Z}_t^0 = \sum_{i=1}^{N(\Omega_{h_z})} z_i \mathbf{1}_{A_i}(Z_t^0),$$

where $N(\Omega_{h_z}) := Card\{z \text{ s.t. } z \in \Omega_{h_z}\}$ and $(A_i)_{1 \leq i \leq N(\Omega_{h_z})}$ is a Borel partition of $\bar{\mathcal{S}}$ defined by

$$A_i = \left\{ z \in \bar{\mathcal{S}} \text{ s.t. } |z_i - z| = \min_{1 \leq j \leq N(\Omega_{h_z})} |z_j - z| \right\}.$$

$|.|$ denotes the canonical Euclidean norm, and we take $v_n(t, Z_t^0) \approx v_n(t, \hat{Z}_t^0)$.

Case 2: If $Z_t \notin [-M_1, M_1] \times [-M_2, M_2] \times [-M_3, M_3]$, $|Z_t^0 - \hat{Z}_t^0|$ could be large. To approximate $v_n(t, Z_t^0)$, we use the growth condition of the value function

$$v(t,z) \leq \frac{e^{\rho(T-t)}}{\gamma}(x + \frac{p}{\lambda}(1 - e^{-\lambda y}))^\gamma \tag{5.11}$$

where ρ is a positive constant s.t. $\rho > \frac{\gamma}{1-\gamma}\frac{b^2+r^2+\sigma^2 r(1-\gamma)}{\sigma^2}$ (See Propopsition 4.1 in Ly Vath, Mnif and Pham [10]).
The approximation of $v_n(t, Z_t^0)$ is given by

$$v_n(t, Z_t^0) \approx v_n(t, \hat{Z}_t^0)\frac{\left(X_t^0 + \frac{P_t^0(1-e^{-\lambda y})}{\lambda}\right)^\gamma}{\left(\hat{X}_t^0 + \frac{\hat{P}_t^0(1-e^{-\lambda y})}{\lambda}\right)^\gamma} \tag{5.12}$$

The discrete time approximation for the value function v_n is given by :

$$v_{n+1}(t, z) \quad = \quad \sup_{\tau \in \mathcal{S}_{m,t,T}} E\left[e^{-r(T-\tau)}\mathcal{H}v_n(\tau, \hat{X}_\tau^{0,t,x}, y, \hat{P}_\tau^{0,t,p})\right], \quad (t, z) \in \mathbf{T}_m \times \Omega_{h_z}.$$

The Snell envelop is computed by backward induction :

$$v_{n+1}(t_m, z) \quad = \quad \mathcal{H}v_n(t_m, z)$$

and

$$v_{n+1}(t_{l-1}, z) \quad = \quad \max\left\{\mathcal{H}v_n(t_{l-1}, z); e^{-rh_t}E[v_{n+1}(t_l, Z_l^0)|\mathcal{F}_{t_{l-1}}]\right\}, \quad 1 \le l \le m,$$

where $\mathcal{F}_{t_{l-1}} = \sigma(P_{t_j}, j \le l-1)$ is the discrete-time filtration. Hence :

$$E[v_{n+1}(t_l, Z_l^0)|\mathcal{F}_{t_{l-1}}] \quad = \quad E[v_{n+1}(t_l, Z_l^0)|P_{t_{l-1}}] \;=:\; \rho(t_{l-1}, P_{t_{l-1}}^0), \quad 1 \le l \le m.$$

5.2 Estimation of the conditional expectation using Malliavin Calculus

Here, we are interested in computing the conditional expectation $E[v_n(t+h, Z_{t+h}^0)|P_t]$. From the definition of v_n, we have $v_n \le v$. The main idea of the Malliavin method consists in using the Malliavin integration by part formula in order to get rid of the Dirac point masses in the following expression :

$$E[v_n(t+h, Z_{t+h}^0)|P_t = p] \quad = \quad \frac{E[v_n(t+h, Z_{t+h}^0)\delta_p(P_t^0)]}{E[\delta_p(P_t^0)]}. \tag{5.13}$$

