Invertibility and Flatness of Switched Linear Discrete-time Systems

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1 Introduction

The problem of inversion and flat output characterization for switched linear discrete-time systems is tackled through a unified framework. For this class of hybrid systems, we first derive algebraic conditions to conclude on invertibility, which extends the previous results of Sain and Massey [1] dealing with linear time-invariant systems. Then, the structure of the switched inverse system is provided. Finally, based on the connection with the inversion problem and the related notion of inverse dynamics, we derive conditions which enable to check whether a given output is flat. Due to space limitation, the proofs are not provided but can be found in [2].

Notation : For any integer l, $\mathbf{1}_l$ refers to the $l-$dimensional identity matrix and $\mathbf{O}_{l\times l'}$ stands for the $l \times l'$ zero matrix. If irrelevant, the dimension of the zero matrix will be omitted and we shall merely write $\mathbf{O}$. For a matrix X, $X^\dagger$ stands for the Moore-Penrose generalized inverse of X.

2 Problem Statement and Definitions

We examine switching linear discrete-time systems of the form

$$\begin{cases} x_{k+1} &= A_{\sigma(k)}x_k + B_{\sigma(k)}u_k \\ y_k &= C_{\sigma(k)}x_k + D_{\sigma(k)}u_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^p$ are the states, the inputs and the measurements, respectively. All the matrices, namely $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$ belong to the respective finite sets $(A_j)_{1\leq j\leq J}$, $(B_j)_{1\leq j\leq J}$, $(C_j)_{1\leq j\leq J}$ and $(D_j)_{1\leq j\leq J}$. The index j corresponds to the mode of the system at a given time k and results from a switching function $\sigma : k \in \mathbb{N} \mapsto j = \sigma(k) \in \{1, \ldots, J\}$. The function $\sigma \in \Sigma$, where Σ is the set of all possible switching rules, orchestrates the switches which are triggered by exogenous events. No restriction on the time separation between switches ("dwell time") is imposed. For any time-dependent matrix $X_{\sigma(k)}$, $X_{\sigma(k_0)}^{\sigma(k_1)}$ stands for

$$\begin{aligned} X_{\sigma(k_0)}^{\sigma(k_1)} &= X_{\sigma(k_1)} X_{\sigma(k_1-1)} \dots X_{\sigma(k_0)} \text{ if } \ k_1 \geq k_0 \\ &= \mathbf{1}_n \ \text{ if } \ k_1 < k_0 \end{aligned}$$

Let $\mathcal{U}$ be the space of input sequences over $[0, \infty)$ and $\mathcal{Y}$ the corresponding output space. For each initial state $x_0 \in \mathbb{R}^n$, when the system (1) is driven by the input sequence $\{u\}_0^T = \{u_0, \dots, u_T\} \subset \mathcal{U}$, for a given mode sequence $\{\sigma\}_0^T = \{\sigma(0), \dots, \sigma(T)\}$ with $\sigma \in \Sigma$, $\{x(x_0, \sigma, u)\}_0^T$ refers to the solution of (1) in the interval of time $[0, T]$ and $\{y(x_0, \sigma, u)\}_0^T \subset \mathcal{Y}$ refers to the corresponding output in the same interval of time $[0, T]$.

Definition 1. *The system (1) is* invertible for a given switching rule $\sigma \in \Sigma$*, if there exists a positive integer* $r < \infty$ *such that, for two any input sequences* $\{u\}_0^r, \{u'\}_0^r \subset \mathcal{U}$*, the following implication applies:*

$$\{y(x_0, \sigma, u)\}_0^r = \{y(x_0, \sigma, u')\}_0^r \Rightarrow u_0 = u'_0 \quad \forall x_0 \tag{2}$$

The integer r will be called the inherent delay of (1).

Definition 2. *The system (1) is* invertible *if it is invertible for all* $\sigma \in \Sigma$

By *invertibility*, we thereby mean here the ability to infer the input u_0 from a finite number r of measurements y_i $(i = 0, \dots, r)$, the state vector x_0 and the mode sequence $\{\sigma\}_0^r$ both associated to $\{y(x_0, \sigma, u)\}_0^r$ and $\{y(x_0, \sigma, u')\}_0^r$ being identical.

Remark: In the foregoing, the initial condition is considered at the special discrete time $k = 0$. However, we could take any discrete time k and then refer to the interval of time $[k, k + r]$ instead of $[0, r]$.

