Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres

S. Mischler[1], C. Mouhot[2]

Abstract

We consider the spatially homogeneous Boltzmann equation for *inelastic hard spheres*, in the framework of so-called *constant normal restitution coefficients* $\alpha \in [0,1]$. In the physical regime of a small inelasticity (that is $\alpha \in [\alpha_*, 1)$ for some constructive $\alpha_* > 0$) we prove uniqueness of the self-similar profile for given values of the restitution coefficient $\alpha \in [\alpha_*, 1)$, the mass and the momentum; therefore we deduce the uniqueness of the self-similar solution (up to a time translation).

Moreover, if the initial datum lies in L^1_3, and under some smallness condition on $(1-\alpha_*)$ depending on the mass, energy and L^1_3 norm of this initial datum, we prove time asymptotic convergence (with polynomial rate) of the solution towards the self-similar solution (the so-called *homogeneous cooling state*).

These uniqueness, stability and convergence results are expressed in the self-similar variables and then translate into corresponding results for the original Boltzmann equation. The proofs are based on the identification of a suitable elastic limit rescaling, and the construction of a smooth path of self-similar profiles connecting to a particular Maxwellian equilibrium in the elastic limit, together with tools from perturbative theory of linear operators. Some universal quantities, such as the "quasi-elastic self-similar temperature" and the rate of convergence towards self-similarity at first order in terms of $(1-\alpha)$, are obtained from our study.

These results provide a positive answer and a mathematical proof of the Ernst-Brito conjecture [16] in the case of inelastic hard spheres with small inelasticity.

Mathematics Subject Classification (2000): 76P05 Rarefied gas flows, Boltzmann equation [See also 82B40, 82C40, 82D05], 76T25 Granular flows [See also 74C99, 74E20].

Keywords: Inelastic Boltzmann equation; granular gases; hard spheres; self-similar solution; self-similar profile; uniqueness; stability; small inelasticity; elastic limit; degenerated perturbation; spectrum.

[1]CEREMADE, Université Paris IX-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris, France. E-mail: mischler@ceremade.dauphine.fr

[2]CEREMADE, Université Paris IX-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris, France. E-mail: cmouhot@ceremade.dauphine.fr

Contents

1 Introduction and main results

1.1 The model

We consider the spatially homogeneous Boltzmann equation for hard spheres undergoing inelastic collisions with a constant normal restitution coefficient $\alpha \in [0,1)$ (see [17, 8, 23, 24]). More precisely, the gas is described by the distribution density of particles $f = f_t = f(t,v) \geq 0$ with velocity $v \in \mathbb{R}^N$ ($N \geq 2$) at time $t \geq 0$ and it satisfies the evolution equation

$$\frac{\partial f}{\partial t} = Q_\alpha(f,f) \quad \text{in} \quad (0,+\infty) \times \mathbb{R}^N, \tag{1.1}$$

$$f(0,\cdot) = f_{\rm in} \quad \text{in} \quad \mathbb{R}^N. \tag{1.2}$$

The quadratic collision operator $Q_\alpha(f,f)$ models the interaction of particles by means of inelastic binary collisions (preserving mass and momentum but dissipating kinetic energy). We define the collision operator by its action on test functions, or *observables*. Taking $\psi = \psi(v)$ to be a suitably regular test function, we introduce the following weak formulation of the collision operator

$$\int_{\mathbb{R}^N} Q_\alpha(g,f)\,\psi\,dv = \iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} b\,|u|\,g_*\,f\,(\psi' - \psi)\,d\sigma\,dv\,dv_*, \tag{1.3}$$

where we use the shorthand notations $f := f(v)$, $g_* := g(v_*)$, $\psi' := \psi(v')$, etc. Here and below $u = v - v_*$ denotes the relative velocity and v', v'_* denotes the possible post-collisional velocities (which encapsule the inelasticity of the collision operator in terms of α). They are defined by

$$v' = \frac{w}{2} + \frac{u'}{2}, \quad v'_* = \frac{w}{2} - \frac{u'}{2}, \tag{1.4}$$

with

$$w = v + v_*, \qquad u' = \left(\frac{1-\alpha}{2}\right) u + \left(\frac{1+\alpha}{2}\right) |u|\,\sigma.$$

We also introduce the notation $\hat{x} = x/|x|$ for any $x \in \mathbb{R}^N$, $x \neq 0$. The function $b = b(\hat{u}\cdot\sigma)$ in (1.3) is (up to a multiplicative factor) the *differential collisional cross-section*. We assume that

$$b \text{ is Lipschitz, non-decreasing and convex on } (-1,1) \tag{1.5}$$

and that

$$\exists\, b_m, b_M \in (0,\infty) \quad \text{s.t.} \quad \forall\, x \in [-1,1], \quad b_m \leq b(x) \leq b_M. \tag{1.6}$$

Note that the "physical" cross-section for hard spheres is given by (see [17, 13])

$$b(x) = b'_0\,(1-x)^{-\frac{N-3}{2}}, \quad b'_0 \in (0,\infty), \tag{1.7}$$

so that it fulfills the above hypothesis (1.5,1.6) when $N = 3$. These hypothesis are needed in the proof of moments estimates (see [23, Proposition 3.2] and [24, Proposition 3.1]).

We also define the symmetrized (or polar form of the) bilinear collisional operator $\tilde{Q}_\alpha$ by setting

$$(1.8)\qquad \begin{cases} \displaystyle\int_{\mathbb{R}^N} \tilde{Q}_\alpha(g,h)\,\psi\,dv = \frac{1}{2}\iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} b\,|u|\,g_*\,h\,\Delta_\psi\,d\sigma\,dv\,dv_*, \\ \text{with}\quad \Delta_\psi = \big(\psi' + \psi'_* - \psi - \psi_*\big). \end{cases}$$

In other words, $\tilde{Q}_\alpha(g,h) = (Q_\alpha(g,h)+Q_\alpha(h,g))/2$. The formula (1.3) suggests the natural splitting $Q_\alpha = Q_\alpha^+ - Q_\alpha^-$ between gain and loss part. The loss part Q_α^- can be defined in strong form noticing that

$$\langle Q_\alpha^-(g,f),\ \psi\rangle = \iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} b\,|u|\,g_*\,f\,\psi\,d\sigma\,dv\,dv_* =: \langle f\,L(g),\ \psi\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the usual scalar product in L^2 and L is the convolution operator

$$(1.9)\qquad L(g)(v) = (b_0\,|\cdot|*g)(v) = b_0\int_{\mathbb{R}^N} g(v_*)\,|v-v_*|\,dv_*,\ \text{ with }\ b_0 = \int_{S^{N-1}} b(\sigma_1)\,d\sigma.$$

In particular note that L and $Q_\alpha^- = Q^-$ are indeed independent of the normal restitution coefficient α.

The Boltzmann equation (1.1) is complemented with an initial datum (1.2) which satisfies

$$(1.10)\qquad \begin{cases} 0 \le f_{\rm in} \in L^1(\mathbb{R}^N), \qquad \rho(f_{\rm in}) := \displaystyle\int_{\mathbb{R}^N} f_{\rm in}\,dv = \rho \in (0,\infty) \\ \displaystyle\int_{\mathbb{R}^N} f_{\rm in}\,v\,dv = 0, \qquad \mathcal{E}(f_{\rm in}) := \int_{\mathbb{R}^N} f_{\rm in}\,|v|^2\,dv < \infty. \end{cases}$$

As explained in [23, 24], the operator (1.3) preserves mass and momentum, and so does the evolution equation:

$$(1.11)\qquad \frac{d}{dt}\int_{\mathbb{R}^N} f_t \begin{pmatrix} 1 \\ v \end{pmatrix} dv = 0,$$

while kinetic energy is dissipated

$$(1.12)\qquad \frac{d}{dt}\mathcal{E}(f_t) = -(1-\alpha^2)\,D_{\mathcal{E}}(f_t).$$

The *energy dissipation functional* is given by

$$D_{\mathcal{E}}(f) := b_1 \iint_{\mathbb{R}^N\times\mathbb{R}^N} f\,f_*\,|u|^3\,dv\,dv_*,$$

where b_1 is (up to a multiplicative factor) the angular momentum defined by

$$(1.13)\qquad b_1 := \frac{1}{8}\int_{\mathbb{S}^{N-1}} (1-(\hat{u}\cdot\sigma))\ b(\hat{u}\cdot\sigma)\,d\sigma.$$

In order to establish (1.12) we have used (1.8) and the elementary computation

$$\Delta_{|\cdot|^2}(v,v_*,\sigma) = -\frac{1-\alpha^2}{4}\ (1-(\hat{u}\cdot\sigma))\,|u|^2.$$

The study of the Cauchy theory and the cooling process of (1.1)-(1.2) was done in [23]. The equation is well-posed for instance in L^1_2: for $0 \le f_{\rm in} \in L^1_2$, there is a unique global solution in $C(\mathbb{R}_+; L^1_2) \cap L^1(\mathbb{R}_+; L^1_3)$ (see Subsection 1.5 for the notation of functional spaces). This solution preserves mass, momentum and has a positive and decreasing kinetic energy. Moreover, as time goes to infinity, it satisfies:

$$\mathcal{E}(t) \to 0 \qquad \text{and} \qquad f(t,\cdot) \rightharpoonup \delta_{v=0} \ \text{ in } \ M^1(\mathbb{R}^N)\text{-weak } *, \tag{1.14}$$

where $M^1(\mathbb{R}^N)$ denotes the space of probability measures on $\mathbb{R}^N$.

1.2 Introduction of rescaled variables

Let us introduce some rescaled variables (which can be found in [15, 8, 24] for instance), in order to study more precisely the asymptotic behavior (1.14) of the solution. For any solution f to the Boltzmann equation (1.1), we may associate for any $\tau \in (0,\infty)$ the self-similar rescaled solution g by the relation

$$g(t,v) = e^{-N\,\tau\,t}\, f\left(\frac{e^{\tau\,t}-1}{\tau}, e^{-\tau\,t}\,v\right).$$

Using the homogeneity property $Q_\alpha(g(\lambda\cdot), g(\lambda\cdot))(v) = \lambda^{-(N+1)}\, Q_\alpha(g,g)(\lambda v)$, it is straightforward that g satisfies the evolution equation

$$\frac{\partial g}{\partial t} = Q_\alpha(g,g) - \tau\, \nabla_v \cdot (vg). \tag{1.15}$$

Any non-negative steady state $0 \le G = G(v)$ of (1.15), that is G satisfying

$$Q_\alpha(G,G) - \tau\, \nabla_v \cdot (v\,G) = 0, \tag{1.16}$$

is called a *self-similar profile*. It translates into a *self-similar solution* (or *homogeneous cooling state*) F of the original equation (1.1) by setting

$$F(t,v) = (V_0 + \tau\, t)^N\, G((V_0+\tau\, t)v), \tag{1.17}$$

for a given constant $V_0 \in (0,\infty)$. Reciprocally, let us consider a *self-similar solution* F of the original equation (1.1). That means a solution F of (1.1) with the specific shape

$$F(t,v) = V(t)^N\, G(V(t)\, v) \tag{1.18}$$

for some given non-negative distribution $G = G(v)$ and some C^1, positive, increasing time rescaling function $V(t)$. One can easily show (see for instance [24, section 1.2]) that $V(t) = \tau\, t + V_0$ for some constants τ, $V_0 > 0$ and G satisfies (1.16) associated to the velocity rescaling parameter τ. For a given self-similar profile G, associated to a velocity rescaling parameter τ and with mass ρ and energy $\mathcal{E}$, we may associate a new self-similar profile $\tilde G$, associated to a velocity rescaling parameter $\tilde\tau$ and with mass $\tilde\rho$ by setting

$$\tilde G(v) = K\, G(V\, v), \quad V = \frac{\tilde\rho}{\rho}\,\frac{\tau}{\tilde\tau}, \quad K = V^N\, \frac{\tilde\rho}{\rho}.$$

The energy of $\tilde G$ is then $\tilde{\mathcal{E}} = \frac{\tilde\rho}{\rho}\left(\frac{\tau}{\tilde\tau}\right)^2 \mathcal{E}$. We thus see that there exists a two real parameters family of self-similar profiles which can be either parametrized by (ρ,τ) or by $(\rho,\mathcal{E})$. For fixed mass, changing the velocity rescaling parameter τ in (1.16) corresponds to a change of the energy of the profile, or equivalently to an homothetic change of variable of the solution. Therefore it is no restriction to choose arbitrarily this constant. Also note that modifying V_0 just corresponds to a time translation in the self-similar solution F defined by (1.17).

It was proved in [24, Theorem 1.1] that for any inelastic parameter $\alpha \in (0,1)$, mass $\rho \in (0,\infty)$ and (thanks to the preceding discussion) any velocity rescaling parameter $\tau \in (0,\infty)$, there exists at least one positive and smooth self-similar profile G with given mass ρ and vanishing momentum:

$$(1.19) \qquad \begin{cases} Q_\alpha(G,G) - \tau \, \nabla_v \cdot (v\, G) = 0 \quad \text{in} \quad \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} G\, dv = \rho, \quad \int_{\mathbb{R}^N} G\, v\, dv = 0, \quad 0 < G \in \mathcal{S}(\mathbb{R}^N), \end{cases}$$

where $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwartz space of C^∞ functions decreasing at infinity faster than any polynomial.

Finally, for any solution g to the Boltzmann equation in self-similar variables (1.15), we may associate a solution f to the evolution problem (1.1), defining f by the relation

$$(1.20) \qquad f(t,v) = (V_0 + \tau\, t)^N \, g\left(\frac{\ln(V_0+\tau\, t)}{\tau}, (V_0+\tau\, t) v\right).$$

1.3 Rescaled variables and elastic limit $\alpha \to 1$

We now make the choice

$$(1.21) \qquad \tau = \tau_\alpha = \rho\,(1-\alpha),$$

and denote by G_α a solution to the problem (1.19). At a formal level, it is immediate that with this choice of scaling, in the elastic limit $\alpha \to 1$, the equation (1.19) becomes

$$(1.22) \qquad \begin{cases} Q_1(G_1,G_1) = 0 \;\; \text{in} \;\; \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} G_1\, dv = \rho, \quad \int_{\mathbb{R}^N} G_1\, v\, dv = 0, \quad 0 \le G_1 \in \mathcal{S}(\mathbb{R}^N). \end{cases}$$

Moreover, multiplying the first equation of (1.19) by $|v|^2$, integrating in the velocity variable as in (1.12) and taking into account the additional term coming from the additional drift term in (1.15), one gets

$$(1.23) \qquad 2\,(1-\alpha)\,\rho\,\mathcal{E}(G_\alpha) - (1-\alpha^2) D_{\mathcal{E}}(G_\alpha) = 0.$$

Dividing the above equation by $(1-\alpha)$ and passing to the limit $\alpha \to 1$, one obtains

$$(1.24) \qquad \rho\,\mathcal{E}(G_1) - D_{\mathcal{E}}(G_1) = 0.$$

It is straightforward (see Proposition 3.6 below) that the only function satisfying the constraints (1.22) and (1.24) is the Maxwellian function

$$(1.25) \qquad \bar{G}_1 := M_{\bar\theta_1} = M_{\rho,0,\bar\theta_1}$$

where, for any $\rho, \theta > 0$, $u \in \mathbb{R}^N$, the function $M_{\rho,u,\theta}$ denotes the Maxwellian with mass ρ, momentum u and temperature θ given by

$$M_{\rho,u,\theta}(v) := \frac{\rho}{(2\pi\theta)^{N/2}} \, e^{-\frac{|v-u|^2}{2\theta}}, \tag{1.26}$$

and where the temperature $\bar\theta_1 \in (0,\infty)$ is given by (we recall that b_1 is defined in (1.13))

$$\bar\theta_1 = \frac{N^2}{8\, b_1^2} \left(\int_{\mathbb{R}^N} M_{1,0,1}(v) \, |v|^3 \, dv \right)^{-2}. \tag{1.27}$$

For instance in dimension $N = 3$ we obtain

$$\bar\theta_1 = \frac{9\pi}{64 b_1^2}.$$

Moreover, in the particular case of the hard-spheres cross-section (1.7) in dimension 3, we find $b_1 = b_0'(4\pi)/3$ and therefore

$$\bar\theta_1 = \frac{81}{1024\pi \, (b_0')^2}.$$

1.4 Physical and mathematical motivation

For a detailed physical introduction to granular gases we refer to [9]. As can be seen from the references included in the latter, granular flows have become a subject of physical research on their own in the last decades, and for certain regimes of dilute and rapid flows this studies are based on kinetic theory. By contrast, the mathematical kinetic theory of granular gas is rather young and began in the late 1990 decade. We refer to [23, 24] for some (short) mathematical introduction to this theory and a (non exhaustive) list of references. As explained in these papers, granular gases are composed of grains of macroscopic size with contact collisional interactions, when one does not consider other additional possible self-interaction mechanisms such as gravitation – for cosmic clouds for instance – or electromagnetism – for "dusty plasmas" for instance –. Therefore the natural assumption about the binary interaction between grains is that of inelastic hard spheres, with no loss of "tangential relative velocity" (according to the impact direction) and a loss in "normal relative velocity" quantified in some (normal) restitution coefficient. The latter is either assumed to be constant as a first approximation (as in this paper) or can be more intricate: for instance it is a function of the modulus $|v' - v|$ of the normal relative velocity in the case of "visco-elastic hard spheres" for instance (see [9]), which shall be studied in a forthcoming work [25].

Simplified Boltzmann models like inelastic Maxwell molecules or pseudo inelastic hard spheres have been proposed (see [5]) for which existence, uniqueness and global stability of a self-similar profile has been shown (see [7, 3]), see also [2] for similar results in the driven case of a thermal bath. However these models do not capture some crucial physical features of the cooling process of granular gas, like the tail behavior of the velocity distribution of the rate of decay of temperature (the so-called Haff's law). For (spatially homogeneous) inelastic hard spheres Boltzmann models, the existing mathematical works are:

- the paper [8] which shows *a priori* polynomial and exponential moments bounds on any possible self-similar profile (resp. stationary solutions), whose existence is assumed, for freely cooling (resp. driven by a thermal bath) inelastic hard spheres with constant restitution coefficient;

- the paper [17] which shows existence of stationary solutions for inelastic hard spheres driven by a thermal bath, and improves the estimates on their tails of the previous paper into pointwise ones in this case;

- the paper [23] which provides a Cauchy theory for freely cooling inelastic hard spheres with a broad family of collision kernels (including in particular restitution coefficients possibly depending on the relative velocity and/or the temperature), and studies the question of cooling in finite time or not for these various interactions;

- the paper [24] which shows, for freely cooling inelastic hard spheres with constant restitution coefficient, existence of self-similar profile(s) as well as propagation of regularity and damping with time of singularity.

