

Characterization of polynomial decay rate for the solution of linear evolution equation

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Abstract

In this paper, we study the decay rate of solutions to strongly stable, but not exponentially stable linear evolution equations. It is known that the resolvent operator of such an equation must be unbounded on the imaginary axis. Our main result is an estimate of the decay rate when the unboundedness is of polynomial order. We then apply our main theorem to three strongly stable but not exponentially stable systems to obtain the decay rate, which is not available in the literature.

1 Introduction

We consider a linear evolution equation on Hilbert space \mathcal{H} :

$$\begin{cases} \frac{dx}{dt} = \mathcal{A}x \\ x(0) = x_0. \end{cases} \quad (1.1)$$

Assume that

(H1). \mathcal{A} generates a bounded C_0 semigroup $S(t) = e^{\mathcal{A}t}$ on \mathcal{H} .

(H2). $i\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$.

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The solution to (1.1) is

$$x(t) = S(t)x_0.$$

We say that (1.1) is *strongly stable* if

$$\lim_{t \rightarrow \infty} \|x(t)\|_{\mathcal{H}} = 0 \quad (1.2)$$

for all $x_0 \in \mathcal{H}$; is *exponentially stable* if there exist constants $M, \alpha > 0$ such that

$$\|x(t)\|_{\mathcal{H}} \leq M e^{-\alpha t} \|x_0\|_{\mathcal{H}}, \quad t > 0. \quad (1.3)$$

for all $x_0 \in \mathcal{H}$. There are many systems which are strongly stable, but not exponentially stable. For example, when damping in an elastic system is located only on a subdomain or on part of the boundary, its energy still dissipates, but may not at a uniform rate. In that case, other kind of decay rates had been introduced. If there exists a positive function $f(t)$ with $\lim_{t \rightarrow \infty} f(t) = 0$ such that

$$\|x(t)\|_{\mathcal{H}} \leq f(t) \|x_0\|_{\mathcal{D}(\mathcal{A})}, \quad t > 0, \quad (1.4)$$

we say the solution of (1.1) decays at a rate of $f(t)$ for all $x_0 \in \mathcal{D}(\mathcal{A})$. Note that the norm on the right-hand side of (1.4) can not be the \mathcal{H} -norm. Otherwise, by the semigroup properties, (1.4) implies (1.3).

Several time domain methods for the polynomial decay rate estimate exist in the literature. Energy estimate method combined with multiplier technique can be found in [R],[RR],[RW]. An energy inequality was established in [Ru] as sufficient condition for polynomial decay rate $1/t$. This inequality was generalized in [JTZ]. A Riesz basis method was used in [LbLw] which gives the polynomial decay rate based on the asymptotic relation of the real and imaginary part of the eigenvalues. Another method by using Ingham's inequality can be found in [ALT]. It is not our intention to do a complete review on this subject here. We refer the readers to the references in the mentioned above for more information.

The semigroup $S(t)$ is strongly stable if condition (H2) holds (see [H1],[H3]); is exponentially stable if and only if (H2) and

$$\sup\{\|(i\beta - \mathcal{A})^{-1}\|_{\mathcal{H}} \mid \beta \in \mathbb{R}\} < \infty \quad (1.5)$$

are satisfied (see [H2],[Pr]). Moreover, $S(t)$ is analytic (see [LY], [LZ]) if

$$\|(i\beta - \mathcal{A})^{-1}\|_{\mathcal{H}} = O\left(\frac{1}{\beta}\right), \quad \beta \rightarrow \infty. \quad (1.6)$$

These frequency domain results have been used to study the asymptotic behavior and smoothness of the solution to a given linear evolution equation.

One can observe that the growth rate of the resolvent operator on the imaginary axis is related to the decay rate of the solution to (1.1). If we know that (1.1) is strongly stable but not exponentially stable, then condition (1.5) must fail, i.e., the resolvent operator is unbounded on the imaginary axis. In his study on wave equation with internal or boundary damping, Lebeau [L] had used this approach to get decay rate estimate. Our main goal is to provide a characterization of the decay rate of solution to (1.1) by the order of unboundedness of the resolvent operator on the imaginary axis, particularly for the case of $\|(i\beta - \mathcal{A})^{-1}\|_{\mathcal{H}} = O(\beta^l)$.

This paper is organized as following: section 2 is devoted to the statement and proof of the main theorem. In section 3, we apply our theorem to three strongly stable but not exponentially stable systems. The first one is an elastic string with Kelvin-Voigt damping on a segment of the string and damping coefficient discontinuous at the interface. The second one is a weakly coupled n-dimensional wave equation and heat equation. The last one is a wave equation on a square domain with viscous damping on a subdomain which contains a vertical strip. We are able to estimate the decay rates for these systems, which are not available in the literature.

2 Main Theorem

We further assume that

$$(H3). \sup_{|\beta| \geq 1} \frac{1}{\beta^l} \|(i\beta - \mathcal{A})^{-1}\| \leq M \text{ for some } l > 0.$$

By the resolvent identity

$$\begin{aligned} (-\alpha + i\beta - \mathcal{A})^{-1} &= (i\beta - \mathcal{A})^{-1} [I - \alpha(i\beta - \mathcal{A})^{-1}]^{-1} \\ &= (i\beta - \mathcal{A})^{-1} \left[I - \alpha |\beta|^l \frac{(i\beta - \mathcal{A})^{-1}}{|\beta|^l} \right]^{-1}, \end{aligned} \quad (2.1)$$

we see that $-\alpha + i\beta - \mathcal{A}$ is invertible when $|\alpha\beta^l| \leq \frac{1}{2M}$ and $|\beta| \geq 1$. Moreover, $\sup_{|\beta| \geq 1} \frac{1}{|\beta|^l} \|(-\alpha + i\beta - \mathcal{A})^{-1}\| \leq 2M$. Since \mathcal{A}^{-1} exists, from the continuity of the resolvent operator, there is an constant $\varepsilon > 0$ and a curve

$$\Gamma^\varepsilon = \left\{ z = -\frac{\varepsilon}{\eta^l} + i\eta \mid |\eta| \geq 1 \right\} \cup \left\{ z = -\varepsilon + i\eta \mid |\eta| < 1 \right\} \quad (2.2)$$

such that the region on the right-hand side of Γ^ε (including the boundary) belongs to the resolvent set of \mathcal{A} .