We are concerned with three closely related issues:

i) determining algebraic conditions, in terms of the state matrices description, under which the system (1) is invertible (invertibility)
ii) looking for a second system, which, when cascaded with the system (1) and so driven by its outputs, produces the same input under an identical initial condition and an identical mode sequence (inversion)
iii) providing some algebraic conditions, in terms of the state matrices description, to check whether a given output of (1) is flat (flatness)

We introduce the following matrices and vectors : $M^0_{\sigma(k)} = D_{\sigma(k)}$, $M^{i(>0)}_{\sigma(k)} =$

$$\begin{pmatrix} D_{\sigma(k)} & \mathbf{0}_{p\times m} & \cdots & \cdots & \cdots \\ C_{\sigma(k+1)}B_{\sigma(k)} & D_{\sigma(k+1)} & \mathbf{0}_{p\times m} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ C_{\sigma(k+i)}A^{\sigma(k+i-1)}_{\sigma(k+1)}B_{\sigma(k)} & C_{\sigma(k+i)}A^{\sigma(k+i-1)}_{\sigma(k+2)}B_{\sigma(k+1)} & \cdots & C_{\sigma(k+i)}B_{\sigma(k+i-1)} & D_{\sigma(k+i)} \end{pmatrix}$$

$$\mathcal{O}^i_{\sigma(k)} = \begin{pmatrix} C_{\sigma(k)} \\ C_{\sigma(k+1)}A_{\sigma(k)} \\ \vdots \\ C_{\sigma(k+i)}A^{\sigma(k+i-1)}_{\sigma(k)} \end{pmatrix}, \quad \underline{y}^i_k = \begin{pmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+i} \end{pmatrix}, \quad \bar{I}_m = (\mathbf{1}_m \ \mathbf{0}_{m\times(m\cdot r)})$$

3 Main results

3.1 Invertibility

Theorem 1. *The following statements are equivalent.*

i) The system (1) is invertible
ii) There exists a finite integer r such that the equation

$$Q^r_{\sigma(k)} M^r_{\sigma(k)} = \bar{I}_m \tag{3}$$

has a solution in $Q^r_{\sigma(k)}$ for all $\sigma \in \Sigma$
iii) There exists a finite integer r such that for all $\sigma \in \Sigma$

$$rank \begin{pmatrix} M^r_{\sigma(k)} \\ \bar{I}_m \end{pmatrix} - rank\ M^r_{\sigma(k)} = 0 \tag{4}$$

As a matter of fact, $\bar{I}_m$ can be replaced by any matrix of the form $(F\ \mathbf{0}_{m\times(m\cdot r)})$ where F is any rank m square matrix of size m. An explicit solution $Q^r_{\sigma(k)}$ of (3) is:

$$Q^r_{\sigma(k)} = \bar{I}_m M^{r\dagger}_{\sigma(k)} \tag{5}$$

3.2 Inversion

Definition 3. *A system is a r-delayed inverse for (1) if, under an identical initial condition x_0 and an identical switching rule σ, when driven by $\underline{y}^r_k$, its output fulfills $\hat{u}_{k+r} = u_k$ for all $k \geq 0$*

Theorem 2. *Assume that (1) is* invertible *with inherent delay r. The system*

$$\begin{cases} \hat{x}_{k+r+1} = P^r_{\sigma(k)} \hat{x}_{k+r} + B_{\sigma(k)} Q^r_{\sigma(k)} \underline{y}^r_k \\ \hat{u}_{k+r} \quad = -Q^r_{\sigma(k)} \mathcal{O}^r_{\sigma(k)} \hat{x}_{k+r} + Q^r_{\sigma(k)} \underline{y}^r_k \end{cases} \tag{6}$$

with

$$P^r_{\sigma(k)} = A_{\sigma(k)} - B_{\sigma(k)} Q^r_{\sigma(k)} \mathcal{O}^r_{\sigma(k)} \tag{7}$$

is a r-delayed inverse system for (1).

Actually, the system (6) enables to retrieve not only the input u_k but the state x_k of (1) as well. Thus, when x_0 is unknown, it can act as an unknown input observer by incorporating an extra gain.

3.3 Flatness

Let us recall that flatness was introduced by Fliess *and al.* [3] in 1995.

Definition 4. *A system with input u_k and state x_k, assumed to be square ($m = p$), is said to be* flat *if there exists a set of independent variables y_k, referred to as flat outputs, such that all system variables can be expressed as a function of the flat outputs and a finite number of its backward and/or forward iterates. In particular, there exist two functions $\mathcal{F}$ and $\mathcal{G}$, $\mathbb{Z}$-valued integers $k_{\mathcal{F}}$, $k'_{\mathcal{F}}$, $k_{\mathcal{G}}$ and $k'_{\mathcal{G}}$ such that*

$$\begin{cases} x_k = \mathcal{F}(y_{k+k_{\mathcal{F}}}, \ldots, y_{k+k'_{\mathcal{F}}}) \\ u_k = \mathcal{G}(y_{k+k_{\mathcal{G}}}, \ldots, y_{k+k'_{\mathcal{G}}}) \end{cases} \tag{8}$$

Theorem 3. *A componentwise independent output y_k of the system (1) assumed to be square ($m = p$), invertible for a given switching rule $\sigma \in \Sigma$ and with inherent delay r, is a* flat output *if there exists a positive integer $K < \infty$ such that the following equality applies for all $k \geq 0$*

$$P^{\sigma(k+K-1)}_{\sigma(k)} = \mathbf{0} \tag{9}$$

References

1. Sain M.K., Massey J.L.: Invertibility of linear time-invariant dynamical systems. IEEE Trans. Automatic Control **14** (1969) 141–149
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3. Fliess M., Levine J., Martin P., Rouchon P.: Flatness and defect of non-linear systems: introductory theory and examples. Int. Jour. of Control **61**(6) (1995) 1327–1361