INDECOMPOSABLE MODULES AND GELFAND RINGS

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Abstract. It is proved that a commutative ring is clean if and only if it is Gelfand with a totally disconnected maximal spectrum. It is shown that each indecomposable module over a commutative ring R satisfies a finite condition if and only if R_P is an artinian valuation ring for each maximal prime ideal P. Commutative rings for which each indecomposable module has a local endomorphism ring are studied. These rings are clean and elementary divisor rings. It is shown that each commutative ring R with a Hausdorff and totally disconnected maximal spectrum is local-global. Moreover, if R is arithmetic then R is an elementary divisor ring.

In this paper R is a commutative ring with unity and modules are unitary.

In [12, Proposition 2] Goodearl and Warfield proved that each zero-dimensional ring R satisfies the second condition of our Theorem 1.1, and this condition plays a crucial role in their paper. In Section 1, we show that a ring R enjoys this condition if and only if it is clean, if and only if it is Gelfand with a totally disconnected maximal spectrum. So we get a generalization of results obtained by Anderson and Camillo in [1] and by Samei in [21]. We deduce that every commutative ring R with a Hausdorff and totally disconnected maximum prime spectrum is local-global, and moreover, R is an elementary divisor ring if, in addition, R is arithmetic. One can see in [8] that local-global rings have very interesting properties.

In Section 3 we give a characterization of commutative rings for which each indecomposable module satisfies a finite condition: finitely generated, finitely cogenerated, cyclic, cocyclic, artinian, noetherian or of finite length. We deduce that a commutative ring is Von Neumann regular if and only if each indecomposable module is simple. This last result was already proved in [5]. We study commutative rings for which each indecomposable module has a local endomorphism ring. These rings are clean and elementary divisor rings. It remains to find valuation rings satisfying this property to give a complete characterization of these rings. We also give characterizations of Gelfand rings and clean rings by using properties of indecomposable modules. Similar results are obtained in Section 4, for commutative rings for which each prime ideal contains only one minimal prime ideal.

We denote respectively Spec R, Max R and Min R, the space of prime ideals, maximal ideals, and minimal prime ideals of R, with the Zariski topology. If A a subset of R, then we denote

$$V(A) = \{P \in \text{Spec } R \mid A \subseteq P\},\ D(A) = \{P \in \text{Spec } R \mid A \nsubseteq P\},$$

$$V_M(A) = V(A) \cap \text{Max } R \text{ and } D_M(A) = D(A) \cap \text{Max } R.$$

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1. Local-global Gelfand rings

As in [19] we say that a commutative ring R is **Gelfand** if each prime ideal is contained in only one maximal ideal. In this case, we put $\mu : \text{Spec } R \to \text{Max } R$ the map defined by $\mu(J)$ is the maximal ideal containing J for each prime ideal J. Then μ is continuous and Max R is Hausdorff by [7, Theorem 1.2].

In [12], Goodearl and Warfield proved that every zero-Krull-dimensional commutative ring satisfies the second condition of the following theorem. This property is used to show cancellation, n-root and isomorphic refinement theorems for finitely generated modules over algebras over a commutative ring which is Von Neumann regular modulo its Jacobson radical. So, the following theorem allows us to extend these results to each ring with a Hausdorff and totally disconnected maximal spectrum. As in [20] we say that a ring R is **clean** if each element of R is the sum of a unit with an idempotent. In [20, Proposition 1.8 and Theorem 2.1] Nicholson proved that commutative clean rings are exactly the exchange rings defined by Warfield in [24]. In [21] Samei proved that the conditions (1), (3) and (4) are equivalent when R is semiprimitive and in [1] Anderson and Camillo showed that each clean ring is Gelfand. We can also see [18, Theorem 3]. If P is a prime ideal we denote by 0_P the kernel of the natural map $R \to R_P$.