Lemma 2.1 *The following identity holds for all integer $k \geq 1$ and all $x \in \mathcal{H}$:*

$$e^{t\mathcal{A}}(I - \mathcal{A})^{-k}x = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\zeta I - \mathcal{A})^{-1}x}{(1 - \zeta)^k} e^{t\zeta} d\zeta = \frac{1}{2\pi i} \int_{\Gamma^\varepsilon} \frac{(\zeta I - \mathcal{A})^{-1}x}{(1 - \zeta)^k} e^{t\zeta} d\zeta \quad (2.3)$$

where $\Gamma = \left\{ \zeta = \frac{1}{2} + i\eta \mid \eta \in \mathbb{R} \right\}$.

Proof. Let $B = \mathcal{A} - I$. Then by equation (1.13) in [Na],

$$e^{tB}B^{-k}x = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{(\lambda I - B)^{-1}x}{\lambda^k} e^{t\lambda} d\lambda \quad (2.4)$$

for every $x \in \mathcal{H}$ and $t > 0$. Returning to the original operator \mathcal{A} and letting $\zeta = \lambda + 1$, we obtain the first equality. Using the following estimates:

$$\begin{cases} \|(\zeta I - \mathcal{A})^{-1}x\| \leq C\|x\|, \\ \frac{1}{|1-\zeta|} \leq \frac{2}{|\zeta|} \end{cases} \quad (2.5)$$

the second equality in (2.3) can be easily established by a contour integration. \square

Theorem 2.1 *Suppose that the hypotheses (H1)-(H3) hold. Then, for any positive integer k there exists a constant $C_k > 0$ such that*

$$\|e^{t\mathcal{A}}z\|_{\mathcal{H}} \leq C_k \left(\frac{\ln t}{t} \right)^{\frac{k}{l}} (\ln t) \|x\|_{\mathcal{D}(\mathcal{A}^k)} \quad (2.6)$$

for all $x \in \mathcal{D}(\mathcal{A}^k)$.

Remark: Estimate (2.6) is not optimal. In the papers [R] and [RW], they got the best decay rate $\frac{1}{t^{1/2}}$ for the solution of an hybrid system. But using (2.6), we can only get the decay rate $\frac{(\ln t)^{3/2}}{t^{1/2}}$. Our effort to remove the $\ln t$ term in (2.6) has not been successful.

Proof of Theorem 2.1. Inserting a weight function to the identities in (2.3), we have

$$\begin{aligned} e^{t\mathcal{A}}(I - \mathcal{A})^{-k}x &= \frac{1}{2\pi i} \int_{\Gamma} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\lambda+i\zeta)^2}{2}} d\lambda \frac{(\zeta - \mathcal{A})^{-1}x}{(1-\zeta)^k} e^{t\zeta} d\zeta \\ &= I_1 + I_2 \end{aligned} \quad (2.7)$$

where

$$I_1 = \int_{\Gamma^\varepsilon} \int_{|\lambda| \leq c_1 \left(\frac{t}{\ln t}\right)^{1/l}} (\cdots) d\lambda d\zeta, \quad I_2 = \int_{\Gamma} \int_{|\lambda| > c_1 \left(\frac{t}{\ln t}\right)^{1/l}} (\cdots) d\lambda d\zeta.$$

Estimation of I_1

For $\zeta \in \Gamma^\varepsilon$, we have

$$\begin{aligned} (\lambda + i\zeta)^2 &= \begin{cases} (\lambda - \eta)^2 - \frac{\varepsilon^2}{\eta^{2l}} - 2(\lambda - \eta)\frac{\varepsilon}{\eta^l}i & |\eta| \geq 1 \\ (\lambda - \eta)^2 - \varepsilon^2 - 2(\lambda - \eta)\varepsilon i & |\eta| < 1, \end{cases} \\ |1 - \zeta| &\geq \begin{cases} C(1 + |\eta|) & |\eta| \geq 1 \\ C & |\eta| < 1, \end{cases} \\ \|(\zeta - \mathcal{A})^{-1}x\| &\leq \begin{cases} \frac{C}{\varepsilon} |\eta|^l \|x\| & |\eta| \geq 1 \\ \frac{C}{\varepsilon} \|x\| & |\eta| < 1. \end{cases} \end{aligned}$$

