Dynamic Random Walks on Heisenberg groups

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Abstract

We prove a Guivarc'h law of large numbers and a central limit theorem for dynamic random walks on Heisenberg groups. The limiting distribution is explicitely given. To our knowledge this is the first study of dynamic random walks on non-commutative Lie groups.

1 Introduction

Random walks on Lie groups have been extensively studied over the last decades ([11, 12, 15]). Among these groups Heisenberg groups play a special role. These groups have their origin in quantum mechanics where they can be interpreted as the Lie algebras generated by the location operator, the momentum operator, and the identity operator. They are simply connected nilpotent Lie groups of rank 2 and one-dimensional center. Heisenberg groups are often considered as the simplest non-commutative Lie groups. The geometry of these groups has been investigated by A. Korányi ([13, 14]). Limit theorems for random walks on Heisenberg groups have been proved by P. Crepel, B. Roynette ([4]) and D. Neuenschwander ([15]) in connection with the resolution of Kesten's conjecture on the classification of recurrent and transient groups. A central limit theorem for nilpotent Lie groups has been proved by P. Crepel and A. Raugi ([3]). The novelty of this paper is in the dynamic model of random walks which we define on Heisenberg groups. The theory of dynamic random walks has been done by the first author in a commutative setting ([6, 7, 8, 9]). This paper is a first step/attempt in extending this theory to non-commutative algebraic structures.

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The organization is as follows: Section 2 provides some introductory material on Heisenberg groups and dynamic random walks. Section 3 is devoted to the proof of a strong law of large numbers and a central limit theorem. The limiting distribution is explicitely calculated. Our strong law of large numbers extends Guivarc'h's result [10] to dynamic random walks on Heisenberg groups.

2 Generalities on Heisenberg groups and dynamic random walks

The continuous Heisenberg group H_d is the group with underlying manifold $\mathbb{R}^{2d+1}$ and group operation

$$X \cdot Y = \left(x_1 + y_1, x_2 + y_2, \ldots, x_{2d} + y_{2d}, z_1 + z_2 + \frac{1}{2}\Big(\sum_{i=1}^{2d-1} (x_i y_{i+1} - x_{i+1} y_i)\Big) \right)$$

where $X = (x_1, \ldots, x_{2d}, z_1) \in \mathbb{R}^{2d+1}$ and $Y = (y_1, \ldots, y_{2d}, z_2) \in \mathbb{R}^{2d+1}$.

H_d is a nilpotent Lie group of rank 2 and with one-dimensional center. In this paper we identify the Heisenberg group H_d with its Lie algebra $\mathcal{H}_d$.

For simplicity we focus on H_1 but all results of this paper remain true for H_d. Let us remember that another representation H_1 is as a group of upper triangular (3×3)-matrices with 1's on the diagonal:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R}$$

If x, y, z are integers (in the ring Z) then we get the discrete Heisenberg group which is known to be the simplest non-abelian nilpotent group.
Let $g = (x, y, z) \in H_1$ and

$$\mid g \mid = [(x^2 + y^2)^2 + z^2]^{1/4}.$$

Let $\delta_r(g) = (rx, ry, r^2 z)$ where $r > 0$. The mapping δ_r is called dilation of ratio r on H_1. The mapping $g \to \mid g \mid$ from H_1 into $\mathbb{R}^+$ is an homogeneous norm. This means that:
i) $\mid g \mid = 0$ if and only if $g = 0$,
ii) $\mid g \mid = \mid -g \mid$,
iii) $\mid \delta_r(g) \mid = r \mid g \mid$.

Remark.
Let V be a compact neighborhood of the identity in H_1. The mapping defined by $\mid g \mid = inf\{n \in \mathbb{N},\ g \in V^n\}$ is also an homogeneous norm.
It is known [10] that all homogeneous norms on H_1 are equivalent.
$B(0, r) = \{g \in H_1; \mid g \mid \leq r\}$ $(r > 0)$ is called a Korányi ball. It replaces the traditional euclidean ball in the geometry developed by A. Korányi ([13, 14]).

We introduce now our model of dynamic random walks on H_1.
Let $(X_n, Y_n, Z_n)_{n\geq 1}$ be a sequence of independent random variables with values in H_1 defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(E, \mathcal{A}, \mu, T)$ be a dynamical system where $(E, \mathcal{A}, \mu)$ is a probability space and T is a transformation defined on E preserving the measure μ. Let $f_i, i = 1, 2, 3$ be functions defined on E with values in $[0, \frac{1}{3}]$. Let $x \in E$ and $(e_j)_{1\leq j\leq 3}$ be the unit coordinate vectors of $\mathbb{R}^3$. For each $i \geq 1$, the distribution of the random vector $M_i := (X_i, Y_i, Z_i)$ is given by

$$\mathbb{P}(M_i = z) = \begin{cases} f_j(T^i x) & \text{if } \ z = e_j \\ \frac{1}{3} - f_j(T^i x) & \text{if } \ z = -e_j \\ 0 & \text{otherwise} \end{cases}$$

We are interested in the right dynamic random walk

$$S_n := (X_1, Y_1, Z_1) \cdot (X_2, Y_2, Z_2) \cdots (X_n, Y_n, Z_n), n \geq 1.$$

We prove in the next sections a strong law of large numbers and a central limit theorem for $(S_n)_{n\geq 1}$.

3 Limit theorems

3.1 Law of large numbers

Y. Guivarc'h [10] proved a strong law of large numbers for random walks on Lie groups. We prove a similar result but for dynamic random walks on Heisenberg groups.
In this section, we assume that the dynamical system $(E, \mathcal{A}, \mu, T)$ is *ergodic* i.e. there exists an unique invariant measure μ. The functions $f_i, i = 1, 2, 3$ are all with integral $1/6$. Let $d\Big(0, \frac{S_n}{n}\Big) = \frac{1}{n} \mid S_n \mid$, d will be called Korányi distance.