Theorem 1.1. *Let R be a ring. The following conditions are equivalent:*

1. *R is a Gelfand ring and* Max R *is totally disconnected.*
2. *Each R-algebra S satisfies this condition: let $f_1, \dots, f_k$ be polynomials over S in noncommuting variables $x_1, \dots, x_m, y_1, \dots, y_n$. Let $a_1, \dots, a_m \in S$. Assume that $\forall P \in \text{Max } R$ there exists $b_1, \dots, b_n \in S_P$ such that $f_i(a_1, \dots, a_m, b_1, \dots, b_n) = 0$ $\forall i$, $1 \leq i \leq k$. Then there exist $d_1, \dots, d_n \in S$ such that $f_i(a_1, \dots, a_m, d_1, \dots, d_n) = 0$ $\forall i$, $1 \leq i \leq k$.*
3. *R is a clean ring.*
4. *R is Gelfand and $\forall P \in \text{Max } R$, 0_P is generated by a set of idempotents.*

Proof. (1) $\Rightarrow$ (2). By [11, Theorem 16.17] Max R has a base of clopen subsets. Since μ is continuous, each clopen subset of Max R is of the form $D_M(e)$ for some idempotent e. So we can do the same proof as in [12, Proposition 2] where we replace Spec R with Max R.

(2) $\Rightarrow$ (3). Let $a \in R$. We consider the following equations: $x^2 = x$ and $y(a - x) = 1$. Since each local ring is clean, these equations have a solution in R_P for each maximal ideal P. We conclude that there is also a solution in R and that R is clean.

(3) $\Rightarrow$ (1). Let $P, P' \in \text{Max } R$, $P \neq P'$. Then there exist $a \in P$ and $a' \in P'$ such that $a + a' = 1$. We have $a = u + e$ where u is a unit and e is an idempotent. Since $a \in P$ and $u \notin P$ we get that $e \notin P$. We have $a' = 1 - a = -u + (1 - e)$. So $1 - e \notin P'$. Consequently P and P' have disjoint clopen neighbourhoods. Since Max R is quasi-compact, we deduce that this space is compact and totally disconnected. The equality $e(1-e) = 0$ implies that $P \cap P'$ contains no prime ideal. Hence R is Gelfand.

(1) $\Rightarrow$ (4). Let P be a maximal ideal and $a \in 0_P$. Then there exists $s \in R \setminus P$ such that $sa = 0$. Since Max R is totally disconnected there is a clopen subset U such that $U \subseteq D_M(s)$. Because of μ is continuous, there exists an idempotent e such that $P \in D(e) = \mu^{\leftarrow}(U) \subseteq \mu^{\leftarrow}(D_M(s)) \subseteq D(s)$. Then $e \in Rs$, $ea = 0$, $a = a(1 - e)$ and $1 - e \in 0_P$.

(4) $\Rightarrow$ (1). Let $P, P' \in \text{Max } R$, $P \neq P'$. Since R is Gelfand, by [7, Theorem 1.2] there exist $a \in 0_P \setminus P'$. Then there exists an idempotent $e \in 0_P \setminus P'$. Clearly $1 - e \notin P$. Consequently P and P' have disjoint clopen neighbourhoods. □

We say that R is **local-global** if each polynomial over R in finitely many indeterminates which admits unit values locally, admits unit values. Recall that most of the results of [12] about commutative rings which are Von Neumann regular modulo their Jacobson radicals, have been extended to local-global rings by Estes and Guralnick in [8]. We have the following corollary:

Corollary 1.2. *Let R be a ring such that* Max R *is Hausdorff and totally disconnected. Then R is local-global.*

Proof. Let J be the Jacobson radical of R. Then R is local-global if and only if R/J is local-global. So we may assume that R is semiprimitive. From the remark that follows [7, Theorem 1.2] and from Theorem 1.1 we deduce that R is clean. Let f be a polynomial over R in finitely many indeterminates $X_1, \dots, X_n$, which admits unit values locally. Then, we apply theorem 1.1 by taking $S = R$ to the polynomial $Yf(X_1, \dots, X_n) - 1$. □

Remark 1.3. If R is the ring of algebraic integers, then R is local-global by [6] and semi-primitive. But this ring is not Gelfand.

