Multidimensional bifractional Brownian motion: Itô and Tanaka formulas

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Abstract

Using the Malliavin calculus with respect to Gaussian processes and the multiple stochastic integrals we derive Itô's and Tanaka's formulas for the d-dimensional bifractional Brownian motion.

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1 Introduction

The stochastic calculus with respect to the fractional Brownian motion (fBm) has now a long enough history. Since the nineties, many authors used different approaches to develop a stochastic integration theory with respect to this process. We refer, among of course many others, to [1], [8], [12] or [9]. The reason for this tremendous interest in the stochastic analysis of the fBm comes from its large amount of applications in practical phenomena such as telecommunications, hydrology or economics.

Nevertheless, even fBm has its limits in modeling certain phenomena. Therefore, several authors introduced recently some generalizations of the fBm which are supposed to fit better in concrete situations. For example, we mention the multifractional Brownian motion (see e.g.

[2]), the subfractional Brownian motion (see e.g. [5]) or the multiscale fractional Brownian motion (see [3]).

Here our main interest consists in the study of the *bifractional Brownian motion (bifBm).* The bifBm has been introduced by Houdré and Villa in [13] and a stochastic analysis for it can be found in [19]. Other papers treated different aspects of this stochastic process, like sample pats properties, extension of the parameters or statistical applications (see [6], [4], [22] or [11]. Recall that the bifBm $B^{H,K}$ is a centered Gaussian process, starting from zero, with covariance function

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right) \tag{1}$$

where the parameters H, K are such that $H \in (0,1)$ and $K \in (0,1]$. In the case $K = 1$ we retrieve the fractional Brownian motion while the case $K = 1$ and $H = \frac{1}{2}$ corresponds to the standard Brownian motion.

The process $B^{H,K}$ is HK-selfsimilar but it has no stationary increments. It has Hölder continuous paths of order $\delta < HK$ and its paths are not differentiable. An interesting property of it is the fact that its quadratic variation in the case $2HK = 1$ is similar to that of the standard Brownian motion, i.e. $[B^{H,K}]_t = cst. \times t$ and therefore especially this case ($2HK = 1$) is very interesting from the stochastic calculus point of view.

In this paper, our purpose is to study multidimensional bifractional Brownian motion and to prove Itô and Tanaka formulas. We start with the one dimensional bifBm and we first derive an Itô and an Tanaka formula for it when $2HK \geq 1$. We mention that the Itô formula has been already proved by [15] but here we propose an alternative proof based on the Taylor expansion which appears to be also useful in the multidimensional settings. The Tanaka formula is obtained from the Itô formula by a limit argument and it involves the so-called *weighted local time* extending the result in [7]. In the multidimensional case we first derive an Itô formula for $2HK > 1$ and we extend it to Tanaka by following an idea by Uemura [20], [21]; that is, since $|x|$ is twice the kernel of the one-dimensional Newtonian potential, i.e. $\frac{1}{2}\Delta|x|$ is equal to the delta Dirac function $\delta(x)$, we will chose the function $U(z), z \in \mathbb{R}^d$ which is twice of the kernel of d-dimensional Newtonian (or logarithmic if $d = 2$) potential to replace $|x|$ in the d-dimensional case. See the last section for the definition of the function U. Our method is based on the Wiener-Itô chaotic expansion into multiple stochastic integrals following ideas from [14] or [10]. The multidimensional Tanaka formula also involves a generalized local time. We note that the terms appearing in our Tanaka formula when $d \geq 2$ are not random variables and they are understood as distributions in the Watanabe spaces.

2 Preliminaries: Deterministic spaces associated and Malliavin calculus

Let $\left(B_t^{H,K}, t \in [0,T]\right)$ be a bifractional Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$.

Being a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to $B^{H,K}$. We refer to [1], [17] for a complete description of stochastic calculus with respect to Gaussian processes. Here we recall only the basic elements of this theory.

The basic ingredient is the canonical Hilbert space $\mathcal{H}$ associated to the bifractional Brownian motion (bifBm). This space is defined as the completion of the linear space $\mathcal{E}$ generated by the indicator functions $1_{[0,t]}, t \in [0,T]$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t,s).$$

The application $\varphi \in \mathcal{E} \to B(\varphi)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B^{H,K}$ and it can be extended to $\mathcal{H}$.

Let us denote by $\mathcal{S}$ the set of smooth functionals of the form

$$F = f(B(\varphi_1), \ldots, B(\varphi_n))$$

where $f \in C_b^\infty(\mathbb{R}^n)$ and $\varphi_i \in \mathcal{H}$. The Malliavin derivative of a functional F as above is given by

$$D^{B^{H,K}} F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i$$

and this operator can be extended to the closure $\mathbb{D}^{m,2}$ $(m \geq 1)$ of $\mathcal{S}$ with respect to the norm

$$\|F\|^2_{m,2} := E\,|F|^2 + E\|D^{B^{H,K}} F\|^2_{\mathcal{H}} + \ldots + E\|D^{B^{H,K},m} F\|^2_{\mathcal{H}^{\hat{\otimes} m}}$$

where $\mathcal{H}^{\hat{\otimes} m}$ denotes the m fold symmetric tensor product and the mth derivative $D^{B^{H,K},m}$ is defined by iteration.

The divergence integral $\delta^{B^{H,K}}$ is the adjoint operator of $D^{B^{H,K}}$. Concretely, a random variable $u \in L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator $(Dom(\delta^{B^{H,K}}))$ if

$$E\left|\langle D^{B^{H,K}} F, u\rangle_{\mathcal{H}}\right| \leq c\|F\|_{L^2(\Omega)}$$

for every $F \in \mathcal{S}$. In this case $\delta^{B^{H,K}}(u)$ is given by the duality relationship

$$E(F\delta^{B^{H,K}}(u)) = E\langle D^{B^{H,K}} F, u\rangle_{\mathcal{H}}$$

for any $F \in \mathbb{D}^{1,2}$. It holds that

$$E\delta^{B^{H,K}}(u)^2 = E\|u\|^2_{\mathcal{H}} + E\langle D^{B^{H,K}} u, (D^{B^{H,K}} u)^* \rangle_{\mathcal{H}\otimes\mathcal{H}} \tag{2}$$

where $(D^{B^{H,K}} u)^*$ is the adjoint of $D^{B^{H,K}} u$ in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

Sometimes working with the space $\mathcal{H}$ is not convenient; once, because this space may contain also distributions (as, e.g. in the case $K = 1$, see [18]) and twice, because the norm in this space is not always tractable. We will use the subspace $|\mathcal{H}|$ of $\mathcal{H}$ which is defined as the set of measurable function f on $[0,T]$ with

$$\|f\|^2_{|\mathcal{H}|} := \int_0^T \int_0^T |f(u)|\,|f(v)|\,\frac{\partial^2 R}{\partial u \partial v}(u,v)dudv < \infty. \tag{3}$$

It follows actually from [15] that the space $|\mathcal{H}|$ is a Banach space for the norm $\|\cdot\|_{|\mathcal{H}|}$ and it is included in $\mathcal{H}$. In fact,

$$L^2([0,T]) \subset L^{\frac{1}{HK}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}. \tag{4}$$

and

$$E\delta^{B^{H,K}}(u)^2 \leq E\|u\|^2_{|\mathcal{H}|} + E\|D^{B^{H,K}}u\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} \tag{5}$$

where, if $\varphi : [0,T]^2 \to \mathbb{R}$

$$\|\varphi\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} = \int_{[0,T]^4} |\varphi(u,v)| \left|\varphi(u',v')\right| \frac{\partial^2 R}{\partial u \partial u'}(u,u') \frac{\partial^2 R}{\partial v \partial v'}(v,v') du dv du' dv'. \tag{6}$$

We will use the following formulas of the Malliavin calculus: the integration by parts

$$F\delta^{B^{H,K}}(u) = \delta^{B^{H,K}}(Fu) + \langle D^{B^{H,K}}F, u\rangle_{\mathcal{H}} \tag{7}$$

for any $u \in Dom(\delta^{B^{H,K}})$, $F \in \mathbb{D}^{1,2}$ such that $Fu \in L^2(\Omega;\mathcal{H})$; and the chain rule

$$D^{B^{H,K}}\varphi(F) = \sum_{i=1}^{n} \partial_i \varphi(F) D^{B^{H,K}} F^i \tag{8}$$

if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and $F = (F^1, \ldots, F^m)$ is a random vector with components in $\mathbb{D}^{1,2}$.