Theorem 3.1 *For $\mu-$almost every $x \in E$, as $n \to +\infty$,*

$$d\Big(0, \frac{S_n}{n}\Big) \to 0 \ \ \mathbb{P} - a.s..$$

Proof:
From the definition of the Korányi distance,

$$d\Big(0, \frac{S_n}{n}\Big) = \left(\left(\Big(\frac{S_n^{(1)}}{n}\Big)^2 + \Big(\frac{S_n^{(2)}}{n}\Big)^2\right)^2 + \Big(\frac{S_n^{(3)}}{n^2}\Big)^2\right)^{1/4}.$$

The two first coordinates of S_n are given by

$$S_n^{(1)} = X_1 + X_2 + \ldots + X_n$$

and

$$S_n^{(2)} = Y_1 + Y_2 + \ldots + Y_n.$$

¿From Birkhoff's theorem, the sequence

$$\frac{\mathbb{E}(S_n^{(1)})}{n} = \frac{1}{n}\sum_{k=1}^{n}(2f_1(T^k x) - \frac{1}{3})$$

converges for μ-almost every $x \in E$ to $\int_E (2f_1 - \frac{1}{3})\ d\mu = 0$. With the same arguments, the sequence

$$\frac{\mathbb{E}(S_n^{(2)})}{n} = \frac{1}{n}\sum_{k=1}^{n}(2f_2(T^k x) - \frac{1}{3})$$

is proved to converge for μ-almost every $x \in E$ to $\int_E (2f_2 - \frac{1}{3})\ d\mu = 0$. Thus, from Kolmogorov's theorem, the sequences $(\frac{S_n^{(1)}}{n})_n$ and $(\frac{S_n^{(2)}}{n})_n$ converge $\mathbb{P}$-almost surely to 0 for μ-almost every $x \in E$.

The third component of the dynamic random walk on H_1 is given by

$$S_n^{(3)} = Z_1 + Z_2 + \ldots + Z_n + \frac{1}{2}\{X_1Y_2 + (X_1 + X_2)Y_3 + \ldots + (X_1 + \ldots + X_{n-1})Y_n$$
$$-Y_1X_2 - (Y_1 + Y_2)X_3 - \ldots - (Y_1 + \ldots + Y_{n-1})X_n\}.$$

For μ-almost every $x \in E$, the sequence

$$\frac{1}{n^2}(Z_1 + Z_2 + \ldots + Z_n)$$

converges $\mathbb{P}$-almost surely to 0 so we are only interested in the asymptotic behavior of the sequence

$$\frac{T_n}{n^2} = \frac{1}{2n^2}(T_n^{(1)} - T_n^{(2)})$$

where

$$T_n^{(1)} = X_1Y_2 + (X_1 + X_2)Y_3 + \ldots + (X_1 + \ldots + X_{n-1})Y_n$$
$$T_n^{(2)} = Y_1X_2 + (Y_1 + Y_2)X_3 + \ldots + (Y_1 + \ldots + Y_{n-1})X_n.$$

Let us prove that $T_n^{(1)}/n^2$ converges almost surely to 0. Since the Y's can only take values 0 and ± 1, we can bound $|\frac{T_n^{(1)}}{n^2}|$ by

$$\frac{1}{\left(\sum_{i=2}^{n}(i-1)|Y_i|\right)}\sum_{i=2}^{n}(i-1)|Y_i|\frac{1}{i-1}\Big|\sum_{j=1}^{i-1}X_j\Big|.$$

We conclude by using Toeplitz's Theorem (see for instance Theorem 2.34 in [5]) since the sequence $\sum_{j=1}^{n} X_j/n$ converges $\mathbb{P}$-almost surely to 0 for μ-almost every $x \in E$ as n goes to infinity. By inverting the random variables X_j's and Y_j's, we evidently get that $T_n^{(2)}/n^2$ converges $\mathbb{P}$-almost surely to 0 for μ-almost every $x \in E$ as n goes to infinity. Consequently, T_n/n^2 converges $\mathbb{P}$-almost surely to 0 for μ-almost every $x \in E$ as n goes to infinity as well as the third component of the dynamic random walk. So the theorem is proved.

3.2 Central limit theorem

In order to prove a central limit theorem for the dynamic random walk on the Heisenberg group H_1 we need to add assumptions on the functions $f_i, i = 1, 2, 3$. Let $\mathcal{C}$ denote the class of functions $f \in L^1(E, \mu)$ satisfying the following condition

$$\sup_{x\in E}\left|\sum_{i=1}^{n}\Big(f(T^i x) - \int_E f(x)d\mu(x)\Big)\right| = o\Big(\sqrt{n}\Big).$$

We denote by $A = (a_{ij})_{1\le i,j\le 3}$ the matrix with coefficients

$$\begin{aligned} a_{jj} &= \frac{4}{9}\int_E (1-9f_j^2(x))d\mu(x) \\ a_{ij} &= a_{ji} = \frac{1}{9}\int_E (1-36f_i(x)f_j(x))d\mu(x). \end{aligned}$$

Let us assume that $f_i,\ f_if_j \in \mathcal{C}$ for every $i, j = 1, 2, 3$ and $\int_E f_i\ d\mu = \frac{1}{6}$. For every $x \in E$, the two first coordinates of S_n converge $\mathbb{P}$-almost surely to 0 (see the proof of Theorem 3.1). In the third component given by

$$Z_1 + Z_2 + \ldots + Z_n + \frac{1}{2}\{X_1Y_2 + (X_1+X_2)Y_3 + \ldots + (X_1+\ldots+X_{n-1})Y_n$$
$$-Y_1X_2 - (Y_1+Y_2)X_3 - \ldots - (Y_1+\ldots+Y_{n-1})X_n\},$$

for every $x \in E$, the sequence

$$\frac{1}{n}(Z_1 + Z_2 + \ldots + Z_n)$$

also converges $\mathbb{P}$-almost surely to 0 so we are only interested in the asymptotic distribution of the sequence

$$T_n = \frac{1}{2}\{X_1Y_2 + (X_1+X_2)Y_3 + \ldots + (X_1+\ldots+X_{n-1})Y_n - Y_1X_2 - (Y_1+Y_2)X_3 - \ldots - (Y_1+\ldots+Y_{n-1})X_n\}.$$