We focus on the calculation of $E[v_n(t+h, Z_{t+h}^0)\delta_p(P_t^0)]$. We recall that $P_t^0 = p_0 e^{(b-\frac{\sigma^2}{2})t+\sigma W_t}$.
We now define

$$\hat{v}_{n,t+h,x,y}(Br) \quad := \quad v_n(t+h, x, y, e^{(b-\frac{\sigma^2}{2})(t+h)+\sigma Br}),$$

and

$$\hat{p}_t \quad := \quad \frac{1}{\sigma}(ln\frac{p}{p_0} - (b - \frac{\sigma^2}{2})t).$$

We obtain :

$$E[v_n(t+h, Z_{t+h}^0)\delta_p(P_t^0)] \quad = \quad E[\hat{v}_{n,t+h,X_{t+h}^0,y}(W_{t+h})\delta_{\hat{p}_t}(W_t)].$$

By the independence of Brownian motion's increments, we have :

$$E[\hat{v}_{n,t+h,X_{t+h},y}(W_{t+h})\delta_{\hat{p}_t}(W_t)] \quad = \quad \int\int \hat{v}_{n,t+h,X_{t+h},y}(w_1 + w_2)\delta_{\hat{p}_t}(w_1)\varphi(\frac{w_1}{\sqrt{t}})\varphi(\frac{w_2}{\sqrt{h}})\frac{dw_1}{\sqrt{t}}\frac{dw_2}{\sqrt{h}},$$

where φ is the density of standard one dimensional normal distribution. Using the growth condition of the value function v (5.11) we obtain

$$E[|v_n(t+h, Z^0_{t+h})|^2] < \infty. \tag{5.14}$$

Recalling that $\delta_x(w_1)$ is a derivative of $\mathbf{1}_{w_1 \geq x}$, using (5.14) and by integration by parts formula with respect to w_1 variable and then with respect to variable w_2, we get :

$$\begin{aligned} & E[\hat{v}_{n,t+h,X_{t+h},y}(W_{t+h})\delta_{\hat{p}_t}(W_t)] \\ = \; & E\left[\hat{v}_{n,t+h,X_{t+h},y}(W_{t+h})\mathbf{1}_{[\hat{p}_t,\infty)}(W_t)\left(-\frac{W_t}{t} + \frac{W_{t+h}-W_t}{h}\right)\right]. \end{aligned} \tag{5.15}$$

By denoting $A_h := \frac{W_t}{t} - \frac{W_{t+h}-W_t}{h}$, it follows that :

$$E[v_n(t+h, Z_{t+h})\delta_p(P_t)] \;=\; E\left[v_n(t+h, Z_{t+h})\mathbf{1}_{[p,\infty)}(P_t)A_h\right].$$

5.3 Variance reduction by localization

By using Monte Carlo Method, we recover a convergence rate of the order $\sqrt{N}$ for the conditional expectation estimator where N is the simulation number. However, the variance of the estimator explodes as h tends to zero since $\limsup_{h \longrightarrow 0} A_h = \infty$ and $\liminf_{h \longrightarrow 0} A_h = -\infty$. To find a remedy to this problem, we introduce localizing functions. Such functions catch the idea that the relevant information for the computation of $E[g(S_{t+h})|S_t = x]$ is located in the neighborhood of x. Let φ be an arbitrary localizing function. By definition, φ is smooth, bounded and it satisfies $\varphi(0) = 1$. Recalling the same arguments as in (5.15) and using (5.14), we obtain a family of alternative representations of the conditional expectation given by (5.13) :

$$\begin{aligned} E[v_n(t+h, Z^0_{t+h})\delta_p(P^0_t)] &= E\left[v_n(t+h, Z_{t+h})\delta_{\hat{p}_t}(W_t)\varphi(W_t - \hat{p}_t)\right] \\ &= E\left[\mathbf{1}_{W_t > \hat{p}_t}\; v_n(t+h, Z^0_{t+h})(\varphi(W_t - \hat{p}_t)A_h - \varphi'(W_t - \hat{p}_t))\right]. \end{aligned}$$