By the duality between $D^{B^{H,K}}$ and $\delta^{B^{H,K}}$ we obtain the following result for the convergence of divergence integrals: if $u_n \in Dom(\delta^{B^{H,K}})$ for every n, $u_n \underset{n\to\infty}{\to} u$ in $L^2(\Omega;\mathcal{H})$ and $\delta^{B^{H,K}}(u_n) \underset{n\to\infty}{\to} G$ in $L^2(\Omega)$ then

$$u \in Dom(\delta^{B^{H,K}}) \text{ and } \delta^{B^{H,K}}(u) = G. \tag{9}$$

It is also possible to introduce multiple integrals $I_n(f_n)$, $f \in \mathcal{H}^{\otimes n}$ with respect to $B^{H,K}$. Let

$$F = \sum_{n\geq 0} I_n(f_n) \tag{10}$$

where for every $n \geq 0$, $f_n \in \mathcal{H}^{\otimes n}$ are symmetric functions. Let L be the Ornstein-Uhlenbeck operator

$$LF = -\sum_{n\geq 0} n I_n(f_n)$$

if F is given by (10).

For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(I-L)^{\frac{\alpha}{2}}\|_{L^p(\Omega)}$$

where I represents the identity. In this way, a random variable F as in (10) belongs to $\mathbb{D}^{\alpha,2}$ if and only if

$$\sum_{n\geq 0}(1+n)^{\alpha}\|I_n(f_n)\|^2_{L^2(\Omega)}<\infty.$$

Note that the Malliavin derivative operator acts on multiple integral as follows

$$D_t^{B^{H,K}}F=\sum_{n=1}^{\infty}nI_{n-1}(f_n(\cdot,t)),\qquad t\in[0,T].$$

The operator $D^{B^{H,K}}$ is continuous from $\mathbb{D}^{\alpha-1,p}$ into $\mathbb{D}^{\alpha,p}(\mathcal{H})$. The adjoint of $D^{B^{H,K}}$ is denoted by $\delta^{B^{H,K}}$ and is called the divergence (or Skorohod) integral. It is a continuous operator from $\mathbb{D}^{\alpha,p}(\mathcal{H})$ into $\mathbb{D}^{\alpha,p}$. For adapted integrands, the divergence integral coincides to the classical Itô integral. We will use the notation

$$\delta^{B^{H,K}}(u)=\int_0^T u_s\delta B_s.$$

Recall that if u is a stochastic process having the chaotic decomposition

$$u_s=\sum_{n\geq 0}I_n(f_n(\cdot,s))$$

where $f_n(\cdot,s)\in\mathcal{H}^{\otimes n}$ for every s, and it is symmetric in the first n variables, then its Skorohod integral is given by

$$\int_0^T u_s dB_s=\sum_{n\geq 0}I_{n+1}(\tilde{f}_n)$$

where $\tilde{f}_n$ denotes the symmetrization of f_n with respect to all $n+1$ variables.

3 Tanaka formula for unidimensional bifractional Brownian motion

This paragraph is consecrated to the proof of Itô formula and Tanaka formula for the one-dimensional bifractional Brownian motion with $2HK\geq 1$. Note that the Itô formula has been already proved in [15]; here we propose a different approach based on the Taylor expansion which be also used in the multidimensional settings.

We start by the following technical lemma.

Lemma 1 *Let us consider the following function on* $[1,\infty)$

$$h(y)=y^{2HK}+(y-1)^{2HK}-\frac{2}{2^K}\left(y^{2H}+(y-1)^{2H}\right)^K.$$

where $H \in (0,1)$ *and* $K \in (0,1)$*. Then,*

$$h(y) \quad \text{converges to } 0 \text{ as } y \text{ goes to } \infty. \tag{11}$$

Moreover if $2HK = 1$ *we obtain that*

$$\lim_{y \to +\infty} y h(y) = \frac{1}{4}(1 - 2H). \tag{12}$$

Proof: Let $y = \frac{1}{\varepsilon}$, hence

$$h(y) = h\left(\frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon^{2HK}}\left[1 + (1-\varepsilon)^{2HK} - \frac{2}{2^K}\left(1 + (1-\varepsilon)^{2H}\right)^K\right]$$

Using Taylor's expansion, as ε close to 0, we obtain

$$h\left(\frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon^{2HK}}\left(H^2K(K-1)\varepsilon^2 + o(\varepsilon^2)\right) \tag{13}$$

Thus

$$\lim_{y \to +\infty} h(y) = \lim_{\varepsilon \to 0} h(1/\varepsilon) = 0.$$

For the case $2HK = 1$ we replace in (13), we have

$$\frac{1}{\varepsilon} h\left(\frac{1}{\varepsilon}\right) = \frac{1}{4}(1 - 2H) + \frac{1}{\varepsilon^2} o(\varepsilon^2)$$

Thus (12) is satisfied. Which completes the proof. ■

Theorem 1 *Let* f *be a function of class* C^2 *on* $\mathbb{R}$*. Suppose that* $2HK \geq 1$*, then*

$$f\left(B_t^{H,K}\right) = f(0) + \int_0^t f'\left(B_s^{H,K}\right)\delta B_s^{H,K} + HK\int_0^t f''\left(B_s^{H,K}\right) s^{2HK-1} ds. \tag{14}$$

Proof: We first prove the case $2HK > 1$. Let us fix $t > 0$ and let be $\pi := \{t_j = \frac{jt}{n}; j = 0, ..., n\}$ a partition of $[0,t]$. By the localization argument and the fact that the process $B^{H,K}$ is continuous, We can assume that f has compact support, and so f, f' and f'' are bounded. Using Taylor expansion, we have

$$\begin{aligned} f\left(B_t^{H,K}\right) &= f(0) + \sum_{j=1}^{n} f'\left(B_{t_{j-1}}^{H,K}\right)\left(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\right) + \frac{1}{2}\sum_{j=1}^{n} f''\left(\overline{B}_j^{H,K}\right)\left(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\right)^2 \\ &:= f(0) + I^n + J^n. \end{aligned} \tag{15}$$

where $\overline{B}_j^{H,K} = B_{t_{j-1}}^{H,K} + \theta_j \left(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K} \right)$ with θ_j is a r.v in $(0,1)$.
Since $B^{H,K}$ is a quasi-helix (see [19]), we can bound the term J^n as follows:

$$\begin{aligned} E|J^n|^2 &\leq nC/4 \sum_{j=1}^{n} E\left(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K} \right)^4 \\ &\leq 2^{-2K} nC \sum_{j=1}^{n} |t_j - t_{j-1}|^{4HK} \\ &\leq 2^{-2K} nC t^{4HK} \sup_{j=1}^{n} |t_j - t_{j-1}|^{4HK-1} = 2^{-2K} C \frac{t^{4HK}}{n^{4HK-2}} \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

where C a constant depends de f''. Then

$$J^n \underset{n\to\infty}{\longrightarrow} 0 \text{ in } L^2(\Omega). \tag{16}$$

On the other hand, we apply (7) and we get

$$\begin{aligned} I^n &= \sum_{j=1}^{n} f'\left(B_{t_{j-1}}^{H,K} \right) \left(\delta^{B^{H,K}} (1_{(t_{j-1},t_j]}) \right) \\ &= \delta^{B^{H,K}} \left(\sum_{j=1}^{n} f'\left(B_{t_{j-1}}^{H,K} \right) 1_{(t_{j-1},t_j]}(.) \right) + \sum_{j=1}^{n} f''\left(B_{t_{j-1}}^{H,K} \right) \langle 1_{(0,t_{j-1}]}, 1_{(t_{j-1},t_j]} \rangle_{\mathcal{H}} \\ &= I_1^n + I_2^n. \end{aligned}$$

Next

$$\begin{aligned} I_2^n &= \sum_{j=1}^{n} f''\left(B_{t_{j-1}}^{H,K} \right) \left(R(t_{j-1}, t_j) - R(t_{j-1}, t_{j-1}) \right) \\ &= \sum_{j=1}^{n} \left[f''\left(B_{t_{j-1}}^{H,K} \right) \left(\frac{1}{2^K} \left((t_j^{2H} + t_{j-1}^{2H})^K - (t_j - t_{j-1})^{2HK} \right) - t_{j-1}^{2HK} \right) \right] \end{aligned}$$

We denote by

$$A_t := HK \int_0^t s^{2HK-1} ds = \frac{1}{2} t^{2HK}$$

To prove that I_2^n converges to $HK \int_0^t f''\left(B_s^{H,K} \right) s^{2HK-1} ds$ in $L^2(\Omega)$ as $n \to \infty$, it suffices to show that

$$C_n := \left(E \left| I_2^n - \sum_{j=1}^{n} f''\left(B_{t_{j-1}}^{H,K} \right) \left(A_{t_j} - A_{t_{j-1}} \right) \right|^2 \right)^{1/2} \underset{n\to\infty}{\longrightarrow} 0.$$