In this paper we want to study the self-similarity properties of Boltzmann equation for inelastic hard spheres. Therefore as a natural first step we consider constant restitution coefficient α in order to have a self-similar scaling, which translates the study of self-similar solutions (often called homogeneous cooling states) to the study of stationary solutions for a rescaled equation. We also reduce to the case of restitution coefficients α close to 1, that is, of small inelasticity. There are several physical as well as mathematical motivations for such a choice:

- the first reason is related to the physical regime of the validity of kinetic theory: as explained in [9, Chapter 6] for instance, the more inelasticity, the more correlations between grains are created during the binary collisions, and therefore the molecular chaos assumption, which is at the basis of the valdidity of Boltzmann's theory, suggests weak inelasticity to be the most effective;

- second as emphasized in [9] again, the case of restitution coefficient α close to 1 has been widely considered in physics or mathematical physics since it allows to use expansions around the elastic case, and since conversely it is an interesting question to understand the connection of the inelastic case (dissipative at the microscopic level) to the elastic case ("hamiltonian" at the microscopic level);

- finally this case of a small inelasticity is reasonable from the viewpoint of applications, since it applies to interstellar dust clouds in astrophysics, or sands and dusts in earth-bound experiments, and more generally to visco-elastic hard spheres whose restitution coefficient is not constant but close to 1 on the average.

In this framework we shall show uniqueness and attractivity of self-similar solutions (in a suitable sense), and thus give a complete answer to the Ernst-Brito conjecture [16] (stated there for the simplified inelastic Maxwell model), for inelastic hard spheres with a small inelasticity. Moreover we give precise results about the elastic limit and deduce some quantitative informations about the weakly inelastic case.

1.5 Notation

Throughout the paper we shall use the notation $\langle\cdot\rangle = \sqrt{1+|\cdot|^2}$. We denote, for any $p \in [1,+\infty]$, $q \in \mathbb{R}$ and weight function $\omega : \mathbb{R}^N \to \mathbb{R}_+$, the weighted Lebesgue space $L^p_q(\omega)$ by

$$L^p_q(\omega) := \left\{ f : \mathbb{R}^N \mapsto \mathbb{R} \text{ measurable} \; ; \; \|f\|_{L^p_q(\omega)} < +\infty \right\},$$

with, for $p < +\infty$,

$$\|f\|_{L^p_q(\omega)} = \left[\int_{\mathbb{R}^N} |f(v)|^p \, \langle v\rangle^{pq} \, \omega(v) \, dv \right]^{1/p}$$

and, for $p = +\infty$,

$$\|f\|_{L^\infty_q(\mathbb{R}^N)} = \sup_{v\in\mathbb{R}^N} |f(v)| \, \langle v\rangle^q \omega(v).$$

We shall in particular use the exponential weight functions

$$m = m_{s,a}(v) := e^{-a\,|v|^s} \quad \text{for} \quad a \in (0,\infty), \; s \in (0,1), \tag{1.28}$$

or a smooth version $m(v) := e^{-\zeta(|v|^2)}$ with $\zeta \in C^\infty$ is a positive function such that $\zeta(r) = r^{s/2}$ for any $r \geq 1$, with $s \in (0,1)$.

In the same way, the weighted Sobolev space $W^{k,p}_q(\omega)$ ($k \in \mathbb{N}$) is defined by the norm

$$\|f\|_{W^{k,p}_q(\omega)} = \left[\sum_{|s|\leq k} \|\partial^s f(v)\|^p_{L^p_q(\omega)} \right]^{1/p},$$

and as usual in the case $p = 2$ we denote $H^k_q(\omega) = W^{k,2}_q(\omega)$. The weight ω shall be omitted when it is 1. Finally, for $g \in L^1_{2k}$, with $k \geq 0$, we introduce the following notation for the homogeneous moment of order $2k$

$$\mathbf{m}_k(g) := \int_{\mathbb{R}^N} g \, |v|^{2\,k} \, dv,$$

and we also denote by $\rho(g) = \mathbf{m}_0(g)$ the mass of g, $\mathcal{E}(g) = \mathbf{m}_1(g)$ the energy of g and by $\theta(g) = \mathcal{E}(g)/(\rho(g)\,N)$ the temperature associated to g (when the distribution g has 0 mean). For any $\rho, \mathcal{E} \in (0,\infty)$, $u \in \mathbb{R}^N$ we then introduce the subsets of L^1 of functions of given mass, mean velocity and energy

$$\begin{aligned} \mathcal{C}_{\rho,u} &:= \{h \in L^1_1; \int_{\mathbb{R}^N} h \, dv = \rho, \; \int_{\mathbb{R}^N} h \, v \, dv = \rho \, u\}, \\ \mathcal{C}_{\rho,u,\mathcal{E}} &:= \{h \in L^1_2; \int_{\mathbb{R}^N} h \, dv = \rho, \; \int_{\mathbb{R}^N} h \, v \, dv = \rho \, u, \; \int_{\mathbb{R}^N} h \, |v|^2 \, dv = \mathcal{E}\}. \end{aligned}$$

For any (smooth version of) exponential weight function m we introduce the Banach space

$$\mathbb{L}^1(m^{-1}) = L^1(m^{-1}) \cap \mathcal{C}_{0,0}.$$

1.6 Main results in self-similar variables

Our main result, that we state now, deals with the evolution equation in self-similar variables

$$\frac{\partial g}{\partial t} = Q_\alpha(g,g) - \tau_\alpha \nabla_v \cdot (vg), \quad g(0,.) = g_{\rm in} \in \mathcal{C}_{\rho,0} \tag{1.29}$$

and the associated stationary equation, namely the self-similar profile equation

$$Q_\alpha(G,G) - \tau_\alpha \nabla_v \cdot (v\,G) = 0, \quad G \in \mathcal{C}_{\rho,0}. \tag{1.30}$$

Theorem 1.1 *There is some constructive $\alpha_* \in (0,1)$ such that for $\alpha \in [\alpha_*, 1]$, and any given mass $\rho \in (0,\infty)$, we have:*

(i) *For any $\tau > 0$, the equation (1.15) admits a unique non-negative stationary solution with mass ρ and vanishing momentum. We denote by $\bar G_\alpha$ the self-similar profile obtained by fixing $\tau = \tau_\alpha$ (defined by (1.21)).*

(ii) *Let define $\bar G_1 = M_{\rho,0,\bar\theta_1}$ the Maxwellian distribution with mass ρ, momentum 0 and "*quasi-elastic self-similar temperature*" $\bar\theta_1$ defined in (1.27). The path of self-similar profiles $\alpha \to \bar G_\alpha$ parametrized by the normal restitution coefficient is C^1 from $[\alpha_*, 1]$ into $W^{k,1} \cap L^1(e^{a\,|v|})$ for any $k \in \mathbb{N}$ and some $a \in (0,\infty)$.*

(iii) *For any $\alpha \in [\alpha_*, 1]$, the linearized collision operator*

$$h \mapsto \mathcal{L}_\alpha\, h := 2\,\tilde Q_\alpha(\bar G_\alpha, h) - \tau_\alpha \nabla_v \cdot (v\,h) \tag{1.31}$$

is well-defined and closed on $\mathbb{L}^1(m^{-1})$ for any exponential weight function m with exponent $s \in (0,1)$ (defined in (1.28)). Its spectrum decomposes between a part which lies in the half-plane $\{\mathrm{Re}\,\xi \le \bar\mu\}$ for some constructive $\bar\mu < 0$, and some remaining discrete eigenvalue μ_α. This eigenvalue is real negative and satisfies

$$\mu_\alpha = -\rho\,(1-\alpha) + \mathcal{O}(1-\alpha)^2 \quad \text{when} \quad \alpha \to 1. \tag{1.32}$$

The associated eigenspace is of dimension 1 and then denoting by $\phi_\alpha = \phi_\alpha(v)$ the unique associated eigenfunction such that $\|\phi_\alpha\|_{L^1_2} = 1$ and $\phi_\alpha(0) < 0$, there holds $\phi_\alpha \in \mathcal{S}(\mathbb{R}^N)$ (with bounds of regularity independent of α) and

$$\phi_\alpha \to \phi_1 := c_0\,(|v|^2 - N\,\bar\theta_1)\,\bar G_1 \quad \text{as} \quad \alpha \to 1, \tag{1.33}$$

where c_0 is the positive constant such that $\|\phi_1\|_{L^1_2} = 1$. Finally one has constructive decay estimates on the semigroup associated to this spectral decomposition in this Banach space (see the key Theorem 5.2 and the following point).

(iv) *The self-similar profile $\bar G_\alpha$ is globally attractive on bounded subsets of L^1_3 under some smallness condition on the inelasticity in the following sense. For any $\rho, \mathcal{E}_0, M_0 \in (0,\infty)$ there exists $\alpha_{**} \in (\alpha_*, 1)$, $C_* \in (0,\infty)$ and $\eta \in (0,1)$, such that for any initial datum satisfying*

$$0 \le g_{\rm in} \in L^1_3 \cap \mathcal{C}_{\rho,0,\mathcal{E}_0}, \qquad \|g_{\rm in}\|_{L^1_3} \le M_0,$$

the solution g to (1.29) satisfies

$$\|g_t - \bar G_\alpha\|_{L^1_2} \le e^{(1-\eta)\,\mu_\alpha\,t}. \tag{1.34}$$

(v) *Moreover, under smoothness condition on the initial datum one may prove a more precise asymptotic decomposition, and construct Liapunov functional for the equation (1.29). More precisely, there exists $k_* \in \mathbb{N}$ and, for any exponential weight m as defined in (1.28) and any $\rho, \mathcal{E}_0, M_0 \in (0,\infty)$, there exists $\alpha_{**} \in (\alpha_*, 1)$ and a constructive functional $\mathcal{H} : H^{k_*} \cap L^1(m^{-1}) \to \mathbb{R}$ such that, first, for any initial datum $0 \le g_{\rm in} \in H^{k_*} \cap L^1(m^{-1}) \cap \mathcal{C}_{\rho,0,\mathcal{E}_0}$ satisfying*

$$\|g_{\rm in}\|_{H^{k_*} \cap L^1(m^{-1})} \le M_0,$$

the solution g to (1.29) satisfies

$$g(t,\cdot) = \bar G_\alpha + c_\alpha(t)\,\phi_\alpha + r_\alpha(t,\cdot), \tag{1.35}$$

with $c_\alpha(t) \in \mathbb{R}$ and $r_\alpha(t,\cdot) \in L^1_2(\mathbb{R}^N)$ such that

$$|c_\alpha(t)| \le C_*\, e^{\mu_\alpha\, t}, \qquad \|r_\alpha(t,\cdot)\|_{L^1_2} \le C_*\, e^{(3/2)\,\mu_\alpha\, t}. \tag{1.36}$$

And second when the initial datum satisfies additionally

$$g_{\rm in} \ge M_0^{-1}\, e^{-M_0\,|v|^8},$$

the solution satisfies also

$$t \mapsto \mathcal{H}(g(t,\cdot)) \quad \textit{is strictly decreasing}$$

(up to reach the stationary state $\bar G_\alpha$).

Remarks 1.2 *1) All the constants appearing in this theorem are contructive, which means that they can be made explicit, and in particular that the proof does not use any compactness argument. Unless otherwise mentioned, these constants will depend on b, on the dimension N, and on some bounds on the initial datum but never on the inelasticity parameter $\alpha \in (0,1]$.*

2) Theorem 1.1 establishes that conjectures 1 and 2 in [24, Section 5] holds true at least for weak inelastic model (that means for α close enough to 1).

3) In point (iv), the condition on the restitution coefficient depends on the mass, temperature and L^1_3 norm of the initial distribution, but this dependence is not *a perturbative condition of closeness to the self-similar profile. This fact relies on the so-called "entropy-entropy production" estimates which yields "overlinear" Gronwall-type estimates, and the decoupling of the timescales of energy dissipation and entropy production.*

4) In (1.36) one can prove $\|r_\alpha(t,\cdot)\|_{L^1_2} \le C_\zeta\, e^{\zeta\,\mu_\alpha\,t}$ for any $\zeta \in (1,2)$. Remark that here we do not have the decay rate $e^{\bar\lambda t}$ on the remaining part when one "removes" from $g_t - \bar G_\alpha$ the projection on the energy eigenvalue, where $\bar\lambda < 0$ would be some constant independent of α related to the second non-zero eigenvalue of $\mathcal{L}_\alpha$. This is due to the coupling effect of the bilinear term, which mixes the different part of the spectral decomposition.

5) As a subproduct the above result provides an alternative argument to the one of [24, Section 3] to show uniform (in time and inelasticity parameter) non-concentration bounds on the rescaled equation, in the case of α close to 1 and a general initial datum $g_{in} \in L^1_3$ (whereas the proof of [24, Section 3] was valid for all $\alpha \in (0,1)$ but for some initial datum $g_{in} \in L^1_3 \cap L^p$, $p \in (1,\infty]$).

6) Our results show that no bifurcation occurs for the self-similar profile for α close to 1. We do not know at now if some bifurcations occur for other values of the inelasticity parameter. Therefore we do not know if there is a continuous branch of self-similar profiles parametrized by $\alpha \in [0,1]$ (even if we know from [24] that self-similar profiles exist for all values of the inelasticity paramaters). The best one could say in terms of "connectivity" from the estimates we have proved on the profile together with the classical theory of topological degree (see [29] for instance) is that there is a set $K \subset [0,1] \times \mathcal{F}$ (where $\mathcal{F}$ is for instance the set of positive functions in the Schwartz space with given mass) which is compact, connected, and such that for any $\alpha \in [0,1]$, the intersection $K \cap \{\alpha\} \times \mathcal{F}$ is not empty.

1.7 Coming back to the original equation

When coming back to the original equation (1.1) with the help of (1.17) and (1.20), Theorem 1.1 translates into the

Theorem 1.3 *There is a constructive $\alpha_* \in (0,1)$ such that for $\alpha \in [\alpha_*, 1]$, and any given mass $\rho \in (0,\infty)$, we have*

(i) Up to a translation of time there exists a unique self-similar solution $\bar F_\alpha$ of the equation (1.1) with mass ρ, and it is given by

$$\bar F_\alpha(t,v) = (1+\tau_\alpha\, t)^N\, \bar G_\alpha((1+\tau_\alpha\, t)\, v), \qquad \tau_\alpha = \rho\,(1-\alpha),$$

where $\bar G_\alpha$ was obtained in Theorem 1.1. More precisely, if F_α is a solution of (1.1) of the form (1.18) and of mass ρ, there exists $t_0 \in \mathbb{R}$ such that $F_\alpha(t,v) = \bar F_\alpha(t+t_0, v)$ for any $t \ge \max\{0, -t_0\}$ and any $v \in \mathbb{R}^N$.

(ii) The self-similar solution $\bar F_\alpha$ is globally attractive on bounded subsets of L^1_3 under some smallness condition on $(1-\alpha_*)$ in the following sense. For any $\rho, \mathcal{E}_0, M_0 \in (0,\infty)$ there exists $\alpha_{**} \in (\alpha_*, 1)$ and $\eta \in (0,1)$ such that for any $q \in \mathbb{N}$ there is $c_q \in (0,\infty)$ such that for any initial datum satisfying

$$0 \le f_{\rm in} \in L^1_3 \cap \mathcal{C}_{\rho,0,\mathcal{E}_0}, \qquad \|f_{\rm in}\|_{L^1_3} \le M_0,$$

the solution $f(t,\cdot)$ to (1.1) satisfies

$$\|f(t,\cdot) - \bar F_\alpha(t,\cdot)\|_{L^1(|v|^q)} \le c_q\ (1+\tau_\alpha\, t)^{(1-\eta)\,\mu_\alpha/\tau_\alpha - q} = c_q\,(1+\tau_\alpha\, t)^{-(1-\eta)-q+\mathcal{O}(1-\alpha)}.$$

(iii) Moreover, there exists $k_* \in \mathbb{N}$ and, for any exponential weight m as defined in (1.28) and any $\rho, \mathcal{E}_0, M_0 \in (0,\infty)$, there exists $\alpha_{**} \in (\alpha_*, 1)$ such that, for any initial datum $0 \le f_{\rm in} \in H^{k_*} \cap L^1(m^{-1}) \cap \mathcal{C}_{\rho,0,\mathcal{E}_0}$ satisfying

$$\|f_{\rm in}\|_{H^{k_*} \cap L^1(m^{-1})} \le M_0$$

the solution f to (1.29) satisfies

$$f(t,\cdot) = \bar F_\alpha(t,\cdot) + \tilde c_\alpha(t)\, \psi_\alpha(t,\cdot) + \tilde r_\alpha(t,\cdot) \tag{1.37}$$

where

$$\psi_\alpha(t,v) = (1+\tau_\alpha\, t)^N\, \phi_\alpha\Big((1+\tau_\alpha\, t)\, v\Big), \quad \tilde c_\alpha(t) = c_\alpha \left(\frac{\ln(1+\tau_\alpha\, t)}{\tau_\alpha}\right).$$

In this expansion, the different terms have the following asymptotic behaviors (for any given $q \ge 0$):

$$\|\bar F_\alpha(t,\cdot)\|_{L^1(|v|^q)} = (1+\tau_\alpha\, t)^{-q} \|\bar G_\alpha\|_{L^1(|v|^q)},$$

$$\|\psi_\alpha(t,\cdot)\|_{L^1(|v|^q)} = (1+\tau_\alpha\, t)^{-q} \|\bar G_\alpha\|_{L^1(|v|^q)},$$

$$|\tilde c_\alpha(t)| \le C_*\,(1+\tau_\alpha\, t)^{\mu_\alpha/\tau_\alpha} = C_*\,(1+\tau_\alpha\, t)^{-1+\mathcal{O}(1-\alpha)},$$

$$\exists\, C_q > 0\ ; \quad \|\tilde r_\alpha\|_{L^1(|v|^q)} \le C_q\,(1+\tau_\alpha\, t)^{(3/2)\,\mu_\alpha/\tau_\alpha - q} = C_q\,(1+\tau_\alpha\, t)^{-(3/2)-q+\mathcal{O}(1-\alpha)}.$$

Hence the leading term in the expansion (1.37) is, as expected, the self-similar solution, and the first order correction behond self-similarity is given by the second term, that is the projection onto the eigenspace of the "energy eigenvalue".