We assume that the matrix C defined by $(a_{ij})_{1\le i,j\le 2}$ is diagonal, namely

$$C = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$$

with $a_{11} = 4[\frac{1}{9} - \int_E f_1^2\ d\mu]$ and $a_{22} = 4[\frac{1}{9} - \int_E f_2^2\ d\mu]$.
Since our random variables $(X_i)_{i\ge 1}$ and $(Y_i)_{i\ge 1}$ are not centered, we need to make the following assumption $(\mathbf{A})$:

$$\sup_{x\in E}\left|\sum_{i=2}^{k}\sum_{j=1}^{i-1}(f_2(T^i x) - \frac{1}{6})(f_1(T^j x) - \frac{1}{6})\right| = o(k).$$

Theorem 3.2 *For every $x \in E$, for every $t > 0$, the sequence*

$$\left(\frac{T_{[nt]}}{n}\right)_{n\geq 1}$$

converges in distribution to the random variable

$$A(t) = \frac{1}{2}\left\{\int_0^t B_s^{(1)} dB_s^{(2)} - \int_0^t B_s^{(2)} dB_s^{(1)}\right\}$$

where $B_t = (B_t^{(1)}, B_t^{(2)})$ is a two-dimensional Brownian motion with zero mean and Covariance matrix Ct. The density of $A(t)$ is given by

$$d_t(x) = \frac{1}{t\sqrt{a_{11}a_{22}}\cosh\left(\frac{\pi\ x}{t\sqrt{a_{11}a_{22}}}\right)}.$$

The random variable $A(t)$ is usually called the Lévy stochastic area. In our context $A(t)$ is driven by dynamic random walks, therefore we will call $A(t)$, by analogy, the dynamic Lévy stochastic area.

Remarks:

1- The hypotheses on the functions f_1, f_2 and $f_1 f_2$ imply that the assumption $(\mathbf{A})$ is satisfied as soon as for every $i, j \geq 1$, for every $x \in E$,

$$(f_2(T^i x) - \frac{1}{6})(f_1(T^j x) - \frac{1}{6}) = (f_2(T^j x) - \frac{1}{6})(f_1(T^i x) - \frac{1}{6})$$

i.e., for instance, in the following cases:
a)- $f_1 = f_2$ (but not necessarily equal to $1/6$).
b)- f_1 or f_2 equal to $1/6$.
c)- $f_2 = \frac{1}{3} - f_1$.
Moreover, this assumption is necessary. Given a dynamical system $(E, \mathcal{A}, \mu, T)$, the fact that the functions $f_1, f_2, f_1 f_2$ belong to the class $\mathcal{C}$, $\int_E f_1\, d\mu = \int_E f_2\, d\mu = 1/6$ and $\int_E f_1 f_2\, d\mu = 1/36$ does not imply that the assumption $(\mathbf{A})$ is necessarily verified. For instance, take the rotation T on the one-dimensional torus $\mathbb{T}^1$ with rational angle equal to $1/4$ and the particular functions $f_1 = \frac{1}{3}\mathbf{1}_{[0,\frac{1}{4}]} + \frac{1}{6}\mathbf{1}_{[\frac{1}{2},1]}$ and $f_2 = \frac{1}{3}\mathbf{1}_{[0,\frac{1}{2}]}$. The functions f_1, f_2 and $f_1 f_2$ are clearly integrable, $\int f_1\, dx = \int f_2\, dx = 1/6$ and $\int f_1 f_2\, dx = 1/36$ and they belong to the class $\mathcal{C}$ associated to the dynamical system. A calculation also gives that for every $x \in \mathbb{T}^1$,

$$\sum_{i=2}^{k}\sum_{j=1}^{i-1}(f_2(T^i x) - \frac{1}{6})(f_1(T^j x) - \frac{1}{6}) \sim \frac{k}{4}\frac{1}{36}, \quad k \to +\infty.$$

So assumption $(\mathbf{A})$ is not satisfied and for these exotic cases, Theorem (3.2) holds by adding a drift to the limit process.
2- Given a dynamical system $(E, \mathcal{A}, \mu, T)$, the class $\mathcal{C}$ is quite difficult to determine. However, for

particular cases, it is possible to find a large sub-class of $\mathcal{C}$. We refer to papers [6, 7, 8] where this question has been studied in details. When T is an irrational rotation on the one-dimensional torus $\mathbb{T}^1$, it was proved that every function with bounded variation belongs to the class $\mathcal{C}$. For example, we can choose $f_1(x) = 1/6$ and $f_2(x) = 1/3\cos^2(2\pi x)$ which takes its values in $[0, 1/3]$. The hypothesis $(\mathbf{A})$ is clearly satisfied thanks to remark 1. The integral of f_2 is equal to $1/6$ so $a_{12} = a_{21} = 0$. After simple computations, $a_{11} = 1/3$ and $a_{22} = 5/18$. In this particular example, the density of the limit distribution $A(t)$ in Theorem 3.2 is then given by

$$d_t(x) = \frac{1}{t\sqrt{5/54}} \cosh\left(\frac{\pi\ x}{t\sqrt{5/54}}\right),\ x \in \mathbb{R}.$$

3.2.1 A preliminary result

In order to prove Theorem 3.2, we shall need a central limit theorem for the dynamic $\mathbb{Z}^2$-random walk $(S_n^{(1)}, S_n^{(2)})_{n\geq 1}$.