By Minkowski inequality, we have

$$\begin{aligned}
C_n &\leq C\sum_{j=1}^{n}\left|\left(\frac{1}{2^K}((t_j^{2H}+t_{j-1}^{2H})^K-(t_j-t_{j-1})^{2HK})-t_{j-1}^{2HK}\right)-\frac{1}{2}(t_j^{2HK}-t_{j-1}^{2HK})\right| \\
&\leq Ct^{2HK}/2\left[\frac{1}{n^{2HK}}\sum_{j=1}^{n}|h(j)|+\frac{2}{2^K}\frac{1}{n^{2HK-1}}\right] \\
&= Ct^{2HK}/2\left[C_n^1+C_n^2\right]
\end{aligned}$$

where C is a generic constant.
Since $2HK>1$ then $C_n^2:=\frac{2}{2^K}\frac{1}{n^{2HK-1}}\underset{n\to\infty}{\longrightarrow}0$. According to (11), we obtain

$$C_n^1:=\frac{1}{n^{2HK}}\sum_{j=1}^{n}h(j)\leq\frac{C}{n^{2HK-1}}\underset{n\to\infty}{\longrightarrow}0.$$

Thus

$$I_2^n\underset{n\to\infty}{\longrightarrow}HK\int_0^t f''\left(B_s^{H,K}\right)s^{2HK-1}ds \text{ in } L^2(\Omega)$$

We show now that

$$\left(\sum_{j=1}^{n}f'\left(B_{t_{j-1}}^{H,K}\right)1_{(t_{j-1},t_j]}(.)\right)\underset{n\to\infty}{\longrightarrow}f'\left(B_.^{H,K}\right)1_{(0,t]}(.) \text{ in } L^2(\Omega;\mathcal{H}).$$

Indeed, using the quasi-helix property of $B^{H,K}$, we obtain

$$\begin{aligned}
&E\left|\,||\sum_{j=1}^{n}\left[f'\left(B_{t_{j-1}}^{H,K}\right)-f'\left(B_.^{H,K}\right)\right]1_{(t_{j-1},t_j]}(.)||_{\mathcal{H}}^2\right| \\
&= E\sum_{j,l=1}^{n}\int_{t_{j-1}}^{t_j}\int_{t_{l-1}}^{t_l}\left|f'\left(B_{t_{j-1}}^{H,K}\right)-f'\left(B_u^{H,K}\right)\right|\left|f'\left(B_{t_{l-1}}^{H,K}\right)-f'\left(B_v^{H,K}\right)\right|\frac{\partial^2R}{\partial u\partial v}(u,v)dudv \\
&\leq 2^{1-K}(\sup_{x\in\mathbb{R}}|f''(x)|)^2\sup_{i=1,\ldots,n}|t_i-t_{i-1}|^{2HK}\sum_{j,l=1}^{n}\int_{t_{j-1}}^{t_j}\int_{t_{l-1}}^{t_l}\frac{\partial^2R}{\partial u\partial v}(u,v)dudv \\
&= 2^{1-K}(\sup_{x\in\mathbb{R}}|f''(x)|)^2\sup_{i=1,\ldots,n}|t_i-t_{i-1}|^{2HK}R(t,t)\rightarrow_{n\to\infty}0
\end{aligned}$$

This with (15) and (16) implies that I_1^n converges to $\delta^{B^{H,K}}\left(f'\left(B_.^{H,K}\right)1_{(0,t]}(.)\right)$ in $L^2(\Omega)$. Therefore (14) is established due to (9). ■

The proof of the case $2HK=1$ is based on a preliminary result concerning the quadratic variation of the bifractional Brownian motion. It was proved in [19] using the stochastic calculus via regularization.

Lemma 2 *Suppose that $2HK = 1$, then*

$$V_t^n := \sum_{j=1}^{n} \left(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K} \right)^2 \underset{n\to\infty}{\longrightarrow} \frac{1}{2^{k-1}} t \quad \text{in } L^2(\Omega).$$

Proof: A straightforward calculation shows that,

$$EV_t^n = \frac{t}{n} \sum_{j=1}^{n} h(j) + \frac{t}{2^{k-1}} \underset{n\to\infty}{\longrightarrow} \frac{t}{2^{k-1}}.$$

To obtain the conclusion it suffices to show that

$$\lim_{n\to\infty} E(V_t^n)^2 = (\frac{t}{2^{k-1}})^2.$$

In fact we have,

$$E(V_t^n)^2 = \sum_{i,j=1}^{n} E\left((B_{t_i}^{H,K} - B_{t_{i-1}}^{H,K})(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}) \right)^2$$

Denote by

$$\mu_n(i,j) = E\left((B_{t_i}^{H,K} - B_{t_{i-1}}^{H,K})(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}) \right)^2$$

It follows by linear regression that

$$\mu_n(i,j) = E\left(N_1^2 \left| \theta_n(i,j) N_1 + \sqrt{\delta_n(i,j) - (\theta_n(i,j))^2} N_2 \right|^2 \right)$$

where N_1 and N_2 two independent normal random variables,

$$\begin{aligned} \theta_n(i,j) &:= E\left((B_{t_i}^{H,K} - B_{t_{i-1}}^{H,K})(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}) \right) \\ &= \frac{t}{2^K n} \left[(i^{2H} + j^{2H})^K - 2|j-i| - (i^{2H} + (j-1)^{2H})^K + |j-i-1| \right. \\ &\quad \left. -((i-1)^{2H} + j^{2H})^K + |j-i+1| + ((i-1)^{2H} + (j-1)^{2H})^K \right] \end{aligned}$$

and

$$\delta_n(i,j) := E\left(B_{t_i}^{H,K} - B_{t_{i-1}}^{H,K} \right)^2 E\left(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K} \right)^2.$$

Hence

$$\mu_n(i,j) = 2(\theta_n(i,j))^2 + \delta_n(i,j)$$

For $1 \le i < j$, we define a function $f_j : (1, \infty) \to \mathbb{R}$, by

$$\begin{aligned} f_j(x) &= ((x-1)^{2H} + j^{2H})^K - ((x-1)^{2H} + (j-1)^{2H})^K \\ &\quad -(x^{2H} + j^{2H})^K + (x^{2H} + (j-1)^{2H})^K \end{aligned}$$

We compute

$$\begin{aligned} f_j'(x) &= \left(\frac{(x-1)^{2H} + j^{2H}}{(x-1)^{2H}}\right)^{K-1} - \left(\frac{(x-1)^{2H} + (j-1)^{2H}}{(x-1)^{2H}}\right)^{K-1} \\ &\quad - \left(\frac{x^{2H} + j^{2H}}{x^{2H}}\right)^{K-1} + \left(\frac{x^{2H} + (j-1)^{2H}}{x^{2H}}\right)^{K-1} \\ &:= g(x-1) - g(x) \ge 0 \end{aligned}$$

Hence f_j is increasing and positive, since the function

$$g(x) = \left(1 + \frac{j^{2H}}{x^{2H}}\right)^{K-1} - \left(1 + \frac{(j-1)^{2H}}{x^{2H}}\right)^{K-1}$$

is decreasing on $(1, \infty)$. This implies that for every $1 \le i < j$

$$|\theta_n(i,j)| = \frac{t}{2^K n} f_j(i) \le \frac{t}{2^K n} f_j(j) \le \frac{t}{n} |h(j)|$$

and $|\theta_n(i,i)| = \frac{t}{n}|h(i) + 2|$ for any $i \ge 1$.
Thus

$$\sum_{i,j=1}^{n} \theta_n(i,j)^2 \le \frac{2t^2}{n^2} \sum_{\substack{i<j \\ i,j=1}}^{n} h(j)^2 + \frac{t^2}{n^2} \sum_{i=1}^{n} (h(i)+2)^2.$$

Combining this with (12), we obtain that $\sum_{i,j=1}^n \theta_n(i,j)^2$ converges to 0 as $n \to \infty$.
On the other hand, by (12)

$$\sum_{i,j=1}^{n} \delta_n(i,j) = \frac{t^2}{n^2} \sum_{i,j=1}^{n} \left(h(i) + \frac{1}{2^{K-1}}\right)\left(h(j) + \frac{1}{2^{K-1}}\right) \underset{n\to\infty}{\longrightarrow} \left(\frac{t}{2^{K-1}}\right)^2.$$

Consequently, $E(V_t^n)^2$ converges to $\left(\frac{t}{2^{K-1}}\right)^2$ as $n \to \infty$, and the conclusion follows. ■