(iv) *We may make more precise Haff's law on the asymptotic behavior of the granular temperature (see [24]) in the following way. Under the assumptions of point (iii), the solution $f = f(t,v)$ to (1.1) satisfies*

$$\mathcal{E}(f(t.,)) = \frac{\mathcal{E}(\bar G_\alpha)}{(1+\tau_\alpha\, t)^2} + \mathcal{O}\left(\frac{1}{(1+\tau_\alpha\, t)^{3+\mathcal{O}(1-\alpha)}}\right). \tag{1.38}$$

(v) *Under the assumptions of point (iii) the rescaling by the square root of the energy familiar to physicists is rigorously justified in the sense: the solution $f = f(t,v)$ to (1.1) satisfies for $t \to +\infty$*

$$\mathcal{E}(f_t)^{N/2}\, f\Big(t, \mathcal{E}(f_t)^{1/2}\, v\Big) \to \mathcal{E}(\bar G_\alpha)^{N/2} \bar G_\alpha\Big(\mathcal{E}(\bar G_\alpha)^{1/2}\, v\Big) \quad in \quad L^1.$$

Remark 1.4 *We see from this theorem that the convergence towards the self-similar solution in indeed faster than the convergence towards the Dirac mass (hence justifying its interest), but also that the speed of convergence towards this self-similar solution degenerates to 0 as $\alpha \to 1$ (because $\tau_\alpha \to 0$ when $\alpha \to 1$). This fact is surprising, since the self-similar solution converges towards a stationary Maxwellian distribution in the elastic limit, and the latter is known to be exponentially attractive for the elastic equation (see [27] for instance). As we shall see this is related to the fact that a birfurcation occurs in the spectrum of the linearized collision operator at $\alpha = 1$ (namely the eigenvalue corresponding the kinetic energy vanishes at $\alpha = 1$ whereas it is non-zero for $\alpha \in [\alpha_*, 1)$). This remark may explain the fact that in the quasi-elastic limit considered – in dimension 1 – in [10], it is proved that the rate of relaxation towards the self-similar solution is worse than any polynomial.*

Proof of Theorem 1.3. Except for points (i) and (v) this theorem is an obvious translation of Theorem 1.1. In order to prove (i), one first remarks that for two given self-similar solutions F and $\tilde F$, there holds

$$F(t,v) = (V_0 + A\, t)^N\, G_A((V_0 + A\, t)\, v), \quad \tilde F(t,v) = (\tilde V_0 + \tilde A\, t)^N\, G_{\tilde A}((\tilde V_0 + \tilde A\, t)\, v),$$

and thus from Theorem 1.1

$$G_{\tilde{A}}(v) = \left(\frac{A}{\tilde{A}}\right)^N \, G_A\left(\frac{A}{\tilde{A}}\, v\right).$$

We deduce

$$\tilde{F}(t,v) = \left(\tilde{V}_0\,\frac{A}{\tilde{A}} + A\,t\right)^N \, G_A\left(\left(\tilde{V}_0\,\frac{A}{\tilde{A}} + A\,t\right)\, v\right) = F(t+t_0, v)$$

with

$$t_0 = \frac{V_0}{A}\,\left(\frac{\tilde{V}_0}{\tilde{A}} - 1\right).$$

In order to prove (v), we introduce the function $\xi(t) = \mathcal{E}(\bar{G}_\alpha)^{1/2}/[\mathcal{E}(f_t)^{1/2}\,(1+\tau_\alpha\, t)]$ and we compute

$$\begin{aligned} &\left\| \mathcal{E}(f_t)^{N/2}\, f(t, \mathcal{E}(f_t)^{1/2}\ \cdot\,) - \mathcal{E}(\bar{G}_\alpha)^{N/2} \bar{G}_\alpha(\mathcal{E}(\bar{G}_\alpha)^{1/2}\,\cdot) \right\|_{L^1} = \\ &= \|g(\tau_\alpha^{-1}\ln(1+\tau_\alpha t), \cdot) - \xi(t)^N\, \bar{G}_\alpha(\xi(t)\,\cdot)\|_{L^1} \\ &\le \|g(\tau_\alpha^{-1}\ln(1+\tau_\alpha t), \cdot) - \bar{G}_\alpha\|_{L^1} + |\xi(t)^N - 1|\, \|\bar{G}_\alpha\|_{L^1} + \xi(t)^N \|\bar{G}_\alpha(\xi(t)\,\cdot) - \bar{G}_\alpha\|_{L^1}. \end{aligned}$$

Using now (1.32), (1.35), (1.36), (1.38) and the fact that $\bar{G}_\alpha$ is bounded in $W^{1,1}_1$ uniformly in $\alpha \in (\alpha_*, 1)$ from Theorem 1.1 (ii), we deduce

$$\left\| \mathcal{E}(f_t)^{N/2}\, f(t, \mathcal{E}(f_t)^{1/2}\ \cdot\,) - \mathcal{E}(\bar{G}_\alpha)^{N/2} \bar{G}_\alpha(\mathcal{E}(\bar{G}_\alpha)^{1/2}\,\cdot) \right\|_{L^1} \le C\,(1+\tau_\alpha\, t)^{-1+\mathcal{O}(1-\alpha)},$$

for some constant $C \in (0,\infty)$ (which depends in particular on the upper bound on $\|\bar{G}_\alpha\|_{W^{1,1}_1}$), from which (v) follows. □

Remark 1.5 *Let us emphasize that the temperature $\bar{\theta}_1$ of the limit Maxwellian $\bar{G}_1$ is "universal" in the sense that it depends only on the collisional cross-section b (through its angular momentum), and not for instance on the density distribution.*

The temperature of the self-similar solution $\bar{F}_\alpha = F_\alpha(t,v)$ associated to a self-similar profile $\bar{G}_\alpha$ decreases like

$$\theta(\bar{F}_\alpha(t,\cdot)) = \frac{\theta(\bar{G}_\alpha)}{(1+\rho(1-\alpha)t)^2}.$$

Hence when α is close to 1 (small inelasticity) we obtain

$$\theta(\bar{F}_\alpha(t,\cdot)) \approx \frac{\bar{\theta}_1}{(1+\rho(1-\alpha)t)^2}.$$

Therefore, as soon as the self-similar solutions correctly describe the asymptotic (at least in the framework of point (ii) of Theorem 1.3), which is conjectured by physicists, generic solutions satisfy

$$\theta(f_\alpha(t,\cdot)) \sim_{t\to\infty} \left(\frac{\bar{\theta}_1}{\rho^2(1-\alpha)^2}\right)\, t^{-2}$$

for an inelasticity coefficient α *close to* 1.

Hence we shall denote the universal quantity $\bar{\theta}_1$ *as a "quasi-elastic self-similar temperature". Remark that its definition as the temperature of* $\bar{G}_1$ *seems to depend on the choice of the scaling. However changing this scaling by some asymptotically equivalent one, as* $\alpha \to 1$*, would only adds a factor which would then disappear when coming back to the solution to the original equation (1.1). Therefore a more "canonical" way to define this quasi-elastic self-similar temperature could be*

$$\bar{\theta}_1 = \rho^2 \lim_{\alpha \to 1} \left((1-\alpha^2) \lim_{t \to +\infty} \theta(f_\alpha(t,\cdot))\, t^2 \right)$$

where f_α *denotes a generic solution with mass* ρ *to equation (1.1).*

1.8 Method of proof and plan of the paper

The first main idea of our method is to consider the rescaled equations (1.15) and (1.16) with an inelasticity dependent anti-drift coefficient τ_α which exactly "compensates" the loss of elasticity of the collision operator (in the sense that it compensates its loss of kinetic energy). This scaling allows by some technical estimates to prove uniform bounds according to α for the family of self-similar profiles G_α to the equation (1.30). The second main idea consists in decoupling the variations along the "energy direction" and its "orthogonal direction". This decoupling makes possible to identify the limit of different objects as $\alpha \to 1$ (among them the limit of G_α). The third main idea is to use systematically the knowledges on the elastic limit problem, once it has been identified thanks to the previous arguments. In particular we use the spectral study of the linearized problem and the dissipation entropy-entropy inequality for the elastic problem. This allows to argue by perturbative method. Let us emphasize that this perturbation is singular in the classical sense because of the addition of a (limit vanishing) first-order derivative operator, but also because of the gain of one more conservative quantity at the limit (which implies in particular at the linearized level that the "energy eigenvalue" μ_α is negative for $\alpha \neq 1$ but converges to $\mu_1 = 0$ in the limit $\alpha \to 0$).

In Section 2, we use the regularity properties of the collision operator in order to establish on the one hand that the family (G_α) is bounded in $H^\infty \cap L^1(m^{-1})$ uniformly according to the inelastic parameter α (the key argument being the use of the entropy functional which provides uniform lower bound on the energy of G_α) and on the other hand that the difference of two self-similar profiles in any strong norm may be bounded by the difference of these ones in weak norm (the key idea is a bootstrap argument). This last point shall allow to deal with the loss of derivatives and weights in the operator norms used in the sequel of the paper.

In Section 3, we prove that $\alpha \mapsto Q^+_\alpha$ is Hölder continuous in the norm of its graph and is Hölder differentiable in a weaker norm. As a consequence we deduce that $G_\alpha \to \bar{G}_1$ when $\alpha \to 1$ with explicit "Hölder" rate, which (partially) proves point (ii) Theorem 1.1. The cornerstone of the proof is the decoupling of the variation $G_\alpha - \bar{G}_1$ between the "energy direction" and its "orthogonal direction".

In Section 4, we prove uniqueness of the profile $\bar{G}_\alpha$ for small inelasticity (point (i) of Theorem 1.1) by a variation around the implicit function theorem. We also deduce that $\alpha \mapsto \bar{G}_\alpha$ is differentiable at $\alpha = 1$.

Section 5 is devoted to the study of the linearized operator $\mathcal{L}_\alpha$, and we partially inspire from the method of [27]. We prove point (iii) of Theorem 1.1 and we end the proof of point (ii) of Theorem 1.1. We obtain information on the localization of the spectrum and we establish some decay estimates on the associated semigroup. Let us emphasize that for technical reasons we state our results in an L^1 framework (because mainly we are not able to generalize Lemma 5.8 to an L^2 framework), which makes the spectral analysis more intricate. The proof proceeds as follows (the cornerstone idea is again the decoupling of the variations in the "energy direction" and its "orthogonal direction"). First, we localize the essential spectrum in the half plan $\Delta^c_{\bar\mu} = \{z \in \mathbb{C},\ \Re e\, z \le \bar\mu < 0\}$ with the help of Weyl's theorem, the compactness properties of $\mathcal{L}_\alpha$ and the "rough" (Hölder type) convergence of $Q^+_\alpha(\bar G_\alpha, \cdot)$ to $Q^+_1(\bar G_1, \cdot)$ in the "good" norm of the graph. Second, we localize the discrete spectrum lying in $\Delta_{\bar\mu} = \{z \in \mathbb{C},\ \Re e\, z \ge \bar\mu\}$ in the disc $\{z \in \mathbb{C},\ |z| \le C\,(1-\alpha)\}$, thanks to estimates on the resolvent of $\mathcal{L}_\alpha$. Third we establish that the spectrum $\Sigma(\mathcal{L}_\alpha)$ of $\mathcal{L}_\alpha$ satisfies $\Sigma(\mathcal{L}_\alpha) \cap \Delta_{\bar\mu} = \{\mu_\alpha\}$, where μ_α has multiplicity 1 (the proof mainly takes advantage of the "precise" convergence of $Q^+_\alpha(\bar G_\alpha, \cdot)$ to $Q^+_1(\bar G_1, \cdot)$ in "bad" norm, together with a regularity estimate holding on the discrete eigenspace). Last we establish the expansion (1.32) using the energy equation associated to the eigenvalue μ_α. The decay properties of the linear semigroup are then deduced from resolvent estimates and the above localization of the spectrum.

Section 6 is devoted to the proof of points (iv) and (v) in Theorem 1.1 which is split in several steps. First we establish a "linearized asymptotic stability result" by decoupling the evolution equation (1.29) along the "energy direction" and its "orthogonal direction", and using the semigroup decay estimates and the quadratic structure of the collision operator. Second we establish a "non-linear stability result" by decoupling the evolution equation (1.29), using the energy dissipation equation along the "energy direction" and the entropy production method on its "orthogonal direction" (let us mention that this method follows closely the physical idea that for small inelasticity the "molecular" timescale of thermalization of velocity distribution decouples from the "cooling" timescale of dissipation of energy). Third we prove the asymptotic decomposition and we exhibit a Liapunov functional for smooth initial data (point (v)) by gathering (and slightly modifying) the two preceding steps. Fourth and last, we prove point (iv) for general initial data, gathering the previous arguments with the decomposition of solutions between a smooth part and a small remaining part as introduced in [28].

2 *A posteriori* estimates on the self-similar profiles

In this section we prove various *a posteriori* regularity and decay estimates on the self-similar profiles (or the differences of self-similar profiles), uniform as $\alpha \to 1$, which shall be useful in the sequel.

2.1 Uniform estimates on the self-similar profiles

For any $\alpha \in (0,1)$ we consider $\mathcal{G}_\alpha$ the set of all the self-similar profiles of the inelastic Boltzmann equation (1.1) with inelasticity coefficient α, with given mass $\rho \in (0, +\infty)$ and finite energy. More precisely, we define $\mathcal{G}_\alpha$ as the following set of functions

$$\mathcal{G}_\alpha := \Big\{ 0 \le G \in L^1_2 \quad \text{satisfying} \quad (1.30) \Big\}.$$

For some fixed $\alpha_0 \in (0,1)$, we also define

$$\mathcal{G} = \cup_{\alpha\in[\alpha_0,1)} \mathcal{G}_\alpha.$$

The fact that for any $\alpha \in (0,1)$, $\mathcal{G}_\alpha$ is not empty was proved in [24], where a solution of (1.30) was built within the class of radially symmetric functions belonging to the Schwartz space. Here we show that any self-similar profile $G_\alpha \in \mathcal{G}$ belongs to the Schwartz space and that decay estimates, pointwise lower bound and regularity estimates can be made uniform according to the inelasticity coefficient $\alpha \in [\alpha_0, 1)$. Let us emphasize once again that the choice of the velocity rescaling parameter $\tau_\alpha = \rho\,(1-\alpha)$ in (1.30) is fundamental in order to get that uniformity in the limit $\alpha \to 1$. Let us also mention that our choice of scaling for the equation (1.30) is mass invariant, that is G with density $\rho(G)$ satisfies the equation if and only if $G/\rho(G)$ satisfies the equation with $\rho = 1$. Therefore all the estimates on the profiles are homogeneous in terms of the density ρ.

Proposition 2.1 *Let us fix $\alpha_0 \in (0,1)$. There exists $a_1, a_2, a_3, a_4 \in (0,\infty)$ and, for any $k \in \mathbb{N}$, there exists $C_k \in (0,\infty)$ such that*

$$\forall\,\alpha \in [\alpha_0, 1),\ \forall\, G_\alpha \in \mathcal{G}_\alpha, \quad \begin{cases} \|G_\alpha\|_{L^1(e^{a_1\,|v|})} \le a_2, \quad \|G_\alpha\|_{H^k(\mathbb{R}^N)} \le C_k, \\ G_\alpha \ge a_3\, e^{-a_4\,|v|^8}. \end{cases} \tag{2.1}$$

We first recall the following geometrical lemma extracted (in a slightly specified form) from [23, Lemma 2.3 & Lemma 4.4], that we shall use several times in the sequel.

Lemma 2.2 *For any $\alpha \in (0,1]$ and $\sigma \in \mathbb{S}^{N-1}$ we define*

$$\phi^*_\alpha = \phi^*_{\alpha,v,\sigma} : \mathbb{R}^N \to \mathbb{R}^N, \quad v_* \mapsto v'$$
$$\phi_\alpha = \phi_{\alpha,v_*,\sigma} : \mathbb{R}^N \to \mathbb{R}^N, \quad v \mapsto v'$$

and the Jacobian functions $J^*_\alpha = \det(D\,\phi^*_{\alpha,v,\sigma})$, $J_\alpha = \det(D\,\phi_{\alpha,v_*,\sigma})$, as well as the cone

$$\Omega_\delta = \Omega_{\delta,\sigma} = \{u \in \mathbb{R}^N,\ \hat{u}\cdot\sigma > \delta - 1\},$$

for any $\delta \in (0,2)$ and $\sigma \in \mathbb{S}^{N-1}$.

For any $\delta \in (0,2)$, ϕ^*_α defines a C^∞-diffeomorphism from $v + \Omega_\delta$ onto $v + \Omega_{\omega^*(\delta)}$ with $\omega^*(\delta) = 1 + \sqrt{\delta/2}$ and ϕ_α defines a C^∞-diffeomorphism from $v_* + \Omega_\delta$ onto $v_* + \Omega_{\omega_\alpha(\delta)}$ with

$$\omega_\alpha(\delta) = 1 + \frac{\delta - 1 + r_\alpha}{\left(1 + 2(\delta-1)r_\alpha + r_\alpha^2\right)^{1/2}}$$

and $r_\alpha = (1+\alpha)/(3-\alpha)$.

Moreover, there exist $C \in (0,\infty)$ such that with $C_\delta = C/\delta$

$$C_\delta^{-1}\,|v - v_*| \le |\phi_\alpha(v) - v_*| \le 2\,|v - v_*|, \tag{2.2}$$
$$|\phi_\alpha^{-1}(v') - \phi_{\alpha'}^{-1}(v')| \le C_\delta\,|\alpha' - \alpha|\,|v' - v_*|, \tag{2.3}$$
$$|J_\alpha| \le C_\delta, \quad |J_\alpha^{-1}| \le C_\delta, \quad |J_\alpha^{-1} - J_{\alpha'}^{-1}| \le C_\delta^2\,|\alpha' - \alpha| \tag{2.4}$$

on $v_* + \Omega_\delta$, uniformly with respect to the parameters $\alpha, \alpha' \in [0,1]$, $\sigma \in \mathbb{S}^{N-1}$ and $v_* \in \mathbb{R}^N$. The same estimate holds for ϕ^*_α on $v + \Omega_\delta$. Finally, for any $\alpha, \alpha' \in [0,1]$, $\sigma \in \mathbb{S}^{N-1}$, $v_* \in \mathbb{R}^N$ and $t \in [0,1]$, there holds

$$t\,\phi_\alpha^{-1} + (1-t)\,\phi_{\alpha'}^{-1} = \phi_{\alpha_t}^{-1} \tag{2.5}$$

for some α_t belonging to the segment with extremal points α and α'. The same result holds for ϕ^*_α.