Break I_1 as follows

$$\begin{aligned} I_1 &= \int_{|\lambda| \leq \alpha_1} \int_{|\eta| < 1} (\cdots) d\zeta d\lambda + \int_{|\lambda| \leq \alpha_1} \int_{\alpha_2 \geq |\eta| \geq 1} (\cdots) d\zeta d\lambda + \int_{|\lambda| \leq \alpha_1} \int_{|\eta| > \alpha_2} (\cdots) d\zeta d\lambda \\ &= I_{1,1} + I_{1,2} + I_{1,3}, \end{aligned} \quad (2.8)$$

where $\alpha_i = (c_i \frac{t}{\ln t})^{\frac{1}{l}}$ and $c_i > 0$ to be chosen. Then

$$|I_{1,1}| \leq C \int_{|\lambda| \leq \alpha_1} \int_{|\eta| < 1} e^{-\varepsilon t} d\eta d\lambda \|x\| \leq 2C\alpha_1 e^{-\varepsilon t} \|x\| = O(t^{-\infty}) \|x\|, \quad (2.9)$$

and

$$\begin{aligned} |I_{1,2}| &\leq C \int_{|\lambda| \leq \alpha_1} \int_{\alpha_2 \geq |\eta| \geq 1} e^{-\frac{\varepsilon t}{|\eta|^l} + \frac{\varepsilon^2}{2|\eta|^{2l}}} \frac{|\eta|^l}{(1+|\eta|)^k} d\eta d\lambda \|x\| \\ &\leq C\alpha_1 \alpha_2 e^{-\frac{\varepsilon t}{2\alpha_2^l}} (\alpha_2)^{l-k} \|x\| \\ &= C \left(\frac{t}{\ln t}\right)^{\frac{2+l-k}{l}} t^{-\frac{\varepsilon}{c_2}} \|x\| = O(t^{-\infty}) \|x\|, \end{aligned} \quad (2.10)$$

provided that c_2 is small enough. Once c_2 is fixed, we choose $c_1 \in (0, c_2)$ such that

$$(\lambda - \eta)^2 \geq 2\delta(\eta^2 + \lambda^2) \geq 2\delta\eta^2$$

for some $\delta > 0$ when $|\lambda| \leq \alpha_1$ and $|\eta| \geq \alpha_2$. Therefore,

$$\begin{aligned} |I_{1,3}| &\leq \int_{|\lambda| \leq \alpha_1} \int_{|\eta| > \alpha_2} \frac{|\eta|^l}{(1+|\eta|)^k} e^{-\frac{1}{2}[(\lambda-\eta)^2 + \frac{2\epsilon t}{|\eta|^l} - \frac{\epsilon^2}{\eta^{2l}}]} d\eta d\lambda \|x\| \\ &\leq C\alpha_1 \int_{|\eta| > \alpha_2} \frac{|\eta|^l}{(1+|\eta|)^k} e^{-\delta\eta^2} d\eta \|x\| = O(t^{-\infty}) \|x\| \end{aligned} \quad (2.11)$$

Combining (2.9)-(2.11), we have

$$|I_1| \leq O(t^{-\infty}) \|x\|. \quad (2.12)$$

Estimation of I_2

We rewrite I_2 as

$$I_2 = e^{tA} J$$

where

$$J = \frac{1}{2\pi i} \int_{\Gamma} \int_{|\lambda| > \alpha_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\lambda+i\zeta)^2}{2}} d\lambda \frac{(\zeta - \mathcal{A})^{-1} x}{(1-\zeta)^k} d\zeta.$$

Define the set on complex λ plane

$$\Omega_t = \left\{ \lambda = \alpha_1 - \frac{i}{2} + \mu e^{-i\theta} \mid \mu \geq 0, \theta \in [0, \frac{\pi}{8}] \right\}.$$

For any $\lambda \in \Omega_t$, we construct a contour on ζ plane

$$\begin{aligned} D_{\lambda}^+ &= \{ \zeta = i(1+\eta) - Im\lambda \mid \eta \geq 0 \}, \\ D_{\lambda}^- &= \{ \zeta = i(1+\eta) + \frac{1}{2} \mid \eta \leq 0 \}, \\ D_{\lambda}^0 &= \{ \zeta = i + \frac{1}{2} + \eta \mid 0 \leq \eta \leq -Im\lambda - \frac{1}{2} \}. \end{aligned}$$

Let

$$H_{\lambda}(x, \zeta) = \frac{1}{2\pi i} \frac{1}{\sqrt{2\pi}} \frac{(\zeta - \mathcal{A})^{-1} x}{(1-\zeta)^k} e^{-\frac{1}{2}(\lambda+i\zeta)^2}, \quad \lambda \in \Omega_t.$$

Then $J = J^+ + J^-$ where

$$J^{\pm} = \int_{\pm\lambda \geq \alpha_1} \int_{\Gamma} H_{\lambda}(x, \zeta) d\zeta d\lambda = \int_{\pm\alpha - \frac{i}{2}}^{\pm\infty - \frac{i}{2}} \int_{\Gamma} H_{\lambda}(x, \zeta) d\zeta d\lambda = \int_{\pm\alpha_1 - \frac{i}{2}}^{\pm\infty - \frac{i}{2}} \int_{D_{\lambda}^- \cup D_{\lambda}^0 \cup D_{\lambda}^+} H_{\lambda}(x, \zeta) d\zeta d\lambda. \quad (2.13)$$

Note that the contour of integration for λ and ζ have been shifted. In what follows, we will carry out the estimation of J^+ . J^- can be estimated similarly.

For $\lambda \in \Omega_t$ and $\zeta \in D_\lambda^0$,

$$\lambda + i\zeta = (\alpha_1 + \mu \cos \theta - 1) + i(\eta - \mu \sin \theta), \quad \eta \leq \mu \sin \theta.$$

Then

$$\operatorname{Re}(\lambda + i\zeta)^2 = (\alpha_1 + \mu \cos \theta - 1)^2 - (-\eta + \mu \sin \theta)^2 \geq 2\delta(\mu^2 + \eta^2 + \alpha_1^2).$$

For $\lambda \in \Omega_t$ and $\zeta \in D_\lambda^-$,

$$\lambda + i\zeta = (\alpha_1 + \mu \cos \theta - \eta - 1) + i\mu \sin \theta, \quad \eta \leq 0.$$