Proof: [Proof of the Theorem 1 in the case $2HK = 1$.] In this case we shall prove that

$$I_1^n \underset{n\to\infty}{\longrightarrow} \int_0^t f'\left(B_s^{H,K}\right) \delta B_s^{H,K} \text{ in } L^2(\Omega), \tag{17}$$

$$I_2^n \underset{n\to\infty}{\longrightarrow} \left(\frac{1}{2} - \frac{1}{2^K}\right) \int_0^t f''\left(B_s^{H,K}\right) ds \text{ in } L^2(\Omega), \tag{18}$$

and

$$J^n \underset{n\to\infty}{\longrightarrow} \frac{1}{2^K} \int_0^t f''\left(B_s^{H,K}\right) ds \text{ in } L^2(\Omega). \tag{19}$$

To prove (18), it is enough to establish that

$$E_n := \left(E\left|I_2^n - \left(\frac{1}{2} - \frac{1}{2^K}\right)\sum_{j=1}^n f''\left(B_{t_{j-1}}^{H,K}\right)(t_j - t_{j-1})\right|^2\right)^{1/2} \underset{n\to\infty}{\longrightarrow} 0.$$

Appliquing the Minkowsky inequality and (12), we obtain

$$\begin{aligned}
E_n &\leq C\sum_{j=1}^n \left|\frac{1}{2^K}(t_j^{2H} + t_{j-1}^{2H})^K - \frac{1}{2}(t_j + t_{j-1})\right| \\
&\leq \frac{C}{2n}\sum_{j=1}^n \left|2j - 1 - \frac{2}{2^K}(j^{2H} + (j-1)^{2H})^K\right| \\
&= \frac{C}{2n}\sum_{j=1}^n h(j) \leq \frac{C}{2n}\sum_{j=1}^n \frac{1}{j} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

By using Lemma 2, we conclude that

$$J^n \underset{n\to\infty}{\longrightarrow} \frac{1}{2^K} \int_0^t f''\left(\overline{B}_s^{H,K}\right) ds \ \text{ in } L^2(\Omega).$$

The rest of the proof is same as in the case $2HK > 1$. ■

Let us regard now the Tanaka formula. As in the case of the standard fractional Brownian motion, it will involve the so-called *weighted local time* L_t^x ($x \in \mathbb{R}$, $t \in [0,T]$) of $B^{H,K}$ defined as the density of the occupation measure

$$A \in \mathcal{B}(\mathbb{R}) \longrightarrow 2HK \int_0^t 1_A(B_s^{H,K}) s^{2HK-1} ds.$$

Theorem 2 *Let* $\left(B_t^{H,K}, t \in [0,T]\right)$ *be a bifractional Brownian motion with* $2HK \geq 1$. *Then for each* $t \in [0,T]$, $x \in \mathbb{R}$ *the following formula holds*

$$\left|B_t^{H,K} - x\right| = |x| + \int_0^t sign(B_s - x)\delta B_s^{H,K} + L_t^x. \tag{20}$$

Proof: Let $p_\varepsilon(y) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{y^2}{2\varepsilon}}$ be the Gaussian kernel and put

$$F'_\varepsilon(z) = 2\int_{-\infty}^{z} p_\varepsilon(y)dy - 1,$$

and

$$F_\varepsilon(z) = \int_0^z F'_\varepsilon(y)dy.$$

By the Theorem 1 we have

$$\begin{aligned} F_\varepsilon\left(B_t^{H,K} - x\right) &= F_\varepsilon(-x) + \int_0^t F'_\varepsilon\left(B_s^{H,K} - x\right)\delta B_s^{H,K} \\ &\quad + HK\int_0^t p_\varepsilon\left(B_s^{H,K} - x\right)s^{2HK-1}ds. \end{aligned} \tag{21}$$

Using (a slightly adaptation of) Proposition 9 in [19], one can prove that

$$L_t^x = \lim_{\varepsilon\to 0} 2HK\int_0^t p_\varepsilon(B_s^{H,K} - x)s^{2HK-1}ds \quad \text{in } L^2(\Omega). \tag{22}$$

and L_t^x admits the following chaotic representation into multiple stochastic integrals (here I_n represents the multiple integral with respect to the bifBm)

$$L_t^x = 2HK\sum_{n=0}^{\infty}\int_0^t \frac{p_{s^{2HK}}(x)}{s^{(n-2)HK+1}} H_n\left(\frac{x}{s^{HK}}\right) I_n(1_{[0,s]}^{\otimes n})ds \tag{23}$$

where H_n is the nth Hermite polynomial defined as

$$H_n(x) = \frac{(-1)^n}{n!}e^{x^2/2}\frac{d^n}{dx^n}(e^{-x^2/2}) \quad \text{for every } n \geq 1.$$

We have $F_\varepsilon(x) \to |x|$ as $\varepsilon \to 0$ and since $F_\varepsilon(x) \leq |x|$, then by Lebesgue's dominated convergence theorem we obtain that $F_\varepsilon(B_t^{H,K} - x)$ converges to $|B_t^{H,K} - x|$ in $L^2(\Omega)$ as $\varepsilon \to 0$.

On the other hand, since $0 \leq F'_\varepsilon(x) \leq 1$ and $F'_\varepsilon(x) \to sign(x)$ as ε goes to 0 the Lebesgue's dominated convergence theorem in $L^2(\Omega\times[0,T]^2; P\otimes\frac{\partial^2 R}{\partial u\partial v}(u,v)dudv)$ implies that $F'_\varepsilon\left(B_\cdot^{H,K} - x\right)$ converges to $sign\left(B_\cdot^{H,K} - x\right)$ in $L^2(\Omega;\mathcal{H})$ as ε goes to 0 because

$$\begin{aligned} & E\|F'_\varepsilon\left(B_\cdot^{H,K} - x\right) - sign\left(B_\cdot^{H,K} - x\right)\|_{|\mathcal{H}|}^2 \\ = \; & E\int_0^T\int_0^T \left|F'_\varepsilon\left(B_u^{H,K} - x\right) - sign\left(B_u^{H,K} - x\right)\right|\left|F'_\varepsilon\left(B_v^{H,K} - x\right) - sign\left(B_v^{H,K} - x\right)\right| \\ & \times\frac{\partial^2 R}{\partial u\partial v}(u,v)dudv. \end{aligned}$$

Consequently, from the above convergences and (9)

$$\int_0^t F'_\varepsilon\left(B_s^{H,K} - x\right)\delta B_s^{H,K} \underset{\varepsilon\to 0}{\longrightarrow} \int_0^t sign\left(B_s^{H,K} - x\right)\delta B_s^{H,K} \quad \text{in } L^2(\Omega).$$

Then the conclusion follows. ∎

4 Tanaka formula for multidimensional bifractional Brownian motion

Given two vectors $H = (H_1, \ldots, H_d) \in [0,1]^d$ and $K = (K_1, \ldots, K_d) \in (0,1]^d$, we introduce the d-dimensional bifractional Brownian motion

$$B^{H,K} = \left(B^{H_1,K_1}, ..., B^{H_d,K_d}\right)$$

as a centered Gaussian vector whose component are independent one-dimensional bifractional Brownian motions.

We extend the Itô formula to the multidensional case.

Theorem 3 *Let $B^{H,K} = \left(B^{H_1,K_1}, ..., B^{H_d,K_d}\right)$ be a d-dimensional bifractional Brownian motion, and let f be a function of class $C^2\left(\mathbb{R}^d, \mathbb{R}\right)$. We assume that $2H_iK_i > 1$ for any $i = 1, ...n$, then*

$$f\left(B_t^{H,K}\right) = f(0) + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}\left(B_s^{H_i,K_i}\right) \delta B_s^{H_i,K_i} + \sum_{i=1}^{d} H_iK_i \int_0^t \frac{\partial^2 f}{\partial x_i^2}\left(B_s^{H,K}\right) s^{2H_iK_i-1} ds. \quad (24)$$

Proof: Let us fix $t > 0$ and a partition $\{t_j = \frac{jt}{n}; j = 0, ..., n\}$ of $[0,t]$. As in above we may assume that f has compact support, and so f, f' and f'' are bounded. Using Taylor expansion, we have

$$\begin{aligned} f\left(B_t^{H,K}\right) &= f(0) + \sum_{j=1}^{n} \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}\left(B_{t_{j-1}}^{H,K}\right)\left(B_{t_j}^{H_i,K_i} - B_{t_{j-1}}^{H_i,K_i}\right) \\ &+ \frac{1}{2} \sum_{j=1}^{n} \sum_{i,l=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_l}\left(\overline{B}_j^{H,K}\right)\left(B_{t_j}^{H_i,K_i} - B_{t_{j-1}}^{H_i,K_i}\right)\left(B_{t_j}^{H_l,K_l} - B_{t_{j-1}}^{H_l,K_l}\right) \\ &:= f(0) + I^n + J^n. \end{aligned}$$

where $\overline{B}_j^{H,K} = B_{t_{j-1}}^{H,K} + \theta_j\left(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\right)$, and θ_j is a random variable in $(0,1)$.
We show that J^n converges to 0 in $L^2(\Omega)$ as $n \to \infty$.