2. Arithmetic Gelfand rings

We say that a module is **uniserial** if its set of submodules is totally ordered by inclusion, we say that a ring R is a **valuation ring** if it is uniserial as R-module and we say that R is **arithmetic** if R_P is a valuation ring for each maximal ideal P. Recall that R is a **Bézout ring** if each finitely generated ideal is principal and R is an **elementary divisor ring** if each finitely presented module is a direct sum of cyclic submodules.

Theorem 2.1. *Let R be an arithmetic local-global ring . Then R is an elementary divisor ring. Moreover, for each $a, b \in R$, there exist $d, a', b', c \in R$ such that $a = a'd$, $b = b'd$ and $a' + cb'$ is a unit of R.*

Proof. Since every finitely generated ideal is locally principal R is Bézout by [8, Corollary 2.7]. Let $a, b \in R$. Then there exist $a', b', d \in R$ such that $a = a'd, b = b'd$ and $Ra + Rb = Rd$. Consider the following polynomial $a' + b'T$. If P is a maximal ideal, then we have $aR_P = dR_P$ or $bR_P = dR_P$. So, either a' is a unit of R_P and $a' + b'r$ is a unit of R_P for each $r \in PR_P$, or b' is a unit of R_P and $a' + b'(1 - a'/b')$ is a unit of R_P. We conclude that the last assertion holds. Now, let $a, b, c \in R$ such that $Ra + Rb + Rc = R$. We set $Rb + Rc = Rd$. Let b', c', s and q such that $b = b'd$, $c = c'd$ and $b' + c'q$ and $a + sd$ are units. Then $(b' + c'q)(a + sd) = (b' + c'q)a + s(b + qc)$ is a unit. We conclude by [10, Theorem 6]. □

We deduce the following corollary which is a generalization of [4, Theorem III.6] and [9, Theorem 5.5].

Corollary 2.2. *Let R be an arithmetic ring with a Hausdorff and totally disconnected maximal spectrum. Then R is an elementary divisor ring. Moreover, for each $a, b \in R$, there exist $d, a', b', c \in R$ such that $a = a'd$, $b = b'd$ and $a' + cb'$ is a unit of R.*

Corollary 2.3. *Let R be an arithmetic Gelfand ring such that* Min R *is compact. Then R is an elementary divisor ring. Moreover, for each $a, b \in R$, there exist $d, a', b', c \in R$ such that $a = a'd$, $b = b'd$ and $a' + cb'$ is a unit of R.*

Proof Let μ' be the restriction of μ to Min R. Since R is arithmetic each prime ideal contains only one minimal prime ideal. Then μ' is bijective and it is an homeomorphism because Min R is compact. One can apply corollary 2.2 since Min R is totally disconnected . □

Remark 2.4. In [9] there is an example of a Gelfand Bézout ring R which is not an elementary divisor ring. Consequently Min R is not compact.

3. Indecomposable modules and maximal ideals

In the two next propositions we give a characterization of Gelfand rings and clean rings by using properties of indecomposable modules.

Proposition 3.1. *Let R be a ring. The following conditions are equivalent:*

(1) *For each R-algebra S and for each left S-module M for which* $\mathrm{End}_S(M)$ *is local,* Supp M *contains only one maximal ideal.*
(2) *R is a Gelfand ring.*
(3) *$\forall P \in \mathrm{Max}\ R$ the natural map $R \rightarrow R_P$ is surjective.*

When these conditions are satisfied, $M = M_P$ for each left S-module M for which $\mathrm{End}_S(M)$ *is local, where P is the unique maximal ideal of* Supp M *and where S is an algebra over R.*

Proof. Assume that R is Gelfand. Let S be an R-algebra and let M be a left S-module such that $\mathrm{End}_S(M)$ is local. Let P be the prime ideal which is the inverse image of the maximal ideal of $\mathrm{End}_S(M)$ by the canonical map $R \rightarrow \mathrm{End}_S(M)$ and let $Q = \mu(P)$. Since M is an R_P-module, $0_Q \subseteq \mathrm{ann}_R(M)$. So, Supp $M \subseteq V(0_Q)$ and Q is the only maximal ideal belonging to $V(0_Q)$ since R is Gelfand.