We will also need the following elementary result in order to estimate the convolution operator L defined in (1.9).

Lemma 2.3 *For any function $g \in L^1_3(\mathbb{R}^N)$ there exists some constants $c_1, c_2 \in (0,\infty)$ such that*

$$c_1\,(1+|v|) \le L(g) \le c_2\,(1+|v|). \tag{2.6}$$

Moreover, if g satisfies $\mathcal{E}(g) \ge a_1\,\rho$ and $\mathbf{m}_{3/2}(g) \le a_2\,\rho$, for some constants $a_1, a_2 > 0$, we can take $c_1 = C^{-1}\,\rho$, $c_2 = C\,\rho$ in (2.6) for some explicit constant $C > 0$ depending only on $a_1, a_2 > 0$.

Proof of Lemma 2.3. The upper bound in (2.6) is immediate. As for the lower bound, we have, on the one hand, by Jensen's inequality,

$$\int_{\mathbb{R}^N} g_*\,|u|\,dv_* \ge \rho\,|v|. \tag{2.7}$$

On the other hand, by triangular inequality,

$$\int_{\mathbb{R}^N} g_*\,|u|\,dv_* \ge \mathbf{m}_{1/2} - |v|\,\mathbf{m}_0.$$

By Hölder's inequality we have $\mathbf{m}_{1/2} \ge \mathcal{E}^2\,\mathbf{m}_{3/2}^{-1} \ge C_0\,\rho$ for some explicit constant $C_0 > 0$ depending only on a_1, a_2. As a consequence

$$\int_{\mathbb{R}^N} g_*\,|u|\,dv_* \ge \rho\,(C_0 - |v|). \tag{2.8}$$

These two lower bounds (2.7, 2.8) imply immediately that

$$\int_{\mathbb{R}^N} g_*\,|u|\,dv_* \ge C^{-1}\,\rho\,(1+|v|).$$

for some explicit constant $C > 0$ depending only on C_0. □

Proof of Proposition 2.1. We split the proof into several steps. In Steps 1, 2 and 3, we establish the smoothness for any profile $G_\alpha \in \mathcal{G}$ as well as upper and lower bounds on its tail. In Steps 4, 5, 6, 7, 8 and 9, we show that these estimates actually are uniform with respect to the choice of the profile $G_\alpha \in \mathcal{G}_\alpha$ and $\alpha \in [\alpha_0, 1)$. Thanks to Steps 1, 2 and 3 the computations then performed are rigorously justified.

We fix $\alpha \in [\alpha_0, 1)$ and G_α a solution of (1.30) for which we will establish the announced bounds. From now we omit the subscript "α" when no confusion is possible.

Step 1. Moment bounds. From [24, Proposition 3.1], by taking $g_{\rm in} = G$ in the evolution equation (1.15), we get that $G \in L^1_k$ for any $k \in \mathbb{N}$.

Step 2. L^2 a posteriori *bound.* We aim to prove that $G \in L^2$. Let us fix $A > 0$ and let us introduce the C^1 function

$$\Lambda_A(x) := \frac{x^2}{2}\,\mathbf{1}_{x\le A} + \left(A\,x - \frac{A^2}{2}\right)\,\mathbf{1}_{x>A}.$$

We multiply the equation (1.30) by $\Lambda_A'(G) = \min\{G, A\} := T_A(G)$. Once again we shall omit the subscript "A" when no confusion is possible. After some straightforward computation we get

$$\int_{\mathbb{R}^N} \Big(T(G)\,G\,L(G) + \rho\,(1-\alpha)\,N\,T(G)^2/2\Big)\,dv = \int_{\mathbb{R}^N} T(G)\,Q^+(G,G)\,dv.$$

Since $L(G) \ge c_1\,(1+|v|)$ thanks to Lemma 2.3 and $\Lambda(G) \le G\,T(G)$ we have

$$\text{(2.9)} \qquad c_1 \int_{\mathbb{R}^N} \Lambda(G)\,(1+|v|)\,dv \;\le\; \int_{\mathbb{R}^N} T(G)\,G\,L(G)\,dv \;\le\; \int_{\mathbb{R}^N} T(G)\,Q^+(G,G)\,dv \le I_1 + I_2 + I_3 + I_4,$$

where the terms I_k are defined in the following way, splitting the collision kernel into some smooth and non-smooth parts. Let $\Theta : \mathbb{R} \to \mathbb{R}_+$ be an even C^∞ function such that support $\Theta \subset (-1,1)$, and $\int_{\mathbb{R}} \Theta = 1$. Let $\widetilde\Theta : \mathbb{R}^N \to \mathbb{R}_+$ be a radial C^∞ function such that support $\widetilde\Theta \subset B(0,1)$ and $\int_{\mathbb{R}^N} \widetilde\Theta = 1$. Introduce the regularizing sequences

$$\Theta_m(z) = m\,\Theta(mz), \quad z \in \mathbb{R}, \qquad \widetilde\Theta_n(x) = n^N \widetilde\Theta(nx), \quad x \in \mathbb{R}^N.$$

As a convention, we shall use subscripts S for "smooth" and R for "remainder". We denote $\Phi(u) := |u|$. First, we set

$$\Phi_{S,n} = \widetilde\Theta_n * (\Phi\,\mathbf{1}_{\mathcal{A}_n}), \qquad \Phi_{R,n} = \Phi - \Phi_{S,n},$$

where $\mathcal{A}_n$ stands for the annulus $\mathcal{A}_n = \left\{x \in \mathbb{R}^N \,;\, \frac{2}{n} \le |x| \le n\right\}$. Similarly, we set

$$b_{S,m}(z) = \Theta_m * (b\,\mathbf{1}_{\mathcal{I}_m})\,(z), \qquad b_{R,m} = b - b_{S,m},$$

where $\mathcal{I}_m$ stands for the interval $\mathcal{I}_m = \left\{x \in \mathbb{R} \,;\, -1 + \frac{2}{m} \le |x| \le 1 - \frac{2}{m}\right\}$ (b is understood as a function defined on $\mathbb{R}$ with compact support in $[-1,1]$). We then define

$$I_1 = \int_{\mathbb{R}^N} T(G)\,Q^+_R(G,G)\,dv,$$

where Q^+_R is the gain term associated to the cross-section $B_R := |u|\,b_{R,m}$,

$$I_2 = \int_{\mathbb{R}^N} T(G)\,Q^+_{RS}(G,G)\,dv,$$

where Q^+_{RS} is the gain term associated to the cross-section $B_{RS} := \Phi_{R,n}\,b_{S,m}$,

$$I_3 = \int_{\mathbb{R}^N} T(G)\,\Big[\tilde Q^+_S(\chi(G), G) + \tilde Q^+_S(T(G), \chi(G)\Big]\,dv,$$

where Q_S^+ is the gain term associated to the smooth cross-section $B_S := \Phi_{S,n}\, b_{S,m}$ and $\chi(G) := G - T(G)$ and finally

$$I_4 = \int_{\mathbb{R}^N} T(G)\, Q_S^+(T(G), T(G))\, dv.$$

We estimate each term separately. We omit the subscripts m and n when there is no confusion. For I_1 we proceed along the line of the proof of the estimate for the term I^r in [23, Proof of Theorem 2.1]. Using Young's inequality $x\, T(y) \le \Lambda(x) + \Lambda(y)$ we have

$$\begin{aligned} I_1 &= \iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} G\, G_*\, T(G')\, b_{R,m}\, |u|\, dv\, dv_*\, d\sigma \\ &\le \iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} G\, [\Lambda(G_*) + \Lambda(G')]\, b_{R,m}\, \mathbf{1}_{\hat u\cdot\sigma\le 0}\, |u|\, dv\, dv_*\, d\sigma \\ &\quad + \iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} G_*\, [\Lambda(G) + \Lambda(G')]\, b_{R,m}\, \mathbf{1}_{\hat u\cdot\sigma\ge 0}\, |u|\, dv\, dv_*\, d\sigma = I_{1,1} + ... + I_{1,4}. \end{aligned}$$

We just deal with the term $I_{1,2}$, the others may be handled in a similar (or even simpler) way. Making the change of variables $v_* \to v' = \phi_\alpha^*(v_*)$ (for some fixed v, σ) and using the elementary inequality $|u| \le 4\, |v' - v|$ valid when $\sigma \cdot \hat u \le 0$, there holds

$$\begin{aligned} I_{1,2} &= \iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} G\, \Lambda(G')\, b_{R,m}\, \mathbf{1}_{\hat u\cdot\sigma\le 0}\, |u|\, dv\, dv_*\, d\sigma \\ &= 2^{N+2} \iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} G\, \Lambda(G')\, b_{R,m}\, \mathbf{1}_{\hat u\cdot\sigma\le 0}\, |v - v'|\, dv'\, dv\, d\sigma \\ &\le 2^{N+2}\, \|b_{R,m}\|_{L^1}\, \|G\|_{L^1_1} \int_{\mathbb{R}^N} \Lambda(G)\, (1 + |v|)\, dv. \end{aligned}$$

Since the same estimates hold for all the terms $I_{1,k}$, we obtain

$$I_1 \le \varepsilon(m)\, \|G\|_{L^1_1} \int_{\mathbb{R}^N} \Lambda(G)\, \langle v\rangle\, dv \qquad \text{with} \qquad \varepsilon(m) \underset{m\to\infty}{\longrightarrow} 0. \tag{2.10}$$

For I_2 we proceed along the line of the proof of the estimate for the term I in [24, Proof of Proposition 2.5]. Using again Young's inequality $x\, T(y) \le \Lambda(x) + \Lambda(y)$ and the trivial estimate $\Phi_{R,n} \le C\, n^{-1}\, (|v|^2 + |v_*|^2)$ we get

$$\begin{aligned} I_2 &= \iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} G\, G_*\, T(G')\, b_{S,m}\, \Phi_{R,n}\, dv\, dv_*\, d\sigma \\ &\le \frac{C}{n} \iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} G\, |v|^2\, [\Lambda(G_*) + \Lambda(G')]\, b_{S,m}\, dv\, dv_*\, d\sigma \\ &\quad + \frac{C}{n} \iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} G_*\, |v_*|^2\, [\Lambda(G) + \Lambda(G')]\, b_{S,m}\, dv\, dv_*\, d\sigma = I_{2,1} + ... + I_{2,4}. \end{aligned}$$

Because of the truncation on b of frontal and grazing collisions, both changes of variables $v \to v' = \phi_\alpha(v)$ (for fixed v_*, σ) and $v_* \to v' = \phi_\alpha^*(v_*)$ (for fixed v, σ) are allowed (and the jacobian of their inverse is bounded). Hence in a similar way as for the term I_1 we obtain

$$I_2 \le \frac{C(m)}{n}\, \|G\|_{L^1_2} \int_{\mathbb{R}^N} \Lambda(G)\, dv. \tag{2.11}$$

For I_3, using again Young's inequality, plus $T(G) \leq G$ and the fact that both changes of variables $v \to v' = \phi_\alpha(v)$ (for fixed v_*, σ) and $v_* \to v' = \phi^*_\alpha(v_*)$ (for fixed v, σ) are allowed, we have

$$\begin{aligned} I_3 &= \iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} (T(G') + T(G'_*))\, [G\,\chi(G_*) + \chi(G)\, T(G_*)]\, b_{S,m}\, \Phi_{S,n}\, dv\, dv_*\, d\sigma \\ &\leq C(n) \iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} \Big\{ \chi(G_*)\, [\Lambda(G') + \Lambda(G'_*) + 2\,\Lambda(G)] \\ &\qquad\qquad + \chi(G)\, [\Lambda(G') + \Lambda(G'_*) + 2\,\Lambda(G_*)] \Big\}\, b_{S,m}\, dv\, dv_*\, d\sigma. \end{aligned}$$

We deduce as before

$$I_3 \leq C_{m,n}\, \|\chi(G)\|_{L^1} \int_{\mathbb{R}^N} \Lambda(G)\, dv \tag{2.12}$$

for some constant $C_{m,n} > 0$.

Finally for I_4, we argue as in the proof of [24, Proposition 2.6] for the treatment of the term involving Q^+_S, and we get for some $\theta \in (0,1)$

$$I_4 \leq C_{m,n}\, \|T(G)\|_{L^1}^{1+2\theta}\, \|T(G)\|_{L^2}^{2\,(1-\theta)}, \tag{2.13}$$

for some constant $C_{m,n} > 0$.

Gathering (2.9), (2.10), (2.11), (2.12), (2.13) and taking m, next n and finally $A \geq A(G)$ large enough we may control the terms I_1, I_2 and I_3 by the half of the left hand side term of (2.9) (for I_3 we use that $\|\chi_A(G)\|_{L^1} \to 0$ when $A \to \infty$). Note that the condition $A \geq A(G)$ depends on the distribution G (by the mean of some non-concentration bound), but shall play no role since we shall take the limit $A \to +\infty$ in the end. We obtain

$$\forall\, A \geq A(G), \qquad \frac{c_1}{2} \int_{\mathbb{R}^N} \Lambda_A(G)\, (1 + |v|)\, dv \leq C_{b,\rho,\mathcal{E}(G)}\, \|T_A(G)\|_{L^2}^{2\,(1-\theta)}$$

for some constant $C_{b,\rho,\mathcal{E}(G)} > 0$ depending on the cross-section b and on the profile G *via* its energy. Using that $T_A(G)^2/2 \leq \Lambda_A(G)$ we deduce

$$\forall A \geq A(G), \qquad \frac{c_1}{4}\, \|T_A(G)\|_{L^2}^{2\,\theta} \leq C_{b,\rho,\mathcal{E}(G)}$$

and we then conclude that $G \in L^2$ passing to the limit $A \to \infty$ in the preceding estimate, with the bound

$$\|G\|_{L^2} \leq \left(\frac{4\, C_{b,\rho,\mathcal{E}(G)}}{c_1} \right)^{\frac{1}{2\theta}}. \tag{2.14}$$

Remark 2.4 *Note that the L^2 bound (2.14) only depends on the distribution G by the mean of the energy $\mathcal{E}(G)$ and the constant c_1. Therefore, thanks to Lemma 2.3, this bound only depends on a lower bound on the energy $\mathcal{E}(g)$ and an upper bound on the third moment $\mathbf{m}_{3/2}(g)$.*

Step 3. Smoothness and positivity. Thanks to [24, Theorem 1.3] and [8, Theorem 1], taking $g_{\rm in} = G$ as an initial condition in (1.15) we have that G belongs to the Schwartz space of C^∞ functions decreasing faster than any polynomials, and that $G \geq a_1\, e^{-a_2\,|v|}$ for some constant $a_1, a_2 > 0$.

So far the estimates in Step 3 may be not uniform on the elasticity coefficient $\alpha \in [\alpha_0, 1)$ and on the profile G_α. The aim of the following steps is to prove that they actually are uniform. Note however that estimates of the previous steps shall ensure that the following computations are rigorously justified.

Step 4. Upper bound on the energy using the energy dissipation term. We prove that

$$\forall\, \alpha \in (0,1] \qquad \mathcal{E} \le \frac{4}{b_1^2}\, \rho. \tag{2.15}$$

From equation (1.23) on the energy of the profile G there holds

$$(1+\alpha)\, b_1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\, G_*\, |u|^3\, dv\, dv_* = 2\,\rho \int_{\mathbb{R}^N} G\, |v|^2\, dv. \tag{2.16}$$

From Jensen's inequality

$$\int_{\mathbb{R}^N} |u|^3\, G_*\, dv_* \ge \rho\, |v|^3,$$

and Hölder's inequality

$$\int_{\mathbb{R}^N} |v|^3\, G\, dv \ge \rho^{-1/2} \left(\int_{\mathbb{R}^N} |v|^2\, G\, dv \right)^{3/2},$$

we get

$$(1+\alpha)\, b_1\, \rho^{1/2}\, \mathcal{E}^{3/2} \le 2\, \rho\, \mathcal{E}$$

from which the bound (2.15) follows.

Step 5. Lower bound on the energy using the entropy. We prove

$$\forall\, \alpha \in (0,1] \qquad \mathcal{E} \ge \frac{N\, \alpha^4}{8}\, \rho. \tag{2.17}$$

Remark 2.5 *The choice of scaling we have made for the evolution equation in self-similar variable becomes clear from this computation: it is chosen such that the energy of the self-similar profile does not blow up nor vanishes for $\alpha \to 1$. The restriction $\alpha \in [\alpha_0, 1)$, $\alpha_0 > 0$, is then made in order to get a uniform estimate from below on the energy.*

By integrating the equation satisfied by G against $\log G$ we find

$$\int_{\mathbb{R}^N} Q(G,G) \log G\, dv - \rho\,(1-\alpha) \int_{\mathbb{R}^N} \log G\ \nabla_v \cdot (v\, G)\, dv = 0.$$

Then we write the first term as in [17, Section 1.4] to find

$$\frac{1}{2} \iiint_{\mathbb{R}^{2N} \times S^{N-1}} G\, G_* \left(\log \frac{G' G'_*}{G G_*} - \frac{G' G'_*}{G G_*} + 1 \right) B\, dv\, dv_*\, d\sigma$$
$$+ \frac{1}{2} \iiint_{\mathbb{R}^{2N} \times S^{N-1}} \left(G' G'_* - G G_* \right) B\, dv\, dv_*\, d\sigma + \rho\,(1-\alpha) \int_{\mathbb{R}^N} v \cdot \nabla_v G\, dv = 0.$$

If we denote

$$D_{H,\alpha}(g) = \frac{1}{2} \iiint_{\mathbb{R}^{2N} \times S^{N-1}} g\, g_* \left(\frac{g' g'_*}{g g_*} - \log \frac{g' g'_*}{g g_*} - 1 \right) B\, dv\, dv_*\, d\sigma \ge 0, \tag{2.18}$$

(recall that in this formula the post-collisional velocities v', v'_* are computed according to the inelastic formula (1.4) with normal restitution coefficient $\alpha \in (0,1]$), we can write

$$(2.19) \qquad - D_{H,\alpha}(G) + \left(\frac{1}{\alpha^2} - 1 \right) b_2 \iint_{\mathbb{R}^{2N}} G\, G_* \, |u| \, dv \, dv_* - (1-\alpha) \, N \, \rho^2 = 0,$$

with $b_2 := \|b\|_{L^1}$, and thus we get

$$\iint_{\mathbb{R}^{2N}} G\, G_* \, |u| \, dv \, dv_* = \frac{\alpha^2}{1+\alpha} \left(N \, \rho^2 + \frac{1}{1-\alpha} D_{H,\alpha}(G) \right) \geq \frac{N\,\alpha^2}{2} \, \rho^2.$$

On the other hand, from Cauchy-Schwarz's inequality

$$\iint_{\mathbb{R}^{2N}} G\, G_* \, |u| \, dv \, dv_* \leq$$
$$\leq \left(\iint_{\mathbb{R}^{2N}} G\, G_* \, dv dv_* \right)^{1/2} \left(\iint_{\mathbb{R}^{2N}} G\, G_* |u|^2 \, dv dv_* \right)^{1/2} = \sqrt{2} \, \rho^{3/2} \, \mathcal{E}^{1/2},$$

and then the bound (2.17) follows gathering the two preceding estimates.