Proposition 3.1 *For every $x \in E$, the sequence of random vectors $\frac{1}{\sqrt{n}}(S_n^{(1)}, S_n^{(2)}),\quad n \geq 1$ converges in distribution, as $n \to +\infty$, to the centered Gaussian random variable G_0 with Covariance matrix C.*

Proof:
Let us introduce the characteristic function ϕ_n of

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k, Y_k)\,,\quad n \geq 1.$$

By independence of the random vectors $(X_k, Y_k), k \geq 1$,

$$\begin{aligned}\phi_n(u) &= \prod_{k=1}^{n}\mathbb{E}\left(\exp\left(i\frac{u_1X_k + u_2Y_k}{\sqrt{n}}\right)\right)\\ &= \prod_{k=1}^{n}Q_n^{(k)}(u_1, u_2)\end{aligned}$$

where

$$Q_n^{(k)}(u_1, u_2) = \frac{1}{3}(1 + \sum_{j=1}^{2}\cos(\frac{u_j}{\sqrt{n}})) + 2i\sum_{j=1}^{2}(f_j(T^kx) - \frac{1}{6})\sin(\frac{u_j}{\sqrt{n}})$$

A direct calculation gives

$$|Q_n^{(k)}(u)|^2 \quad = \quad \frac{1}{9}(1 + \sum_{j=1}^{2}(1 - \frac{u_j^2}{2n}) + \mathcal{O}(n^{-2}))^2 + \frac{1}{9}\sum_{j=1}^{2}(6f_j(T^kx) - 1)^2\frac{u_j^2}{n}$$

$$+ \quad \frac{2}{9}(6f_1(T^kx)-1)(6f_2(T^kx)-1))\frac{u_1u_2}{n} + \mathcal{O}(n^{-2})$$
$$= \quad 1 - \frac{1}{3n}\sum_{j=1}^{2} u_j^2 + \frac{1}{9n}\sum_{j=1}^{2} u_j^2(6f_j(T^kx)-1)^2$$
$$+ \quad \frac{2}{9n}(6f_1(T^kx)-1)(6f_2(T^kx)-1))u_1u_2 + \mathcal{O}(n^{-2})$$

and then

$$\begin{aligned} |\phi_n(u)| &= \prod_{k=1}^{n} |Q_n^{(k)}(u)| \\ &= \exp(-\frac{1}{2} < u, Cu > + o(1)) \end{aligned}$$

The imaginary part of the characteristic function can be rewritten as

$$\prod_{k=1}^{n} \exp\left(i \arctan\left(\frac{\sum_{j=1}^{2}(6f_j(T^kx)-1)\sin(\frac{u_j}{\sqrt{n}})}{(1+\sum_{j=1}^{2}\cos(\frac{u_j}{\sqrt{n}}))}\right)\right)$$
$$= \exp\left(\frac{i}{3}\sum_{k=1}^{n}\sum_{j=1}^{2}(6f_j(T^kx)-1)\frac{u_j}{\sqrt{n}} + o(1)\right) = 1 + o(1)$$

using the fact that f_1, f_2 belong to the class $\mathcal{C}$ and for every $j = 1, 2$, the integral of f_j is equal to $1/6$.

3.2.2 Proof of the central limit theorem 3.2

We only prove Theorem 3.2 for the particular value t equal to 1. The proof can easily be adapted to get the result for every $t > 0$. The proof of the theorem is decomposed into four parts.

a) Straightforward calculations give us

$$\begin{aligned} 2T_{nk} &= [N_0^{(k)} + \ldots + N_{n-1}^{(k)}] - [\tilde{N}_0^{(k)} + \ldots + \tilde{N}_{n-1}^{(k)}] \\ &+ [M_0^{(k)}(Q_1^{(k)} - Q_0^{(k)}) + \ldots + M_{n-2}^{(k)}(Q_{n-1}^{(k)} - Q_{n-2}^{(k)})] \\ &- [Q_0^{(k)}(M_1^{(k)} - M_0^{(k)}) + \ldots + Q_{n-2}^{(k)}(M_{n-1}^{(k)} - M_{n-2}^{(k)})] \end{aligned}$$

where

$$N_p^{(k)} = X_{pk+1}Y_{pk+2} + (X_{pk+1} + X_{pk+2})Y_{pk+3} + \ldots + (X_{pk+1} + \ldots + X_{(p+1)k-1})Y_{(p+1)k}$$
$$\tilde{N}_p^{(k)} = Y_{pk+1}X_{pk+2} + (Y_{pk+1} + Y_{pk+2})X_{pk+3} + \ldots + (Y_{pk+1} + \ldots + Y_{(p+1)k-1})X_{(p+1)k}$$
$$M_p^{(k)} = X_1 + \ldots + X_{(p+1)k}$$

$$Q_p^{(k)} = Y_1 + \ldots + Y_{(p+1)k}.$$

The choice of this decomposition can appear artificial but it will be justified in the sequel.

b) Let $x \in E$ fixed. ¿From Proposition 3.1, for every $p \geq 0$, the sequence of random vectors

$$W_p^{(k)} := \frac{1}{\sqrt{k}} \sum_{j=pk+1}^{(p+1)k} (X_j, Y_j), \quad k \geq 1$$

converges in distribution, as $k \to +\infty$, to the centered Gaussian random vector G_0 with covariance matrix C. Since the random vectors $(W_p^{(k)})_{p\geq 0}$ are independent, we deduce that $(W_0^{(k)}, W_1^{(k)}, \ldots, W_{n-1}^{(k)})$ converges in distribution, as $k \to +\infty$, to the centered Gaussian random vector $(G_0, G_1, \ldots, G_{n-1})$ where the G_i's are independent copies of G_0. Since C is diagonal, each random vector G_i can be decomposed in two independent centered Gaussian random variables $G_i^{(1)}$ and $G_i^{(2)}$ with variance a_{11} and a_{22} respectively. So it implies that $(W_0^{(k)}, W_1^{(k)}, \ldots, W_{n-1}^{(k)})$ converges in distribution, as $k \to +\infty$, to the centered Gaussian random vector $(G_0^{(1)}, G_0^{(2)}, G_1^{(1)}, G_1^{(2)}, \ldots, G_{n-1}^{(1)}, G_{n-1}^{(2)})$. Now, $W_p^{(k)} = (M_p^{(k)} - M_{p-1}^{(k)}, Q_p^{(k)} - Q_{p-1}^{(k)})/\sqrt{k}$ (with the convention $M_{-1}^{(k)} = Q_{-1}^{(k)} = 0$). The convergence in distribution being preserved by linear transformation we get that