Conversely, if P is a prime ideal then $R_P = \mathrm{End}_R(R_P)$. It follows that P is contained in only one maximal ideal.

By [7, Theorem 1.2] R is Gelfand if and only if, $\forall P \in \mathrm{Max}\ R$, P is the only maximal ideal containing 0_P. This is equivalent to $R/0_P$ is local, $\forall P \in \mathrm{Max}\ R$. It is obvious that $R_P = R/0_P$ if $R/0_P$ is local. (When R is semi-primitive we can see [3, Proposition 1.6.1]).

Recall that the diagonal map $M \rightarrow \Pi_{P' \in \mathrm{Max}(R)} M_{P'}$ is monic. Since R is Gelfand, we have $M_P = M/0_P M$ where P is the only maximal ideal of Supp M. Hence $M = M_P$. □

Proposition 3.2. *Let R be a ring. The following conditions are equivalent:*

(1) *For each R-algebra S and for each indecomposable left S-module M,* Supp M *contains only one maximal ideal.*
(2) *R is clean.*

When these conditions are satisfied, $M = M_P$ for each indecomposable left S-module M, where P is the unique maximal ideal of Supp M *and S is an R-algebra.*

Proof. (2) $\Rightarrow$ (1). By Theorem 1.1 Max R is totally disconnected. So, if P and P' are two different maximal ideals such that $P \in$ Supp M then there exists an idempotent $e \in P \setminus P'$ because μ is continuous. Since $(1-e) \notin P$ and $M_P \neq 0$, we have $(1-e)M \neq 0$. We deduce that $eM = 0$ and $M_{P'} = 0$.

(1) $\Rightarrow$ (2). R is Gelfand by Proposition 3.1. Let A be an ideal such that $V(A)$ is the inverse image of a connected component of Max R by μ. Then $V(A)$ is connected too, whence R/A is indecomposable. So there is only one maximal ideal in $V(A)$. Since each connected component contains only one point, Max R is totally disconnected. □

This lemma is needed to prove the main results of this section.

Lemma 3.3. *Let R be a local ring which is not a valuation ring. Then there exists an indecomposable non-finitely generated R-module whose endomorphism ring is not local.*

Proof. Since R is not a valuation ring there exist $a,\ b \in R$ such that neither divides the other. By taking a suitable quotient ring, we may assume that $Ra \cap Rb = 0$ and $Pa = Pb = 0$. Let F be a free module generated by $\{e_n \mid n \in \mathbb{N}\}$, let K be the submodule generated by $\{ae_n - be_{n+1} \mid n \in \mathbb{N}\}$ and let $M = F/K$. Clearly $M/PM \cong F/PF$. We will show that M is indecomposable and $S := \mathrm{End}_R(M)$ is not local. Let us observe that M is defined as in proof of [13, Theorem 2.3]. But, since R is not necessarily artinian, we do a different proof to show that M is indecomposable. We shall prove that S contains no trivial idempotents. Let $s \in S$. Then s is induced by an endomorphism $\tilde{s}$ of F which satisfies $\tilde{s}(K) \subseteq K$. For each $n \in \mathbb{N}$ there exists a finite family $(\alpha_{p,n})$ of elements of R such that:

$$\tilde{s}(e_n) = \sum_{p\in\mathbb{N}} \alpha_{p,n} e_p \tag{1}$$

Since $\tilde{s}(K) \subseteq K,\ \forall n \in \mathbb{N},\ \exists$ a finite family $(\beta_{p,n})$ of elements of R such that:

$$\tilde{s}(ae_n - be_{n+1}) = \sum_{p\in\mathbb{N}} \beta_{p,n}(ae_p - be_{p+1}) \tag{2}$$