Step 6. Upper bound on (exponential) moments using Povzner inequality. There exists $A, C > 0$ such that

$$\forall \, \alpha \in [0,1), \qquad \int_{\mathbb{R}^N} G(v) \, e^{A|v|} \, dv \leq C \, \rho.$$

We refer to [8] where that bound is obtained as an immediate consequence of the following sharp moment estimates: there exists $X > 0$ such that

$$(2.20) \qquad \forall \, \alpha \in [0,1), \qquad \mathbf{m}_k = \int_{\mathbb{R}^N} G \, |v|^k \, dv \leq \Gamma(k+1/2) \, X^{k/2} \, \rho.$$

It is worth noticing that in [8] the Povzner inequality used in order to get (2.20) is uniform in the normal restitution coefficient $\alpha \in [0,1]$ and that the factor ρ comes from our choice of the scaling variables (in which ρ is involved).

Step 7. Uniform upper bound on the L^2 norm. From (2.17), (2.20) and Remark 2.4, the L^2 bound (2.14) is uniform on $\alpha \in [\alpha_0, 1)$ and $G \in \mathcal{G}_\alpha$.

Step 8. Smoothness. It is enough to show some uniform bounds from above and below on the energy together with uniform non-concentration bounds on the self-similar profiles in $\mathcal{G}$, in the form of upper bounds on the L^2 bounds for instance. Indeed the proofs of [24, Proposition 3.1, Proposition 3.2, Proposition 3.4, Theorem 3.5 and Theorem 3.6] then apply straightforwardly (in these proofs we did not use the part associated with the anti-drift in the semigroup). Therefore the uniform bounds on the H^k norms for all $k \geq 0$ follows from these results.

Step 9. Pointwise Lower bound. It is a consequence of the following lemma. ☐

Lemma 2.6 *Let $g \in C([0,\infty); L^1_3)$ be a solution of the rescaled equation (1.29) with inelasticity parameter $\alpha \in (0,1)$ and assume that for some $p > 1$ and $C, T \in (0,\infty)$*

$$\sup_{[0,T]} \|g\|_{L^p \cap L^1_3} \leq C.$$

(i) For any $t_1 \in (0,T)$ there exists $a_1 \in (0,\infty)$ (depending on C, ρ and t_1 but not on T) such that

$$\forall\, t \in [t_1, T], \ \ \forall\, v \in \mathbb{R}^N, \quad g(t,v) \ge a_1^{-1}\, e^{-a_1\,|v|^8}. \tag{2.21}$$

(ii) If furthermore, $g_{\rm in}$ satisfies

$$g_{\rm in}(v) \ge a_0^{-1}\, e^{-a_0\,|v|^8},$$

then (2.21) holds with $t_0 = 0$ and some constant $a_1 \in (0,\infty)$ (depending on C, ρ, a_0 but not on T).

Proof of Lemma 2.6. We only prove (i), the proof of (ii) being similar. Let us fix $t_1 \in (0,1)$. We closely follow the proof of the Maxwellian lower bound for the solutions of the elastic Boltzmann equation (see [11, 30]) taking advantage of some technical results established in its extension to the solutions of the inelastic Boltzmann equation (see [24, Theorem 4.9]). The starting point is again the evolution equation satisfied by g written in the form

$$\partial_t g + \tau_\alpha\, v \cdot \nabla_v g + (\tau_\alpha\, N + C + C\,|v|)\, g = Q^+_\alpha(g,g) + (C + C\,|v| - L(g))\, g,$$

where the last term in the right hand side term is non-negative for some well-chosen numerical constant $C \in (0,\infty)$ thanks to Lemma 2.3, (2.20) and (2.17). Let us introduce the semigroup U_t associated to the operator $\tau_\alpha\, v \cdot \nabla_v + \lambda(v)$, where $\lambda(v) := \tau_\alpha\, N + C + C\,|v|$, which action is given by

$$(U_t\, h)(v) = h(v\, e^{-\tau_\alpha\, t})\, \exp\left(- \int_0^t \lambda(v\, e^{-s})\, ds \right).$$

Thanks to the Duhamel formula, we have

$$\forall\, t > 0, \ \forall\, \tau \ge 0, \quad g(t+\tau,.) \ge \int_0^t U_{t-s} Q^+(g(s+\tau,.), g(s+\tau,.))\, ds. \tag{2.22}$$

Noticing that

$$\left(- \int_0^t \lambda(v\, e^{-s})\, ds \right) \ge -(C\,|v| + \tau_\alpha\, N\, t + C\, t),$$

and repeating the arguments of Steps 2 and 3 in the proof of [24, Theorem 4.9], we get that

$$\forall\, t \ge \tau, \quad g(t,.) \ge \eta\, \mathbf{1}_{B(0,\delta)}(v) \tag{2.23}$$

with $\tau = \tau_1 = t_1/2$ and some constant $\eta = \eta_1 > 0$, $\delta = \delta_1 > 1$. Let us emphasize that here we make use of Lemma 4.6, Lemma 4.7 and Lemma 4.8 in [24] where the constants exhibited in these ones are uniform in $\alpha \in [\alpha_0, 1)$ thanks to the uniform $L^p \cap L^1_3$ estimates assumed on g.

Now, on the one hand, from [24, Lemma 4.8], there exists $\kappa \in (0,\infty)$ such that

$$Q^+_\alpha(\mathbf{1}_{B(0,1)}, \mathbf{1}_{B(0,1)}) \ge \kappa\, \mathbf{1}_{B(0,\sqrt{5}/2)}$$

which in turns implies

$$\forall\, \delta > 0, \quad Q^+_\alpha(\mathbf{1}_{B(0,\delta)}, \mathbf{1}_{B(0,\delta)}) \ge \kappa\, \delta^{-N-1}\, \mathbf{1}_{B(0,\sqrt{5}/2\,\delta)}. \tag{2.24}$$

On the other hand, there exists $\kappa' \in (0,\infty)$ such that

$$\forall\,\delta>0,\quad \forall\,s\in[0,1],\quad U_s(\mathbf{1}_{B(0,\delta)}) \ge \kappa'\,e^{-C\,\delta}\,\mathbf{1}_{B(0,\delta)}. \tag{2.25}$$

From (2.23) with $\eta=\eta_1$, $\delta=\delta_1$, and making use of (2.22), (2.24), (2.25), we get that (2.23) holds with

$$\tau=\tau_2=\tau_1+\frac{t_1}{2^2},\quad \delta=\delta_2=\frac{\sqrt{5}}{2}\,\delta_1\quad\text{and}\quad \eta=\eta_2=(\tau_2-\tau_1)\,\kappa''\,\eta_1^2\,e^{-C'\,\delta_1},$$

where $\kappa''=\kappa\,\kappa'$ and C' depends on C and N. Iterating the argument we get that (2.23) holds with $\tau=\tau_k=\tau_{k-1}+t_1\,2^{-k}=(1-2^{-k})\,t_1$, $\delta=\delta_{k+1}=\left(\sqrt{5}/2\right)^{k+1}$ and

$$\eta_{k+1}=(\kappa''\,t_1)^{1+2+\ldots+2^{k-1}}\,\eta_1^{2^k}\,e^{-C'\,(\delta_k+2\,\delta_{k-1}+\ldots+2^{k-1}\,\delta_1)}\,2^{-[k+2\,(k-1)+\ldots+2^{k-1}\,1]}\ge A^{2^{k+1}},$$

with $A:=\sqrt{\kappa''\,t_1}\,\sqrt{\eta_1}\,e^{-C'\,\delta_1}/2$. In other words, using that $\left(\frac{\sqrt{5}}{2}\right)^8>2$, we have proved

$$\forall\,t\ge t_1,\ \forall\,k\in\mathbb{N},\quad g(t,v)\ge A^{2^k}\,\mathbf{1}_{B(0,2^{k/8}\,\delta_1)}(v),$$

from which we easily conclude. □

2.2 Estimates on the difference of two self-similar profiles

In this subsection we take advantage of the mixing effects of the collision operator in order to show that the L^1 norm of their difference of two self-similar profiles (corresponding to the same inelasticity coefficient) indeed controls the $H^k\cap L^1(m^{-1})$ norm of their difference for any $k\in\mathbb{N}$ and for some exponential weight function m, uniformly in terms of $\alpha\in[\alpha_0,1)$.

Proposition 2.7 *For any $k>0$, there is $m=\exp(-a\,|v|)$, $a\in(0,\infty)$ and $C_k>0$ such that for any $\alpha\in[\alpha_0,1)$ and any $G_\alpha,H_\alpha\in\mathcal{G}_\alpha$ there holds*

$$\|H_\alpha-G_\alpha\|_{H^k\cap L^1(m^{-1})}\le C_k\,\|H_\alpha-G_\alpha\|_{L^1}. \tag{2.26}$$

Proof of Proposition 2.7. We proceed in three steps. It is worth mentioning that all the constants in the proof are uniform in terms of the normal restitution coefficient $\alpha\in[\alpha_0,1)$, as they only depend on the uniform bounds of Proposition 2.1 and some uniform bounds on the collision kernel.

Step 1. Control of the L^1 moments. We prove first that there exists $A,C\in(0,\infty)$ such that

$$\forall\,\alpha\in[\alpha_0,1),\qquad \int_{\mathbb{R}^N}|H_\alpha-G_\alpha|\,e^{A\,|v|}\,dv\le C\int_{\mathbb{R}^N}|H_\alpha-G_\alpha|\,dv.$$

Let us consider some normal restitution coefficient $\alpha\in[\alpha_0,1)$ and two self-similar profiles $G,H\in\mathcal{G}_\alpha$ (here again, we omit the subscript α when there is no confusion). We denote $D=G-H$, $S=G+H$ and $\varphi=|v|^{2p}\,\mathrm{sgn}(D)$, $p\in\frac12\mathbb{N}$, $p\ge 3/2$, where $\mathrm{sgn}(D)$ denotes the sign of D. The equation for D reads

$$\begin{aligned} 0 &= Q_\alpha(G,G)-Q_\alpha(H,H)-\rho\,(1-\alpha)\,\nabla_v\cdot(v\,D)\\ &= 2\,\tilde{Q}_\alpha(D,S)-\rho\,(1-\alpha)\,\nabla_v\cdot(v\,D). \end{aligned} \tag{2.27}$$

Multiplying equation (2.27) by φ, we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} B\,D\,S_*\left[\varphi'_* + \varphi' - \varphi_* - \varphi\right] dv\,dv_*\,d\sigma \\ &\qquad\qquad -\rho\,(1-\alpha)\int_{\mathbb{R}^N} \nabla_v(vD)\,|v|^{2p}\,\mathrm{sgn}(D)\,dv \\ &\le \int_{\mathbb{R}^N\times\mathbb{R}^N} |u|\,|D|\,S_*\,K_p\,dv\,dv_* + 2\int_{\mathbb{R}^N\times\mathbb{R}^N} |u|\,|D|\,S_*\,|v_*|^{2p}\,dv\,dv_* \\ &\qquad\qquad +\rho\int_{\mathbb{R}^N} |D|\,v\cdot\nabla(|v|^{2p})\,dv \end{aligned}$$

with

$$K_p(v,v_*) := \int_{\mathbb{S}^{N-1}} (|v'|^{2p} + |v'_*|^{2p} - |v|^{2p} - |v_*|^{2p})\,b(\sigma\cdot u)\,d\sigma.$$

From [8, Corollary 3, Lemma 2], there holds

$$K_p(v,v_*) \le \gamma_p\,\Sigma_p - (1-\gamma_p)\,(|v|^{2p} + |v_*|^{2p})$$

where $(\gamma_p)_{p=3/2,2,...}$ is a decreasing sequence of real numbers such that

$$0 < \gamma_p < \min\left\{1, \frac{4}{p+1}\right\}, \tag{2.28}$$

and Σ_p is defined by

$$\Sigma_p := \sum_{k=1}^{k_p} \binom{p}{k} \left(|v|^{2k}\,|v_*|^{2p-2k} + |v|^{2p-2k}\,|v_*|^{2k}\right),$$

with $k_p := [(p+1)/2]$ is the integer part of $(p+1)/2$ and $\binom{p}{k}$ stands for the binomial coefficient. As a consequence,

$$\begin{aligned} (1-\gamma_{3/2})\int_{\mathbb{R}^N\times\mathbb{R}^N} |v|^{2p}\,|u|\,S_*\,|D|\,dv\,dv_* &\le \gamma_p\int_{\mathbb{R}^N\times\mathbb{R}^N} |u|\,|D|\,S_*\,\Sigma_p\,dv\,dv_* \\ +2\int_{\mathbb{R}^N\times\mathbb{R}^N} |u|\,|D|\,S_*\,|v_*|^{2p}\,dv\,dv_* &+ 2\,\rho\,p\int_{\mathbb{R}^N} |D|\,|v|^{2p}\,dv. \end{aligned}$$

Using Lemma 2.3 in order to estimate $L(S)$ from below, the inequality $|u| \le |v| + |v_*|$ and introducing the notations

$$\mathbf{d}_k := \int_{\mathbb{R}^N} |D|\,|v|^{2k}\,dv, \quad \mathbf{s}_k := \int_{\mathbb{R}^N} S\,|v|^{2k}\,dv,$$

we get, for some numerical constant $C \in (0,\infty)$,

$$\frac{\rho}{C}\,\mathbf{d}_{p+1/2} \le \gamma_p\,S_p + \left(\mathbf{d}_0\,\mathbf{s}_{p+1/2} + \mathbf{d}_{1/2}\,\mathbf{s}_p\right) + 2\,\rho\,p\,\mathbf{d}_p, \tag{2.29}$$

with

$$S_p := \sum_{k=1}^{k_p} \binom{p}{k} \left(\mathbf{d}_{k+1/2}\,\mathbf{s}_{p-k} + \mathbf{d}_k\,\mathbf{s}_{p-k+1/2} + \mathbf{d}_{p-k+1/2}\,\mathbf{s}_k + \mathbf{d}_{p-k}\,\mathbf{s}_{k+1/2}\right).$$

From Proposition 2.1, or more precisely (2.20), we know that $\mathbf{s}_k \le \rho\,\Gamma(k+1/2)\,x^k$ for any $k \ge 1$ and for some $x \in (1,\infty)$. By Hölder's inequality, we also have

$$\mathbf{d}_p^{1+\frac{1}{2p}} \le \mathbf{d}_{p+\frac{1}{2}}\,\mathbf{d}_0^{\frac{1}{2p}}.$$

Repeating the proof of [8, Lemma 4], for any $a \ge 1$, there exists $A > 0$ such that

$$S_p \le A\,\rho\,(\mathbf{d}_0 + \mathbf{d}_{1/2})\,\Gamma(a\,p + a/2 + 1)\,Z_p$$

with

$$Z_p := \max_{k=1,..,k_p} \{\delta_{k+1/2}\,\sigma_{p-k}, \delta_k\,\sigma_{p-k+1/2}, \delta_{p-k+1/2}\,\sigma_k, \delta_{p-k}\,\sigma_{k+1/2}\},$$

and

$$\delta_k := \frac{\mathbf{d}_k}{(\mathbf{d}_0 + \mathbf{d}_{1/2})\,\Gamma(a\,k+1/2)}, \qquad \sigma_k := \frac{\mathbf{s}_k}{\rho\,\Gamma(a\,k+1/2)}.$$

We may then rewrite (2.29) as

$$\Gamma(a\,p+1/2)^{1/2p}\,\delta_p^{1+1/2p} \le A\,\gamma_p\,\frac{\Gamma(a\,p+a/2+1)}{\Gamma(ap+1/2)}\,Z_p + (\sigma_{p+1/2} + \sigma_p) + 2\,\rho\,p\,\delta_p.$$

On the one hand, from (2.28), there exists A' such that

$$A\,\gamma_p\,\frac{\Gamma(ap+a/2+1)}{\Gamma(ap+1/2)} \le A'\,p^{a/2-1/2} \qquad \forall\,p = 3/2, 2, \ldots$$

On the other hand, thanks to Stirling's formula $n! \sim n^n\,e^{-n}\,\sqrt{2\pi n}$ when $n \to \infty$ and the estimate (2.28), there exists $A'' > 0$ such that

$$(1-\gamma_p)\,\Gamma(a\,p+1/2)^{1/2p} \ge A''\,p^{a/2} \qquad \forall\,p = 3/2, 2, \ldots$$

Therefore,

$$p^{a/2}\,\delta_p^{1+1/2p} \le p^{a/2-1/2}\,Z_p + (\sigma_{p+1/2} + \delta_1\,\sigma_p) + 2\,\rho\,p\,\delta_p.$$

We finally obtain

$$\mathbf{d}_k \le x^k\,\Gamma(ak+1/2)\,(\mathbf{d}_0 + \mathbf{d}_{1/2}),$$

and we easily conclude as in [8, Proof of Theorem 1] or in [23, Proof of Proposition 3.2, Step 2].