$$\begin{aligned} E|J^n|^2 &\leq n/4 \sum_{j=1}^{n} E\left(\sum_{i,l=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_l}\left(\overline{B}_j^{H,K}\right)\left(B_{t_j}^{H_i,K_i} - B_{t_{j-1}}^{H_i,K_i}\right)\left(B_{t_j}^{H_l,K_l} - B_{t_{j-1}}^{H_l,K_l}\right)\right)^2 \\ &\leq \frac{Cd^2}{4} n \sum_{j=1}^{n} \sum_{i,l=1}^{d} E\left(B_{t_j}^{H_i,K_i} - B_{t_{j-1}}^{H_i,K_i}\right)^2 E\left(B_{t_j}^{H_l,K_l} - B_{t_{j-1}}^{H_l,K_l}\right)^2 \\ &\leq Cd^2 n \sum_{i,l=1}^{d} \sum_{j=1}^{n} |t_j - t_{j-1}|^{2(H_iK_i + H_lK_l)} = Cd^2 \sum_{i,l=1}^{d} \frac{t^{4HK}}{n^{2(H_iK_i + H_lK_l - 1)}} \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

According to (7), we get

$$\begin{aligned} I^n &= \sum_{j=1}^{n}\sum_{i=1}^{d}\frac{\partial f}{\partial x_i}\left(B^{H,K}_{t_{j-1}}\right)\left(\delta^{B^{H_i,K_i}}(1_{(t_{j-1},t_j]})\right) \\ &= \sum_{i=1}^{d}\left[\delta^{B^{H_i,K_i}}\left(\sum_{j=1}^{n}\frac{\partial f}{\partial x_i}\left(B^{H,K}_{t_{j-1}}\right)1_{(t_{j-1},t_j]}(.)\right)\right. \\ &\quad \left. + \sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_i^2}\left(B^{H,K}_{t_{j-1}}\right)\langle 1_{(0,t_{j-1}]},1_{(t_{j-1},t_j]}\rangle_{\mathcal{H}}\right] \\ &= \sum_{i=1}^{d}\left[I_1^{n,i}+I_2^{n,i}\right] \end{aligned}$$

As the similar way in the above theorem, we obtain that for every $i = 1, ..., d$

$$I_2^{n,i} \underset{n\to\infty}{\longrightarrow} H_iK_i\int_0^t \frac{\partial^2 f}{\partial x_i^2}\left(B^{H,K}_s\right)s^{2H_iK_i-1}ds \text{ in } L^2(\Omega).$$

We show that for every $i \in \{1, ..., d\}$

$$I_1^{n,i} \underset{n\to\infty}{\longrightarrow} \int_0^t \frac{\partial f}{\partial x_i}\left(B^{H,K}_s\right)\delta B^{H_i,K_i}_s \text{ in } L^2(\Omega).$$

We set

$$u_s^{n,i} = \sum_{j=1}^{n}\frac{\partial f}{\partial x_i}\left(B^{H,K}_{t_{j-1}}\right)1_{(t_{j-1},t_j]}(s) - \frac{\partial f}{\partial x_i}\left(B^{H,K}_s\right)1_{(0,t]}(s).$$

By inequality (5), we have

$$E\left(\delta^{B^{H_i,K_i}}(u^{n,i})\right)^2 \le E\|u^{n,i}\|^2_{|\mathcal{H}^i|} + E\|Du^{n,i}\|^2_{|\mathcal{H}^i|\otimes|\mathcal{H}^i|}$$

where $\mathcal{H}^i$ is the Hilbert space associated to B^{H_i,K_i} and R_i its covariance function.
For every $r, s \le t$

$$D_r u_s^{n,i} = \sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_i^2}\left(B^{H,K}_{t_{j-1}}\right)1_{(0,t_{j-1}]}(r)1_{(t_{j-1},t_j]}(s) - \frac{\partial^2 f}{\partial x_i^2}(B^{H,K}_s)1_{(0,s]}(r)$$

we remark that $D_r u_s^{n,i}$ and $u_s^{n,i}$ converge to zero as $n \longrightarrow \infty$ for any $r, s \le t$. Since the first and second partial derivatives of f are bounded, then by using the Lebesgue dominated convergence theorem and the expression of the norm $\left|\mathcal{H}^i\right| \otimes \left|\mathcal{H}^i\right|$ we obtain that

$$\delta^{B^{H_i,K_i}}(u^{n,i}) \underset{n\to\infty}{\longrightarrow} 0 \text{ in } L^2(\Omega).$$

The proof is thus complete.

■

One can easily generalize the above theorem to the case when the function f depends on time.

Theorem 4 *Let $f \in C^{1,2}\left([0,T] \times \mathbb{R}^d, \mathbb{R}\right)$ and $B^{H,K} = \left(B^{H_1,K_1}, ..., B^{H_d,K_d}\right)$ be a d-dimensional bifBm with $2H_iK_i > 1$ for any $i = 1, ..., n$. Then*

$$\begin{aligned} f\left(t, B_t^{H,K}\right) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, B_s^{H,K})ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}\left(s, B_s^{H,K}\right)\delta B_s^{H_i,K_i} \\ &\quad + \sum_{i=1}^d H_iK_i \int_0^t \frac{\partial^2 f}{\partial x_i^2}\left(s, B_s^{H,K}\right) s^{2H_iK_i-1}ds. \end{aligned} \tag{25}$$

We consider twice of the kernel of the $d-$dimensional Newtonian potential

$$U(z) = \begin{cases} -\frac{\Gamma(d/2-1)}{2\pi^{d/2}} \frac{1}{|z|^{d-2}} & \text{if } d \geq 3 \\ \frac{1}{\pi} log|z| & \text{if } d = 2. \end{cases}$$

Set

$$\bar{U}(s,z) = \frac{1}{\prod_{j=1}^d \sqrt{2H_jK_j}} s^\theta U\left(\frac{(z_1 - x_1)}{\sqrt{2H_1K_1}} s^{1/2-H_1K_1}, ..., \frac{(z_d - x_d)}{\sqrt{2H_dK_d}} s^{1/2-H_dK_d}\right) \tag{26}$$

where $x = (x_1, ..., x_d) \in \mathbb{R}^d$ and $0 < \gamma := \frac{1}{2}(2-d) + \theta + (d-2)(HK)^* - \sum_{i=1}^d H_iK_i$ with $(HK)^* = max\{H_1K_1, \ldots, H_dK_d\}$.

We shall prove the following Tanaka formula. It will involve a multidimensional weighted local time which is an extension of the one-dimensional local time given by (23). Note for any dimension $d \geq 2$ the local time is not a random variable anymore and it is a distribution in the Watanabe's sense.

Theorem 5 *Let $\bar{U}$ as above and let $B^{H,K} = \left(B^{H_1,K_1}, ..., B^{H_d,K_d}\right)$ be a d-dimensional bifBm with $2H_iK_i > 1$ for any $i = 1, ...d$. Then the following formula holds in the Watanabe space $\mathbb{D}_2^{\alpha-1}$ for any $\alpha < \frac{1}{2(HK)^*} - d/2$.*

$$\bar{U}(t, B_t^{H,K}) = \bar{U}(0,0) + \int_0^t \partial_s \bar{U}(s, B_s^{H,K})ds + \sum_{i=1}^d \int_0^t \frac{\partial \bar{U}(s, B_s^{H,K})}{\partial x_i}\delta B_s^{H_i,K_i} + L^\theta(t,x) \tag{27}$$

where the generalized weighted local time $L^\theta(t,x)$ is defined as

$$L^\theta(t,x) = \sum_{n=(n_1,...,n_d)} \int_0^t \prod_{i=1}^d \frac{p_{s^{2H_iK_i}}(x_i)}{s^{\frac{1}{2}+(n_i-1)H_iK_i}} H_{n_i}\left(\frac{x_i}{\sqrt{s^{2H_iK_i}}}\right) I_{n_i}^i(1_{[0,s]}^{\otimes n_i})s^\theta ds.$$

Proof: We regularize the function $\bar{U}$ by standard convolution. Put $\bar{U}_\varepsilon = p_\varepsilon^d * \bar{U}$, with p_ε^d is the Gaussian kernel on $\mathbb{R}^d$ given by

$$p_\varepsilon^d(x) = \prod_{i=1}^d p_\varepsilon(x_i) = \prod_{i=1}^d \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x_i^2}{2\varepsilon}}, \quad \forall x = (x_1, ..., x_d) \in \mathbb{R}^d.$$