From 1 and 2 if follows that:

$$\sum_{p\in\mathbb{N}} (a\alpha_{p,n} - b\alpha_{p,n+1})e_p = a\beta_{0,n}e_0 + \sum_{p\in\mathbb{N}^*} (a\beta_{p,n} - b\beta_{p-1,n})e_p$$

Since $Pa = Pb = Ra \cap Rb = 0$ we deduce that

$$\alpha_{0,n+1} \equiv 0\ [P],\quad \alpha_{p,n} \equiv \beta_{p,n}\ [P]\ \text{ and }\ \alpha_{p,n+1} \equiv \beta_{p-1,n}\ [P]$$

It follows that

$$(i)\ \alpha_{p,n} \equiv \alpha_{p+1,n+1}\ [P],\ \forall p,n \in \mathbb{N},\ \text{ and }\ (ii)\ \alpha_{p,p+k+1} \equiv 0\ [P],\ \forall p,k \in \mathbb{N} \tag{3}$$

Now we assume that s is idempotent. Let $x_n = e_n + K,\ \forall n \in \mathbb{N}$. Let $\bar{s}$ be the endomorphism of M/PM induced by s. If L is an R-module and x an element of L, we put $\bar{x} = x + PL$. From $s^2(x_0) = s(x_o)$ we get the following equality:

$$\sum_{n\in\mathbb{N}} (\sum_{p\in\mathbb{N}} \alpha_{n,p}\alpha_{p,0})x_n = \sum_{n\in\mathbb{N}} \alpha_{n,0}x_n \tag{4}$$

Then $\bar{\alpha}_{0,0} = \sum_{p\in\mathbb{N}} \bar{\alpha}_{0,p}\bar{\alpha}_{p,0} = \bar{\alpha}_{0,0}^2$, since $\bar{\alpha}_{0,p} = 0$ by $3(ii)$, $\forall p > 0$. So, we have $\bar{\alpha}_{0,0} = 0$ or $\bar{\alpha}_{0,0} = 1$. If $\bar{\alpha}_{0,0} = 1$ then we replace s with $\mathbf{1}_M - s$. So we may assume that $\bar{\alpha}_{0,0} = 0$. By $3(i)$ $\bar{\alpha}_{n,n} = 0$, $\forall n \in \mathbb{N}$. By using 4 and $3(ii)$ we get that

$$\bar{\alpha}_{n,0} = \sum_{p=0}^{n-1} \bar{\alpha}_{n,p}\bar{\alpha}_{p,0} + \bar{\alpha}_{n,n}\bar{\alpha}_{n,0}$$

Hence, if $\bar{\alpha}_{p,0} = 0,\ \forall p < n$ then $\bar{\alpha}_{n,0} = 0$ too. By induction we obtain that $\bar{\alpha}_{n,0} = 0$, $\forall n \in \mathbb{N}$. We deduce that

$$\alpha_{p,n} \in P,\ \forall p, n \in \mathbb{N} \tag{5}$$

Let $A = \text{Im } s,\ B = \text{Ker } s$ and let A' and B' be the inverse image of A and B by the natural map $F \to M$. If $x \in A'$ then $\tilde{s}(x) = x + y$ for some $y \in K$. By 5 and $Pa = Pb = 0$ it follows that $\tilde{s}(y) = 0$ and $\tilde{s}^2(x) = \tilde{s}(x)$. If $x \in B'$ then $\tilde{s}(x) \in K$. So $\tilde{s}^2(x) = 0$. We deduce that $(\tilde{s}^2)^2 = \tilde{s}^2$. Let $C = \text{Im } \tilde{s}^2$. Then C is projective and $C = PC$ by 5. By [2, Proposition 2.7] $C = 0$. So $s = 0$ (or $\mathbf{1}_M - s = 0$).