$$\frac{1}{\sqrt{k}}(M_0^{(k)}, M_1^{(k)}, \ldots, M_{n-1}^{(k)}, Q_0^{(k)}, Q_1^{(k)}, \ldots, Q_{n-1}^{(k)})$$

converges in distribution, as $k \to +\infty$, to the centered Gaussian random vector $(G_0^{(1)}, G_0^{(1)} + G_1^{(1)}, \ldots, G_0^{(1)} + \ldots + G_{n-1}^{(1)}, G_0^{(2)}, G_0^{(2)} + G_1^{(2)}, \ldots, G_0^{(2)} + \ldots + G_{n-1}^{(2)})$. Then,

$$\begin{aligned} & \frac{1}{k}[M_0^{(k)}(Q_1^{(k)} - Q_0^{(k)}) + \ldots + M_{n-2}^{(k)}(Q_{n-1}^{(k)} - Q_{n-2}^{(k)})] \\ - & \frac{1}{k}[Q_0^{(k)}(M_1^{(k)} - M_0^{(k)}) + \ldots + Q_{n-2}^{(k)}(M_{n-1}^{(k)} - M_{n-2}^{(k)})] \end{aligned}$$

converges in distribution, as $k \to +\infty$, to the random variable

$$\begin{aligned} & [G_0^{(1)}G_1^{(2)} + (G_0^{(1)} + G_1^{(1)})G_2^{(2)} + \ldots + (G_0^{(1)} + \ldots + G_{n-2}^{(1)})G_{n-1}^{(2)}] \\ - & [G_0^{(2)}G_1^{(1)} + (G_0^{(2)} + G_1^{(2)})G_2^{(1)} \ldots + (G_0^{(2)} + \ldots + G_{n-2}^{(2)})G_{n-1}^{(1)}]. \end{aligned}$$

c) Let us now prove that

$$\mathrm{Var}\ \left(N_0^{(k)} + \ldots + N_{n-1}^{(k)}\right) = \mathcal{O}(nk^2) \tag{1}$$

and

$$\mathrm{Var}\ \left(\tilde{N}_0^{(k)} + \ldots + \tilde{N}_{n-1}^{(k)}\right) = \mathcal{O}(nk^2). \tag{2}$$

The above result is quite evident in the classical case studied in [11] where the random variables X_i and Y_i are centered and uncorrelated but due to the temporal inhomogeneity of our model **it is not at all the case** and we need the following lemma.

Lemma 3.1 *Uniformly in $p \geq 0$,*

$$\text{Var } (N_p^{(k)}) = \mathcal{O}(k^2) \tag{3}$$

and

$$\text{Var } (\tilde{N}_p^{(k)}) = \mathcal{O}(k^2). \tag{4}$$

Proof:
Let us recall that

$$N_p^{(k)} = \sum_{l=1}^{k-1} Y_{pk+l+1}\Big(\sum_{i=1}^{l} X_{pk+i}\Big).$$

For every random variable X such that $\mathbb{E}(|X|) < +\infty$, we define $\bar{X} = X - \mathbb{E}(X)$. The random variable $N_p^{(k)}$ can be rewritten

$$N_p^{(k)} = \Sigma_1(k) + \Sigma_2(k) + \Sigma_3(k) - \Sigma_4(k)$$

where

$$\begin{aligned}
\Sigma_1(k) &= \sum_{l=1}^{k-1} \bar{Y}_{pk+l+1}\Big(\sum_{i=1}^{l} \bar{X}_{pk+i}\Big) \\
\Sigma_2(k) &= \sum_{l=1}^{k-1} \mathbb{E}(Y_{pk+l+1})\Big(\sum_{i=1}^{l} X_{pk+i}\Big) \\
\Sigma_3(k) &= \sum_{l=1}^{k-1} Y_{pk+l+1}\Big(\sum_{i=1}^{l} \mathbb{E}(X_{pk+i})\Big) \\
\Sigma_4(k) &= \sum_{l=1}^{k-1} \mathbb{E}(Y_{pk+l+1})\Big(\sum_{i=1}^{l} \mathbb{E}(X_{pk+i})\Big).
\end{aligned}$$

Using that the random variables $\bar{X}_i$ and $\bar{Y}_j$ are independent when $i \neq j$ and centered, a direct computation gives

$$\text{Var } (\Sigma_1(k)) \quad \leq \quad Ck^2$$

where $C > 0$ is a constant independent of p. Furthermore, the random variable $\Sigma_2(k)$ can be rewritten as

$$\Sigma_2(k) = \sum_{l=1}^{k-1} X_{pk+l}\Big(\sum_{i=l}^{k-1} \mathbb{E}(Y_{pk+i+1})\Big)$$

thus

$$\begin{aligned}
\text{Var } (\Sigma_2(k)) &= \sum_{l=1}^{k-1} \Big(\sum_{i=l}^{k-1} \mathbb{E}(Y_{pk+i+1})\Big)^2 \text{Var } (X_{pk+l}) \\
&= 4\sum_{l=1}^{k-1} \Big(\sum_{i=l}^{k-1} (f_2(T^{pk+i+1}x) - \frac{1}{6})\Big)^2 \text{Var } (X_{pk+l})
\end{aligned}$$

Since the function f_2 belongs to the class $\mathcal{C}$ and $\int_E f_2 \, d\mu = 1/6$, there exists a constant C which does not depend on p and k such that for every $l \geq 1$,

$$\Big|\sum_{i=1}^{k-l} \Big(f_2(T^{pk+l+i}x) - \frac{1}{6}\Big)\Big| \leq C\sqrt{k}.$$

Then,

$$\text{Var }(\Sigma_2(k)) \leq C\sum_{l=1}^{k-1}\Big|\sum_{j=1}^{k-l}\Big(f_2(T^{pk+l+i}x) - \frac{1}{6}\Big)\Big|^2 = \mathcal{O}(k^2).$$

The variance of the third sum $\Sigma_3(k)$ is estimated in the same manner using the fact that the function f_1 belongs to the class $\mathcal{C}$ and $\int_E f_1 \, d\mu = 1/6$. So, we get

$$\text{Var }(\Sigma_3(k)) = \mathcal{O}(k^2).$$

The variance of the last sum is evidently zero.
The following inequality holds for square integrable random variables X and Y

$$\text{Var }(X+Y) \leq 2(\text{Var }(X) + \text{Var }(Y)).$$

So, by applying twice this inequality, we get

$$\text{Var }(N_p^{(k)}) \leq 4\sum_{i=1}^{4}\text{Var }(\Sigma_i(k)) = \mathcal{O}(k^2)$$

uniformly in p. The estimation (4) is obtained in the same manner by inverting the X's and the Y's.