Moreover, it is possible to reduce the Monte Carlo estimator variance by a convenient choice of the localizing function. We consider the integrated mean square error :

$$J(\varphi) \;:=\; \int_{I\!R} E\left[\mathbf{1}_{W_t > \hat{p}_t}\; v_n^2(t+h, Z^0_{t+h})A^2_{h,\varphi}\right] dx, \tag{5.16}$$

where we adopted the following notation : $A_{h,\varphi} := \varphi(W_t - \hat{p}_t)A_h - \varphi'(W_t - \hat{p}_t)$ and we are interested in minimizing J respect to the subset $\{\, \varphi$ smooth, bounded and $\varphi(0) = 1\}$. Following [2], we prove that the optimal localizing function is given by :

$$\varphi(x) = e^{\nu_h x} \quad where \quad \nu_h := \left(\frac{E[v_n^2(t+h, Z^0_{t+h})A_h^2]}{E[v_n^2(t+h, Z^0_{t+h})]}\right)^{\frac{1}{2}}.$$

In conclusion, we obtain

$$E[v_n(t+h, Z_{t+h})\delta_p(P_t)] \;=\; E\left[v_n(t+h, Z_{t+h})\mathbf{1}_{[p,\infty)}e^{\nu_h(W_t - \hat{p}_t)}(A_h - \nu_h)\right].$$

5.4 Algorithm and discrete value function formula

The algorithm computes two sequences $\{v_n, \zeta_n\}_{n\geq 1}$ by performing the following steps.
Parameters: ϵ, λ, k, L_1, L_2, L_3, N the number simulation, T, M_1, M_2, M_3 and m.
Initialisation: $v_0 = (v_0(t,z))_{(t,z)\in \mathbf{T}_m\times\Omega_{h_z}}$, $n = 0$.
Step 1: Compute $\mathcal{H}v_n$ and ζ_n on $\mathbf{T}_m \times \Omega_{h_z}$ defined by

$$\mathcal{H}v_n(t,z) = \sup_{\zeta\in\hat{C}(z)} v_n(t,\hat{\Gamma}(z,\zeta)),\ (t,z)\in \mathbf{T}_m\times\Omega_{h_z},$$

where $\hat{\Gamma}(z,\zeta) = (\hat{x},\hat{y},\hat{p}) = \sum_{i=1}^{N(\Omega_{h_z})} z_i \mathbf{1}_{A_i}(x-\zeta p e^{\lambda y} - k, y+\zeta, p e^{\lambda\zeta})$, $z_i \in \Omega_{h_z}$ and

$$\hat{C}(z) = \{\zeta \in I\!R \text{ s.t. } \hat{L}(\hat{\Gamma}(z,\zeta)) := \max\left[\hat{L}_0(z), L_1(z)\right]\mathbf{1}_{y\geq 0} + \hat{L}_0(z)1_{y<0} \geq 0\},$$

$\hat{L}_0(z)$ is the closest point of the grid $(ih_x)_{-M_1L_1\leq i\leq M_1L_1}$ to the point $x - l(y,p) - k$.
Step 2: According to the previous section, we are able to calculate the value function :

$$v_{n+1}(t_l,z) = \max\left\{\mathcal{H}v_n(t_l,z); e^{-rh_t}\hat{\rho}_n(t_l,z)\right\},\ 0\leq l\leq m-1,\ z\in\Omega_{h_z},$$

Let us denote $P^{(i)}$ the i-th price simulation such that $1 \leq i \leq N$, where N is the simulation number. Then, we define the estimators of ρ_n by :

$$\tilde{\rho}_n(t_l,z) = \frac{\frac{1}{N}\sum_{i=1}^{N} v_n(t_{l+1}, Z^{0(i)}_{t_{l+1}})\mathbf{1}_{[p,\infty)} e^{\nu_h(W^{(i)}_{t_{l+1}} - \hat{p}^{(i)}_{t_{l+1}})}(A^{(i)}_h - \nu_h)}{\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{[p,\infty)} e^{\nu_h(W^{(i)}_{t_{l+1}} - \hat{p}^{(i)}_{t_{l+1}})}(A^{(i)}_h - \nu_h)},$$