Using the above Itô formula we have

$$\begin{aligned} \bar{U}_\varepsilon\left(t, B_t^{H,K}\right) &= \bar{U}_\varepsilon(0,0) + \int_0^t \frac{\partial \bar{U}_\varepsilon}{\partial s}\left(s, B_s^{H,K}\right) ds + \sum_{i=1}^d \int_0^t \frac{\partial \bar{U}_\varepsilon}{\partial x_i}\left(s, B_s^{H,K}\right) \delta B_s^{H_i,K_i} \\ &\quad + \sum_{i=1}^d H_i K_i \int_0^t \frac{\partial^2 \bar{U}_\varepsilon}{\partial x_i^2}\left(s, B_s^{H,K}\right) s^{2H_iK_i - 1} ds. \\ &= \bar{U}_\varepsilon(0,0) + I_1^\varepsilon(t) + I_2^\varepsilon(t). \end{aligned}$$

On the other hand, if $V(z) = U(a_1 z_1, ..., a_d z_d)$ and $V_\varepsilon = p_\varepsilon^d * V$ we have

$$\frac{1}{2}\sum_{i=1}^d \frac{1}{a_i^2} \frac{\partial^2 V_\varepsilon}{\partial z_i^2}(z) = p_\varepsilon^d(a_1 z_1, ..., a_d z_d).$$

Hence

$$I_2^\varepsilon(t) = \frac{1}{\prod_{j=1}^d \sqrt{2H_jK_j}} \int_0^t p_\varepsilon^d\left(c_1(s)(B_s^{H_1,K_1} - x_1), ..., c_d(s)(B_s^{H_d,K_d} - x_d)\right) s^\theta ds$$

where $c_i(s) = \frac{s^{1/2 - H_iK_i}}{\sqrt{2H_iK_i}}$ for every $i = 1, ..., d$. The next step is to find the chaotic expansion of the last term I_2^ε. By Strook formula, we have

$$p_\varepsilon\left(c_i(s)(B_s^{H_i,K_i} - x_i)\right) = \sum_{n=0}^\infty \frac{1}{n!} I_n^i\left(ED_.^n p_\varepsilon\left(c_i(s)(B_s^{H_i,K_i} - x_i)\right)\right).$$

and

$$ED_.^n p_\varepsilon\left(c_i(s)(B_s^{H_i,K_i} - x_i)\right) = c_i(s)^n E p_\varepsilon^{(n)}\left(c_i(s)(B_s^{H_i,K_i} - x_i)\right) 1_{[0,s]}^{\otimes n}(.)$$

$$\begin{aligned} &= c_i(s)^n n! \left(\frac{\varepsilon}{c_i(s)^2} + s^{2H_iK_i}\right)^{-n/2} p_{s^{2H_iK_i} + \frac{\varepsilon}{c_i(s)^2}}(x_i) \frac{H_n\left(\frac{x_i}{\sqrt{s^{2H_iK_i} + \frac{\varepsilon}{c_i(s)^2}}}\right)}{c_i(s)^{n+1}} 1_{[0,s]}^{\otimes n}(.) \\ &= \frac{n!}{c_i(s)} \left(\frac{\varepsilon}{c_i(s)^2} + s^{2H_iK_i}\right)^{-n/2} p_{s^{2H_iK_i} + \frac{\varepsilon}{c_i(s)^2}}(x_i) H_n\left(\frac{x_i}{\sqrt{s^{2H_iK_i} + \frac{\varepsilon}{c_i(s)^2}}}\right) 1_{[0,s]}^{\otimes n}(.) \\ &:= \frac{n!}{c_i(s)} \beta_{n,\varepsilon}^i(s) 1_{[0,s]}^{\otimes n}(.) \end{aligned}$$

Consequently

$$p_\varepsilon^d\left(c_1(s)(B_s^{H_1,K_1}-x_1),...,c_d(s)(B_s^{H_d,K_d}-x_d)\right)=\sum_{n=(n_1,...,n_d)\in\mathbb{N}^d}\prod_{i=1}^d\frac{\beta^i_{n_i,\varepsilon}(s)}{c_i(s)}I^i_{n_i}(1^{\otimes n_i}_{[0,s]})$$

and that

$$\begin{aligned}I_2^\varepsilon(t) &= \sum_{n=(n_1,...,n_d)\in\mathbb{N}^d}\int_0^t\prod_{i=1}^d\frac{\beta^i_{n_i,\varepsilon}(s)}{s^{\frac12-H_iK_i}}I^i_{n_i}(1^{\otimes n_i}_{[0,s]})s^\theta ds\\ &= \sum_{n=(n_1,...,n_d)}\int_0^t\prod_{i=1}^d\frac{p_{s^{2H_iK_i}+\frac{\varepsilon}{c_i(s)^2}}(x_i)}{(\frac{\varepsilon}{c_i(s)^2}+s^{2H_iK_i})^{n_i/2}s^{\frac12-H_iK_i}}H_{n_i}\left(\frac{x_i}{\sqrt{s^{2H_iK_i}+\frac{\varepsilon}{c_i(s)^2}}}\right)I^i_{n_i}(1^{\otimes n_i}_{[0,s]})s^\theta ds\end{aligned}$$

This term (in fact, slightly modified) appeared in some other papers such as Proposition 12 in [10], or in [22]. Using standard arguments we obtain that the last term converges in $\mathbb{D}_2^\alpha$ to $L^\theta(t,x)$ as ε goes to 0, with $\alpha<\frac{1}{2(HK)^*}-d/2$.

The rest of the proof is to show that the following convergences are holds:
For every $i=1,...,d$

$$\int_0^t\partial_i\bar{U}_\varepsilon\left(s,B_s^{H,K}\right)\delta B_s^{H_i,K_i}\xrightarrow[\varepsilon\to0]{\mathbb{D}_2^{\alpha-1}}\int_0^t\partial_i\bar{U}\left(s,B_s^{H,K}\right)\delta B_s^{H_i,K_i}. \tag{28}$$

$$\int_0^t\partial_s\bar{U}_\varepsilon(s,B_s^{H,K})ds\xrightarrow[\varepsilon\to0]{\mathbb{D}_2^{\alpha}}\int_0^t\partial_s\bar{U}(s,B_s^{H,K})ds \tag{29}$$

and

$$\bar{U}_\varepsilon(t,B_t^{H,K})\xrightarrow[\varepsilon\to0]{\mathbb{D}_2^{\alpha}}\bar{U}(t,B_t^{H,K}). \tag{30}$$

We start with the convergence (28). Fix $i\in\{1,...,d\}$, we note $g^i_\varepsilon(s,z)=\partial_i\bar{U}_\varepsilon(s,z)$. By the formal relation (δ is the Dirac distribution)

$$\int_{\mathbb{R}}f(y)\delta(x-y)dy=f(x)$$

we can write (this is true in the sense of Watanabe distributions)

$$g^i_\varepsilon(s,B_s^{H,K})=\int_{\mathbb{R}^d}g^i_\varepsilon(s,y)\delta\left(B_s^{H,K}-y\right)dy.$$

Furthermore (see [10] but it can be also derived from a general formula in [16])

$$
\begin{aligned}
\delta\left(B_s^{H,K}-y\right) &= \prod_{i=1}^{d}\delta\left(B_s^{H_i,K_i}-y_i\right)\\
&= \prod_{i=1}^{d}\left(\sum_{n\geq 0}\frac{1}{(R_i(s))^{n/2}}p_{R_i(s)}(y_i)H_n(\frac{y_i}{R_i(s)^{1/2}})I_n^i\left(1_{[0,s]}^{\otimes n}\right)\right)\\
&= \sum_{n=(n_1,\ldots,n_d)} A_n(s,y)I_n(1_{[0,s]}^{\otimes|n|})
\end{aligned}
$$

where $R_i(s)=R_i(s,s)=s^{2H_iK_i}$, $A_n(s,y)=\prod_{i=1}^{d}\frac{1}{(R_i(s))^{n_i/2}}p_{R_i(s)}(y_i)H_{n_i}(\frac{y_i}{R_i(s)^{1/2}})$ and $I_n(1_{[0,s]}^{\otimes|n|}):=\prod_{i=1}^{d}I_{n_i}^i\left(1_{[0,s]}^{\otimes n_i}\right)$ for every $n=(n_1,...,n_d)$.
Hence