¿From Lemma 3.1 we can now establish (1) as follows: the random variables $N_p^{(k)}$ and $N_{p'}^{(k)}$ are independent when $p \neq p'$, so we have

$$\text{Var}\Big(N_0^{(k)} + \ldots + N_{n-1}^{(k)}\Big) = \sum_{p=0}^{n-1}\text{Var}\Big(N_p^{(k)}\Big) \leq Cnk^2$$

using Lemma 3.1, so (1) is proved and (2) can be obtained by inverting the X's and the Y's.

Let $\varepsilon > 0$,

$$\begin{aligned} &\mathbb{P}\Big(\frac{1}{2nk}\Big|\sum_{p=0}^{n-1}[N_p^{(k)} - \mathbb{E}(N_p^{(k)})] - \sum_{p=0}^{n-1}[\tilde{N}_p^{(k)} - \mathbb{E}(\tilde{N}_p^{(k)})]\Big| \geq \varepsilon\Big) \\ \leq\ &\mathbb{P}\Big(\frac{1}{nk}\Big|\sum_{p=0}^{n-1}[N_p^{(k)} - \mathbb{E}(N_p^{(k)})]\Big| \geq \varepsilon\Big) + \mathbb{P}\Big(\frac{1}{nk}\Big|\sum_{p=0}^{n-1}[\tilde{N}_p^{(k)} - \mathbb{E}(\tilde{N}_p^{(k)})]\Big| \geq \varepsilon\Big) \\ \leq\ &\frac{Cnk^2}{\varepsilon^2 n^2 k^2} = \frac{C}{\varepsilon^2 n} \end{aligned}$$

Consequently,

$$\mathbb{P}\Big(\frac{1}{2nk}\Big|\sum_{p=0}^{n-1}[N_p^{(k)}-\mathbb{E}(N_p^{(k)})]-\sum_{p=0}^{n-1}[\tilde{N}_p^{(k)}-\mathbb{E}(\tilde{N}_p^{(k)})]\Big|\geq\varepsilon\Big)\to 0$$

as n goes to infinity, uniformly in k. Using that $|e^{i\theta}-1|\leq|\theta|\wedge 2$, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\Big|\exp\Big\{i\frac{\theta}{2nk}\Big(\sum_{p=0}^{n-1}[N_p^{(k)}-\mathbb{E}(N_p^{(k)})]-\sum_{p=0}^{n-1}[\tilde{N}_p^{(k)}-\mathbb{E}(\tilde{N}_p^{(k)})]\Big)\Big\}-1\Big|\Big) \qquad (5)\\
\leq\ & \varepsilon+2\mathbb{P}\Big(\frac{1}{2nk}\Big|\sum_{p=0}^{n-1}[N_p^{(k)}-\mathbb{E}(N_p^{(k)})]-\sum_{p=0}^{n-1}[\tilde{N}_p^{(k)}-\mathbb{E}(\tilde{N}_p^{(k)})]\Big|\geq\varepsilon\Big)\to 0
\end{aligned}$$

as n goes to infinity, uniformly in k.
Let us remark that for every $\theta\in\mathbb{R}$,

$$\begin{aligned}
&\Big|\mathbb{E}\Big(e^{i\theta\frac{T_{nk}}{nk}}\Big)-\mathbb{E}\Big(\exp\Big[i\frac{\theta}{2nk}\Big\{M_0^{(k)}(Q_1^{(k)}-Q_0^{(k)})+\ldots+M_{n-2}^{(k)}(Q_{n-1}^{(k)}-Q_{n-2}^{(k)})\\
-\ & Q_0^{(k)}(M_1^{(k)}-M_0^{(k)})-\ldots-Q_{n-2}^{(k)}(M_{n-1}^{(k)}-M_{n-2}^{(k)})+\sum_{p=0}^{n-1}[\mathbb{E}(N_p^{(k)})-\mathbb{E}(\tilde{N}_p^{(k)})]\Big\}\Big]\Big)\Big|\\
\leq\ & \mathbb{E}\Big(\Big|\exp\Big\{i\frac{\theta}{2nk}\Big(\sum_{p=0}^{n-1}[N_p^{(k)}-\mathbb{E}(N_p^{(k)})]-\sum_{p=0}^{n-1}[\tilde{N}_p^{(k)}-\mathbb{E}(\tilde{N}_p^{(k)})]\Big)\Big\}-1\Big|\Big)
\end{aligned}$$

converging to 0 as n goes to infinity, uniformly in k.
Let $\varepsilon>0$, we can choose n_0 such that for every k,

$$\begin{aligned}
&\Big|\mathbb{E}\Big(e^{i\theta\frac{T_{n_0k}}{n_0k}}\Big)-\mathbb{E}\Big(\exp\Big[i\frac{\theta}{2n_0k}\Big\{M_0^{(k)}(Q_1^{(k)}-Q_0^{(k)})+\ldots+M_{n_0-2}^{(k)}(Q_{n_0-1}^{(k)}-Q_{n_0-2}^{(k)})\\
-\ & Q_0^{(k)}(M_1^{(k)}-M_0^{(k)})-\ldots-Q_{n_0-2}^{(k)}(M_{n_0-1}^{(k)}-M_{n_0-2}^{(k)})+\sum_{p=0}^{n_0-1}[\mathbb{E}(N_p^{(k)})-\mathbb{E}(\tilde{N}_p^{(k)})]\Big\}\Big]\Big)\Big|\leq\varepsilon.
\end{aligned}$$