Step 2. Control of the L^2 norms. For $k=0$, the propagation of the L^2 norm is immediate using the result [24, Corollary 2.3]. Indeed one just has to split the collision kernel as in [24, Section 2.4]. For the truncated and regularized part Q_S^+ (we use the notation introduced in step 2 the proof of Proposition 2.1), [24, Corollary 2.3] together with some basic interpolation yield the following control:

$$\int_{\mathbb{R}^N} \left(Q_S^+(S,D) + Q_S^+(D,S)\right) D\,dv \le C\,\rho^{1+2\theta}\,\|D\|_{L^2}^{2-2\theta}$$

for some explicit $C > 0$ and $\theta \in (0,1)$. For the remaining term Q_R^+, we use the same control as in [24, Proof of Proposition 2.5] to get

$$\int_{\mathbb{R}^N} \left(Q_R^+(S,D) + Q_R^+(D,S)\right)\,D\,dv \le \varepsilon\,\left(\|D\|_{L^1_2} + \|D\|_{L^2_{1/2}}\right)\|D\|_{L^2_{1/2}}$$

for some ε which can be taken as small as wanted by the truncation. Gathering these estimates, we get

$$\forall\,\varepsilon > 0 \qquad \int_{\mathbb{R}^N} \tilde{Q}^+(S,D)\,D\,dv \le \varepsilon\,\|D\|^2_{L^2_{1/2}} + C_\varepsilon$$

where C_ε depends on weighted L^1 and L^2 norms of S, on L^1 norms on D and on ε. Using equation (2.30) with $i = 0$, Lemma 2.3 to treat the term $L(D)$, and some elementary interpolation, we deduce that

$$\|D\|_{L^2_{1/2}} \le C\,\|D\|_{L^1_2}$$

for some constant $C > 0$, which concludes the proof for $k = 0$ using the previous step on the L^1 moments.

Step 3. Control of the H^k norms. From the previous step and some interpolation, in order to conclude it is enough to prove (2.26) for any $k \in \mathbb{N}$ and $m \equiv 1$. We proceed by induction on k. For any $i \in \mathbb{N}^N$, the equation satisfied by $\partial^i D$ is

$$\begin{aligned} &\partial^i Q^+(S,D) + \partial^i Q^+(D,S) - \partial^i (L(D)\,S) - L(S)\,\partial^i D \\ &\quad - \sum_{0<i'\le i} \begin{pmatrix} i' \\ i \end{pmatrix} \partial^{i'} L(S)\,\partial^{i-i'} D - \rho\,(1-\alpha)\,\partial^i \nabla\cdot(v\,D) = 0. \end{aligned}$$

We deduce that

$$\begin{aligned} C \int_{\mathbb{R}^N} (\partial^i D)^2\,(1+|v|)\,dv &\le \int_{\mathbb{R}^N} \left(\partial^i Q^+(S,D) + \partial^i Q^+(D,S)\right) \partial^i D\,dv \\ &\quad - \sum_{0\le i'\le i} \begin{pmatrix} i' \\ i \end{pmatrix} \int_{\mathbb{R}^N} \partial^{i'} L(D)\,\partial^{i-i'} S\,\partial^i D\,dv \\ &\quad - \sum_{0<i'\le i} \begin{pmatrix} i' \\ i \end{pmatrix} \int_{\mathbb{R}^N} \partial^{i'} L(S)\,\partial^{i-i'} D\,\partial^i D\,dv \end{aligned} \tag{2.30}$$

droping the non-positive term.

The induction is initialized by Step 2. Let us assume the induction step $k \ge 0$ to be proved, and let us consider some $i \in \mathbb{N}^N$ such that $|i| = k+1$. Using equation (2.30) and [24, Theorem 2.5] to estimate the gain term, we find easily

$$\|\partial^i D\|_{L^2} \le C\,\left(\|D\|_{L^1_q} + \|D\|_{H^{k+(3-N)/2}_q}\right)$$

for some $q > 0$. Therefore we obtain by interpolation (since $(3-N)/2 < 1$ for $N \ge 2$), for another q' possibly larger:

$$\|D\|_{H^{k+1}} \le C\,\left(\|D\|_{L^1_{q'}} + \|D\|_{H^k}\right).$$

This concludes the proof, using interpolation, the induction hypothesis k, and the Step 1 on the L^1 moments. ∎

3 The elastic limit $\alpha \to 1$

3.1 Dependency of the collision operator according to the inelasticity

In this subsection we show that the collision operator depends continuously on the inelasticity coefficient $\alpha \in [0,1]$. Since it is an unbounded operator, this continuous dependency is expressed in the norm of the graph of the operator or in some weaker norm. We start showing that this dependency of the collision operator is Lipschitz, and even $C^{1,\eta}$ for any $\eta \in (0,1)$, when allowing a loss (in terms of derivatives and weight) in the norm they are expressed. Let define the formal derivative of the collision operator according to α by

$$Q'_\alpha(g,f) := \nabla_v \cdot \left(\int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} g({}'v_*(\alpha))\, f({}'v(\alpha))\, b\, |u| \left(\frac{u - |u|\,\sigma}{4\,\alpha^2} \right) d\sigma\, dv_* \right)$$

or by duality

$$\langle Q'_\alpha(g,f), \psi \rangle := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} g_*\, f\, b\, |u| \left(\frac{|u|\,\sigma - u}{4} \right) \nabla \psi(v'_\alpha)\, d\sigma\, dv_*\, dv.$$

Proposition 3.1 *Let us fix a smooth exponential weight $m = \exp(-a\,|v|^s)$, $a \in (0,+\infty)$, $s \in (0,1)$. Then*

(i) For any $k, q \in \mathbb{N}$ the exists $C \in (0,\infty)$ such that for any smooth functions f, g (say in $\mathcal{S}(\mathbb{R}^N)$) and any $\alpha \in [0,1]$ there holds

$$\|Q^\pm_\alpha(g,f)\|_{W^{k,1}_q(m^{-1})} \le C_{k,m}\, \|f\|_{W^{k,1}_{q+1}(m^{-1})}\, \|g\|_{W^{k,1}_{q+1}(m^{-1})} \tag{3.1}$$

$$\|Q'_\alpha(g,f)\|_{W^{k,1}_q(m^{-1})} \le C_{k,m}\, \|f\|_{W^{k+1,1}_{q+2}(m^{-1})}\, \|g\|_{W^{k+1,1}_{q+2}(m^{-1})}. \tag{3.2}$$

(ii) Moreover, for any smooth functions f, g and for any $\alpha, \alpha' \in [0,1]$, there holds

$$\begin{aligned} &\|Q^+_\alpha(g,f) - Q^+_{\alpha'}(g,f) - (\alpha - \alpha')\, Q'_\alpha(g,f)\|_{W^{-2,1}_q(m^{-1})} \\ &\qquad \le |\alpha - \alpha'|^2\, \|f\|_{L^1_{q+3}(m^{-1})}\, \|g\|_{L^1_{q+3}(m^{-1})}. \end{aligned} \tag{3.3}$$

(iii) As a consequence, there holds

$$\|Q^+_{\alpha'}(g,f) - Q^+_\alpha(g,f)\|_{W^k_q(m^{-1})} \le C\, |\alpha - \alpha'|\, \|f\|_{W^{2k+3,1}_{q+3}(m^{-1})}\, \|g\|_{W^{2k+3,1}_{q+3}(m^{-1})}, \tag{3.4}$$

and for any $\eta \in (1,2)$, there exists $k_\eta \in \mathbb{N}$, $q_\eta \in \mathbb{N}$ and $C_\eta \in (0,\infty)$ such that

$$\begin{aligned} &\|Q^+_\alpha(g,f) - Q^+_{\alpha'}(g,f) - (\alpha - \alpha')\, Q'_\alpha(g,f)\|_{L^1(m^{-1})} \\ &\qquad \le C_\eta\, |\alpha - \alpha'|^\eta\, \|f\|_{W^{k_\eta,1}_{q_\eta}(m^{-1})}\, \|g\|_{W^{k_\eta,1}_{q_\eta}(m^{-1})}. \end{aligned} \tag{3.5}$$

Proof of Proposition 3.1. First by classical convolution-like estimates (see for instance [28] in the elastic case, and [17] in the inelastic case, as well as the proof of Proposition 3.2 below) we easily have (3.1) and (3.1).

Next, in order to prove (3.3) we proceed by duality. Let us consider $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and define $\psi := \varphi \, \langle v \rangle^q \, m^{-1}$. We compute

$$\begin{aligned} I &:= \int_{\mathbb{R}^N} \left[Q^+_\alpha(g,f) - Q^+_{\alpha'}(g,f) - (\alpha - \alpha') \, Q'_\alpha(g,f) \right] \, \psi(v) \, dv \\ &= \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} |u| \, b \, g_* \, f \, \left[\psi(v'_\alpha) - \psi(v'_{\alpha'}) - (\alpha - \alpha') \left(\frac{|u| \, \sigma - u}{4} \right) \cdot \nabla \psi(v'_\alpha) \right] \, dv \, dv_* \, d\sigma. \end{aligned}$$

Hence, if one denotes by $\xi_{v,v_*,\sigma}(\alpha) := \psi(v'_\alpha)$ (for given fixed values of v, v_*, σ), we obtain (omitting the subscripts for clarity)

$$\begin{aligned} |I| &= \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} |u| \, b \, g_* \, f \, \left[\xi(\alpha) - \xi(\alpha') - (\alpha - \alpha') \, \xi'(\alpha) \right] \, dv \, dv_* \, d\sigma \\ &\le (\alpha - \alpha')^2 \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} |u| \, b \, g_* \, f \, \sup_{\alpha \in (0,1)} |\xi''(\alpha)| \, dv \, dv_* \, d\sigma. \end{aligned}$$

We then easily conclude that (3.3) holds using that $\langle v' \rangle^q \, (m')^{-1} \le C \, \langle v \rangle^q \, (m)^{-1} \, \langle v_* \rangle^q \, (m_*)^{-1}$ for some constant $C \in (0, \infty)$.

Last, we prove (3.4) by using the following interpolation on $J = Q^+_\alpha(g,f) - Q^+_{\alpha'}(g,f) - (\alpha - \alpha') \, Q'_\alpha(g,f)$:

$$\|J\|_{W^{k,1}_q(m^{-1})} \le \|J\|_{W^{-2,1}_q(m^{-1})} \, \|J\|_{W^{2(k+1),1}_q(m^{-1})}$$

and using (3.3) on the first term in the right-hand side, and (3.1,3.2) on the second term in the right-hand side. It yields

$$\left\| Q^+_\alpha(g,f) - Q^+_{\alpha'}(g,f) - (\alpha - \alpha') \, Q'_\alpha(g,f) \right\|_{W^{k,1}_q(m^{-1})} \le C \, |\alpha - \alpha'| \, \|f\|_{W^{2k+3,1}_{q+3}(m^{-1})} \, \|g\|_{W^{2k+3,1}_{q+3}(m^{-1})}$$

and (3.4) follows by using (3.2) again.

Then the proof of (3.5) is done in the same way using suitable interpolation. □

We next state a mere (Hölder) continuity dependency on α, which is however stronger than Proposition 3.1 in some sense, since it is written in the norm of the graph of the operator for one the argument.

Proposition 3.2 *For any $\alpha, \alpha' \in (0,1]$, and any $g \in L^1_1(m^{-1})$, $f \in W^{1,1}_1(m^{-1})$, there holds*

$$(3.6) \qquad \begin{cases} \|Q^+_\alpha(g,f) - Q^+_{\alpha'}(g,f)\|_{L^1(m^{-1})} \le \varepsilon(\alpha - \alpha') \, \|f\|_{W^{1,1}_1(m^{-1})} \, \|g\|_{L^1_1(m^{-1})}, \\ \|Q^+_\alpha(f,g) - Q^+_{\alpha'}(f,g)\|_{L^1(m^{-1})} \le \varepsilon(\alpha - \alpha') \, \|f\|_{W^{1,1}_1(m^{-1})} \, \|g\|_{L^1_1(m^{-1})}. \end{cases}$$

where $\varepsilon(r) = C \, r^{\frac{1}{3+4/s}}$ for some constant C (depending only on b).

Proof of Proposition 3.2. For any given $v, v_* \in \mathbb{R}^N$, $w = v + v_* \neq 0$ and $\sigma \in S^{N-1}$ we define $\chi \in [0, \pi/2]$, $\cos \chi := |\sigma \cdot \hat{w}|$. Let us fix $\delta \in (0,1)$, $R \in (1, \infty)$ and let us define $\theta_\delta \in W^{1,\infty}(-1,1)$ such that $\theta_\delta(s) = 1$ on $(-1 + 2\delta, 1 - 2\delta)$, $\theta_\delta(s) = 0$ on $(-1 + \delta, 1 - \delta)^c$, $0 \le \theta_\delta \le 1$, $|\theta'_\delta(s)| \le 3/\delta$, $\Theta_R(u) = \Theta(|u|/R)$ with $\Theta(x) = 1$ on $[0,1]$, $\Theta(x) = 1 - x$ for $x \in [1,2]$ and $\Theta(x) = 0$ on $[2, \infty)$, $A(\delta) := \{\sigma \in S^{N-1}; \, \sin^2 \chi \ge \delta\}$,

$B(\delta) := \{\sigma \in S^{N-1};\ \cos\theta \in (-1+2\delta, 1-2\delta)^c \text{ or } \sin^2\chi \le \delta\}$. We then split Q^+ in three terms, namely

$$Q^+_\alpha = Q^{+,a}_\alpha + Q^{+,v}_\alpha + Q^{+,r}_\alpha$$

where $Q^{+,r}_\alpha$ is defined by (1.3) with b replaced by $b^r := b\,\theta_\delta(\sigma\cdot\hat u)\,\Theta_R(u)$, where $Q^{+,v}_\alpha$ is defined by (1.3) with b replaced by $b^v := b\,\mathbf{1}_{A(\delta)}\,(1-\Theta_R(u))$ and where $Q^{+,a}_\alpha$ is defined by (1.3) with b replaced by $b^a := b\,(1-\theta_\delta(\sigma\cdot\hat u))\,\Theta_R(u) + b\,(1-\Theta_R(u))\,\mathbf{1}_{A^c(\delta)}$. We split the proof into three steps.

Step 1. Treatment of small angles. There exists a constant $C \in (0,\infty)$ such that for any $\alpha \in (0,1]$ and $\delta \in (0,1)$ there holds

$$\|Q^{+,a}_\alpha(\psi,\varphi)\|_{L^1(m^{-1})} \le C\,\delta\,\|\psi\|_{L^1_1(m^{-1})}\|\varphi\|_{L^1_1(m^{-1})}.$$

Indeed let us consider some $\ell \in L^\infty$ and let us proceed by duality. We estimate

$$\begin{aligned}\int_{\mathbb{R}^N} Q^{+,a}_\alpha(\psi,\varphi)\,\ell(v)\,m^{-1}(v)\,dv &= \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} |u|\,b^a\,\psi_*\,\varphi\,\ell'\,(m')^{-1}\,dv\,dv_*\,d\sigma \\ &\le \|b^a\|_{L^\infty_{v,v_*}(L^1(\mathbb{S}^{N-1}))}\,\|\ell\|_{L^\infty}\,\|\psi\|_{L^1_1(m^{-1})}\|\varphi\|_{L^1_1(m^{-1})},\end{aligned}$$

and we conclude using that $\|b^a\|_{L^\infty_{v,v_*}(L^1(\mathbb{S}^{N-1}))} \le C\,(\delta + \max_{v,v_*}|B(\delta)|) \le C\,\delta$.

Step 2. Treatment of large relative velocities. There exists a constant $C = C_{a,s,b} \in (0,\infty)$ such that for any $\alpha \in (0,1]$ and $\delta \in (0,1)$ there holds

$$\|Q^{+,v}_\alpha(\psi,\varphi)\|_{L^1(m^{-1})} \le \frac{C}{R\,\delta^{2/s}}\,\|\psi\|_{L^1_1(m^{-1})}\,\|\varphi\|_{L^1_1(m^{-1})} \tag{3.7}$$

We need the following lemma, which we state below and prove at the end of the subsection.

Lemma 3.3 *For any $\delta > 0$ and $\alpha \in (0,1)$, there holds*

$$\sigma \in S^{N-1},\ \sin^2\chi \ge \delta \quad \textit{implies} \quad m^{-1}(v') \le m^{-k}(v)\,m^{-k}(v_*), \tag{3.8}$$

with $k = (1-\delta/160)^{s/2}$.

In order to prove (3.7) we fix $\ell \in L^\infty$ and we argue by duality again. We estimate thanks to Lemma 3.3

$$\begin{aligned}\int_{\mathbb{R}^N} Q^{+,v}_\alpha(\psi,\varphi)\,\ell(v)\,m^{-1}(v)\,dv &= \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} |u|\,b^v\,\psi_*\,\varphi\,\ell'\,(m')^{-1}\,dv\,dv_*\,d\sigma \\ &\le \frac{1}{R}\int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} |u|^2\,b^v\,\psi_*\,\varphi\,\ell'\,(m)^{-k}\,(m_*)^{-k}\,dv\,dv_*\,d\sigma \\ &\le \frac{1}{R}\,\|\ell\|_{L^\infty}\,\|\psi\|_{L^1_2(m^{-k})}\|\varphi\|_{L^1_2(m^{-k})} \\ &\le \frac{1}{R}\,\|\ell\|_{L^\infty}\,\||.|\,m^{1-k}(.)\|^2_{L^\infty}\,\|\psi\|_{L^1_1(m^{-1})}\|\varphi\|_{L^1_1(m^{-1})},\end{aligned}$$

from which we easily conclude since $x \mapsto x\,m^{1-k}(x)$ is uniformly bounded by $C_{a,s}\,(1-k)^{-1/s}$, $C_{a,s} \in (0,\infty)$.