It remains to prove that S is not local. Let $f, g \in S$ defined in the following way: $f(x_n) = x_{n+1}$ and $g(x_n) = x_n - x_{n+1},\ \forall n \in \mathbb{N}$. We easily check that $x_0 \notin \text{Im } f \cup \text{Im } g$. So f and g are not units of S and $f + g = \mathbf{1}_M$ is a unit. Hence S is not local. □

A module is **cocyclic** (respectively **finitely cogenerated**) if it is a submodule of the injective hull of a simple module (respectively of a finite direct sum of injective hulls of simple modules).

Now we give a characterization of commutative rings for which each indecomposable module satisfies a finite condition.

Theorem 3.4. *Let R be a ring. The following conditions are equivalent:*

1. *Each indecomposable R-module is of finite length.*
2. *Each indecomposable R-module is noetherian.*
3. *Each indecomposable R-module is finitely generated.*
4. *Each indecomposable R-module is artinian.*
5. *Each indecomposable R-module is finitely cogenerated.*
6. *Each indecomposable R-module is cyclic.*
7. *Each indecomposable R-module is cocyclic.*
8. *For each maximal ideal P, R_P is an artinian valuation ring.*
9. *R is an arithmetic ring of Krull-dimension 0 and its Jacobson ideal J is T-nilpotent.*

Proof. The following implications are obvious: (1) ⇒ (2) ⇒ (3), (1) ⇒ (4) ⇒ (5), (6) ⇒ (3) and (7) ⇒ (5).

(8) ⇒ (1), (6) and (7). R has Krull dimension 0. Hence R is clean. So, if M is an indecomposable module, by proposition 3.2 there is only one maximal ideal P such that $M_P \neq 0$. Moreover $M \cong M_P$. We conclude by [13, Theorem 4.3].

(3) ⇒ (8). Let P be a maximal ideal and E the injective hull of R/P. Then each submodule of E is indecomposable. It follows that E is a noetherian module. By [23, Proposition 3] E is a module of finite length, and by [23, Theorem 3] R_P is artinian. We conclude by [13, Theorem 2.3] or Lemma 3.3.

(5) ⇒ (8). Let P be a maximal ideal. Then each factor of R_P modulo an ideal of R_P is finitely cogenerated. It follows that R_P is artinian. We conclude as above.

(8) ⇒ (9). Let $x_1, \dots, x_n, \dots$ be a sequence of elements of J. Then for each maximal ideal P $\exists s_P \notin P$ and a positive integer n_P such that $s_P x_1 \dots x_{n_P} = 0$. There is a finite family of open sets $D(s_{n_{P_1}}), \dots, D(s_{n_{P_m}})$ that cover Spec R. We set $n = max\{n_{P_1}, \dots, n_{P_m}\}$. Then $x_1 \dots x_n = 0$.

(9) ⇒ (8). $\forall P \in \text{Max } R$, R_P is a valuation ring and PR_P is a nilideal. Then for every $r \in P$ there exists $s \in R \setminus P$ such that sr is nilpotent. So we get that $PR_P = JR_P$, whence PR_P is T-nilpotent. We easily prove that R_P is artinian. □

From this theorem it is easy to deduce the two following corollaries. Another proof of the second corollary is given in [5, Theorem 2.13].

Corollary 3.5. *Let n be a positive integer, R a ring and J its Jacobson radical. Then the following conditions are equivalent:*

1. *Each indecomposable module has a length $\leq n$.*
2. *For each maximal ideal P, R_P is a valuation ring and $(PR_P)^n = 0$.*
3. *R is an arithmetic ring of Krull-dimension 0 and $J^n = 0$.*

Corollary 3.6. *A ring R is Von Neumann regular if and only if every indecomposable module is simple.*

The next theorem gives a partial characterization of commutative rings for which each indecomposable module has a local endomorphism ring.