$$
\begin{aligned}
g_\varepsilon^i(s,B_s^{H,K}) &= \sum_{n=(n_1,\ldots,n_d)}\int_{\mathbb{R}^d} g_\varepsilon^i(s,y)A_n(s,y)dyI_n(1_{[0,s]}^{\otimes|n|})\\
&:= \sum_{n=(n_1,\ldots,n_d)} B_n^{\varepsilon,i}(s)I_n(1_{[0,s]}^{\otimes|n|})
\end{aligned}
$$

and using the chaotic form of the divergence integral

$$
\begin{aligned}
J_i^\varepsilon(t) &:= \int_0^t g_\varepsilon^i\left(s,B_s^{H,K}\right)\delta B_s^{H_i,K_i}\\
&= \sum_{n=(n_1,\ldots,n_d)} I_{n_i+1}^i\left[B_n^{\varepsilon,i}(s)1_{[0,s]}^{\otimes n_i}(s_1,...,s_{n_i})1_{[0,t]}(s)\prod_{\substack{j=1\\ j\neq i}}^{d}I_{n_j}^j\left(1_{[0,s]}^{\otimes n_j}\right)\right]^{(s)}\\
&= \sum_{n=(n_1,\ldots,n_d)} I_{n_i+1}^i\left[f_{i,n}^{\varepsilon,t}(s_1,...,s_{n_i},s)\right]\\
&= \sum_{n_i\geq 0} I_{n_i+1}^i\left[\sum_{n=(n_1,\ldots,\hat{n}_i,\ldots,n_d)} f_{i,n}^{\varepsilon,t}(s_1,...,s_{n_i},s)\right]
\end{aligned}
$$

where the superscript (s) denoted the symmetrization with respect to $s_1,\ldots,s_{n_i},s$, and

$$
f_{i,n}^{\varepsilon,t}(s_1,...,s_{n_i+1})=\sum_{l=1}^{n_i+1}\frac{1}{n_i+1}B_n^{\varepsilon,i}(s_l)1_{[0,s_l]}^{\otimes n_i}(s_1,...,\widehat{s_l},...,s_{n_i+1})1_{[0,t]}(s_l)\prod_{\substack{j=1\\ j\neq i}}^{d}I_{n_j}^j\left(1_{[0,s_l]}^{\otimes n_j}\right).
$$

Observe here that, since the components of the vector $B^{H,K}$ are independent, the term $\prod_{\substack{j=1\\ j\neq i}}^{d}I_{n_j}^j\left(1_{[0,s_l]}^{\otimes n_j}\right)$ is viewed as a deterministic function for the integral $I_{n_i}^i$. The convergence

(28) is satisfied if the conditions i) and ii) of Lemma 3 in [10] hold. It is easy to verify the condition i), we will prove only the condition ii).
Fixing $i \in \{1, ..., d\}$, we can write,

$$\begin{aligned}
\|J_i^\varepsilon(t)\|_{\alpha-1,2}^2 &\leq \sum_{m\geq 1}(m+1)^{\alpha-1}\sum_{|n|=n_1+...+n_d=m-1}(n_i+1)!E\left\|f_{i,n}^{\varepsilon,t}\right\|_{\mathcal{H}^{\otimes n_i+1}}^2 \\
&= \sum_{m\geq 1}(m+1)^{\alpha-1}\sum_{|n|=n_1+...+n_d=m-1}(n_i+1)! \\
&\times \int_{[0,T]^{n_i+1}}\int_{[0,T]^{n_i+1}}\sum_{l,k=1}^{n_i+1}\frac{1}{(n_i+1)^2}|B_n^{\varepsilon,i}(s_l)||B_n^{\varepsilon,i}(r_k)|1_{[0,t]}(s_l)1_{[0,t]}(r_k) \\
&\times 1_{[0,s_l]}^{\otimes n_i}(s_1,...,\widehat{s_l},...,s_{n_i+1})1_{[0,r_k]}^{\otimes n_i}(r_1,...,\widehat{r_l},...,r_{n_i+1})\prod_{\substack{j\neq i\\ j=1}}^{d} n_j!R_j(s_l,r_k)^{n_j} \\
&\times \prod_{q=1}^{n_i+1}\frac{\partial^2 R_i}{ds_q dr_q}(s_q,r_q)dr_1....dr_{n_i+1}ds_1....ds_{n_i+1}
\end{aligned}$$

Since $\alpha < 0$ then $(m+2)^{\alpha-1} \leq (m+1)^{\alpha-1}$ and $(n_i+1) \leq (m+1)$. This implies that

$$\begin{aligned}
\|J_i^\varepsilon(t)\|_{\alpha-1,2}^2 &\leq \sum_{m\geq 0}(m+1)^{\alpha}\sum_{|n|=n_1+...+n_d=m}(n_i)!\Bigg[(1-\frac{1}{(n_i+1)}) \\
&\times \int_{[0,T]^2}\int_{[0,T]^2}|B_n^{\varepsilon,i}(s_1)||B_n^{\varepsilon,i}(r_2)|1_{[0,t]}(s_1)1_{[0,t]}(r_2)R_i(s_1,r_2)^{n_i-1} \\
&\times \prod_{\substack{j\neq i\\ j=1}}^{d} n_j!R_j(s_1,r_2)^{n_j}1_{[0,s_1]}(s_2)1_{[0,r_2]}(r_1)\frac{\partial^2 R_i}{ds_1 dr_1}(s_1,r_1)\frac{\partial^2 R_i}{ds_2 dr_2}(s_2,r_2)ds_1ds_2dr_1dr_2 \\
&+ \frac{1}{(n_i+1)}\int_{[0,T]^2}\int_{[0,T]^2}|B_n^{\varepsilon,i}(s_1)||B_n^{\varepsilon,i}(r_1)|1_{[0,t]}(s_1)1_{[0,t]}(r_1)R_i(s_1,r_1)^{n_i-1} \\
&\times \prod_{\substack{j\neq i\\ j=1}}^{d} n_j!R_j(s_1,r_1)^{n_j}1_{[0,s_1]}(s_2)1_{[0,r_1]}(r_2)\frac{\partial^2 R_i}{ds_1 dr_1}(s_1,r_1)\frac{\partial^2 R_i}{ds_2 dr_2}(s_2,r_2)ds_1ds_2dr_1dr_2\Bigg]
\end{aligned}$$

By integration we obtain

$$
\begin{aligned}
\|J_i^\varepsilon(t)\|_{\alpha-1,2}^2 &\le \sum_{m\ge 0}(m+1)^\alpha \sum_{|n|=n_1+\ldots+n_d=m}(n_i)!\Bigg[(1-\frac{1}{(n_i+1)})\\
&\times \int_{[0,t]^2}|B_n^{\varepsilon,i}(s_1)||B_n^{\varepsilon,i}(r_2)|R_i(s_1,r_2)^{n_i-1}\\
&\times \prod_{\substack{j\neq i\\ j=1}}^d n_j!R_j(s_1,r_2)^{n_j}\frac{\partial R_i}{ds_1}(s_1,r_2)\frac{\partial R_i}{dr_2}(s_1,r_2)ds_1dr_2\\
&+ \frac{1}{(n_i+1)}\int_{[0,t]^2}|B_n^{\varepsilon,i}(s_1)||B_n^{\varepsilon,i}(r_1)|R_i(s_1,r_1)^{n_i}\\
&\times \prod_{\substack{j\neq i\\ j=1}}^d n_j!R_j(s_1,r_1)^{n_j}\frac{\partial^2 R_i}{ds_1dr_1}(s_1,r_1)ds_1dr_1\Bigg].
\end{aligned}
$$

We have for any $1/4\le\beta\le 1/2$

$$
\begin{aligned}
B_n^{\varepsilon,i}(s) &= \int_{\mathbb{R}^d} g_\varepsilon^i(s,y)A_n(s,y)dy\\
&= \int_{\mathbb{R}^d} g_\varepsilon^i(s,y)\prod_{j=1}^d H_{n_j}\left(\frac{y_j}{\sqrt{R_j(s)}}\right)e^{\frac{-\beta y_j^2}{R_j(s)}}\frac{1}{\sqrt{R_j(s)}^{n_j}}\frac{e^{-(\frac12-\beta)\frac{y_j^2}{R_j(s)}}}{\sqrt{2\pi R_j(s)}}dy\\
&= \int_{\mathbb{R}^d} p_\varepsilon^d(z)dz\int_{\mathbb{R}^d}\partial_i\bar U\left(s,y-z\right)\prod_{j=1}^d H_{n_j}\left(\frac{y_j}{\sqrt{R_j(s)}}\right)\frac{e^{\frac{-\beta y_j^2}{R_j(s)}}}{\sqrt{2\pi}}\frac{e^{-(\frac12-\beta)\frac{y_j^2}{R_j(s)}}}{\sqrt{R_j(s)}^{n_j+1}}dy.
\end{aligned}
$$