Step 3. The truncated operator. Let us prove that there exists a constant $C \in (0,\infty)$ such that for any $\delta \in (0,1)$, $\alpha, \alpha' \in (0,1]$ and $R \in (1,\infty)$ there holds

$$\|Q^{+,r}_\alpha(g,f) - Q^{+,r}_{\alpha'}(g,f)\|_{L^1(m^{-1})} \le C\,|\alpha-\alpha'|\,\left(\frac{R^2}{\delta}+\frac{R}{\delta^3}\right)\,\|g\|_{L^1(m^{-1})}\,\|f\|_{W^{1,1}(m^{-1})}.$$

We closely follow the proof of [23, Proposition 4.3]. We consider some $\ell \in L^\infty$, $f, g \in \mathcal{D}(\mathbb{R}^N)$, we proceed by duality and next conclude thanks to a density argument. We have

$$\begin{aligned} I &:= \int_{\mathbb{R}^N} [Q^{+,r}_\alpha(g,f) - Q^{+,r}_{\alpha'}(g,f)]\, m^{-1}\,\ell\,dv \\ &= \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} |u|\,\Theta_R(u)\,b_\delta\,g_*\,f\,\left[\ell(v'_\alpha)\,m^{-1}(v'_\alpha) - \ell(v'_{\alpha'})\,m^{-1}(v'_{\alpha'})\right]\,dv\,dv_*\,d\sigma. \end{aligned}$$

With the notations of Lemma 2.2 we perform the changes of variables $v \mapsto v'_\alpha = \phi_\alpha(v)$ and $v \mapsto v'_{\alpha'} = \phi_{\alpha'}(v)$ (for fixed v_* and σ) with jacobians J_α and $J_{\alpha'}$. Observing that without restriction we may assume $\alpha \le \alpha'$ and therefore $\mathcal{O}_\alpha = v_* + \Omega_{\omega_\alpha(\delta)} \subset \mathcal{O}_{\alpha'} = v_* + \Omega_{\omega_{\alpha'}(\delta)}$ since $s \mapsto \omega_s(0)$ is an increasing function, we get

$$\begin{aligned} I &= \int_{\mathbb{R}^N\times\mathbb{S}^{N-1}}\int_{\mathcal{O}_\alpha\setminus\mathcal{O}_{\alpha'}} g_*\,\ell'\,(m^{-1})'\,F(\phi_\alpha^{-1})\,J_\alpha^{-1}\,dv'\,dv_*\,d\sigma \\ &+ \int_{\mathbb{R}^N\times\mathbb{S}^{N-1}}\int_{\mathcal{O}_{\alpha'}} g_*\,\ell'\,(m^{-1})'\,F(\phi_\alpha^{-1})\left[J_\alpha^{-1} - J_{\alpha'}^{-1}\right]dv'\,dv_*\,d\sigma \\ &+ \int_{\mathbb{R}^N\times\mathbb{S}^{N-1}}\int_{\mathcal{O}_{\alpha'}} g_*\,\ell'\,(m^{-1})'\left[F(\phi_\alpha^{-1}) - F(\phi_\alpha^{-1})\right]J_\alpha^{-1}\,dv'\,dv_*\,d\sigma \\ &= I_1 + I_2 + I_3, \end{aligned}$$

with $F(w) := |w-v_*|\,\Theta_R(w-v_*)\,f(w)\,b_\delta(\sigma\cdot\widehat{w-v_*})$. For the first term I_1 we use the backward change of variables $v' \mapsto v = \phi_\alpha^{-1}(v')$ (for fixed v_* and σ) and we get

$$I_1 = \int_{\mathbb{R}^N\times\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |u|\,\Theta_R(u)\,f\,g_*\,\ell(v'_\alpha)\,m^{-1}(v'_\alpha)\,b_\delta\,\mathbf{1}_{0\le\hat{u}\cdot\sigma\le\eta}\,dv_*\,dv\,d\sigma$$

with $\eta := \omega_\alpha^{-1}\circ\omega_{\alpha'}(\delta) \le C\,\delta^{-3/2}\,|\alpha-\alpha'|$ for some constant $C \in (0,\infty)$. Since $v \mapsto |v|^{s/2}$ is an increasing subadditive function, we also have $|v'_\alpha|^s \le (|v|^2+|v_*|^2)^{s/2} \le |v|^s + |v_*|^s$, which implies $m(v'_\alpha) \le C\,m^{-1}\,m_*^{-1}$ for some constant $C \in (0,\infty)$ (depending of ζ). As a consequence, we obtain

$$|I_1| \le C\,R\,\delta^{-3/2}\,|\alpha-\alpha'|\,\|b\|_{L^\infty}\,\|\ell\|_{L^\infty}\,\|f\|_{L^1(m^{-1})}\,\|g\|_{L^1(m^{-1})}.$$

For the term I_2, using the backward change of variable $v' \mapsto v = \phi_{\alpha'}^{-1}(v')$ (for some fixed v_* and σ) and using the bounds (2.4) on J_α and $|J_\alpha^{-1} - J_{\alpha'}^{-1}|$, we obtain

$$|I_1| \le C\,R\,\delta^{-3}\,|\alpha-\alpha'|\,\|b\|_{L^\infty}\,\|\ell\|_{L^\infty}\,\|f\|_{L^1(m^{-1})}\,\|g\|_{L^1(m^{-1})}.$$

In order to estimate I_3, we introduce $\alpha_t := (1-t)\,\alpha + t\,\alpha'$ and, thanks to (2.3)-(2.2), we get

$$|I_3| \le \frac{C}{\delta}\,|\alpha-\alpha'|\,\int_0^1\int_{\mathbb{R}^N\times\mathbb{S}^{N-1}}\int_{\mathcal{O}_{\alpha_t}} |g_*|\,\,|\ell'|\,(m^{-1})'\,|v'-v|\left|\nabla_w F(\phi_{\alpha_t}^{-1}(v'))\right|dv'dv_*d\sigma dt.$$

Using finally the backward change of variable $v' \mapsto v = \phi^{-1}_{\alpha_t}(v')$ and the uniform bound (2.4) on J_{α_t}, $t \in [0,1]$, on $v_* + \Omega_\delta$, we get

$$|I_3| \le C \left(\frac{R^2}{\delta} + \frac{R}{\delta^2} \right) |\alpha - \alpha'| \, \|b\|_{W^{1,\infty}} \, \|\ell\|_{L^\infty} \, \|g\|_{L^1(m^{-1})} \, \|f\|_{W^{1,1}(m^{-1})}.$$

Gathering the estimates established in Steps 1, 2 and 3, we deduce the first inequality in (3.6). The second inequality in (3.6) is proved in a similar way (using symmetric changes of variable, allowed by the truncation). □

Proof of Lemma 3.3. We proceed in three steps.
Step 1. Assume first that $(2/\sqrt{5})\,|v_*| \le |v| \le (\sqrt{5}/2)\,|v_*|$. Using the fact that $x \mapsto x^{s/2}$ is an increasing and subadditive function, there holds

$$|v'|^s \le (|v|^2 + |v_*|^2)^{s/2} \le (9/4)^{s/2}\, |v_*|^s,$$

and then by symmetry and because $s \le 1$

$$|v'|^s \le \frac{1}{2}\,(9/4)^{s/2}\,(|v|^s + |v_*|^s) \le \frac{3}{4}\,(|v|^s + |v_*|^s).$$

In that case, (3.8) holds with $k = 3/4$.
Step 2. We shall first show that for any $v, v_* \in \mathbb{R}^N$ and $\sigma \in S^{N-1}$, there holds

$$|v'|^2,\ |v'_*|^2 \le |v|^2 + |v_*|^2 - \frac{1+\alpha}{8}\,\sin^2\chi\,|v+v_*|^2. \tag{3.9}$$

We recall the formula

$$v' := \frac{v+v_*}{2} + \frac{1}{2}\left[\frac{1-\alpha}{2}\,u + \frac{1+\alpha}{2}\,|u|\,\sigma\right], \qquad v'_* := \frac{v+v_*}{2} + \frac{1}{2}\left[\frac{1-\alpha}{2}\,u - \frac{1+\alpha}{2}\,|u|\,\sigma\right].$$

Straightforward computations yield (denoting $S = v + v_*$)

$$|v'|^2 \le \frac{|S|^2}{4} + \frac{1}{4}\left[\frac{1+\alpha^2}{2}\,|u|^2 + \frac{1-\alpha^2}{2}\,|u|^2\,\cos\theta\right] + \frac{1-\alpha}{4}\,(S\cdot u) + \frac{1+\alpha}{4}\,|S|\,|u|\,\cos\chi.$$

We deduce the bound from above

$$|v'|^2 \le \frac{|S|^2}{4} + \frac{|u|^2}{4} + \frac{1-\alpha}{4}\,|S|\,|u| + \frac{1+\alpha}{4}\,|S|\,|u|\,\cos\chi.$$

Then by applying twice Young's inequality

$$\begin{aligned}
|v'|^2 &\le |S|^2\left(\frac{1}{4} + \frac{1-\alpha}{8}\right) + |u|^2\left(\frac{1}{4} + \frac{1-\alpha}{8}\right) + \frac{1+\alpha}{4}\,|S|\,|u|\,\cos\chi,\\
&\le |S|^2\left(\frac{1}{4} + \frac{1-\alpha}{8}\right) + |u|^2\left(\frac{1}{4} + \frac{1-\alpha}{8} + \frac{1+\alpha}{8}\right) + \frac{1+\alpha}{8}\,|S|^2\,\cos^2\chi,\\
&\le \frac{|S|^2}{2} + \frac{|u|^2}{2} + \frac{1+\alpha}{8}\,|S|^2\,(\cos^2\chi - 1),
\end{aligned}$$

from which we deduce (3.9).

Step 3. Assume that $\sin^2\chi \geq \delta$ and that either $(2/\sqrt{5})\,|v_*| \geq |v|$ or $|v| \geq (\sqrt{5}/2)\,|v_*|$. In the first case, we have

$$|v+v_*| \geq (1-(2/\sqrt{5}))\,|v_*| + (2/\sqrt{5})\,|v_*| - |v| \geq (1-(2/\sqrt{5}))\,|v_*|,$$

which then implies

$$|v+v_*| \geq (1-(2/\sqrt{5}))\,(\sqrt{5}/2)\,|v| \geq (1-(2/\sqrt{5}))\,|v|.$$

The same inequalities are proved in a similar way in the second case. We deduce

$$|v+v_*|^2 \geq \frac{1}{2}(1-(2/\sqrt{5}))\,(|v|^2+|v_*|^2).$$

We then deduce from (3.9) that $|v'|^2 \leq (1-\delta/160)\,(|v|^2+|v_*|^2)$ and we conclude that (3.8) holds as in Step 1. □

3.2 Quantification of the elastic limit $\alpha \to 1$

We begin with a simple consequence of Proposition 3.1.

Corollary 3.4 *There exists $k_0, q_0 \in \mathbb{N}$ such that for any $a_i \in (0,\infty)$ $i = 1, 2, 3$, there exists an explicit constant $C \in (0,\infty)$ such that for any function g satisfying*

$$\|g\|_{H^{k_0}\cap L^1_{q_0}} \leq a_1, \quad g \geq a_2\, e^{-a_3\,|v|^8},$$

there holds

$$|D_{H,\alpha}(g) - D_{H,1}(g)| \leq C\,(1-\alpha),$$

where we recall that $D_{H,\alpha}$ is defined in (2.18).

Proof of Corollary 3.4. We write

$$\begin{aligned} D_{H,\alpha}(g) - D_{H,1}(g) &= \iiint b\,|u|\,\Big[g'_\alpha\, g'_{*\alpha} - g'\, g'_*\Big]\,dv\,dv_*\,d\sigma \quad (=: I_1) \\ &+ \iiint b\,|u|\,g\,g_*\,[\log g'_\alpha + \log g'_{*\alpha} - \log g' - \log g'_*]\,dv\,dv_*\,d\sigma \quad (=: I_2). \end{aligned}$$

For the first term, thanks to Proposition 3.1, we have

$$|I_1| \;\leq\; \|Q^+_\alpha(g,g) - Q^+_1(g,g)\|_{L^1} \leq C\,(1-\alpha)\,\|g\|^2_{W^{3,1}_3}.$$

For the second term, we write

$$\begin{aligned} |I_2| &= 2\left|\langle (Q^+_\alpha(g,g) - Q^+_1(g,g))\,\langle v\rangle^8, \langle v\rangle^{-8}\,\log g\rangle\right| \\ &\leq 2\,\|Q^+_\alpha(g,g) - Q^+_1(g,g)\|_{L^1_8}\,\|\langle v\rangle^{-8}\,\log g\|_{L^\infty} \\ &\leq C\,(1-\alpha)\,\|g\|^2_{W^{3,1}_{11}}\,(|\log\,\|g\|_{L^\infty}| + |\log a_2| + a_3) \\ &\leq C\,(1-\alpha)\,a_1^2\,(|\log a_1| + |\log a_2| + C\,a_3), \end{aligned}$$

thanks to Proposition 3.1 and the bounded embedding $H^{k_0} \cap L^1_{q_0} \subset L^\infty \cap W^{3,1}_{11}$ for k_0, q_0 large enough (see Proposition B.1). We conclude the proof gathering these two estimates. □

Let us recall now two famous inequalities, namely the Csiszár-Kullback-Pinsker inequality (see [14, 22]) and the so-called entropy-entropy production inequalities (the version we present here is established in [31]) that we will use several time in the sequel.

Theorem 3.5 *(i) For a given function $g \in L^1_2$, let us denote by $M[g]$ the Maxwellian function with the same mass, momentum and temperature as g. For any $0 \le g \in L^1_2(\mathbb{R}^N)$, there holds*

$$\|g - M[g]\|^2_{L^1} \le 2\,\rho(g) \int_{\mathbb{R}^N} g \ln \frac{g}{M[g]}\,dv. \tag{3.10}$$

(ii) For any $\varepsilon > 0$ there exists k_ε, $q_\varepsilon \in \mathbb{N}$ and for any $A \in (0,\infty)$ there exists $C_\varepsilon = C_{\varepsilon,A} \in (0,\infty)$ such that for any $g \in H^{k_\varepsilon} \cap L^1_{q_\varepsilon}$ such that

$$g(v) \ge A^{-1}\, e^{-A\,|v|^8}, \quad \|g\|_{H^{k_\varepsilon} \cap L^1_{q_\varepsilon}} \le A,$$

there holds

$$C_\varepsilon\,\rho(g)^{1-\varepsilon} \left(\int_{\mathbb{R}^N} g \ln \frac{g}{M[g]}\,dv \right)^{1+\varepsilon} \le D_{H,1}(g). \tag{3.11}$$

We have then the following estimate on the distance between G_α and $\bar{G}_1$ for any self-similar profile G_α.

Proposition 3.6 *For any $\varepsilon > 0$ there exists C_ε (independent of the mass ρ) such that*

$$\forall\,\alpha \in [\alpha_0, 1) \qquad \sup_{G_\alpha \in \mathcal{G}_\alpha} \|G_\alpha - \bar{G}_1\|_{L^1_2} \le C_\varepsilon\,\rho\,(1-\alpha)^{\frac{1}{2+\varepsilon}} \tag{3.12}$$

where we recall that $\bar{G}_1$ is the Maxwellian function defined by (1.25)–(1.27).

Proof of Proposition 3.6. On the one hand, for any inelasticity coefficient $\alpha \in [\alpha_0, 1)$ and profile G_α, there holds from (2.19) together with Corollary 3.4 and the uniform estimates of Proposition 2.1

$$D_{H,1}(G_\alpha) \le D_{H,\alpha}(G_\alpha) + \rho^2\,\mathcal{O}(1-\alpha) \le \rho^2\,\mathcal{O}(1-\alpha). \tag{3.13}$$

On the other hand, introducing the Maxwellian function M_θ with the same mass, momentum and temperature as G_α, that is M_θ given by (1.26) with $u = 0$ and $\theta = \mathcal{E}(G_\alpha)/\rho$, and gathering (3.13), (3.11), (3.10) with the uniform estimates of Proposition 2.1 and interpolation inequality, we obtain that for any $q, \varepsilon > 0$ there exists $C_{q,\varepsilon}$ such that

$$\forall \alpha \in [\alpha_0, 1) \qquad \|G_\alpha - M_\theta\|^{2+\varepsilon}_{L^1_q} \le C_{q,\varepsilon}\,\rho^{2+\varepsilon}\,(1-\alpha). \tag{3.14}$$

Next, from (2.16), we have

$$b_1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G_\alpha\, G_{\alpha*}\,|u|^3\,dv\,dv_* - \rho \int_{\mathbb{R}^N} G_\alpha\,|v|^2\,dv = (1-\alpha)\,\frac{b_1}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G_\alpha\, G_{\alpha*}\,|u|^3\,dv\,dv_*$$

and then

$$|\Psi(\theta)| \le C_1 \, \|G_\alpha - M_\theta\|_{L^1_3} + C_2 \, \rho^2 \, (1-\alpha), \tag{3.15}$$

where we have used that G_α and M_θ are bounded thanks to Proposition 2.1 and we have defined

$$\Psi(\theta) = \rho \int_{\mathbb{R}^N} M_\theta \, |v|^2 \, dv - b_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M_\theta \, M_{\theta *} \, |u|^3 \, dv \, dv_*. \tag{3.16}$$

By elementary changes of variables, this formula simplifies into

$$\Psi(\theta) = k_1 \, \theta - k_2 \, \theta^{3/2}$$

with $k_1 = \rho^2 \, N$ and, using (A.3),

$$k_2 = \rho^2 \, b_1 \int_{\mathbb{R}^N \times \mathbb{R}^N} M_{1,0,1} \, (M_{1,0,1})_* \, |u|^3 \, dv \, dv_* = 2^{3/2} \, \rho^2 \, b_1 \, \mathbf{m}_{3/2}(M_{1,0,1}).$$

We next observe that $\Psi \in C^\infty(0,\infty)$ and Ψ is strictly concave. It is also obvious that the equation $\Psi(\theta) = 0$ for $\theta > 0$ has a unique solution which is $\bar\theta_1$ defined in (1.27), and that we have

$$\Psi(\theta) \le \Psi'(\bar\theta_1) \, (\theta - \bar\theta_1) = -k_1 \, (\theta - \bar\theta_1)/2$$

as well as

$$\Psi(\theta) = \theta \, [k_1 - k_2 \, \theta^{1/2}] = k_2 \, \theta \, [\bar\theta_1^{1/2} - \theta^{1/2}]. \tag{3.17}$$

Plugging this expression for Ψ into (3.15) and using the lower bound (2.17) on the temperature θ and the estimate (3.14) we obtain that for any $\varepsilon > 0$ there is $C_\varepsilon \in (0,\infty)$ such that

$$\forall \, \alpha \in (\alpha_0, 1) \qquad \left| \theta^{1/2} - \bar\theta_1^{1/2} \right|^{2+\varepsilon} \le C_\varepsilon \, (1-\alpha). \tag{3.18}$$

Namely, we have thus proved that the temperature of $\bar G_\alpha$ converge (with rate) to the expected temperature $\bar\theta_1$. In order to come back to the norm of $G_\alpha - \bar G_1$, we first write, using Cauchy-Schwarz's inequality,

$$\begin{aligned} \|G_\alpha - \bar G_1\|_{L^1_{-N}} &\le \|G_\alpha - M_\theta\|_{L^1_{-N}} + \|M_\theta - \bar G_1\|_{L^1_{-N}} \\ &\le \|G_\alpha - M_\theta\|_{L^1} + C_N \, \|M_\theta - \bar G_1\|_{L^2}, \end{aligned} \tag{3.19}$$

and we remark that

$$\|M_\theta - \bar G_1\|^2_{L^2} \le C \, \rho^2 \, |\theta^{1/2} - \bar\theta_1^{1/2}|. \tag{3.20}$$

Gathering (3.19) with (3.20), (3.18) and (3.14) we deduce that for any $\varepsilon > 0$ there is $C_\varepsilon \in (0,\infty)$ such that

$$\forall \, \alpha \in (\alpha_0, 1) \qquad \|G_\alpha - \bar G_1\|^{2+\varepsilon}_{L^1_{-N}} \le C_\varepsilon \, \rho^{2+\varepsilon} \, (1-\alpha),$$

and (3.12) follows by interpolation again. ☐

4 Uniqueness and continuity of the path of self-similar profiles

4.1 The proof of uniqueness

Theorem 4.1 *There exists a constructive $\alpha_1 \in (0,1)$ such that the solution G_α of (1.30) is unique for any $\alpha \in [\alpha_1, 1]$. We denote by $\bar G_\alpha$ this unique self-similar profile.*

That is an immediate consequence of the following result.