Theorem 3.7. *Let R be a ring for which $\mathrm{End}_R(M)$ is local for each indecomposable module M. Then R is a clean elementary divisor ring.*

Proof. Let P be a prime ideal. Then $R/P = \mathrm{End}_R(R/P)$ is local. Hence R is Gelfand. We prove that Max R is totally disconnected as in proof of proposition 3.2. If P is a maximal ideal, each indecomposable R_P-module M is also indecomposable over R and $\mathrm{End}_R(M) = \mathrm{End}_{R_P}(M)$. By Lemma 3.3 R_P is a valuation ring. □

Example 3.8. If R is a ring satisfying the equivalent conditions of Theorem 3.4, then each indecomposable R-module has a local endomorphism ring. But, by [15, Corollary 2 p.52] and [22, Corollary 3.4], each complete discrete rank one valuation ring enjoys this property too. So, we consider a complete discrete rank one valuation ring D, Q its ring of fractions and R the subring of $Q^{\mathbb{N}}$ defined as in [20, Example 1.7]: $x = (x_n)_{n\in\mathbb{N}} \in R$ if $\exists p \in \mathbb{N}$ and $s \in D$ such that $x_n = s$, $\forall n > p$. Since D is local, R is clean and semi-primitive by [18, Theorem 2]. We put $\mathbf{1} = (\delta_{n,n})_{n\in\mathbb{N}}$ and $\forall p \in \mathbb{N}$, $\mathbf{e}_p = (\delta_{p,n})_{n\in\mathbb{N}}$ where $\delta_{n,p}$ is the Kronecker symbol. Let J be the maximal ideal of D. If P is a maximal ideal of R, then either $\mathbf{e}_p \in P$, $\forall p \in \mathbb{N}$, whence $P = J\mathbf{1} + \oplus_{p\in\mathbb{N}} R\mathbf{e}_p$ and $R_P \cong R/\oplus_{p\in\mathbb{N}} R\mathbf{e}_p \cong D$, or $\exists p \in \mathbb{N}$ such that $\mathbf{e}_p \notin P$, whence $P = R(\mathbf{1} - \mathbf{e}_p)$ and $R_P \cong R/P \cong Q$. Thus R is arithmetic and each indecomposable R-module has a local endormorphism ring. Observe that each indecomposable R-module is uniseriel and linearly compact and its endomorphism ring is commutative.

4. Indecomposable modules and minimal prime ideals

In this section we study rings R for which each prime ideal contains only one minimal prime ideal. In this case, if $P \in$ Spec R, let $\lambda(P)$ be the only minimal prime ideal contained in P. We shall see that λ is continuous if and only if Min R is compact. (See [16, Theorem 2] when R is semi-prime). But, since λ is surjective, the set of minimal primes can be endowed with the quotient topology induced by the Zariski topology of Spec R. We denote this topologic space by QMin R. Then we have the following:

Proposition 4.1. *Let R be a ring such that each prime ideal contains a unique minimal prime ideal and N its nilradical. Then* QMin *R is compact. Moreover,* QMin *R and* Min *R are homeomorphic if and only if* Min *R is compact.*

The following lemma is needed to prove this proposition. This lemma is a generalization of [14, Lemma 2.8]. We do a similar proof.

Lemma 4.2. *Let R be a ring, N its nilradical and $a \in R \setminus N$. Let P be a prime ideal such that $P/(N : a)$ is minimal in $R/(N : a)$. Then P is a minimal prime ideal.*

Proof. First we show that $a + (N : a)$ is a non-zerodivisor in $R/(N : a)$ and consequently $a \notin P$. Let $b \in R$ such that $ab \in (N : a)$. Then $a^2b \in N$. We easily deduce that $ab \in N$, whence $b \in (N : a)$. Let $r \in P$. Then there exist a positive integer n and $s \in R \setminus P$ such that $sr^n \in (N : a)$. It follows that $asr^n \in N$. Since $as \notin P$ we deduce that PR_P is a nilideal, whence P is a minimal prime. □