Since (see Lemma 11 in [10])

$$
\sup_{z\in\mathbb{R}^d}\prod_{j=1}^d|H_{n_j}(z_j)|e^{-\beta z_j^2}\le C\prod_{j=1}^d\frac{1}{\sqrt{n_j!}(n_j\vee 1)^{\frac{8\beta-1}{12}}}.
$$

and for every $(s,z)\in(0,T]\times\mathbb{R}^d$

$$
\left|\partial_i\bar U(s,z)\right|\le Cs^{\frac12(1-d)+\theta}\left|\left((z_1-x_1)s^{-H_1K1},\ldots,(z_d-x_d)s^{-H_dKd}\right)\right|^{1-d}.
$$

Then, for any $s \in (0,T]$

$$
\begin{aligned}
|B_n^{\varepsilon,i}(s)| &\leq C \prod_{j=1}^{d} \frac{1}{\sqrt{n_j!}(n_j \vee 1)^{\frac{8\beta-1}{12}}} \frac{1}{s^{n_j H_j K_j}} \int_{\mathbb{R}^d} p_\varepsilon^d(z) dz \\
&\times \int_{\mathbb{R}^d} s^{\frac{1}{2}(1-d)+\theta-\sum_{j=1}^d H_j K_j} \frac{e^{-(\frac{1}{2}-\beta)|s^{-HK}y|^2}}{|s^{-HK}(y-z-x)|^{d-1}} dy \\
&\leq C s^{\gamma-\frac{1}{2}} \prod_{j=1}^{d} \frac{1}{\sqrt{n_j!}(n_j \vee 1)^{\frac{8\beta-1}{12}}} \frac{1}{s^{n_j H_j K_j}} \int_{\mathbb{R}^d} p_\varepsilon^d(z) dz \\
&\times \int_{\mathbb{R}^d} \frac{e^{-(\frac{1}{2}-\beta)|s^{-HK}y|^2}}{|(y-z-x)|^{d-1}} dy
\end{aligned}
$$

where $s^{-HK}y := (s^{-H_1K_1}y_1, ..., s^{-H_dK_d}y_d)$.
Let η a positive constant such that, for every $s \in (0,T]$, $j \in \{1,\dots,d\}$ we have $s^{-2H_jK_j} > \eta$. Combining this with for any $a,b \in \mathbb{R}$, $a^2 \geq \frac{1}{2}(a-b)^2 - b^2$, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^d} p_\varepsilon^d(z) dz \int_{\mathbb{R}^d} \frac{e^{-(\frac{1}{2}-\beta)|s^{-HK}y|^2}}{|(y-z-x)|^{d-1}} dy &\leq \int_{\mathbb{R}^d} p_\varepsilon^d(z) dz \int_{\mathbb{R}^d} \frac{e^{-\eta(\frac{1}{2}-\beta)|y|^2}}{|(y-z-x)|^{d-1}} dy \\
&\leq \int_{\mathbb{R}^d} p_\varepsilon^d(z) e^{\eta(\frac{1}{2}-\beta)|z+x|^2} dz \int_{\mathbb{R}^d} \frac{e^{-\frac{\eta}{2}(\frac{1}{2}-\beta)|y-(z+x)|^2}}{|(y-(z+x))|^{d-1}} dy \\
&\leq C e^{2\eta(\frac{1}{2}-\beta)|x|^2} \int_{\mathbb{R}^d} p_\varepsilon^d(z) e^{2\eta(\frac{1}{2}-\beta)|z|^2} dz \\
&\leq C e^{2\eta(\frac{1}{2}-\beta)|x|^2} \int_{\mathbb{R}^d} \frac{e^{-\frac{|v|^2}{2}}}{\sqrt{2\pi}^d} e^{2\eta(\frac{1}{2}-\beta)\varepsilon|v|^2} dv \\
&\leq C e^{2\eta(\frac{1}{2}-\beta)|x|^2} \int_{\mathbb{R}^d} \frac{e^{-\beta|v|^2}}{\sqrt{2\pi}^d} dv < \infty
\end{aligned}
$$

since $2\eta\varepsilon \leq 1$ when ε close to 0.
Thus

$$
|B_n^{\varepsilon,i}(s)| \leq C s^{\gamma-\frac{1}{2}} \prod_{j=1}^{d} \frac{1}{\sqrt{n_j!}(n_j \vee 1)^{\frac{8\beta-1}{12}}} \frac{1}{s^{n_j H_j K_j}}
$$

where C is a constant depending only on d, H, K, T, x and β.
Using this inequality $a^2 + b^2 \leq 2ab$, for every $a, b \in \mathbb{R}_+$, we conclude that there exist a constant $C(H,K)$ positive, such that for every $i = 1, \dots, d$

$$
\left| \frac{\partial R_i}{dr}(r,s) \frac{\partial R_i}{ds}(r,s) \right| \leq C(H,K)(rs)^{2H_iK_i-1},
$$

$$
\left| \frac{\partial^2 R_i}{drds}(r,s) \right| \leq C(H,K)(rs)^{H_iK_i-1}
$$

and

$$\left|\frac{R_i(r,s)}{(rs)^{H_iK_i}}\right| \le C(H,K).$$

It follows by anterior inequalities that

$$\begin{aligned}\|J_i^{\varepsilon}(t)\|_{\alpha-1,2}^2 &\le C\sum_{m\ge 0}(1+m)^{\alpha}\sum_{|n|=n_1+\ldots+n_d=m}\prod_{j=1}^{d}\frac{1}{(n_j\vee 1)^{\frac{8\beta-1}{6}}}\\ &\times \int_{[0,t]^2}\frac{R_i(r,s)^{n_i-1}}{(rs)^{(n_i-1)H_iK_i}}(rs)^{\gamma-\frac{3}{2}+H_iK_i}\prod_{\substack{j=1\\ j\neq i}}^{d}\frac{R_j(r,s)^{n_j}}{(rs)^{n_jH_jK_j}}drds.\end{aligned}$$

We use the selfsimilarity of the covariance kernel $R(r,s) = R(1,\frac{s}{r})r^{2HK}$ and the change of variables $r/s = z$ in the integral respect to dz to obtain

$$\begin{aligned}\|J_i^{\varepsilon}(t)\|_{\alpha-1,2}^2 &\le C\sum_{m\ge 0}(1+m)^{\alpha}\sum_{|n|=n_1+\ldots+n_d=m}\prod_{j=1}^{d}\frac{1}{(n_j\vee 1)^{\frac{8\beta-1}{6}}}\\ &\times \int_0^t r^{2(\gamma-1+H_iK_i)}dr\int_0^1 (z)^{\gamma-\frac{3}{2}+H_iK_i}\left(\frac{R_i(1,z)}{z^{H_iK_i}}\right)^{n_i-1}\prod_{\substack{j=1\\ j\neq i}}^{d}\left(\frac{R_j(1,z)}{z^{H_jK_j}}\right)^{n_j}dz.\end{aligned}$$

Since for each $i\in\{1,\ldots,d\}$, $\gamma-\frac{3}{2}+H_iK_i>-1$, $2(\gamma-1+H_iK_i)>-1$ and from Lemma 12 and the proof of the Proposition 12 in [10], we obtain that

$$\begin{aligned}\|J_i^{\varepsilon}(t)\|_{\alpha-1,2}^2 &\le C\sum_{m\ge 0}(1+m)^{\alpha}m^{-\frac{1}{2(HK)^*}}\sum_{|n|=n_1+\ldots+n_d=m}\prod_{j=1}^{d}\frac{1}{(n_j\vee 1)^{\frac{8\beta-1}{6}}}\\ &\le C\sum_{m\ge 0}(1+m)^{\alpha}(m)^{-\frac{1}{2(HK)^*}-1+d(1-\frac{8\beta-1}{6})}\end{aligned}$$

and since is finite if and only if $\alpha<\frac{1}{2(HK)^*}-\frac{d}{2}$.
On the other hand, for every $(s,z)\in(0,T]\times\mathbb{R}^d$ we have

$$\left|\partial_s\bar{U}(s,z)\right| \le Cs^{\frac{-d}{2}+\theta}\left|\left((z_1-x_1)s^{-H_1K1},...,(z_d-x_d)s^{-H_dKd}\right)\right|^{2-d}.$$

and

$$\left|\bar{U}(s,z)\right| \le Cs^{\frac{1}{2}(2-d)+\theta}\left|\left((z_1-x_1)s^{-H_1K1},...,(z_d-x_d)s^{-H_dKd}\right)\right|^{2-d}$$

and this inequalities imply as in [21] the convergences (29) and (30) in $mathbbD_2^{\alpha}$. ■

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