where $z = (x,y,p)$, $A^{(i)}_h := \frac{W^{(i)}_{t_l}}{t_l} - \frac{W^{(i)}_{t_{l+1}} - W^{(i)}_{t_{l+1}}}{h}$, $\hat{p}^i_{t_l} = \frac{1}{\sigma}(ln\frac{p}{p_0} - (b - \frac{\sigma^2}{2})t_l)$ and $W^{(i)}$ i-th simulation of W.
Taking into account the growth condition of the value function, we truncate the estimator $\tilde{\rho}_n$:

$$\hat{\rho}_n(t_l,z) := \tilde{\rho}_n(t_l,z) \wedge \frac{1}{\gamma} e^{\rho(T-t_l)}(x + \frac{p}{\lambda})^\gamma,$$

which improves the algorithm.
Step 3: Stopping test: If $||v_{n+1} - v_n||_\infty \leq \epsilon$, stop, otherwise go to step 1.

5.5 Numerical results

The computation is achieved with a cluster of 13 Intel Xeon Processors running at 2.8 Ghz with 2 Giga Bytes of RAM. Numerical tests are performed with the following numerical constants

$$\gamma = 0.5,\ r = 0.1,\ \alpha = 0.12,\ \sigma = 0.3.$$

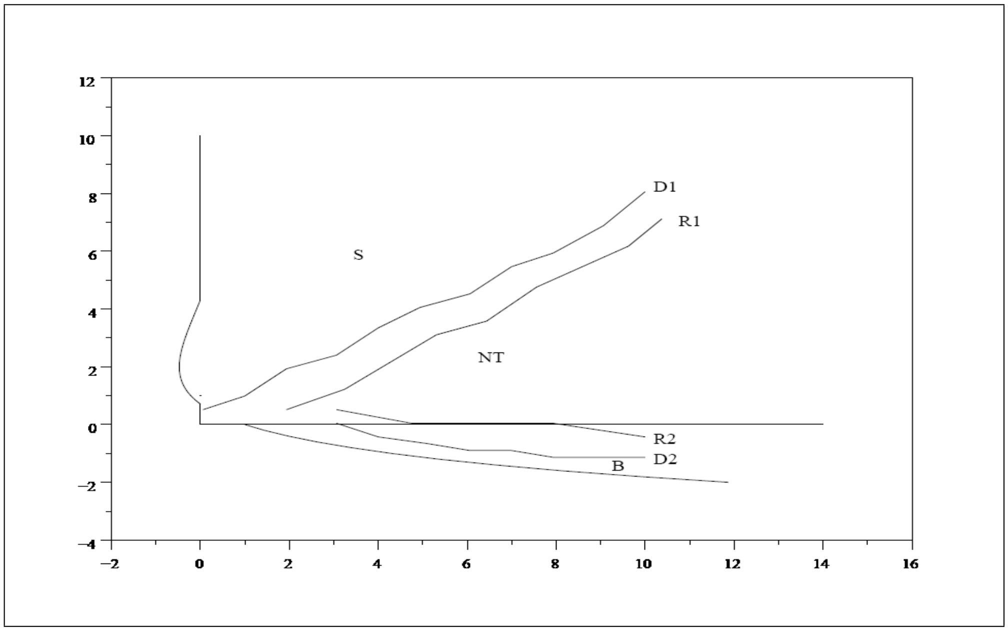

Figure 1: The optimal transaction policy for p=2, λ=0.5 and k=1

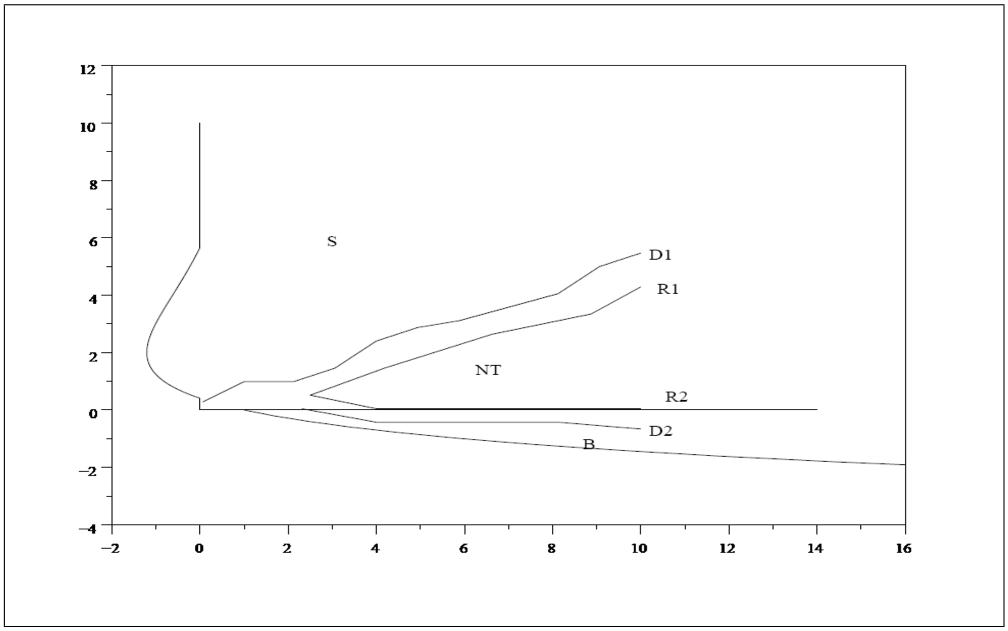

Figure 2: The optimal transaction policy for p=3, λ=0.5 and k=1