Then

$$\begin{aligned} \operatorname{Re}(\lambda + i\zeta)^2 &= (\alpha_1 + \mu \cos \theta - \eta - 1)^2 - \mu^2 \sin^2 \theta \\ &\geq \mu^2 \cos 2\theta + \eta^2 + (\alpha_1 - 1)^2 \\ &\geq 2\delta(\mu^2 + \eta^2 + \alpha_1^2). \end{aligned} \quad (2.14)$$

It follows that

$$\left| \int_{D_\lambda^- \cup D_\lambda^0} H_\lambda(x, \zeta) d\zeta \right| \leq C \|x\| e^{-\delta\alpha_1^2} \int_0^{+\infty} \frac{e^{-\delta(\mu^2 + \eta^2)}}{(1 + |\eta|)^k} d\eta \leq C \|x\| e^{-\delta\alpha_1^2} e^{-\delta\mu^2}. \quad (2.15)$$

For $\lambda \in \Omega_t$ and $\zeta \in D_\lambda^+$, we have

$$(\lambda + i\zeta)^2 = (\alpha_1 + \mu \cos \theta - \eta - 1)^2, \quad (2.16)$$

$$\|(\zeta - \mathcal{A})^{-1}\| \leq \frac{C}{|\operatorname{Re}\zeta|} = \frac{C}{|\operatorname{Im}\lambda|} \leq \frac{C}{\mu \sin \theta + 1}, \quad (2.17)$$

$$\|(\zeta - \mathcal{A})^{-1}x\| \leq \frac{C_x}{1 + |\zeta|} \leq \frac{C_x}{1 + |\eta|} \quad (2.18)$$

Then, it follows from (2.16) and (2.18) that

$$\begin{aligned} \left| \int_{D_\lambda^+} H_\lambda(x, \zeta) d\zeta \right| &\leq C_x \int_0^{+\infty} \frac{e^{-\frac{1}{2}[(\alpha_1 + \mu \cos \theta - 1) - \eta]^2}}{(1 + |\eta|)^{k+1}} d\eta \\ &\leq \frac{C_x}{(\alpha_1 + \mu \cos \theta)^{k+1}} \end{aligned} \quad (2.19)$$

Therefore, by contour integration, we can rewrite

$$J^+ = \int_\gamma \int_{D_\lambda^- \cup D_\lambda^0 \cup D_\lambda^+} H_\lambda(x, \zeta) d\zeta d\lambda,$$

where $\gamma = \{\lambda = \alpha_1 - \frac{i}{2} + \mu e^{-i\frac{\pi}{8}} \mid \mu \geq 0\}$. On the other hand, using (2.16)-(2.17), we have

$$\begin{aligned} \left| \int_{D_\lambda^+} H_\lambda(x, \zeta) d\zeta \right| &\leq \frac{C\|x\|}{|\mu \sin \frac{\pi}{8} + 1|} \int_0^{+\infty} \frac{e^{-\frac{1}{2}[(\alpha_1 + \mu \cos \frac{\pi}{8} - 1) - \eta]^2}}{(1 + |\eta|)^k} d\eta \\ &\leq \frac{C\|x\|}{|\mu \sin \frac{\pi}{8} + 1|} \cdot \frac{1}{(\alpha_1 + \mu \cos \frac{\pi}{8})^k} \\ &\leq \frac{C\|x\|}{(\mu + 1)(\mu + \alpha_1)^k}. \end{aligned} \quad (2.20)$$

Combining (2.15) and (2.20), we conclude that

$$\begin{aligned} |J^+| &\leq C\|x\| \int_0^{+\infty} \left[e^{-\delta\alpha_1^2} e^{-\delta\mu^2} + \frac{1}{(\mu + 1)(\mu + \alpha_1)^k} \right] d\mu \\ &\leq C\|x\| \left(e^{-\delta\alpha_1^2} + \frac{\ln \alpha_1}{\alpha_1^k} \right) \\ &\leq C\|x\| \ln \left(c_1 \frac{t}{\ln t} \right)^{\frac{1}{t}} \left(c_1 \frac{t}{\ln t} \right)^{-\frac{k}{t}} \\ &\leq C\|x\| \left(\frac{\ln t}{t} \right)^{\frac{k}{t}} \ln t. \end{aligned} \quad (2.21)$$

After a similar estimation of J^- , we obtain

$$|J| \leq C\|x\| \left(\frac{\ln t}{t} \right)^{\frac{k}{t}} \ln t. \quad (2.22)$$

Therefore,

$$|I_2| \leq C\|x\| \left(\frac{\ln t}{t} \right)^{\frac{k}{t}} \ln t. \quad (2.23)$$

Finally, we conclude (2.6) from (2.7), (2.12) and (2.23). \square

3 Applications

In this section, we consider three strongly stable but not exponentially stable systems. By Theorem 2.1, we are able to obtain the decay rates for the solutions to these systems.

Example 1. We consider a 1-d wave equation with local Kelvin-Voigt damping and discontinuous coefficient functions:

$$\begin{cases} \rho(x)u_{tt}(x, t) - [a(x)u'(x, t) + d(x)u'_t(x, t)]' = 0 & \text{in } (0, L) \times \mathbb{R}^+, \\ u(0, t) = u(L, t) = 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \end{cases} \quad (3.1)$$

where $\rho(x), a(x) \geq c_0 > 0$ and $d(x) = 0$ on $(0, \alpha)$, $d(x) \geq c_0$ on (α, L) , $\rho, a \in H^1(0, \alpha) \cup H^1(\alpha, L)$ and $d \in C(\alpha, L)$. The prime denotes the derivatives with respect to x .

