

# Use of an Hourglass Model in Neuronal Coding.

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## Abstract

We study a system of interacting renewal processes which is a model for neuronal activity. We show that the system possesses an exponentially large number (with respect to the number of neurons in the network) of limiting configurations of the "firing neurons". These we call patterns. Furthermore, under certain conditions of symmetry we find an algorithm to control limiting patterns by means of the connection parameters.

*Keywords:* Ergodicity, Hourglass Model, Stochastic Neural Network

*AMS Classification:* 60K20, 60K35

## 1 Introduction

**Definition of the model.** We study here the *hourglass* model. This model is represented by a system of interacting renewal processes, numerated by the sites in a finite subset  $\Lambda \subset \mathbf{Z}^{\nu}$ . Define for any  $z \in \Lambda$  its neighbourhood:

$$D(z) := \{z \pm l_k, k = 1, \dots, K\} \cap \Lambda, \quad (1.1)$$

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where  $\{l_k, k = 1, \dots, K\} \subset \mathbf{Z}^\nu$  is a fixed but arbitrary set of non-zero vectors. Notice by our definition, that  $z \notin D(z)$ , and for any  $z, y \in \Lambda$

$$z \in D(y) \text{ iff } y \in D(z).$$

Also let for any subset  $J \subset \Lambda$

$$D(J) := \{j \pm l_k, k = 1, \dots, K, j \in J\}.$$

Define on some probability space  $(\Omega, \Sigma, \mathbf{P})$  some random variables  $X_i(0)$ ,  $Y_i$ ,  $\theta_{ij}$ ,  $i \in \Lambda$ ,  $j \in D(i)$ , to be independent with densities having a certain number of moments. The variables  $X_i(0)$  and  $Y_i$ ,  $i \in \Lambda$ , are assumed to be positive and to represent the initial state and the self-characteristic of the  $i$ th neuron, respectively. We shall call the  $\theta_{ij}$  the connection parameters. For any fixed  $i \in \Lambda$ ,  $j \in D(i)$ , the distribution of  $\theta_{ij}$  is assumed to be concentrated either on  $\mathbf{R}_-$  or  $\mathbf{R}_+$ . We will specify the sign of  $\theta_{ij}$  in every example we treat below. Assume also  $\theta_{ij} \equiv 0$  for any  $i \in \Lambda$ ,  $j \notin D(i)$ . Finally let  $Y_i^{(n)}$ ,  $Y_i^{(n,z)}$  and  $\theta_{ij}^{(n)}$ ,  $\theta_{ij}^{(n,z)}$ ,  $n \geq 1$ ,  $z \in \Lambda$ , be independent copies of the variables  $Y_i$  and  $\theta_{ij}$ , respectively.

We shall define a Markov process  $X(t) = (X_i(t), i \in \Lambda)$ ,  $t \geq 0$ , with left-continuous trajectories in  $\mathbf{R}_+^\Lambda$  as follows. For all  $i \in \Lambda$  and  $t > 0$  define

$$\begin{aligned} X_i(t) = & X_i(0) - t + \sum_{0 < t_n < t: X_i(t_n)=0} Y_i^{(n)} \tag{1.2} \\ & + \sum_{j \in D(i)} \sum_{0 < t_n < t: X_j(t_n)=0} \left( (Y_i^{(n,j)} - X_i(t_n)) I\{X_i(t_n) \leq \theta_{ji}^{(n)}\} \right. \\ & \quad \left. - \theta_{ji}^{(n)} I\{X_i(t_n) > \theta_{ji}^{(n)}\} \right) \\ & - \sum_{j \in D(D(i)) \setminus \{i\} \cup D(i)} \sum_{z \in D(i) \cap D(j): \theta_{zi} < 0} \sum_{\substack{0 < t_n < t: X_j(t_n) = 0, \\ X_i(t_n) > \theta_{ji}^{(n)}, \text{ and } X_z(t_n) \leq \theta_{jz}^{(n,j)}}} \theta_{zi}^{(n,j)}. \end{aligned}$$

**Biological interpretation.** Any moment  $t$  such that  $X_z(t) = 0$  for some  $z \in \Lambda$  we call *the moment of firing* of the  $z$ th neuron. Assume for a moment that

$\theta_{ji} \equiv 0$  for any  $i, j \in \Lambda$ . Then the last two summation terms in the right-hand side of equation (1.2) turn into zero, meaning that  $X_i(t)$  is merely a renewal process in this case. Thus each component  $X_i(t)$  represents *the duration of time before the next firing of the corresponding  $i$ th neuron assuming no interaction takes place meanwhile*.

In the presence of non-zero interactions the dynamics of  $X(t) = (X_i(t), i \in \Lambda), t \geq 0$ , are described as follows. As long as all the components of  $X(t)$  are strictly positive, they decrease from the initial state  $X(0)$  linearly in time with rate one until the first time  $t_{z_1}$  that one of the components reaches zero for some  $z_1 \in \Lambda$ :  $X_{z_1}(t_{z_1}) = 0$ . We say that at this moment  $t_{z_1}$ , the  $z_1$ th neuron fires and *sends impulse*  $\theta_{z_1j}^{(1)}$  to the  $j$ th neuron, if  $j \in D(z_1)$ . This means the following. At the moment  $t_{z_1}$ , the trajectory  $X_{z_1}(t_{z_1})$  jumps to a random value  $Y_{z_1}^{(1)}$ , i.e.

$$X_{z_1}(t_{z_1}+) = Y_{z_1}^{(1)}, \quad (1.3)$$

which corresponds to the first summation term in (1.2). At the same moment every trajectory  $X_j(t)$  with  $j \in D(z_1)$  receives an increment  $\theta_{z_1j}^{(1)}$ , independent of the other processes, more precisely, according to the second summation term in (1.2)

$$X_j(t_{z_1}+) = \begin{cases} X_j(t_{z_1}) - \theta_{z_1j}^{(1)}, & \text{if } X_j(t_{z_1}) - \theta_{z_1j}^{(1)} > 0, \\ Y_j^{(1)}, & \text{otherwise.} \end{cases} \quad (1.4)$$

Notice, that by (1.4) the negative connections  $\theta_{z_1j}^{(1)}$  delay the moment when  $X_j(t)$  hits zero, i.e. the moment of firing of the  $j$ -th neuron. Whereas positive connections can shorten the time-interval until the next firing. That is why we shall call negative connections *inhibitory* and the positive ones *excitatory*.

**Case 1.** Assume that all the interactions  $\theta_{ij}, j \in D(i)$ , are negative. Then definition (1.2) simply becomes

$$X_i^{inh}(t) = X_i(0) - t + \sum_{0 < t_n < t: X_i(t_n)=0} Y_i^{(n)} + \sum_{j \in D(i)} \sum_{0 < t_n < t: X_j(t_n)=0} |\theta_{ji}^{(n)}|, \quad (1.5)$$

for  $i \in \Lambda$  and  $t \geq 0$ . We use a notation  $X^{inh}(t)$  for this particular case. This model is due to Cottrell (1992) [1] and it has been intensively studied (see an account of the previous results in Section 2 below).

**Case 2.** Assume, some of the interactions  $\theta_{ij}$  are positive. Notice that, only if  $\theta_{z_1 j_1} > 0$ , it may happen that  $X_{j_1}(t_{z_1}) - \theta_{z_1 j_1}^{(1)} \leq 0$ . In this case the trajectory  $X_{j_1}(t)$  is reset instantaneously to an independent value  $Y_j^{(1)}$  according to (1.4). Furthermore, we say that the  $j_1$ th neuron also *fires* at the moment  $t_{z_1}$  and changes instantaneously the trajectories of the neighbouring neurons according to the last term in (1.2), which means the following.

Given  $X_{z_1}(t_{z_1}) = 0$ , define the set of the firing at the same moment  $t_{z_1}$  neurons:

$$F_1(z_1, t_{z_1}) := \{j_1 \in D(z_1) : X_{j_1}(t_{z_1}) - \theta_{z_1 j_1}^{(1)} \leq 0\}.$$

Then at time  $(t_{z_1} +)$  the state of the system is defined as follows:

$$X_j(t_{z_1} +) = \begin{cases} Y_j^{(1)}, & \text{if } j \in \{z_1\} \cup F_1, \\ X_j(t_{z_1}) - \theta_{z_1 j}^{(1)} - \sum_i \theta_{ij}^{(1)}, & \text{if } j \in D(z_1) \setminus F_1, \\ X_j(t_{z_1}) - \sum_i \theta_{ij}^{(1)}, & \text{if } j \in D(F_1) \setminus (D(z_1) \cup F_1 \cup \{z_1\}), \end{cases} \quad (1.6)$$

where  $F_1 = F_1(z_1, t_{z_1})$ , the summation  $\sum_i$  runs over the set  $\{i \in F_1 : i \in D(j) \text{ and } \theta_{ij} < 0\}$ , and any summation over an empty set equals zero. The rest of the trajectories  $X_i(t)$  with  $i \notin D(F_1) \cup F_1 \cup D(z_1) \cup \{z_1\}$  remain unchanged. After moment  $t_{z_1}$  the foregoing dynamics are repeated.