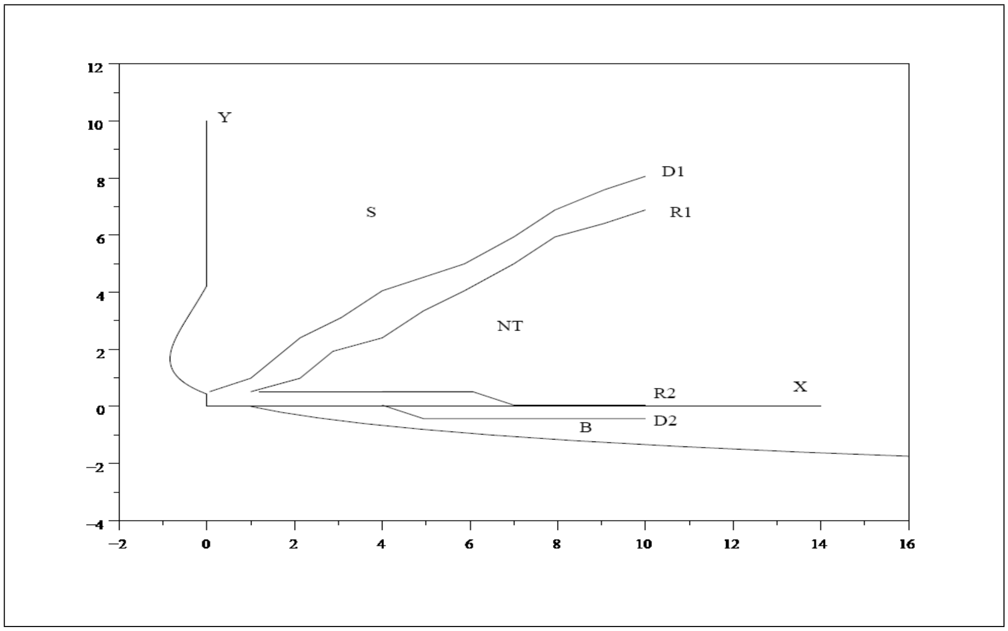

Figure 3: The optimal transaction policy for p=3, λ=0.6 and k=1

We take

$$L_1 = L_2 = L_3 = 10,\ T = 1,\ h_x = 1,\ h_y = 0.5,\ h_z = 1,\ h_t = 0.1.$$

According to Bouchard and Touzi [3], in order to achieve an error estimate of the order of $n^{-1/2}$, we have to choose a number N of simulated trajectories such that $N = O(n^{7/2})$. We choose $N = 10000$. Contrary to the policy-iteration algorithm (named Howard algorithm), the value-iteration algorithm needs more iterations to converge. At each iteration and each step of time we must estimate 1267 conditional expectations (it is the number of points in our domain). This explains that we need several days to achieve the whole computation. We equally mention that by using the probabilistic approach, we do not need to assume any boundary condition as in Chancelier, Øksendal, and Sulem [4]. However when the trajectories are outside our bounded domain, we approximate the value function by taking into account the growth condition (see (5.12)).