The energy of this system is defined by

$$E(t) = \frac{1}{2} \int_0^L [\rho(x)|u_t|^2 + a(x)|u'|^2] dx. \quad (3.2)$$

Let

$$\begin{aligned} H(b, c) &= L^2(b, c), & \|v\|_{H(b,c)}^2 &= \int_b^c \rho(x)|v(x)|^2 dx, \\ V(b, c) &= H_0^1(b, c), & \|v\|_{V(b,c)}^2 &= \int_b^c a(x)|v'(x)|^2 dx, \end{aligned}$$

and

$$\mathcal{H} = V(0, L) \times H(0, L), \quad \|(u, v)\|_{\mathcal{H}}^2 = \|u\|_{V(0,L)}^2 + \|v\|_{H(0,L)}^2.$$

We then define in \mathcal{H} that

$$D(\mathcal{A}) = \{(u, v) \mid v \in V, au' + dv' \in H^1(0, L)\}, \quad (3.3)$$

and

$$\mathcal{A}(u, v) = (v, \frac{1}{\rho}(au' + dv')). \quad (3.4)$$

Thus (3.1) can be rewritten as an abstract evolution equation on \mathcal{H} :

$$\frac{d}{dt}(u, v) = \mathcal{A}(u, v), \quad (u(0), v(0)) = (u_0, u_1). \quad (3.5)$$

It is known that (see [LL]) \mathcal{A} satisfies the hypotheses (H1) and (H2). Moreover, $e^{t\mathcal{A}}$ is not exponentially stable.

In what follows, we show that the hypothesis (H3) holds here for $l = \frac{1}{2}$. If it is false, then there exist $\beta_n \in \mathbb{R}^+$, $(u_n, v_n) \in D(\mathcal{A})$, $n = 1, 2, \dots$, with

$$\|(u_n, v_n)\|_{\mathcal{H}} = 1, \quad \beta_n \rightarrow \infty, \quad (3.6)$$

and

$$\beta_n^{\frac{1}{2}}(\mathbf{i}\beta_n - \mathcal{A})(u_n, v_n) \equiv (f_n, g_n) \rightarrow 0 \quad \text{in } \mathcal{H}, \quad (3.7)$$

i.e.,

$$\beta_n^{\frac{1}{2}}(\mathbf{i}\beta_n u_n - v_n) = f_n \rightarrow 0 \quad \text{in } V(0, L), \quad (3.8)$$

$$\beta_n^{\frac{1}{2}}(\mathbf{i}\beta_n v_n - \frac{1}{\rho}T'_n) = g_n \rightarrow 0 \quad \text{in } H(0, L). \quad (3.9)$$

where $T_n = au'_n + dv'_n$. Our goal is to find a contradiction to (3.6).

We first consider (3.8)–(3.9) on the interval (α, L) . From (3.7), we obtain

$$\beta_n^{\frac{1}{2}} \int_{\alpha}^L d(x) |v'_n|^2 dx = \operatorname{Re} \langle \beta_n^{\frac{1}{2}} (i\beta_n - \mathcal{A})(u_n, v_n), (u_n, v_n) \rangle_{\mathcal{H}} \rightarrow 0, \quad (3.10)$$

which implies that

$$\|\beta_n^{\frac{1}{4}} v'_n\|_{H(\alpha, L)} \rightarrow 0, \quad \|\beta_n^{\frac{5}{4}} u'_n\|_{H(\alpha, L)} \rightarrow 0. \quad (3.11)$$

Thus, we also have

$$\|\beta_n^{\frac{1}{4}} T_n\|_{H(\alpha, L)} \rightarrow 0. \quad (3.12)$$

we take inner product of (3.9) with v_n in $H(\alpha, L)$ to obtain

$$i \|\beta_n^{\frac{3}{4}} v_n\|_{H(\alpha, L)}^2 + \beta_n^{\frac{1}{2}} T_n(\alpha^+) \bar{v}_n(\alpha^+) + \langle \beta_n^{\frac{1}{4}} T_n, \beta_n^{\frac{1}{4}} v'_n \rangle_{L^2(\alpha, L)} \rightarrow 0. \quad (3.13)$$

The third term converges to zero due to (3.11) and (3.12). For the second term, we have the following estimate:

$$\begin{aligned} \beta_n^{\frac{1}{2}} |T_n(\alpha^+) v_n(\alpha^+)| &\leq C \beta_n^{\frac{1}{2}} \|v_n\|_{\frac{1}{2}H(\alpha, L)} \|v'_n\|_{\frac{1}{2}H(\alpha, L)} \|T_n\|_{\frac{1}{2}H(\alpha, L)} \|T'_n\|_{\frac{1}{2}H(\alpha, L)} \\ &= C \|\beta_n^{\frac{1}{4}} v'_n\|_{\frac{1}{2}H(\alpha, L)} \|\beta_n^{\frac{1}{4}} T_n\|_{\frac{1}{2}H(\alpha, L)} \|\beta_n^{\frac{3}{4}} v_n\|_{H(\alpha, L)} + o(1) \\ &= o(1)(1 + \|\beta_n^{\frac{3}{4}} v_n\|_{H(\alpha, L)}), \end{aligned}$$

where we have used $\|\beta_n v_n\|_{H(\alpha, L)} \sim \|T'_n\|_{H(\alpha, L)}$ from (3.9). Now (3.13) leads to

$$\|\beta_n^{\frac{3}{4}} v_n\|_{H(\alpha, L)} \rightarrow 0. \quad (3.14)$$

Next, we divide (3.9) by $\beta_n^{\frac{1}{2}}$, then take inner product with $(L-x)T_n$ in $H(\alpha, L)$ to get

$$\operatorname{Re} \langle \beta_n^{\frac{3}{4}} v_n, (L-x)\beta_n^{\frac{1}{4}} T_n \rangle_{H(\alpha, L)} - \frac{1}{2} (L-\alpha) |T_n(\alpha^+)|^2 + \frac{1}{2} \|T_n\|_{L^2(\alpha, L)}^2 \rightarrow 0.$$