Proof of proposition 4.1. Let A and B be two distinct minimal prime ideals. Since each maximal ideal contains only one minimal prime ideal, we have $A+B = R$. Therefore there exist $a \in A$ and $b \in B$ such that $a + b = 1$. Thus $a \notin B$ and $a \notin N$. But a is a nilpotent element of R_A. Hence $(N : a) \nsubseteq A$. In the same way we show that $B \in D((N : b))$. We have $(N : a) \cap (N : b) = (N : Ra + Rb) = N$. So $D((N : a)) \cap D((N : b)) = \emptyset$. By Lemma 4.2, $D((N : a))$ and $D((N : b))$ are the inverse images of disjoint open subsets of QMin R by λ. We conclude that this space is Hausdorff. Since Spec R is quasi-compact, it follows that QMin R is compact.

Let λ' be the restriction of λ to Min R. It is obvious that $(\lambda')^{-1}$ is continuous if and only if Min R is compact. □

Remark 4.3. If we consider the set of D-components of Spec R, defined in [17], endowed with the quotient topology, we get a topologic space X. Then X is homeomorphic to Max R (respectively QMin R) if R is Gelfand (respectively every prime ideal contains only one minimal prime). But X is not generally Hausdorff: see [17, Propositions 6.2 and 6.3].

Now we can show the two following propositions which are similar to Propositions 3.1 and 3.2. The proofs are similar too.

Proposition 4.4. *Let R be a ring. The following conditions are equivalent:*

1. *For each R-algebra S and for each left S-module M for which* $\mathrm{End}_S(M)$ *is local, there exists only one minimal prime ideal A such that* Supp $M \subseteq V(A)$.
2. *Every prime ideal contains only one minimal prime ideal.*

Proof. (2) ⇒ (1). Let S be an R-algebra and let M be a left S-module such that $\mathrm{End}_S(M)$ is local. Let P be the prime ideal which is the inverse image of the maximal ideal of $\mathrm{End}_S(M)$ by the canonical map $R \to \mathrm{End}_S(M)$, $A = \lambda(P)$ and 0_P the kernel of the natural map $R \to R_P$. Since M is an R_P-module, $0_P \subseteq \mathrm{ann}_R(M)$. It is obvious that $0_P \subseteq A$. On the other hand, AR_P is the nilradical of R_P. It follows that $\mathrm{rad}(0_P) = A$. Hence we get that Supp $M \subseteq V(\mathrm{ann}_R(M)) \subseteq V(0_P) = V(A)$. If B is another minimal prime, it is obvious that $V(A) \cap V(B) = \emptyset$.

(1) ⇒ (2). If P is a prime ideal then $R_P = \mathrm{End}_R(R_P)$. It follows that P contains only one minimal prime ideal. □

Proposition 4.5. *Let R be a ring. The following conditions are equivalent:*

1. *For each R-algebra S and for each indecomposable left S-module M, there is only one minimal prime ideal A such that* Supp $M \subseteq V(A)$.

(2) *Each prime ideal contains a unique minimal prime ideal and* QMin R *is totally disconnected.*

Proof. (1) $\Rightarrow$ (2). By proposition 4.4 each prime ideal contains a unique minimal prime ideal. Let $P \in$ QMin R and C its connected component. There exists an ideal A such that $V(A) = \lambda^{\leftarrow}(C)$. Then $V(A)$ is connected. It follows that R/A is indecomposable. So $V(A) = V(P)$ and $C = \{P\}$.

(2) $\Rightarrow$ (1). Let S be an R-algebra and M be an indecomposable left S-module. Let $P \in$ Supp M, $A = \lambda(P)$, $P' \in$ Spec $R \setminus V(A)$ and $A' = \lambda(P')$. Since QMin R is totally disconnected, there exists an idempotent $e \in A \setminus A'$. We easily deduce that $e \in P \setminus P'$. Now we do as in the proof of Proposition 3.2 to conclude. □

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