To end this description recall the well-known fact from physiology that a neuron does not react to the incoming impulses during a certain period right after it's own firing. This period is called a refractory period. Observe that in our model a neuron receives an impulse only in the moments when it does not fire itself. This reflects the property of the refractory period.

This type of neural network has been proved to be equivalent, in a sense, to the "classical" neural model, which describes the interacting membrane potentials (see [12]).

**Plan of the paper.** Our paper is organized as follows. In Section 2 we find the critical values of the parameters which separate the ergodic and transient cases. Also we provide some historical comments on our model in Section 2. In Section 3 we analyze a fully connected network with inhibitory connections only, in which case we solve a problem related to the memory capacity.

## 2 Critical parameters.

### 2.1 Results.

Here we formulate our results on the critical values of the parameters which separate the ergodic and transient cases of the networks with excitatory and inhibitory connections.

In order to eliminate boundary effects, we shall assume here that  $\Lambda = \{-N, \dots, N\}^\nu$  is a  $\nu$ -dimensional torus, i.e. we identify any two points  $(i_1, \dots, i_\nu)$  and  $(j_1, \dots, j_\nu)$  in  $\mathbf{Z}^\nu$  whenever  $|i_m - j_m| \in \{0, 2N\}$  for every  $1 \leq m \leq \nu$ . Hence, any point in  $\Lambda$  has the same number of neighbours. Further we assume that  $N > 1$  and  $\nu \in \mathbf{N}_+$  are fixed but arbitrary (although only the cases  $\nu = 1, 2, 3$  are relevant for our context). Define for any  $i, j \in \Lambda$

$$\|i - j\|_\Lambda := \sum_{k=1}^{\nu} |i_k - j_k|_{2N},$$

where  $|x - y|_{2N} := \min\{|x - y|, |x - y - 2N|, |x - y + 2N|\}$  for any  $x, y \in \mathbf{Z}$ . Let  $\Lambda_0$  be the subset of the points in  $\Lambda$  such that:

- i*) the origin  $(0, \dots, 0) \in \Lambda_0$ ,
- ii*)  $\|x - y\|_\Lambda \neq 1$  for any  $x, y \in \Lambda_0$ .

Consider now the network  $X(t) = (X_i(t), i \in \Lambda)$  defined in (1.2) with the following connection architecture.

**Assumption 1** *Let the neighbourhood  $D(i)$  defined in (1.1) be such that*

$$D(i) = D_I(i) \cup D_E(i),$$

where

$$D_I(i) = \{j \in \Lambda : \|i - j\|_\Lambda = 1\}$$

and

$$\begin{aligned} D_E(i) &\subseteq \Lambda_0 && \text{iff } i \in \Lambda_0, \\ D_E(i) &\subseteq \Lambda \setminus \Lambda_0 && \text{iff } i \in \Lambda \setminus \Lambda_0, \end{aligned} \tag{2.1}$$

with  $|D_I(i)| = 2\nu$  and  $|D_E(i)| = K_E$ . The constants  $0 < K_E < N$  are fixed but arbitrary.

Then we set

$$\theta_{ij} = \begin{cases} -w_I \eta_1^{ij}, & \text{if } j \in D_I(i), \\ w_E \eta_2^{ij}, & \text{if } j \in D_E(i), \end{cases} \quad (2.2)$$

where  $\eta_k^{ij}, i \neq j$  are independent copies of positive independent random variables  $\eta_k$ , respectively,  $k = 1, 2$ , with  $\mathbf{E}\eta_k = 1$ , and where  $w_I$  and  $w_E$  are positive parameters of the inhibitory and excitatory connections, respectively.

In words the condition (2.2) means that the nearest connections are inhibitory while the more distant ones are excitatory.

Assume further that  $X(0), X_i(0), i \in \Lambda$ , are *i.i.d.*, and also  $Y, Y_i, i \in \Lambda$ , are *i.i.d.* with  $\mathbf{E}Y = 1$ .

**Assumption 2** *Assume, that the distributions of  $Y, \eta_1, \eta_2$ , and  $X(0)$  have densities  $g_0(u), g_1(u), g_2(u)$  and  $g_3(u)$ , respectively, which are positive differentiable functions, such that for some positive constants  $a$  and  $\alpha$*

$$g_k(u) \leq ae^{-\alpha u} \quad \text{for all } u > 0 \text{ and } k = 0, 1, 2, 3. \quad (2.3)$$

We note that the assumption of the exponential decay, though seemingly rather restrictive, arises naturally from the physiology (see, for example [12] and the references therein).

**Theorem 1** *Under Assumptions 1 and 2 for any  $w_E \geq 0$  there exists a positive  $w_I^{cr}(w_E)$  such that the system  $X(t)$  with parameters (2.2) is transient if*

$$w_I > w_I^{cr}(w_E), \quad (2.4)$$

*and ergodic if*

$$w_I < w_I^{cr}(w_E). \quad (2.5)$$

We specify the critical function  $w_I^{cr}(w_E)$  in (2.15) below. Generally speaking, this function depends on  $N$ . However, its asymptotic behaviour when  $w_E \rightarrow 0$  is uniform in  $N$ , as we establish in the following theorem.

**Theorem 2** *There exist positive constants  $C$  and  $C_0$  independent of  $N$  such that*

$$\left| w_I^{cr}(w_E) - \left( \frac{1}{2\nu} - \frac{K_E}{2\nu} w_E \right) \right| \leq C w_E^2 \quad (2.6)$$

*for all  $0 \leq w_E \leq C_0$ .*

We postpone the proofs of these theorems to the next section.

So far only the case  $w_E = 0$  has been studied analytically. Cottrell [1] proved that when  $w_E = 0$ , the network  $X(t) = X^{inh}(t)$  (see (1.5) ) is ergodic, whenever  $w_I < 1/(2\nu)$ , which is, clearly a particular case of (2.6). In [1] sufficient conditions for convergence and transience were found for the finite network  $X(t)$  with  $\Lambda \subset \mathbf{Z}^2$  and a specific structure of the connections. Further Piat [10] (1994) extended these results to a more general connection structure. Karpelevich *et al.* (1995) provided a complete analysis of the evolution of inhibitory networks when the matrix of the expected values of the connections defines a self-adjoint operator.

Paper [2] presents simulated results for the model, in which both inhibitory and excitatory connections were incorporated. Here for the first time we study analytically a rigorous mathematical model for such a network. Our result explains the slope of the diagram given on Fig. 4 [2] in the neighbourhood of the critical value  $(0, 1/(2\nu))$ . The most attractive feature of the hourglass network for neuromodelling is that in the transient case the system splits as  $t \uparrow \infty$  into two subsets. These are a subset of active (i.e. infinitely often firing) neurons and one of inactive neurons, which can be recognized as dark and white areas, respectively, in the simulations, see e.g. figures 3 and 6 of [2]. A rigorous definition of the possible limiting patterns of active and inactive neurons, called *traps* (see Section 2.2.1 below) was given by Karpelevich, Malyshev and Rybko in [7]. Thus the set of all possible traps for a network should be naturally thought of as a system of patterns, which the network can hold, that is, memorize and recognize, hopefully. Recently [8] obtained a result on phase transitions in the thermodynamical limit, which establishes the large memory capacity of these networks.

Further (in Section 3) besides a description of the possible patterns, we solve an inverse problem. This is the problem of how to determine the connections in order to get a system which stores a given system of patterns.

## 2.2 Proofs of Theorem 1 and Theorem 2.

### 2.2.1 Preliminary definitions and results.

We shall use in our proofs the results of [7]. Therefore firstly we introduce the terminology of [7] for our model.