Since the first the third terms converge to zero, we see that

$$|T_n(\alpha^+)| \rightarrow 0. \quad (3.15)$$

On the other hand, by the Trace theorem and (3.11),

$$|\beta_n u_n(\alpha^+)| \rightarrow 0. \quad (3.16)$$

Using the continuity conditions at $x = \alpha$, we arrive at

$$\begin{cases} |\beta_n u_n(\alpha^-)| \rightarrow 0 \\ |a(\alpha^-) u_n'(\alpha^-)| \rightarrow 0 \end{cases} \quad (3.17)$$

Now, we consider (3.8)–(3.9) on the interval $(0, \alpha)$. Eliminating v_n in (3.9) by (3.8) gives

$$-\beta_n^2 u_n - u_n'' = \beta_n^{-\frac{1}{2}} g_n + i \beta_n^{\frac{1}{2}} f_n. \quad (3.18)$$

Take the inner product of (3.18) with $q(x)u_n'$ in $H(0, \alpha)$ where $q(0) = 0$ and $q \in H^1(0, \alpha)$. A straight forward calculation shows that the real part of this inner product leads to the following

$$\int_0^\alpha [(q'\rho + q\rho')|\beta_n u_n|^2 + (q'a - qa')|u_n'|^2] dx = q(\alpha)\rho(\alpha^-)|\beta_n u_n(\alpha^-)|^2 + q(\alpha)a(\alpha^-)|u_n'(\alpha^-)|^2 + o(1). \quad (3.19)$$

The boundary terms on the right-hand side of (3.19) converge to zero which we have obtained in (3.17). Moreover, we can let

$$q(x) = \int_0^x e^{\int_s^x \left(\left| \frac{\rho'}{\rho} \right| + \left| \frac{a'}{a} \right| \right) ds} ds$$

and verify

$$q'\rho + q\rho' \geq \rho, \quad q'a - qa' \geq a.$$

It follows from (3.19) and (3.8) that

$$\|v_n\|_{H(0,\alpha)} \rightarrow 0, \quad \|u_n\|_{V(0,\alpha)} \rightarrow 0. \quad (3.20)$$

Finally, we combine (3.20), (3.11) and (3.14) to obtain the promised contradiction. Thus, we have verified the hypothesis (H3).

Example 2. Consider a weakly coupled wave equation and heat equation on a bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} u_{tt}(x, t) = \Delta u(x, t) - \gamma \theta(x, t) \\ \theta_t(x, t) = \gamma u_t(x, t) + k \Delta \theta(x, t) \\ u|_{\partial\Omega} = \theta|_{\partial\Omega} = 0 \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x) \end{cases}$$

where ρ, k are positive constants.

Let

$$\begin{aligned} V &= H_0^1(\Omega), & \|v\|_V^2 &= \int_{\Omega} |\nabla u|^2 dx \\ H &= L^2(\Omega), & \|v\|_H^2 &= \int_{\Omega} |v|^2 dx \end{aligned}$$

and

$$\mathcal{H} = V \times H \times H, \quad \|(u, v, \theta)\|_{\mathcal{H}}^2 = \int_{\Omega} [|\nabla u|^2 + |v|^2 + |\theta|^2] dx.$$

We then define

$$D(\mathcal{A}) = \{(u, v, \theta) \mid v \in V, \Delta u - \gamma\theta \in H, \gamma v + k\Delta\theta \in H\}$$

and

$$\mathcal{A}(u, v, \theta) = (v, \Delta u - \gamma\theta, v + k\Delta\theta).$$

This coupled system can be rewritten as an abstract evolution equation on \mathcal{H} :

$$\begin{cases} \frac{d}{dt}(u, v, \theta) = \mathcal{A}(u, v, \theta) \\ (u(0), v(0), \theta(0)) = (u_0, u_1, \theta_0) \end{cases}$$

It is known that (see [KBT]) $e^{t\mathcal{A}}$ strongly stable and is not exponentially stable.

\mathcal{A} satisfies the hypotheses (H1) and (H2). Let us verify (H3) for $l = 2$. Suppose it is false, then there exists a sequence $\beta_n \in \mathbb{R}^+, \beta_n \rightarrow \infty$ and $(u_n, v_n, \theta_n) \in \mathcal{D}(\mathcal{A}), n = 1, 2, \dots$ with

$$\|(u_n, v_n, \theta_n)\|_{\mathcal{H}} = 1, \quad \forall n \tag{3.21}$$

and

$$\beta_n^2(i\beta_n - \mathcal{A})(u_n, v_n, \theta_n) \equiv (f_n, g_n, h_n) \rightarrow 0 \text{ in } \mathcal{H} \tag{3.22}$$

i.e.,

$$\beta_n^2(i\beta_n u_n - v_n) = f_n \rightarrow 0 \text{ in } V, \tag{3.23}$$

$$\beta_n^2(i\beta_n v_n - \Delta u_n + \gamma\theta_n) = g_n \rightarrow 0 \text{ in } H, \tag{3.24}$$

$$\beta_n^2(i\beta_n \theta_n - \gamma v_n - k\Delta\theta_n) = h_n \rightarrow 0 \text{ in } H. \tag{3.25}$$

Our goal is to obtain $\|(u_n, v_n, \theta_n)\|_{\mathcal{H}} \rightarrow 0$ as $n \rightarrow \infty$, thus a contradiction.