The characteristic function of the random variable

$$\begin{aligned}
&[G_0^{(1)}G_1^{(2)}+(G_0^{(1)}+G_1^{(1)})G_2^{(2)}+\ldots+(G_0^{(1)}+\ldots+G_{n-2}^{(1)})G_{n-1}^{(2)}]\\
-\ &[G_0^{(2)}G_1^{(1)}+(G_0^{(2)}+G_1^{(2)})G_2^{(1)}+\ldots+(G_0^{(2)}+\ldots+G_{n-2}^{(2)})G_{n-1}^{(1)}]
\end{aligned}$$

can be written as

$$I_n(\theta):=\frac{1}{(2\pi\sqrt{a_{11}a_{22}})^n}\int_{\mathbb{R}^{2n}}e^{g_n(x,y)}\,dx_1\ldots\,dx_ndy_1\ldots dy_n$$

where g_n is the function defined for every $x=(x_1,\ldots,x_n),y=(y_1,\ldots,y_n)\in\mathbb{R}^n$ by

$$\begin{aligned}
g_n(x,y)\ =\ & \frac{i\theta}{2n}[x_1y_2+(x_1+x_2)y_3+\ldots+(x_1+\ldots+x_{n-1})y_n\\
-\ & y_1x_2-(y_1+y_2)x_3-\ldots-(y_1+\ldots+y_{n-1})x_n]\\
-\ & \frac{1}{2a_{11}}(x_1^2+\ldots+x_n^2)-\frac{1}{2a_{22}}(y_1^2+\ldots+y_n^2).
\end{aligned}$$

Now from Lemma 37 in [11], we know that

$$\lim_{n\to+\infty} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{g_n(x\sqrt{a_{11}},y\sqrt{a_{22}})}\, dxdy = \frac{1}{\cosh(\frac{\theta\sqrt{a_{11}a_{22}}}{2})}$$

uniformly in θ on a compact set of $\mathbb{R}$. So we also impose n_0 to be large enough so as to have

$$\left|I_{n_0}(\theta) - \frac{1}{\cosh(\frac{\theta\sqrt{a_{11}a_{22}}}{2})}\right| \leq \varepsilon$$

uniformly in θ on a compact set.
Under the assumption (**A**), it is possible to control the drift coming from the non-centered random variables $N_p^{(k)}$ and $\tilde{N}_p^{(k)}$.

Lemma 3.2 *Under hypothesis* (**A**),

$$\frac{1}{n_0 k} \sum_{p=0}^{n_0-1} [\mathbb{E}(N_p^{(k)}) - \mathbb{E}(\tilde{N}_p^{(k)})] = o(k). \tag{6}$$

Proof:
From the definition of the law of the random variables X_i and Y_i, the above sum can be explicitly calculated and bounded as follows:

$$\begin{aligned} & \left|\frac{1}{n_0 k} \sum_{p=0}^{n_0-1} [\mathbb{E}(N_p^{(k)}) - \mathbb{E}(\tilde{N}_p^{(k)})]\right| \\ \leq \; & \frac{1}{k} \sup_{x\in E} \left| \sum_{l=1}^{k}\sum_{i=1}^{l} [(2f_2(T^l x) - \frac{1}{3})(2f_1(T^i x) - \frac{1}{3}) - (2f_1(T^l x) - \frac{1}{3})(2f_2(T^i x) - \frac{1}{3})] \right| \end{aligned}$$

Using Abel's summation by parts,

$$\begin{aligned} \sum_{l=1}^{k}\sum_{i=1}^{l}(2f_1(T^l x) - \frac{1}{3})(2f_2(T^i x) - \frac{1}{3}) \; &= \; \sum_{l=1}^{k}(2f_1(T^l x) - \frac{1}{3})\sum_{l=1}^{k}(2f_2(T^l x) - \frac{1}{3}) \\ &- \; \sum_{l=2}^{k}(2f_2(T^l x) - \frac{1}{3})\sum_{i=1}^{l-1}(2f_1(T^i x) - \frac{1}{3}) \end{aligned}$$

Then,

$$\begin{aligned} & \sum_{l=1}^{k}\sum_{i=1}^{l}[(2f_2(T^l x) - \frac{1}{3})(2f_1(T^i x) - \frac{1}{3}) - (2f_1(T^l x) - \frac{1}{3})(2f_2(T^i x) - \frac{1}{3})] \\ = \; & 2\sum_{l=2}^{k}(2f_2(T^l x) - \frac{1}{3})\sum_{i=1}^{l-1}(2f_1(T^i x) - \frac{1}{3}) - \sum_{l=1}^{k}(2f_1(T^l x) - \frac{1}{3})\sum_{l=1}^{k}(2f_2(T^l x) - \frac{1}{3}) \\ + \; & \sum_{l=1}^{k}(2f_1(T^l x) - \frac{1}{3})(2f_2(T^l x) - \frac{1}{3}) \end{aligned}$$

In the right hand side the second sum divided by k goes to 0 uniformly in x since f_1 and f_2 belong to the class $\mathcal{C}$ and $\int_E f_i \, d\mu = 1/6, i = 1, 2$. The third one divided by k also goes to 0 uniformly in $x \in E$ since $\int_E f_1 f_2 \, d\mu = 1/36$. So, as soon as $(\mathbf{A})$ is satisfied, (6) holds.