For any non-empty  $W \subset \Lambda$  we call  $X^W(t)$  the *restriction* of process  $X(t)$  on the set  $W$ . This is defined as  $X(t)$ , but with initial conditions

$$X_i^W(0) := \begin{cases} X_i(0), & \text{if } i \in W, \\ \infty, & \text{if } i \notin W, \end{cases}$$

in which case we assume that  $X_i^W(t) = \infty$  if  $i \notin W$ , for all  $t > 0$ , i.e. the nodes  $\Lambda \setminus W$  are "deleted".

Assume, the process  $(X_i^W(t), i \in W)$  is ergodic for some non-empty  $W \subset \Lambda$ . In this case we define for any  $i \in W$  and  $j \in D_E(i) \cap W$  the following limiting frequencies:

$$\pi_i^{W,0} := \lim_{T \rightarrow \infty} \frac{1}{T} \#\{0 < t_n < T : X_i^W(t_n) = 0\}, \quad (2.7)$$

and

$$\pi_{ij}^{W,e} := \lim_{T \rightarrow \infty} \frac{1}{T} \#\{0 < t_n < T : X_j^W(t_n) = 0 \text{ and } X_i^W(t_n) - \theta_{ji}^{(n)} \leq 0\}. \quad (2.8)$$

Thus  $\pi_i^{W,0}$  is the limiting frequency of firing of the  $i$ th neuron due to the hitting 0 by the trajectory  $X_i^W(t)$ , while  $\pi_{ij}^{W,e}$  is the limiting frequency of firing of the  $i$ th neuron due to an immediate excitatory impulse from the  $j$ th neuron. Then it is natural to call the (*total*) *limiting frequency of firing* of the  $i$ th neuron the following sum of the defined above limits:

$$\pi_i^W := \pi_i^{W,0} + \sum_{j \in D_E(i) \cap W} \pi_{ij}^{W,e}, \quad i \in W. \quad (2.9)$$

Next we define the *second vector field*  $v^W = (v_j^W, j \in \Lambda \setminus W)$  by the following formula for its components:

$$v_j^W = -1 - \sum_{i \in D_I(j) \cap W} \mathbf{E} \theta_{ij} \pi_i^W - \sum_{i \in D_E(j) \cap W} \mathbf{E} \theta_{ij} \pi_i^{W,0}, \quad j \in \Lambda \setminus W. \quad (2.10)$$

Formally  $v_j^W$  is the limiting mean drift of the  $j$ th component  $X_j^W(t)$ . Indeed, according to our definition (1.2) the  $i$ th neuron sends inhibitory impulses to the corresponding neighbours with the limiting frequency  $\pi_i^W$ , but it sends excitatory impulses with the limiting frequency  $\pi_i^{W,0}$ . For the details on the application of the theory of the second vector field we refer to [7] and the references therein.

We shall call a *trap* for the process  $X(t)$  any non-empty set  $M \subset \Lambda$  such that the process  $(X_i^{\Lambda \setminus M}(t), i \in \Lambda \setminus M)$  is ergodic, while any coordinate of the second vector field  $v^{\Lambda \setminus M}$  is positive, i.e.

$$v_j^{\Lambda \setminus M} > 0 \quad \text{for all } j \in M. \quad (2.11)$$

For future reference let us rewrite now in our notations the criteria from [7] on inductive ergodicity and transience conditions.

**Theorem A** (See Theorem 2.1, [7].) **I. Inductive ergodicity.** If for any  $W \subset \Lambda$  the process  $(X_i^W(t), i \in W)$  is ergodic and

$$v_j^W < 0 \quad \text{for all } j \in \Lambda \setminus W$$

then the process  $X(t)$  is ergodic.

**II. Sufficient transient conditions.** If there exists a set  $M$  which is a trap for the process  $X(t)$ , then the process  $X(t)$  is transient.

### 2.2.2 Subsystems with excitatory connections only.

Consider now the restriction  $X^{\Lambda_0}(t)$ . Notice that due to the definition of  $\Lambda_0$  and Assumption 1 the only connections between the components of  $(X_i^{\Lambda_0}(t), i \in \Lambda_0)$  are excitatory. Then by the definition of the process  $X^{\Lambda_0}(t)$  we have for all  $i \in \Lambda_0$ :

$$\begin{aligned} X_i^{\Lambda_0}(t) &= X_i(0) - t + \sum_{0 < s_n < t: X_i^{\Lambda_0}(s_n)=0} Y^{(n)} \\ &- \sum_{j \in D_E(i)} \sum_{0 < t_n < t: X_j^{\Lambda_0}(t_n)=0} \left( w_E \eta_2^{(n,j)} I\{X_i^{\Lambda_0}(t_n) > w_E \eta_2^{(n,j)}\} \right. \\ &\left. + (X_i^{\Lambda_0}(t_n) - Y^{(n,j)}) I\{X_i^{\Lambda_0}(t_n) \leq w_E \eta_2^{(n,j)}\} \right), \end{aligned} \quad (2.12)$$

where  $\eta_2^{(n,j)}$ ,  $n \geq 1$ ,  $j \in \Lambda_0$ , are independent copies of the variable  $\eta_2$ . Clearly, for any  $i \in \Lambda_0$  the one-dimensional  $X_i^{\{i\}}(t)$  is ergodic. Further for any  $w_E > 0$  and for any  $W \subset \Lambda_0$  such that  $(X_i^W(t), i \in W)$  is ergodic, the second vector field defined in (2.10) has negative coordinates only:

$$v_j^W = -1 - \sum_{i \in D_E(j) \cap W} w_E \pi_i^{W,0}, \quad j \in \Lambda_0 \setminus W.$$

Hence, by Theorem A on inductive ergodicity  $(X_i^{\Lambda_0}(t), i \in \Lambda_0)$  is ergodic for any fixed  $w_E > 0$ . Taking into account translation invariance, we derive from (2.12) and (2.9)

$$\pi_i^{\Lambda_0} =: \pi^+(w_E) = \pi^{+,0}(w_E) + K_E \pi^{+,e}(w_E) > 0 \quad (2.13)$$

for any  $i \in \Lambda_0$ . In particular,

$$\pi^+(0) = \pi^{+,0}(0) = 1/\mathbf{E}Y = 1, \quad (2.14)$$

since in the case  $w_E = 0$  we have a system of independent renewal processes numerated by the sites of  $\Lambda_0$ .

Thus one can view the system  $X(t)$  as a result of inhibitory interactions between two (independent at initial moment) excitatory networks:  $(X_i(t), i \in \Lambda_0)$  and  $(X_i(t), i \in \Lambda \setminus \Lambda_0)$ .

### 2.2.3 Proof of Theorem 1.

We shall show that under condition (2.4), with

$$w_I^{cr}(w_E) = \frac{1}{2\nu\pi^+(w_E)} \quad (2.15)$$

the set  $\Lambda_0$  is a trap for the system  $X(t)$ . Then the first statement on transience in our theorem follows from Theorem A.

As we have shown above,  $(X^{\Lambda_0}(t), i \in \Lambda_0)$  is ergodic. Computing the second vector field with respect to formula (2.10), and taking into account (2.2) and (2.13) we get

$$v_j^{\Lambda_0} = -1 + \sum_{i \in D_I(j)} w_I \pi_i^{\Lambda_0} = -1 + 2\nu w_I \pi^+(w_E), \quad j \in \Lambda \setminus \Lambda_0. \quad (2.16)$$

Hence condition (2.4) together with (2.15) is equivalent to

$$v_j^{\Lambda_0} > 0, \quad j \in \Lambda \setminus \Lambda_0, \quad (2.17)$$

i.e. inequality in (2.11) is fulfilled for all  $j \in \Lambda \setminus \Lambda_0$ , which implies that  $\Lambda_0$  is a trap. Hence, our statement on transience follows by Theorem A.

Assume now that condition (2.5) with (2.15) holds. Clearly, for any  $i \in \Lambda$  one-dimensional  $X_i^{\{i\}}(t)$  is ergodic. Next we will show that for any ergodic

face  $W$  the components of the second vector field  $v^W$  defined by (2.10) are negative. It obviously follows from our definition of the model, that the limiting firing frequency is the highest when the neuron has the excitatory connections only and the maximal number of them, i.e.

$$\pi_i^W \leq \pi_j^{\Lambda_0} = \pi^+(w_E)$$

for any  $W \in \Lambda$ , any  $i \in \Lambda \setminus W$  and  $j \in \Lambda \setminus \Lambda_0$ . From here and (2.10) we derive for any  $i \in \Lambda \setminus W$

$$v_i^W \leq -1 + 2\nu w_I \pi^+(w_E) < 0,$$

where the last inequality is due to (2.5). Hence, using Theorem A (Theorem 2.1 [7]) on inductive ergodicity, we readily derive our statement on ergodicity. Theorem 1 is proved.