Since, by (3.21),

$$-k\|\beta_n \nabla \theta_n\|_H^2 = \operatorname{Re} \langle \beta_n^2 (i\beta_n - A)(u_n, v_n, \theta_n), (u_n, v_n, \theta_n) \rangle_{\mathcal{H}} \rightarrow 0, \quad (3.26)$$

We obtain

$$\|\theta_n\|_H \rightarrow 0. \quad (3.27)$$

Taking the inner product of (3.25) with v_n in H yields

$$i \langle \beta_n \theta_n, v_n \rangle_H - \|v_n\|_H^2 - \int_{\partial\Omega} v_n \frac{\partial \theta_n}{\partial \nu} dx + \langle \beta_n \nabla \theta_n, \frac{1}{\beta_n} \nabla v_n \rangle_H \rightarrow 0. \quad (3.28)$$

It follows from (3.23) that $\frac{1}{\beta_n} \nabla v_n$ is bounded in H . Thus the third term in (3.28) converges to zero. The boundary term in (3.28) vanishes due to the boundary conditions. The first term in (3.28) also converges to zero due to (3.26) and the boundedness of v_n . We have obtained

$$\|v_n\|_H \rightarrow 0. \quad (3.29)$$

On the other hand, using the sum of the inner product of (3.23) with v_n in H and the inner product of (3.24) with u_n in H ,

$$\|u_n\|_V^2 - \|v_n\|_H^2 \rightarrow 0. \quad (3.30)$$

Combining (3.27), (3.29) and (3.30), we have the promised contradiction.

Example 3. Consider a Wave equation on a square $\Omega = (0, \pi) \times (0, \pi)$ with local viscous damping

$$\begin{cases} u_{tt}(x, t) = \Delta u(x, t) - d(x)u_t(x, t) \\ u|_{\partial\Omega} = 0 \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \end{cases}$$

where $d(x)$ has support $\Omega_0 \subset \Omega$ and is continuous and strictly positive on Ω_0 , and the damping region Ω_0 contains a vertical strip of the square domain, i.e.,

$$\Omega_0 \supset \Omega_s = \{(x_1, x_2) \mid a < x_1 < b, 0 < x_2 < \pi\}.$$

Let

$$\begin{aligned} V &= H_0^1(\Omega), & \|v\|_V^2 &= \int_{\Omega} |\nabla v|^2 dx \\ H &= L^2(\Omega), & \|v\|_H^2 &= \int_{\Omega} |v|^2 dx \end{aligned}$$

and

$$\mathcal{H} = V \times H.$$

Define

$$\begin{aligned} D(\mathcal{A}_d) &= \{(u, v) \in \mathcal{H} \mid v \in V, \Delta u \in H\}, \\ \mathcal{A}_d(u, v) &= (v, \Delta u - d(x)v). \end{aligned}$$

Then the system can be rewritten as an abstract evolution equation on \mathcal{H}

$$\begin{cases} \frac{d}{dt}(u, v) = \mathcal{A}_d(u, v) \\ (u(x, 0), v(x, 0)) = (u_0, u_1) \end{cases}$$

We also use \mathcal{A}_0 to denote \mathcal{A}_d when $d \equiv 0$.

It is well known that this system is strongly stable but not exponentially stable since the “geometric optics” condition is violated [BLR]. Clearly, (H1) and (H2) are satisfied. We verify (H3) for $l = 2$. Suppose it is false, then there exists $\beta_n \in \mathbb{R}^+$, $\beta_n \rightarrow \infty$ as $n \rightarrow \infty$ and $z_n = (u_n, v_n) \in D(\mathcal{A}_d)$ with

$$\|u_n\|_V^2 + \|v_n\|_H^2 = 1 \quad (3.31)$$

such that

$$\beta_n^2(\mathbf{i}\beta_n - \mathcal{A}_d)(u_n, v_n) \rightarrow 0 \quad \text{in } \mathcal{H}. \quad (3.32)$$

Since

$$\operatorname{Re}\langle \beta_n^2 \mathcal{A}_d(u_n, v_n), (u_n, v_n) \rangle_{\mathcal{H}} = \int_{\Omega_0} d(x) |\beta_n v_n|^2 dx,$$

we obtain from (3.32) that

$$\int_{\Omega_0} d(x) |\beta_n v_n|^2 dx \rightarrow 0,$$

which further leads to

$$\lim_{n \rightarrow \infty} \|v_n\|_{L^2(\Omega_s)} = 0 \quad (3.33)$$

and

$$\lim_{t \rightarrow \infty} \|\beta_n(\mathbf{i}\beta_n - \mathcal{A}_0)z_n\| = 0. \quad (3.34)$$

It is easy to see that the normalized eigenvectors of \mathcal{A}_0

$$\phi_{m,l}(x) = \left(\frac{\sqrt{2}}{\pi\sqrt{m^2 + l^2}} \sin mx_1 \sin lx_2, \mathbf{i} \frac{\sqrt{2}}{\pi} \sin mx_1 \sin lx_2 \right) \quad (3.35)$$

for $m, l = 1, 2, \dots$, form an orthonormal basis for \mathcal{H} . The corresponding eigenvalue for $\phi_{m,l}(x)$ is

$$i\lambda_n = i\sqrt{m^2 + l^2}. \quad (3.36)$$

we rearrange the eigenvalues $\{\lambda_k\}$ in the order

$$|\lambda_1| < |\lambda_2| < \dots < |\lambda_k| < \dots$$

with algebraic multiplicity i_k , $1 \leq i_k \leq \dim V_k$, where V_k is the eigenspace of λ_k . We then expand z_n as

$$z_n = \sum_{k, i_k} \langle z_n, \phi_k^{i_k} \rangle \phi_k^{i_k}, \quad (3.37)$$

and substitute it into (3.34). This leads to

$$\lim_{n \rightarrow \infty} \sum_{k, i_k} \beta_n (i\beta_n - i\lambda_k) \langle z_n, \phi_k^{i_k} \rangle \phi_k^{i_k} = 0, \quad (3.38)$$

i.e., for any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

$$\sum_{k, i_k} |\beta_n(\beta_n - \lambda_k)|^2 |\langle z_n, \phi_k^{i_k} \rangle|^2 \leq \varepsilon \quad \text{for all } n \geq N_\varepsilon. \quad (3.39)$$