A partition of the solvency region $\mathcal{S}$ is displayed in figures (1)-(2)-(3) for different values of P and λ. It consists of three regions: Buy (B), Sell (S), and No-Trade (NT) regions. The domain between $R1$ and $R2$ corresponds to the region reached by the state variable after a purchase or a sale of risky asset, dictated by the optimal strategy. Due to the presence of fixed costs, the lines $R1$ and $R2$ do not coincide with $D1$ and $D2$ boundaries of the no-transaction region.

We equally try to see the sensitivity of different parameters and variables.

$\star$ First, there is a reduction in the No-Trade region when the price of the risky asset P increases, i.e. the line $D1$ moves downwards while the line $D2$ marginally moves upwards (see Figures (1)-(2)). The interpretation of this observation is the following :

- in the case where the investor has a significant long position in the risky asset, he is required to reduce his risky asset position when the share price goes up. This phenomenon has also been observed in the Merton model [11].

- in the case where the investor has a significant short position in the risky asset, he is required to buy back shares in order to reduce the risk when the share price goes up.

⋆ Second, we look at the impact of the coefficient of the impact price λ. We notice that when λ increases, the NT region widens (see Figures (2)-(3)). In particular, the line $D1$ significantly moves upwards. Economically, it means that when the liquidity impact increases, the investor should trade less frequently.

References

[1] Barles G. and P.E. Souganidis (1991) :"Convergence of approximation schemes for fully nonlinear second order equations",*Asymptotic analysis*, 4, 271-283.

[2] Bouchard B., I. Ekeland and N. Touzi (2002). "On the Malliavin approach to Monte Carlo approximation of conditional expectations", *Finance and Stochastics*, Vol **8**, 45-72, Springer-Verlag 2004

[3] Bouchard B. and N. Touzi (2004) : "Discrete time approximation and Monte Carlo simulation of Backward Stochastic Differential Equations", *Stochastic Processes and their applications*, Vol **111**,175-206.

[4] Chancelier J.P., Oksendal B. and A. Sulem (2001) : "Combined stochastic control and optimal stopping, and application to numerical approximation of combined stochastic and impulse control", *Stochastic Financial Mathematics*, Proc. Steklov Math. Inst. Moscou, 149-175, ed. A. Shiryaev.

[5] Cvitanic J. and I. Karatzas (1996) : "Hedging and portfolio optimization under transaction costs : a martingale approach", *Mathematical Finance*, 6, 133-165.

[6] Deelstra G., Pham H. and N. Touzi (2001) : "Dual formulation of the utility maximization problem under transaction costs", *Annals of Applied Probability*, 11, 1353-1383.

[7] Fournié E., J.-M. Lasry, J. Lebuchoux, P.-L. Lions and N. Touzi(1999). Applications of Malliavin calculus to Monte Carlo methods.

[8] Korn R. (1998) : "Portfolio optimization with strictly positive transaction costs and impulse control", *Finance and Stochastics*, **2**, 85-114.

[9] Kyle A. (1985) : "Continuous auctions and insider trading", *Econometrica*, **53**, 1315-1335.

[10] Ly Vath V., M. Mnif and H Pham (2005) : "A model of optimal portfolio selection under liquidity risk and price impact", to appear in *Finance and Stochastics*.

[11] Merton, R.C. (1971) : "Optimum consumption and portfolio rules in a continuous-time model", *J. Econ. Theory*, **3**, 373-413.

[12] Morton A. and S. Pliska (1995) : "Optimal portfolio management with fixed transaction costs", *Mathematical Finance*, 5, 337-356.

[13] Oksendal B. and A. Sulem (2002) : "Optimal consumption and portfolio with both fixed and proportional transaction costs", *SIAM J. Cont. Optim.*, **40**, 1765-1790.