#### 2.2.4 Proof of Theorem 2.

Let  $p_A(t, u_A)$ ,  $u_A \in \mathbf{R}_+^{|\Lambda|}$ , denote for any finite  $A \subseteq \Lambda_0$  the density of the process  $X_A^{\Lambda_0}(t) = (X_i^{\Lambda_0}(t), i \in A)$ . In the particular case  $w_E = 0$ , we denote the corresponding density by  $p_A^0(t, u_A)$ . Due to ergodicity the following limit

$$\lim_{t \rightarrow \infty} p_A(t, u_A) =: p_A(u_A) \quad (2.18)$$

exists for any  $A \subseteq \Lambda_0$ . In particular, we have for the limiting density of an independent renewal process:

$$\lim_{t \rightarrow \infty} p_A^0(t, u_A) = p_A^0(u_A) = \prod_{j \in A} p_j^0(u_j) \quad (2.19)$$

where

$$p_j^0(u) = \frac{\int_u^\infty g_0(v) dv}{\mathbf{E}Y} =: p^0(u). \quad (2.20)$$

Further we will use the following lemma stating that the limiting density in (2.18) is a small perturbation of the function in (2.19) when the parameter of interaction  $w_E$  is sufficiently small.

**Lemma 1** *There exist positive and independent of  $N$  constants  $C, \beta$  and  $c$  such that for any  $0 < w_E \leq c$  and for any subset  $A \subseteq \Lambda_0$*

$$|p_A(t, u_A) - p_A^0(t, u_A)| \leq w_E C^{|\Lambda|} e^{-\beta \sum_{i \in A} u_i}, \quad (2.21)$$

for any  $t > 0$  and  $u_A \in \mathbf{R}_+^A$ .

**Proof.** Recall that  $g_0$  and  $g_3$  are the density functions of the variables  $Y$  and  $X(0)$ , respectively. Let  $p(t, u_{\Lambda_0}, v_{\Lambda_0})$ ,  $u_{\Lambda_0}, v_{\Lambda_0} \in \mathbf{R}_+^{\Lambda_0}$ , denote the transition density of the process  $(X_i^{\Lambda_0}(t), i \in \Lambda_0)$ . Then for any  $A \subseteq \Lambda_0$  the density  $p_A(t, u_A)$ ,  $u_A \in \mathbf{R}_+^{|\Lambda_0 \setminus A|}$ , is given by the formula

$$p_A(t, u_A) = \int_{\mathbf{R}_+^{|\Lambda_0 \setminus A|}} \int_{\mathbf{R}_+^{\Lambda_0}} \left( \prod_{z \in \Lambda_0} g_3(u_z) \right) p(t, u_{\Lambda_0}, v_{\Lambda_0}) du_{\Lambda_0} dv_{\Lambda_0 \setminus A}. \quad (2.22)$$

In the particular case  $w_E = 0$ , we denote the transition density by  $p^0(t, u_{\Lambda_0}, v_{\Lambda_0})$ . Clearly, in this case

$$p^0(t, u_{\Lambda_0}, v_{\Lambda_0}) = \prod_{j \in \Lambda_0} p^0(t, u_j, v_j)$$

due to the independence of the components. Notice, that the transition density of each of the independent renewal processes is

$$p^0(t, u, v) = \quad (2.23)$$

$$= \begin{cases} \delta(u - t - v), & \text{if } 0 \leq t < u, \\ g_0(t - u + v) + \sum_{k=1}^{\infty} \int_0^{t-u} p_{S_k}(x) g_0(t - u + v - x) dx, & \text{if } 0 \leq u \leq t, \end{cases}$$

for every  $u, v \in \mathbf{R}_+$  and  $t > 0$ , where  $\delta(u - \cdot)$  is the Dirac distribution concentrated at  $u$ ;  $Y, Y^k$ ,  $k \geq 1$ , are *i.i.d.*;  $S_k := \sum_{l=1}^k Y^l$ ,  $k \geq 1$ , and  $p_{S_k}$  is the density of the distribution  $S_k$ .

Using the results of Stone [11] one can show that under Assumption 2 there exist positive constants  $C'$  and  $\alpha'$  such that

$$|p^0(t + t', 0, v) - p^0(t, 0, v)| \leq C' e^{-\alpha'(t+v)} \quad (2.24)$$

for all  $v \in \mathbf{R}_+$  and  $t, t' > 0$  (for the details see for example, [13] Section 4). Combining (2.24) and (2.19), we get:

$$|p^0(v) - p^0(t, 0, v)| \leq C' e^{-\alpha'(t+v)}. \quad (2.25)$$

(For the general theory of piece-wise linear stochastic processes we refer to the book [3] by Davis, 1993.)

After these preliminaries the proof of Lemma 1 follows by standard techniques using so-called cluster expansions (see, e.g. [13] for a similar model

and [6] for a class of stochastic processes with local weak interactions) as soon as the Kolmogorov equation (see (2.26) below) is written down for the transition density  $p(t, u_{\Lambda_0}, v_{\Lambda_0})$ .

Let  $M(\mathbf{R}_+^{\Lambda_0})$  denote a class of densities  $\mu$  on  $\mathbf{R}_+^{\Lambda_0}$  which possess all continuous partial derivatives. Define also

$$p(t, u_{\Lambda_0}, \Gamma) = \int_{\Gamma} p(t, u_{\Lambda_0}, v_{\Lambda_0}) dv_{\Lambda_0}$$

for any Borel set  $\Gamma \in \mathbf{R}_+^{\Lambda_0}$ . Denote here  $O(i) = D_E(i)$  and  $w_E = \epsilon$ . Then for our model we derive for any density  $\mu \in M(\mathbf{R}_+^{\Lambda_0})$  and for any Borel set  $\Gamma \in \mathbf{R}_+^{\Lambda_0}$

$$\begin{aligned} & \frac{\partial}{\partial t} \int_{\mathbf{R}_+^{\Lambda_0}} \mu(u_{\Lambda_0}) p(t, u_{\Lambda_0}, \Gamma) du_{\Lambda_0} \tag{2.26} \\ &= - \int_{\mathbf{R}_+^{\Lambda_0}} \sum_{i \in \Lambda_0} \mu(u_{\Lambda_0}) \frac{\partial}{\partial u_i} p(t, u_{\Lambda_0}, \Gamma) du_{\Lambda_0} \\ &+ \int_{\mathbf{R}_+^{\Lambda_0}} \mu(u_{\Lambda_0}) \sum_{i \in \Lambda_0} \delta(u_i) \int_{\mathbf{R}_+^{1+2|O(i)|}} p_Y(w_i) \left( \prod_{j \in O(i)} p_Y(y_j) p_{\epsilon \eta_2}(w_j) \right) \\ &\quad \times [p(t, u_{\Lambda_0} + U_{\Lambda_0}(u_{\Lambda_0}, w_i \cup_{O(i)}, y_{O(i)}), \Gamma) - p(t, u_{\Lambda_0}, \Gamma)] \\ &\quad \times dy_{O(i)} dw_{i \cup_{O(i)}} du_{\Lambda_0}, \end{aligned}$$

where  $\delta(\cdot)$  is the Dirac measure concentrated at 0,

$$p_Y(u) = g_0(u), \quad p_{\epsilon \eta_2}(u) = \begin{cases} g_2(u/\epsilon)/\epsilon, & \text{if } \epsilon > 0, \\ \delta(u), & \text{if } \epsilon = 0, \end{cases} \quad u \geq 0;$$

and the vector  $U_{\Lambda_0}(u_{\Lambda_0}, w_i \cup_{O(i)}, y_{O(i)})$  has components:

$$\begin{aligned} & U_j(u_{\Lambda_0}, w_i \cup_{O(i)}, y_{O(i)}) \\ &= \begin{cases} w_i, & \text{if } j = i, \\ -w_j I\{w_j < u_j\} + (y_j - u_j) I\{w_j \geq u_j\}, & \text{if } j \in O(i), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Further for any  $w_i \geq 0$  let the vector  $\bar{w}_i = (\bar{w}_{i,j}, j \in \Lambda_0)$  have the following components:

$$\bar{w}_{i,j} = \begin{cases} w_i, & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

Let us rewrite now formula (2.26) in the following operator form:

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_{\mathbf{R}_+^{\Lambda_0}} \mu(u_{\Lambda_0}) p(t, u_{\Lambda_0}, \Gamma) du_{\Lambda_0} \tag{2.27} \\
&= \int_{\mathbf{R}_+^{\Lambda_0}} \mu(u_{\Lambda_0}) \sum_{i \in \Lambda_0} \left( - \frac{\partial}{\partial u_i} p(t, u_{\Lambda_0}, \Gamma) \right. \\
&\quad \left. + \delta(u_i) \int_{\mathbf{R}_+} p_Y(w_i) [p(t, u_{\Lambda_0} + \bar{w}_i, \Gamma) - p(t, u_{\Lambda_0}, \Gamma)] dw_i \right) du_{\Lambda_0} \\
&+ \int_{\mathbf{R}_+^{\Lambda_0}} \mu(u_{\Lambda_0}) \sum_{i \in \Lambda_0} \delta(u_i) \int_{\mathbf{R}_+^{1+2|O(i)|}} p_Y(w_i) \left( \prod_{j \in O(i)} p_Y(y_j) p_{\epsilon \eta_2}(w_j) \right) \\
&\quad \times [p(t, u_{\Lambda_0} + U_{\Lambda_0}(u_{\Lambda_0}, w_i \cup_{O(i)}, y_{O(i)}), \Gamma) - p(t, u_{\Lambda_0} + \bar{w}_i, \Gamma)] \\
&\quad \times dy_{O(i)} dw_i \cup_{O(i)} du_{\Lambda_0} \\
&=: \int_{\mathbf{R}_+^{\Lambda_0}} \mu(u_{\Lambda_0}) \sum_{i \in \Lambda_0} (H_0^{i, \Lambda_0} + H_1^{i, \Lambda_0}) p(t, u_{\Lambda_0}, \Gamma) du_{\Lambda_0}.
\end{aligned}$$

Notice, that when  $\epsilon = 0$  the operator  $H_1^{i, \Lambda_0} \equiv 0$  for any  $i$ . In this case (2.27) simply becomes

$$\frac{\partial}{\partial t} \int_{\mathbf{R}_+^{\Lambda_0}} \mu(u_{\Lambda_0}) p^0(t, u_{\Lambda_0}, \Gamma) du_{\Lambda_0} = \int_{\mathbf{R}_+^{\Lambda_0}} \mu(u_{\Lambda_0}) \sum_{i \in \Lambda_0} H_0^{i, \Lambda_0} p^0(t, u_{\Lambda_0}, \Gamma) du_{\Lambda_0} \tag{2.28}$$

i.e., the equation for the density of the process whose components are independent renewal processes. Hence the formula (2.23) gives a solution to (2.28):

$$p^0(t, u_{\Lambda_0}, \Gamma) = \int_{\Gamma} p^0(t, u_{\Lambda_0}, v_{\Lambda_0}) dv_{\Lambda_0}.$$

Now we can find the solution to equation (2.27) by solving the equivalent integral equation (for the reference see also [13] and [6]):

$$\int_{\mathbf{R}_+^{\Lambda_0}} \mu(u_{\Lambda_0}) p(t, u_{\Lambda_0}, \Gamma) du_{\Lambda_0} = \int_{\mathbf{R}_+^{\Lambda_0}} \mu(u_{\Lambda_0}) p^0(t, u_{\Lambda_0}, \Gamma) du_{\Lambda_0} \tag{2.29}$$

$$+ \int_0^t \int_{\mathbf{R}_+^{\Lambda_0}} \int_{\mathbf{R}_+^{\Lambda_0}} \mu(u_{\Lambda_0}) p^0(t-s, u_{\Lambda_0}, u_{\Lambda_0}^1) \sum_{i \in \Lambda_0} H_1^{i, \Lambda} p(s, u_{\Lambda_0}^1, \Gamma) du_{\Lambda_0}^1 du_{\Lambda_0} ds.$$

Finally from (2.29) we obtain the formula for the density defined by (2.22):

$$p_A(t, v_A) = p_A^0(t, v_A) \quad (2.30)$$

$$+ \sum_{k=1}^{\infty} \sum_{(i_1, \dots, i_k) \in \Lambda_0^k} \int_0^t \int_{\mathbf{R}_+^{\Lambda_0}} \int_{\mathbf{R}_+^{\Lambda_0}} \left( \prod_{z \in \Lambda_0} g_3(u_z) \right) p^0(t-s_1, u_{\Lambda_0}, u_{\Lambda_0}^1) \\ H_1^{i_1, \Lambda_0} \int_0^{s_1} \int_{\mathbf{R}_+^{\Lambda_0}} p^0(s_1-s_2, u_{\Lambda_0}^1, u_{\Lambda_0}^2) \dots$$

$$H_1^{i_k, \Lambda_0} \int_0^{s_{k-1}} \int_{\mathbf{R}_+^{\Lambda_0}} p^0(s_k, u_{\Lambda_0}^k, v_{\Lambda_0}) dv_{\Lambda_0 \setminus A} du_{\Lambda_0}^k ds_k \dots du_{\Lambda_0}^1 du_{\Lambda_0} ds_1,$$

Making use of the formula (2.23) and the bounds (2.3) and (2.24), one can prove the convergence of the series in (2.30) and get the necessary bounds by following the the formulae in Section 3 of [13] (p.179 and further) with only minor modifications due to (2.26). Therefore for the sake of brevity we skip the rest of the straightforward part of the proof of Lemma 1.

Consider now the embedded Markov chain  $(x_i(n) := X_i^{\Lambda_0}(\tau_n), i \in \Lambda_0)$ ,  $n = 1, 2, \dots$ , where  $\tau_1, \tau_2, \dots$  denote the consecutive moments when at least one of the components of  $X^{\Lambda_0}$  reaches zero, i.e.  $X_j^{\Lambda_0}(\tau_n) = 0$  for some  $j \in \Lambda_0$ . Due to ergodicity of  $(X_i^{\Lambda_0}(t), i \in \Lambda_0)$ , there exists a limiting distribution, call it  $F$ , of  $x(n)$  as  $n \rightarrow \infty$ . Let  $x^\infty(n)$  be the stationary version of the Markov chain  $x(n)$ , i.e. whose initial distribution is  $F$ . Further for any  $i \in \Lambda_0$  and  $j \in D_E(i)$  let  $Z_{ij}$  be a random variable with distribution function

$$F_{ij}(u) := P\{x_i^\infty(n) < u \mid x_j^\infty(n) = 0\}.$$

Clearly, the density function  $f_{ij}$  for each  $F_{ij}$  is given by

$$f_{ij}(u) = \frac{p_{\{ij\}}(u, 0)}{p_j(0)} \quad (2.31)$$

as long as  $p_j(0) > 0$ . Observe that  $p_j^0(0) = \frac{1}{\mathbf{E}Y} = 1$  by (2.20). Hence, according to Lemma 1 condition  $p_j(0) > 0$  is satisfied at least for all small values of  $w_E$ .

We shall derive now an equation for  $\pi_i^{\Lambda_0,0} = \pi^{+,0}(w_E)$  (see definition (2.7) and (2.13) ). From (2.12) and the above arguments on ergodicity we obtain for all  $i \in \Lambda_0$ :

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} X_i^{\Lambda_0}(t) = 0 &= -1 + \pi^{+,0}(w_E) \mathbf{E}Y - K_E \pi^{+,0}(w_E) w_E \\ &- \sum_{j \in D_E(i)} \pi^{+,0}(w_E) \mathbf{E}(Z_{ij} - Y - w_E \eta_2) I\{Z_{ij} \leq w_E \eta_2\}, \end{aligned} \quad (2.32)$$

where  $Z_{ij}$ ,  $Y$  and  $\eta_2$  are independent by their definitions. Therefore taking into account that  $\mathbf{E}Y = 1$ , we obtain from (2.32) the following equation:

$$\begin{aligned} \pi^{+,0}(w_E) &\left( 1 - K_E w_E + \sum_{j \in D_E(i)} \mathbf{P}\{Z_{ij} \leq w_E \eta_2\} \right. \\ &\left. + \sum_{j \in D_E(i)} \mathbf{E}(w_E \eta_2 - Z_{ij}) I\{Z_{ij} \leq w_E \eta_2\} \right) = 1. \end{aligned} \quad (2.33)$$

**Lemma 2** *There exists a positive constant  $C_1$  independent of  $N$  such that for any  $i \in \Lambda_0$  and for any  $j \in D_E(i)$*

$$|\mathbf{P}\{Z_{ij} \leq w_E \eta_2\} - w_E| \leq w_E^2 C_1 \quad (2.34)$$

and

$$\mathbf{E}(w_E \eta_2 - Z_{ij}) I\{Z_{ij} \leq w_E \eta_2\} \leq w_E^2 C_1 \quad (2.35)$$

for all  $0 \leq w_E \leq c$  (with  $c$  defined in Lemma 1).