¿From item b) and the previous lemma, the characteristic function of the random variable

$$\begin{aligned} & \frac{1}{2n_0k}\{M_0^{(k)}(Q_1^{(k)} - Q_0^{(k)}) + \ldots + M_{n_0-2}^{(k)}(Q_{n_0-1}^{(k)} - Q_{n_0-2}^{(k)}) \\ - & \quad Q_0^{(k)}(M_1^{(k)} - M_0^{(k)}) - \ldots - Q_{n_0-2}^{(k)}(M_{n_0-1}^{(k)} - M_{n_0-2}^{(k)}) + \sum_{p=0}^{n_0-1}[\mathbb{E}(N_p^{(k)}) - \mathbb{E}(\tilde{N}_p^{(k)})]\} \end{aligned}$$

converges, as k goes to infinity, to $I_{n_0}(\theta)$. Hence there exists k_0 such that for every $k \geq k_0$,

$$\left|\mathbb{E}\left(e^{i\theta \frac{T_{n_0k}}{n_0k}}\right) - \frac{1}{\cosh(\frac{\theta \sqrt{a_{11}a_{22}}}{2})}\right| \leq 3\varepsilon. \tag{7}$$

d) Let $p = n_0k + q, 0 \leq q < n_0, k \geq k_0$ (remark that $p \geq n_0k_0$) and define

$$\begin{aligned} V_{n_0}^{(k)} &= T_{n_0k+q} - T_{n_0k} \\ &= \frac{1}{2}\{(X_1 + \ldots + X_{n_0k})Y_{n_0k+1} + \ldots + (X_1 + \ldots + X_{n_0k+q-1})Y_{n_0k+q} \\ &- (Y_1 + \ldots + Y_{n_0k})X_{n_0k+1} - \ldots - (Y_1 + \ldots + Y_{n_0k+q-1})X_{n_0k+q}\} \end{aligned}$$

¿From Tchebychev's inequality, for every $\varepsilon > 0$,

$$\mathbb{P}\Big(\Big|\frac{V_{n_0}^{(k)} - \mathbb{E}(V_{n_0}^{(k)})}{n_0k}\Big| \geq \varepsilon\Big) \leq \frac{\mathrm{Var}(V_{n_0}^{(k)})}{\varepsilon^2 n_0^2 k^2}.$$

Now, using the same techniques than the ones developed in the proof of Lemma 3.1, it can be proved that

$$\mathrm{Var}(V_{n_0}^{(k)}) = \mathcal{O}(n_0^2 k).$$

Assumption $(\mathbf{A})$ implies (see the proof of Lemma 3.2) that

$$\lim_{k\to+\infty} \frac{\mathbb{E}(V_{n_0}^{(k)})}{n_0k} = 0.$$

Then, we deduce that

$$\lim_{k\to+\infty} \mathbb{P}\Big(\Big|\frac{V_{n_0}^{(k)}}{n_0k}\Big| \geq \varepsilon\Big) = 0.$$

Straightforward calculations lead to

$$\begin{aligned} & \left|\mathbb{E}\left(e^{i\theta \frac{T_p}{p}}\right) - \mathbb{E}\left(e^{i\theta \frac{T_{n_0k}}{n_0k}}\right)\right| \\ \leq & \quad \left|\mathbb{E}\left(e^{i\theta \frac{n_0k}{n_0k+q} \frac{T_{n_0k}}{n_0k} + \frac{V_{n_0}^{(k)}}{n_0k+q}}\right) - \mathbb{E}\left(e^{i\theta \frac{T_{n_0k}}{n_0k}}\right)\right| \end{aligned}$$

$$
\begin{aligned}
&\le \Big|\mathbb{E}\Big(e^{i\theta\frac{n_0k}{n_0k+q}\frac{T_{n_0k}}{n_0k}+\frac{V_{n_0}^{(k)}}{n_0k+q}}\Big)-\mathbb{E}\Big(e^{i\theta\frac{T_{n_0k}}{n_0k}+\frac{V_{n_0}^{(k)}}{n_0k+q}}\Big)\Big|\\
&+ \Big|\mathbb{E}\Big(e^{i\theta\frac{T_{n_0k}}{n_0k}+\frac{V_{n_0}^{(k)}}{n_0k+q}}\Big)-\mathbb{E}\Big(e^{i\theta\frac{T_{n_0k}}{n_0k}}\Big)\Big|\\
&\le \mathbb{E}\Big(\Big|e^{i\theta\frac{q}{n_0k+q}\frac{T_{n_0k}}{n_0k}}-1\Big|\Big)+\mathbb{E}\Big(\Big|e^{i\theta\frac{V_{n_0}^{(k)}}{n_0k+q}}-1\Big|\Big).
\end{aligned}
$$

Now, we have (since $|e^{i\theta}-1|\le 2\wedge|\theta|$) for k large enough

$$
\mathbb{E}\Big(\Big|e^{i\theta\frac{V_{n_0}^{(k)}}{n_0k+q}}-1\Big|\Big)\le\varepsilon+2\mathbb{P}\Big(\Big|\frac{\theta V_{n_0}^{(k)}}{n_0k+q}\Big|\ge\varepsilon\Big)\le 3\varepsilon.
$$

From (7), for k large enough,

$$
\Big|\mathbb{E}\Big(e^{i\theta\frac{q}{n_0k+q}\frac{T_{n_0k}}{n_0k}}\Big)-1\Big|\le 3\varepsilon. \tag{8}
$$

On a compact set of $\mathbb{R}$ (for θ), for p large enough,

$$
\begin{aligned}
\Big|\mathbb{E}\Big(e^{i\theta\frac{T_p}{p}}\Big)-\frac{1}{\cosh(\frac{\theta\ \sqrt{a_{11}a_{22}}}{2})}\Big| &\le \Big|\mathbb{E}\Big(e^{i\theta\frac{T_p}{p}}\Big)-\mathbb{E}\Big(e^{i\theta\frac{T_{n_0k}}{n_0k}}\Big)\Big|\\
&+ \Big|\mathbb{E}\Big(e^{i\theta\frac{T_{n_0k}}{n_0k}}\Big)-\frac{1}{\cosh(\frac{\theta\ \sqrt{a_{11}a_{22}}}{2})}\Big|\\
&\le 9\varepsilon,
\end{aligned}
$$

so Theorem 3.2 is proved.

3.3 Concluding remarks

As stated in the introduction, this paper is a first attempt in extending the theory of dynamic random walks developed by the first author to non-commutative algebraic structures. More results on limit theorems on Heisenberg groups are expected. The law of the iterated logarithm is presently under investigation.

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