Lemma 3.1 *For each n , there exists some index $k(n)$ and a constant $\gamma > 0$ so that*

$$|\beta_n(\beta_n - \lambda_{k(n)})|^2 < \varepsilon, \quad (3.40)$$

$$|\beta_n(\beta_n - \lambda_k)|^2 \geq \gamma \quad \text{for all } k \neq k(n) \quad (3.41)$$

Proof. If (3.40) does not hold, we have $|\beta_n(\beta_n - \lambda_k)|^2 \geq \varepsilon$ for all k . This contradicts (3.39) since

$$\sum_{k, i_k} |\langle z_n, \phi_k^{i_k} \rangle|^2 = 1. \quad (3.42)$$

It follows from (3.40) that

$$|\beta_n(\beta_n - \lambda_k)| \geq |\beta_n(\beta_n - \lambda_{k(n)}) + \beta_n(\lambda_{k(n)} - \lambda_k)| \geq |\beta_n(\lambda_{k(n)} - \lambda_k)| - \sqrt{\varepsilon}. \quad (3.43)$$

Since β_n goes to infinity, (3.41) holds when $|\lambda_{k(n)} - \lambda_k| \geq 1$ for all $k \neq k(n)$. On the other hand, we always have $|\lambda_{k(n)}^2 - \lambda_k^2| \geq 1$ and $\lambda_{k(n)} \geq 1$. Hence, when $|\lambda_{k(n)} - \lambda_k| < 1$, we obtain

$$\begin{aligned} |\beta_n(\lambda_{k(n)} - \lambda_k)| &= \frac{|\beta_n(\lambda_{k(n)}^2 - \lambda_k^2)|}{|\lambda_{k(n)} + \lambda_k|} \\ &\geq \frac{|\beta_n|}{|\lambda_{k(n)} + \lambda_k|} \\ &\geq \frac{|\beta_n|}{3\lambda_{k(n)}} \geq \frac{\beta_n^2}{3(\beta_n^2 + \sqrt{\varepsilon})}, \quad \text{by (3.40)}. \end{aligned}$$

(3.41) is proved. □

Lemma 3.1 and (3.39) implies that

$$\gamma^2 \sum_{i_k} \sum_{k \neq k(n)} |\langle z_n, \phi_k^{i_k} \rangle|^2 + |\beta_n(\beta_n - \lambda_{k(n)})|^2 \sum_{i_{k(n)}} |\langle z_n, \phi_{k(n)}^{i_{k(n)}} \rangle|^2 \leq \varepsilon \quad \text{for all } n \geq N_\varepsilon. \quad (3.44)$$

Define

$$\psi_n = \sum_{i_{k(n)}} \langle z_n, \phi_{k(n)}^{i_{k(n)}} \rangle \phi_{k(n)}^{i_{k(n)}}. \quad (3.45)$$

Then,

$$\|z_n - \psi_n\|_{\mathcal{H}}^2 = \sum_{i_k} \sum_{k \neq k(n)} |\langle z_n, \phi_k^{i_k} \rangle|^2 \leq \frac{\varepsilon}{\gamma^2} \quad (3.46)$$

and

$$\sum_{i_{k(n)}} |\langle z_n, \phi_{k(n)}^{i_{k(n)}} \rangle|^2 \geq 1 - \frac{\varepsilon}{\gamma^2}. \quad (3.47)$$

From the second component of $z_n - \psi_n$,

$$\|v_n - \mathbf{i} \frac{\sqrt{2}}{\pi} \sum_i \langle z_n, \phi_{k(n)}^i \rangle \sin m_i x_1 \sin l_i x_2\|_{L^2(\Omega_s)} \leq \frac{\varepsilon}{\gamma^2}, \quad (3.48)$$

where

$$m_i^2 + l_i^2 = m_{k(n)}^2 + l_{k(n)}^2, \quad i = 1, \dots, \dim V_{k(n)},$$

$$m_i \neq m_j, \quad l_i \neq l_j \quad \text{for } i \neq j.$$

On the other hand,

$$\begin{aligned} & \frac{2}{\pi^2} \int_a^b \int_0^\pi \left| \sum_i \langle z_n, \phi_{k(n)}^i \rangle \sin m_i x_1 \sin l_i x_2 \right|^2 dx_2 dx_1 \\ &= \frac{1}{\pi} \int_a^b \sum_i |\langle z_n, \phi_{k(n)}^i \rangle|^2 \sin^2 m_i x_1 dx_1 \\ &\geq \delta \sum_i |\langle z_n, \phi_{k(n)}^i \rangle|^2 \\ &\geq \delta \left(1 - \frac{\varepsilon}{\gamma^2}\right) \end{aligned} \quad (3.49)$$

for some constant $\delta > 0$. This contradicts (3.33) and (3.48).

By Theorem 2.1,

$$\|e^{t\mathcal{A}}(u_0, u_1)\|_{\mathcal{H}} \leq C_k \left(\frac{\ln t}{t}\right)^{\frac{k}{2}} (\ln t) \|(u_0, u_1)\|_{\mathcal{D}(\mathcal{A}^k)}.$$

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