**Proof of Lemma 2.** In the case  $w_E = 0$  let  $Z_{ij} = Z_{ij}^0$ . Then the density of  $Z_{ij}^0$  (call it correspondingly  $f_{ij}^0(u)$ ) is given by

$$f_{ij}^0(u) = p^0(u) = \int_u^\infty g_0(y) dy \quad (2.36)$$

due to (2.31) and (2.20).

First let us obtain the following bound as a corollary of Lemma 1:

$$|f_{ij}(u) - f_{ij}^0(u)| \leq w_E C_2, \quad u \in \mathbf{R}_+, \quad (2.37)$$

where  $C_2$  is some positive constant independent of  $N$ . Notice, that from (2.21) and (2.18) we immediately derive:

$$|p_A(u_A) - p_A^0(u_A)| \leq w_E C_3 \exp\{-\beta \sum_{i \in A} u_i\}, \quad (2.38)$$

for any  $u_A \in \mathbf{R}_+^A$ , where  $C_3$  is some positive constant independent of  $N$ . Then the bound (2.37) readily follows from (2.38) and the continuity of the transformation (2.31).

Let us prove now (2.34). Consider

$$\begin{aligned} \mathbf{P}\{Z_{ij} \leq w_E \eta_2\} &= \int_0^\infty g_2(y) \int_0^{w_E y} f_{ij}^0(u) du dy \\ &+ \int_0^\infty g_2(y) \int_0^{w_E y} (f_{ij}(u) - f_{ij}^0(u)) du dy. \end{aligned} \quad (2.39)$$

Substituting formula (2.36) into the first integral in (2.39) we derive after simple calculations:

$$\begin{aligned} &\int_0^\infty g_2(y) \int_0^{w_E y} f_{ij}^0(u) du dy \\ &= \int_0^\infty \mathbf{P}\{\eta_2 \geq y\} w_E (1 - \mathbf{P}\{Y \leq w_E y\}) dy \\ &= w_E - w_E \int_0^\infty \mathbf{P}\{\eta_2 \geq y\} \mathbf{P}\{Y \leq w_E y\} dy, \end{aligned} \quad (2.40)$$

where we used the condition  $\mathbf{E}\eta_2 = 1$ . Taking into account (2.3) it is easy to see that

$$w_E \int_0^\infty \mathbf{P}\{\eta_2 \geq y\} \mathbf{P}\{Y \leq w_E y\} dy \leq C' w_E^2 \quad (2.41)$$

for some positive constant  $C'$ . Next, making use of (2.37) we readily derive

$$|\int_0^\infty g_2(y) \int_0^{w_E y} (f_{ij}(u) - f_{ij}^0(u)) du dy| \leq w_E^2 C_2. \quad (2.42)$$

Combining the bounds (2.42) and (2.41) together with equations (2.40) and (2.39) gives us (2.34).

To prove (2.35) let us consider the following decomposition:

$$\begin{aligned} &\mathbf{E}(w_E \eta_2 - Z_{ij}) I\{Z_{ij} \leq w_E \eta_2\} \\ &= \int_0^\infty g_2(y) \left( \int_0^{w_E y} (w_E y - u) f_{ij}^0(u) du \right) dy \end{aligned}$$

$$+ \int_0^\infty g_2(y) \left( \int_0^{w_E y} (w_E y - u)(f_{ij}(u) - f_{ij}^0(u)) du \right) dy.$$

Using again (2.3) and (2.37) we immediately derive from here:

$$\mathbf{E}(w_E \eta_2 - Z_{ij}) I\{Z_{ij} \leq w_E \eta_2\} \leq w_E^2 C_3 + w_E^3 C_4, \quad (2.43)$$

where  $C_3$  and  $C_4$  are some positive constants, which implies (2.35). The lemma is proved.

Lemma 2 allows us to derive from equation (2.33) that

$$|\pi^{+,0}(w_E) - 1| \leq w_E^2 C_5 \quad (2.44)$$

for some positive constant  $C_5$ .

Finally let us consider  $\pi_{ij}^{+,e}(w_E)$  for  $i \in \Lambda_0, j \in D_E(i)$ . According to definition (2.8) we derive similarly (2.32)

$$\begin{aligned} \pi_{ij}^{W,e} &= \lim_{t \rightarrow \infty} \frac{1}{t} \#\{0 < t_n < t : X_j^W(t_n) = 0 \text{ and } X_i^W(t_n) - \theta_{ji}^{(n)} \leq 0\} \quad (2.45) \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{0 < t_n < t: X_j^{\Lambda_0}(t_n) = 0} I\{X_i^{\Lambda_0}(t_n) \leq w_E \eta_2^{(n)}\} \\ &= \pi^{+,0}(w_E) \mathbf{P}\{Z_{ij} \leq w_E \eta_2\}. \end{aligned}$$

Substituting (2.45) into definition (2.13) we obtain

$$\pi^+(w_E) = \pi^{+,0}(w_E) \left(1 + \sum_{j \in D_E(i)} \mathbf{P}\{Z_{ij} \leq w_E \eta_2\}\right), \quad (2.46)$$

which together with (2.44) and (2.34) gives us the following bound:

$$|\pi^+(w_E) - (1 + K_E w_E)| \leq w_E^2 C_6 \quad (2.47)$$

for all  $w_E \leq C_0$ , where  $C_6$  and  $C_0$  are some positive constants.

Substituting the bound (2.47) into (2.15) we get (2.6). This completes the proof of Theorem 2.

### 3 Fully connected finite networks.

#### 3.1 The model.

Let  $\Lambda = \{1, \dots, N\}$ . Consider an  $N$ -neuron network with inhibitory connections only, i.e.  $X(t) = X^{inh}(t)$  as defined in (1.5). Here we assume that  $D(i) = \Lambda \setminus \{i\}$  for any  $i \in \Lambda$ , which means that our network is *fully connected*.

Let  $N = 2pk$ , where  $k, p \in \mathbb{Z}_+$  are fixed arbitrary. Suppose a network consists of  $2p$  symmetric subnets (blocks), determined by the values of the connection constants  $\mathbf{E}\theta_{ij}$  as follows. We divide set  $\Lambda = \{1, \dots, N\}$  into  $2p$  subsets  $W_i$ ,  $i = 1, \dots, 2p$ , so that

$$|W_i| = k \text{ and } \cup_{i=1}^{2p} W_i = \Lambda. \quad (3.1)$$

Notice that we keep  $k > 1$  fixed but arbitrary. This is simply a size of one block or unit of our network. The case  $k = 1$  is trivial.

Let us divide also the set  $\{1, \dots, 2p\}$  into a set of  $p$  non-intersecting pairs, i.e.,

$$\{1, \dots, 2p\} = \cup_{n=1}^p C_n = \cup_{n=1}^p \{C_n^1, C_n^2\}, \quad (3.2)$$

where

$$|C_n| = 2, \text{ and } C_i \cap C_j = \emptyset.$$

Then we set

$$\begin{aligned} \mathbf{E}Y_x &= a > 0, \\ \mathbf{E}\theta_{xy} &= -c_{ij} < 0, \text{ if } x \in W_i, y \in W_j, \end{aligned} \quad (3.3)$$

for all  $x, y \in \Lambda$  and  $1 \leq i, j \leq 2p$ , where

$$c_{ij} = \begin{cases} c, & \text{if } \{i, j\} = C_n \text{ for some } n, \\ b, & \text{otherwise,} \end{cases} \quad (3.4)$$

for some arbitrary but fixed constants

$$0 < b < a < c. \quad (3.5)$$

Thus our network consists of  $2p$  connected blocks of interacting identical neurons. The connections between the neurons of different blocks are different in general from those between the neurons of the same block. In the following Theorem 3 below, we describe all the possible traps for our network, i.e. all the possible limiting states in the transient case. Notice that

the total number of the patterns is  $|T| = 2^p = 2^{\frac{N}{2k}}$ , which is exponentially large with respect to  $N$ . This description will allow us to find a method to reconstruct or "memorize", in some particular cases, a given system of traps (see Corollary 1 below). More precisely, we will show how to choose the appropriate connection constants.

### 3.2 Description of patterns.

**Theorem 3** *Let  $N = 2pk$ , where  $k > 1, p > 1$ , and decompositions (3.1) and (3.2) be fixed but arbitrary. Call  $\mathcal{L}$  the set of all the decompositions of  $\{1, \dots, 2p\}$  into a pair of subsets such that*

$$\mathcal{L} := \{(L_1, L_2) : L_1 \cup L_2 = \{1, \dots, 2p\}, L_1 \cap L_2 = \emptyset, \quad (3.6)$$

$$L_i \cap C_n = 1, \quad i = 1, 2, \quad n = 1, \dots, p\}.$$

*Then a set  $A \subset \Lambda$  is a trap for the network with the connection constants (3.3)-(3.5), if and only if*

$$A = \{x \in \Lambda : x \in \cup_{i \in B} W_i\}, \quad (3.7)$$

*for some  $B \subset \Lambda$  such that*

$$(B, \{1, \dots, 2p\} \setminus B) \in \mathcal{L}. \quad (3.8)$$

**Proof.** First we will show that any set  $A$  satisfying conditions (3.7) and (3.8) is a trap for the above defined system. Indeed, consider a restriction  $X^{\Lambda \setminus A}(t)$ . In this case  $(X_i^{\Lambda \setminus A}(t), i \in \Lambda \setminus A)$  is a completely connected system, and for any  $x, y \in \Lambda \setminus A, y \neq x$

$$-\mathbf{E}\theta_{xy} = b < a = \mathbf{E}Y_x, \quad (3.9)$$

which implies ergodicity of  $(X_i^{\Lambda \setminus A}(t), i \in \Lambda \setminus A)$  due to Proposition 2.2 in [7].

Assume,  $(X_i^W(t), i \in W)$  is ergodic for some  $W \subset \Lambda$ . Observe that in the case of a fully connected network with only negative connections definition (2.9) becomes

$$\pi_i^W = \pi_i^{W,0}, \quad i \in W. \quad (3.10)$$

Furthermore it follows by ergodicity from (1.5) and definition (2.7) that

$$\pi_i^W = \left( \mathbf{E}Y_i + \sum_{j \in W \setminus \{i\}} \mathbf{E}|\theta_{ji}| \right)^{-1}, \quad i \in W. \quad (3.11)$$

Correspondingly, we derive from (2.10)

$$v_j^W = -1 + \sum_{i \in W} \mathbf{E}|\theta_{ij}| \pi_i^W, \quad j \in \Lambda \setminus W. \quad (3.12)$$

Notice that definitions (3.1)-(3.2) imply  $|B| = p$  for the set  $B$  satisfying conditions (3.7)-(3.8). Then for any  $x \in \Lambda \setminus A$  we derive from (3.11) and condition (3.4) (see also [7]), that

$$\pi_x^{\Lambda \setminus A} = \frac{1}{a + (|\Lambda \setminus A| - 1)b} = \frac{1}{a + (pk - 1)b}. \quad (3.13)$$

Substituting (3.13) into definition (3.12) and taking into account (3.3)-(3.4), we obtain for any  $y \in A$

$$v_y^{\Lambda \setminus A} := -1 + \frac{ck + (p-1)bk}{a + (pk-1)b}, \quad (3.14)$$

which together with (3.5) implies

$$v_y^{\Lambda \setminus A} > 0.$$

Hence, we conclude that  $A$  is a trap.

Next we will show that any trap satisfies (3.7)-(3.8). Clearly, any subset  $A \subseteq \Lambda$  can be represented as

$$A = \{x \in \Lambda : x \in \cup_{i \in B} A_i\}, \quad (3.15)$$

where  $B \subseteq \{1, \dots, 2p\}$  and  $A_i \subseteq W_i$  for any  $i \in B$ . Suppose a subset  $A$  does not satisfy conditions (3.7)-(3.8), which happens if and only if at least one of the following situations (I) or (II) takes place:

(I) the set  $B$  does not satisfy (3.8), which means that

$$B = \{C_n, n \in I\} \cup \{C_n^l, n \in J\} \quad (3.16)$$

and

$$\bar{B} := \{1, \dots, 2p\} \setminus B = \{C_n, n \in I'\} \cup \{C'_n, n \in J\}, \quad (3.17)$$

where  $\{l, l'\} = \{1, 2\}$  and sets  $I$  and  $I'$  are such that

$$I \cup I' \neq \emptyset, \quad I \cap I' = \emptyset, \quad \text{and } I \cup I' \cup J = \{1, \dots, 2p\}; \quad (3.18)$$

(II)  $W_i \setminus A_i \neq \emptyset$  at least for some  $i \in B$ , i.e. (3.7) is not satisfied.

Suppose situation (I) takes place. Without loss of generality let  $l = 1$  and  $l' = 2$ . Consider

$$\bar{A} := \Lambda \setminus A = \{\cup_{i \in \bar{B}} W_i\} \cup \{\cup_{i \in B} W_i \setminus A_i\}. \quad (3.19)$$

Define the subset  $B_1 \subseteq B$  so that

$$W_i \setminus A_i \neq \emptyset \text{ iff } i \in B_1, \quad (3.20)$$

and thus

$$\bar{A} = \{\cup_{i \in \bar{B}} W_i\} \cup \{\cup_{i \in B_1} W_i \setminus A_i\}.$$

Notice that  $B_1$  can be empty. Further define  $I_1$  so that

$$C_n \in \bar{B} \cup B_1 \text{ iff } n \in I_1. \quad (3.21)$$

(a) Suppose  $I_1 \neq \emptyset$ . Let

$$B_0 := \{C_n^1, n \in I_1\} \cup \{\bar{B} \cup B_1 \setminus \{C_n, n \in I_1\}\}. \quad (3.22)$$

Denote

$$\bar{A}^{B_0} = \{\cup_{i \in \bar{B} \cap B_0} W_i\} \cup \{\cup_{i \in B_1 \cap B_0} W_i \setminus A_i\}.$$

Then the subsystem  $(X_i^{\bar{A}^{B_0}}(t), i \in \bar{A}^{B_0})$  is ergodic, since it is completely connected, and (3.9) holds for any  $x, y \in \bar{A}^{B_0}$ ,  $x \neq y$ . Also, we can compute as in (3.13)

$$\pi_x^{\bar{A}^{B_0}} = \frac{1}{a + (|\bar{A}^{B_0}| - 1)b}, \quad (3.23)$$

Taking into account that  $|I_1| \geq 1$ , we easily derive from (3.23) the following upper bound for the components of the second vector field  $v^{\bar{A}^{B_0}}$ :

$$v_z^{\bar{A}^{B_0}} \geq -1 + \frac{c + (|\bar{A}^{B_0}| - 1)b}{a + (|\bar{A}^{B_0}| - 1)b} > 0 \quad (3.24)$$

for any  $z \in \bar{A} \setminus \bar{A}^{B_0}$  due to condition (3.5). Hence,  $(X_i^{\bar{A}}(t), i \in \bar{A})$  is transient according to Theorem A. This contradicts our assumption that  $A$  is a trap.

(b) Suppose now that  $I_1 = \emptyset$ . Then it follows from (3.21) and (3.17) that we also have

$$I' = \emptyset. \quad (3.25)$$

This together with our assumption (3.18) implies that  $I \neq \emptyset$ . Thus in this case we have

$$A = \{x \in \Lambda : x \in \cup_{i \in B} A_i\}, \quad (3.26)$$

where  $B \ni \{C_n, n \in I\}$ , and

$$\bar{A} = \{\cup_{i \in \bar{B}} W_i\} \cup \{\cup_{i \in B_1} W_i \setminus A_i\}, \quad (3.27)$$

where  $\{\bar{B} \cup B_1\} \cap C_n \leq 1$  for any  $n$ , according to (3.21) and (3.25). The latter implies that (3.9) holds for any  $y \in \bar{A}$  which in turn implies ergodicity of  $(X_i^{\bar{A}}(t), i \in \bar{A})$ . Furthermore, we can find  $\pi_x^{\bar{A}}$ , analogously to (3.13), namely:

$$\pi_x^{\bar{A}} = \frac{1}{a + (|\bar{A}| - 1)b}, \quad x \in \bar{A}. \quad (3.28)$$

Let us compute now the  $x$ th component of the second vector field  $v_x^{\bar{A}}$  for  $x \in A_i$  with  $i \in I$ . According to (3.28) and assumption (3.4) we get

$$v_x^{\bar{A}} := -1 + \frac{|\bar{A}|b}{a + (|\bar{A}| - 1)b} < 0. \quad (3.29)$$

The latter contradicts our assumption, that  $A$  is a trap.

Hence we conclude that if  $A$  is a trap then necessarily condition (3.8) holds. Assume now situation (II). More precisely, suppose that  $A$  satisfies (3.8) but does not satisfy (3.7), i.e.

$$A = \{x \in \Lambda : x \in \cup_{i \in B} A_i\}, \quad (3.30)$$

for some  $B \subset \Lambda$  such that (3.8) holds, while  $W_i \setminus A_i \neq \emptyset$  for  $i \in B_1 \subseteq B$  for some nonempty  $B_1$  (see definition (3.20)). The latter implies

$$\bar{A} = \{\cup_{i \in \bar{B}} W_i\} \cup \{\cup_{i \in B_1} W_i \setminus A_i\},$$

where according to the assumption (3.8)

$$|B_1 \cap C_n| = 1 \quad (3.31)$$

for at least one  $C_n$ . The latter implies  $C_n \in \bar{B} \cup B_1$ , i.e. the set  $I_1$  defined in (3.21) is non-empty. But as we have seen, this situation contradicts the assumption that  $A$  is a trap. This finishes our argument that conditions (3.7) and (3.8) are necessary for a set  $A$  to be a trap. This completes the proof of Theorem 3.

### 3.3 Learning rule for the almost symmetric case

We will show here that the Hebbian rule (see [5]) of learning patterns which successfully works in the case of Hopfield networks ([5]), is applicable for our network at least for the following particular case.

For any  $A \subset \Lambda$  define configuration  $\xi(A) = (\xi_i(A), i \in \Lambda) \in \{-1, +1\}^\Lambda$  such that

$$\xi_i(A) = \begin{cases} +1, & \text{if } i \in A, \\ -1, & \text{otherwise.} \end{cases} \quad (3.32)$$

We shall also call a configuration  $\xi$  a trap for  $X(t)$  if and only if  $\xi = \xi(A)$ , where  $A$  is a trap.

Suppose that we are given  $2^p$ ,  $p = \frac{N}{2k}$ , binary vectors (images)  $\xi_1, \dots, \xi_{2^p}$ . We shall find the connection constants  $\mathbf{E}\theta_{xy}$  such that  $X(t)$  satisfying (3.3) with these parameters, possesses a system of traps consisting exactly of the given  $2^p$  vectors.

Notice the difference between this task and the problem of stability of patterns for Hopfield neural model (see for example, [9] for a recent account on the relevant results). Recall, that the capacity of Hopfield network is determined by the number of given *i.i.d.* patterns, which are stable with respect to the dynamics of the system. This means that starting from an arbitrary initial state the system should converge with a large probability to one of the given patterns, which is the closest to the initial state. It was conjectured that the number of such patterns for Hopfield network of  $N$  neurons is at most a fraction of  $N$ . Here we construct a network which possesses given exponentially large (with respect to  $N$ ) number of the limiting patterns and only them. These patterns are stable in a trivial sense, i.e. if the initial state of the system is one of the given traps (patterns), then the system stays at this trap forever. However, we do not predict which state (out of  $p$  possible) the system converges to, starting from an arbitrary initial condition. The problem of determination of the basins of attraction of the limiting patterns for the hourglass model will be a subject of a separate study.

**Corollary 1** Suppose the collection of  $N$ -dim vectors  $\{\xi^\mu, \mu = 1, \dots, M\}$  where  $M = 2^p$ , and  $\xi_x^\mu \in \{-1, 1\}$  for any  $\mu$  and  $1 \leq x \leq N$ , has the following properties:

1.  $\xi_x^\mu \xi_y^\mu = 1$  for any  $\mu$  if  $x, y \in W_n$  for some  $n \in \{1, \dots, 2p\}$ ,
2.  $\sum_{x=1}^N \xi_x^\mu = 0$  for all  $\mu$ ,
3. for any  $n \in \{1, \dots, 2p\}$  there exists unique  $l = l(n) \in \{1, \dots, 2p\} \setminus \{n\}$  such that  $\xi_x^\mu \xi_y^\mu = -1$  for any  $\mu$  whenever  $x \in W_n$  and  $y \in W_l$ .

Then the  $N$ -neuron system with  $a_i = a$  and the connection constants

$$\mathbf{E}\theta_{xy} = \begin{cases} b(x, y), & \text{if } b(x, y) = \min_{(x', y')} b(x', y') \\ \max_{(x', y')} b(x', y'), & \text{otherwise,} \end{cases} \quad (3.33)$$

where

$$b(x, y) := Aa \frac{1}{M} \sum_{\mu=1}^M \xi_x^\mu \xi_y^\mu - Ba, \quad (3.34)$$

with the constants  $A$  and  $B$  satisfying the conditions

$$\begin{aligned} 0 &< B - A < 1, \\ 1 &< B + A, \end{aligned} \quad (3.35)$$

has a system of traps, which is  $\{\xi^\mu, \mu = 1, \dots, 2^p\}$ .

**Proof.** Indeed, having conditions 1-3 of the corollary satisfied, we derive from (3.34) that

$$b(x, y) = \begin{cases} -(A + B)a, & \text{if } x \in W_n, y \in W_{l(n)}, \\ -(B - A)a, & \text{if } x, y \in W_n, \end{cases} \quad (3.36)$$

and

$$-(A + B)a < b(x, y) < -(B - A)a \quad \text{otherwise.} \quad (3.37)$$

Substituting (3.36) and (3.37) into (3.33), and taking into account condition (3.35), we obtain

$$\begin{aligned} -\mathbf{E}\theta_{xy} &= (A + B)a > a, & \text{if } x \in W_n, y \in W_{l(n)}, \\ -\mathbf{E}\theta_{xy} &= (B - A)a < a, & \text{otherwise,} \end{aligned} \quad (3.38)$$

which shows that conditions (3.4)-(3.5) are satisfied. Hence we can use Theorem 3 to conclude that  $\{\xi^\mu, \mu = 1, \dots, 2^p\}$  is a system of traps for the defined network.

**Acknowledgements** We thank the referee for useful remarks.

## References

- [1] M. Cottrell, Mathematical analysis of a neural network with inhibitory coupling, *Stochastic Process. Appl.* **40** (1992) 103-126.
- [2] M. Cottrell, F. Piat and J.-P. Rospars, A stochastic model for interconnected neurons. *Biosystems*, **40** (1997) 29-35.
- [3] M. H. A. Davis, *Markov Models and Optimization* Chapman and Hall, 1993.
- [4] C. Fricker, P. Robert, E. Saada, and D. Tibi, Analysis of some networks with interaction, *Ann. Appl. Prob.* **4** (1994) 1112-1128.
- [5] J.J. Hopfield, Neural networks and physical systems with emergent collective computational abilities. *Proc. Natl. Acad. Sci. USA* **79**, (1982) 2554-2558.
- [6] I.A. Ignatyuk, V.A. Malyshev, and T.S. Turova, Stability of infinite systems of stochastic equations (in Russian), in: *Itogy nauky i tekhniki. Teoriya veroyatnostey*, 27 (VINITI, Moscow, 1990) pp. 79-128. English translation: *Journal of Soviet Mathematics* 61 (1992) 2114-2151.
- [7] F. Karpelevich, V.A. Malyshev, and A.N. Rybko, Stochastic Evolution of Neural Networks, *Markov Processes and Related Fields* **1** (1995) 141-161.
- [8] V.A. Malyshev and T.S. Turova, Gibbs Measures on Attractors in Biological Neural Networks. Gibbs measures on attractors in biological neural networks. *Markov Processes and Related Fields* **3** (1997) 443-464.
- [9] *Mathematical Aspects of Spin Glasses and Neural Networks*. Ed. A. Bovier and P. Picco. Birkhäuser, 1998.
- [10] F. Piat, Modélisation du système olfactif par une approche neuromimétique: aspects cognitifs et théoriques, Ph.D. thesis, Université Paris 1, (1994).
- [11] C. Stone, On moment generating functions and renewal theory, *The Annals of Mathematical Statistics* 36 (1965) 1298-1301.

- [12] T.S. Turova, Analysis of a biologically plausible neural network via an hourglass model. *Markov Processes and Related Fields* **2** (1996) 487-510.
- [13] T.S. Turova, Exponential rate of convergence of an infinite neuron model with local connections. *Stochastic Processes and Their Appl.* **73** (1998) 173-193.