Proposition 4.2 *There is a constructive constant $\eta \in (0,1)$ such that*

$$\left. \begin{array}{l} G,\ H \in \mathcal{G}_\alpha,\ \ \alpha \in (1-\eta, 1) \\ \|G - \bar G_1\|_{L^1_2} \le \eta,\ \ \|H - \bar G_1\|_{L^1_2} \le \eta \end{array} \right\} \quad \textit{implies} \quad G = H.$$

Proof of Theorem 4.1. Let us assume that Proposition 4.2 holds. Then Proposition 3.6 implies that there is some explicit $\varepsilon \in (0,1)$ such that for $\alpha \in (1-\varepsilon, 1]$ one has

$$\sup_{G_\alpha \in \mathcal{G}_\alpha} \|G_\alpha - \bar G_1\|_{L^1_2} \le \eta$$

where η is defined in the statement of Proposition 4.2. Up to reducing η, it is always possible to take $\eta \le \varepsilon$, and the proof is completed by applying Proposition 4.2. □

Proof of Proposition 4.2. Let us consider any exponential weight function m with $s \in (0,1)$, $a \in (0,+\infty)$, or with $s = 1$ and $a \in (0,\infty)$ small enough. Let us also define $\mathcal{O} = \mathcal{C}_{0,0,0} \cap \mathbb{L}^1(m^{-1})$ the subvector space of of $\mathbb{L}^1(m^{-1})$ of functions with zero energy, $\psi = C\,(|v|^2 - N)\,M_{1,0,1}$ such that $\mathcal{E}(\psi) = 1$, and Π the following projection

$$\Pi : \mathbb{L}^1(m^{-1}) \to \mathcal{O}, \qquad \Pi(g) = g - \mathcal{E}(g)\,\psi.$$

Finally, let us introduce Φ the following non-linear functional operator

$$\Phi : [0,1) \times (W^{1,1}_1(m^{-1}) \cap \mathcal{C}_{\rho,0}) \ \to\ \mathbb{R} \times \mathcal{O},$$

and

$$\Phi(1,\cdot) : (L^1_1(m^{-1}) \cap \mathcal{C}_{\rho,0}) \ \to\ \mathbb{R} \times \mathcal{O},$$

by setting

$$\Phi(\alpha, g) = \Big((1+\alpha)\,D_{\mathcal{E}}(g) - 2\,\rho\,\mathcal{E}(g),\ \Pi\Big[Q_\alpha(g,g) - \tau_\alpha\,\mathrm{div}_v(v\,g)\Big]\Big).$$

It is straightforward that $\Phi(\alpha, G_\alpha) = 0$ for any $\alpha \in [\alpha_0, 1]$ and $G_\alpha \in \mathcal{G}_\alpha$, and that the equation

$$\Phi(1, g) = (0,0)$$

has a unique solution, given by $g = \bar G_1 = M_{\rho,0,\bar\theta_1}$ defined in (1.25), (1.27).

The function Φ is linear and quadratic in its second argument by inspection, and easy computations yield the following formal differential according to the second argument at the point $(1, \bar G_1)$:

$$D_2\Phi(1, \bar G_1)\,h = A\,h := \left(4\,\tilde D_{\mathcal{E}}(\bar G_1, h) - 2\rho\,\mathcal{E}(h),\ 2\,\tilde Q_1(\bar G_1, h) \right) \tag{4.1}$$

where $\tilde{Q}_\alpha$ is defined in (1.8) and

$$\tilde{D}_{\mathcal{E}}(g,h) := b_1 \iint_{\mathbb{R}^N\times\mathbb{R}^N} g\, h_*\, |u|^3\, dv\, dv_*.$$

Notice that we can remove the projection on the last argument in (4.1) since the elastic collision operator always has zero energy.

Then we have the

Lemma 4.3 *The linear functional*

$$\begin{array}{rcl} A \;:\; \mathbb{L}^1_1(m^{-1}) & \to & \mathbb{R}\times\mathcal{O} \\ h & \mapsto & A\,h = D_2\Phi(1,\bar{G}_1)\,h \end{array}$$

is invertible: it is bijective with A^{-1} bounded with explicit estimate.

Proof of Lemma 4.3. Since the spectrum of the linear operator $\mathcal{L}_1$ defined on $L^1(m^{-1})$ (with domain $L^1_1(m^{-1})$) includes 0 as a discrete eigenvalue associated with the eigenspace $\mathrm{Ker}\mathcal{L}_1 = \mathrm{Span}\{\bar{G}_1, v_1\,\bar{G}_1, \dots, v_N\,\bar{G}_1, |v|^2\,\bar{G}_1\}$ by [27, Theorem 1.3] and since moreover $\mathcal{O}\cap\mathrm{Ker}\mathcal{L}_1 = \{0\}$, we deduce that it is invertible from $\mathcal{O}\cap\mathbb{L}^1_1(m^{-1})$ onto $\mathcal{O}$. Moreover the work [27, Section 4] provides explicit estimates on the norm of its inverse. We deduce immediately that $\mathcal{L}_1^{-1}$ maps $\mathcal{O}$ onto itself with explicit bound.

For any $h\in\mathbb{L}^1(m^{-1})$, we decompose

$$h = h_1\,\phi_1 + h^\perp, \quad \text{with} \quad h_1 := \frac{\mathcal{E}(h)}{\mathcal{E}(\phi_1)} \in \mathbb{R},\; h^\perp\in\mathcal{O},$$

where we recall that ϕ_1 is defined in (1.33). Then, using the characterization (1.30) of $\bar{G}_1$,

$$A\,h = \Bigg(b_1 \int_{\mathbb{R}^N\times\mathbb{R}^N} \bar{G}_1(v)\,h^\perp(v_*)\,|u|^3\,dv\,dv_* \\ + h_1 \left[b_1 \int_{\mathbb{R}^N\times\mathbb{R}^N} |v|^2\,\bar{G}_1\,\bar{G}_{1*}\,|u|^3\,dv\,dv_* - 2\rho \int_{\mathbb{R}^N} \bar{G}_1\,|v|^4\,dv \right],\; \mathcal{L}_1(h^\perp) \Bigg).$$

The claimed invertibility follows from the fact that $C^* = 2\,N\,\rho^2\,\bar{\theta}_1^2 \neq 0$. Indeed, from (A.2) and (A.4) there holds

$$\begin{aligned} C^* &:= b_1 \int_{\mathbb{R}^N\times\mathbb{R}^N} |v|^2\,\bar{G}_1\,\bar{G}_{1*}\,|u|^3\,dv\,dv_* - 2\rho\int_{\mathbb{R}^N} \bar{G}_1\,|v|^4\,dv \\ &= \rho^2\,\bar{\theta}_1^2 \left[b_1\,\bar{\theta}_1^{1/2} \int_{\mathbb{R}^N\times\mathbb{R}^N} M_{1,0,1}\,(M_{1,0,1})_*\,|v|^2\,|u|^3\,dv\,dv_* - 2\int_{\mathbb{R}^N} M_{1,0,1}\,|v|^4\,dv \right] \\ &= \rho^2\,\bar{\theta}_1^2 \left[b_1\,\bar{\theta}_1^{1/2}\,\sqrt{2}\,(2N+3)\,\mathbf{m}_{3/2}(M_{1,0,1}) - 2\,N(N+2) \right], \end{aligned}$$

and we conclude thanks to formula (1.27). □

Let us come back to the proof of Proposition 4.2. We write

$$\begin{aligned} G_\alpha - H_\alpha &= A^{-1}\Big[A\,G_\alpha - \Phi(\alpha,G_\alpha) + \Phi(\alpha,H_\alpha) - A\,H_\alpha\Big] \\ &= A^{-1}\,(I_1,I_2) \end{aligned} \tag{4.2}$$

with (recall that the bilinear operators $\tilde D_{\mathcal E}$ and $\tilde Q_\alpha$ are symmetric)

$$\begin{cases} I_1 := 4\,\tilde D_{\mathcal E}(\bar G_1, G_\alpha - H_\alpha) - (1+\alpha)\,D(G_\alpha) + (1+\alpha)\,D(H_\alpha) \\ I_2 := \Pi\, I_{2,1} + \Pi\, I_{2,2} \end{cases}$$

and

$$\begin{cases} I_{2,1} := 2\,\tilde Q_1(\bar G_1, G_\alpha - H_\alpha) - Q_\alpha(G_\alpha,G_\alpha) + Q_\alpha(H_\alpha,H_\alpha) \\ I_{2,2} := \rho\,(1-\alpha)\,\nabla_v\cdot(v\,(H_\alpha - G_\alpha)). \end{cases}$$

On the one hand,

$$I_1 = 2\,D(2\bar G_1 - (G_\alpha+H_\alpha), G_\alpha - H_\alpha) + (1-\alpha)\,D(G_\alpha+H_\alpha, G_\alpha - H_\alpha)$$

so that

$$\begin{aligned} |I_1| &\le C_3\Big(\|\bar G_1 - G_\alpha\|_{L^1_3} + \|\bar G_1 - H_\alpha\|_{L^1_3} \\ &\qquad\qquad + (1-\alpha)\,\|G_\alpha\|_{L^1_3} + (1-\alpha)\,\|H_\alpha\|_{L^1_3}\Big)\,\|G_\alpha - H_\alpha\|_{L^1_3} \\ &\le \eta_1(\alpha)\,\|G_\alpha - H_\alpha\|_{L^1_1(m^{-1})} \end{aligned} \tag{4.3}$$

with $\eta_1(\alpha)\to 0$ when $\alpha\to 1$ (with explicit rate, for instance $\eta_1(\alpha) = C_1\,(1-\alpha)^{1/3}$) because of Propositions 2.1 and 3.6.

On the other hand,

$$\begin{aligned} I_{2,1} &= Q_1(\bar G_1, G_\alpha - H_\alpha) - Q_\alpha(\bar G_1, G_\alpha - H_\alpha) + Q_1(G_\alpha - H_\alpha, \bar G_1) - Q_\alpha(G_\alpha - H_\alpha, \bar G_1) \\ &\qquad + Q_\alpha(\bar G_1 - G_\alpha, G_\alpha - H_\alpha) + Q_\alpha(G_\alpha - H_\alpha, \bar G_1 - H_\alpha). \end{aligned}$$

From Proposition 3.2 there holds

$$\|Q_1(\bar G_1, G_\alpha - H_\alpha) - Q_\alpha(\bar G_1, G_\alpha - H_\alpha)\|_{L^1(m^{-1})} \le \varepsilon(\alpha)\,\|G_\alpha - H_\alpha\|_{L^1_1(m^{-1})}$$

$$\|Q_1(G_\alpha - H_\alpha, \bar G_1) - Q_\alpha(G_\alpha - H_\alpha, \bar G_1)\|_{L^1(m^{-1})} \le \varepsilon(\alpha)\,\|G_\alpha - H_\alpha\|_{L^1_1(m^{-1})}$$

with $\varepsilon(\alpha)\to 0$ as $\alpha\to 1$ (with again explicit rate, for instance $\varepsilon(\alpha) = C_1'\,(1-\alpha)^{1/12}$ if $s = 1/2$ in the formula of m). From elementary estimates in $L^1(m^{-1})$ we have

$$\begin{aligned} &\|Q_\alpha(\bar G_1 - G_\alpha, G_\alpha - H_\alpha) + Q_\alpha(G_\alpha - H_\alpha, \bar G_1 - H_\alpha)\|_{L^1(m^{-1})} \\ &\qquad \le C_4\,\Big(\|G_\alpha - \bar G_1\|_{L^1_1(m^{-1})} + \|H_\alpha - \bar G_1\|_{L^1_1(m^{-1})}\Big)\,\|G_\alpha - H_\alpha\|_{L^1_1(m^{-1})}. \end{aligned}$$

Together with Propositions 3.6 we thus obtain

$$\|I_{2,1}\|_{L^1(m^{-1})} \le \eta_2(\alpha)\,\|G_\alpha - H_\alpha\|_{L^1_1(m^{-1})} \tag{4.4}$$

for some $\eta_2(\alpha) \to 0$ as $\alpha \to 1$. Here we can take for instance (when $s = 1/2$ in the formula of m) $\eta_2(\alpha) = C_2\,(1-\alpha)^{1/12}$ for some $C_2 \in (0,\infty)$ by picking a suitable ε and interpolating.

Finally from Proposition 2.7 there holds

$$\|I_{2,2}\|_{L^1(m^{-1})} \le C_5\,(1-\alpha)\,\|G_\alpha - H_\alpha\|_{L^1_1(m^{-1})}. \tag{4.5}$$

Gathering (4.3), (4.4) and (4.5) we obtain from (4.2) and Lemma 4.3

$$\|G_\alpha - H_\alpha\|_{L^1_1(m^{-1})} \le \eta(\alpha)\,\|A^{-1}\|\,\|G_\alpha - H_\alpha\|_{L^1_1(m^{-1})}$$

for some function η such that $\eta(\alpha) \to 0$ as $\alpha \to 1$ (with explicit rate). Hence choosing α_1 close enough to 1 we have $\eta(\alpha)\,\|A^{-1}\| \le 1/2$ for any $\alpha \in [\alpha_1, 1)$. This implies $G_\alpha = H_\alpha$ and concludes the proof. □

4.2 Differentiability of the map $\alpha \mapsto \bar G_\alpha$ at $\alpha = 1$

Lemma 4.4 *The map $[\alpha_1, 1] \to L^1(m^{-1})$, $\alpha \mapsto \bar G_\alpha$ is continuous on $[\alpha_1, 1]$ and differentiable at $\alpha = 1$. More precisely, there exists $\bar G_1' \in L^1(m^{-1})$ and for any $\eta \in (1,2)$ there exists a constructive $C_\eta \in (0,\infty)$ such that*

$$\|\bar G_\alpha - \bar G_1 - (1-\alpha)\,\bar G_1'\|_{L^1(m^{-1})} \le C_\eta\,(1-\alpha)^\eta \qquad \forall\, \alpha \in (\alpha_0, 1). \tag{4.6}$$

Proof of Lemma 4.4. We split the proof into four steps.

Step 1. For the continuity we use a classical stability argument. Let us consider a sequence $(\alpha_n)_{n\ge 0}$ such that $\alpha_n \in [\alpha_1, 1]$ and $\alpha_n \to \alpha$. From the uniform bound (2.1), we may extract a subsequence $(\bar G_{\alpha_{n'}})$ which strongly converges in $L^1(m^{-1})$ to a function G_α. Passing to the limit in the equations (1.30) associated to the normal restitution coefficient α_n and written for $G_{\alpha_{n'}}$, we deduce that G_α satisfies (1.30) associated to the normal restitution coefficient α. From the uniqueness of the solution proved in Theorem 4.1, there holds $G_\alpha = \bar G_\alpha$ and thus the whole sequence $\bar G_{\alpha_n}$ converges to $\bar G_\alpha$.

Step 2. We next prove that there exists an explicit constant C such that

$$\forall\, \alpha \in [\alpha_1, 1] \qquad \|\bar G_\alpha - \bar G_1\|_{L^1(m^{-1})} \le C\,(1-\alpha).$$

We write

$$\begin{aligned} \bar G_\alpha - \bar G_1 &= A^{-1}\,[A\,\bar G_\alpha - \Phi(\alpha, \bar G_\alpha) + \Phi(1, \bar G_1) - A\,\bar G_1] \\ &= A^{-1}\,(J_1, J_2) \end{aligned} \tag{4.7}$$

with

$$\begin{cases} J_1 := 4\,\tilde D_{\mathcal E}(\bar G_1, \bar G_\alpha - \bar G_1) + 2\,\tilde D_{\mathcal E}(\bar G_1, \bar G_1) - (1+\alpha)\,\tilde D_{\mathcal E}(\bar G_\alpha, \bar G_\alpha) \\ J_2 := \Pi\, J_{2,1} + \Pi\, J_{2,2} \end{cases}$$

and

$$\begin{cases} J_{2,1} := Q_1(\bar G_1, \bar G_\alpha) + Q_1(\bar G_\alpha, \bar G_1) - Q_\alpha(\bar G_\alpha, \bar G_\alpha) \\ J_{2,2} := \rho\,(1-\alpha)\,\nabla_v \cdot (v\,(\bar G_\alpha)). \end{cases}$$

On the one hand,

$$J_1 = -2\,\tilde{D}_{\mathcal{E}}(\bar{G}_1 - \bar{G}_\alpha, \bar{G}_1 - \bar{G}_\alpha) + (1-\alpha)\,D(\bar{G}_\alpha, \bar{G}_\alpha)$$

so that

$$|J_1| \le C\,\|\bar{G}_1 - \bar{G}_\alpha\|^2_{L^1_3} + C\,(1-\alpha).$$

On the other hand,

$$J_{2,1} \quad = \quad -Q_1(\bar{G}_1 - \bar{G}_\alpha, \bar{G}_1 - \bar{G}_\alpha) + Q_1(\bar{G}_\alpha, \bar{G}_\alpha) - Q_\alpha(\bar{G}_\alpha, \bar{G}_\alpha).$$

Hence using Propositions 2.7, 3.1, and the bound (2.1), we deduce

$$|J_{2,1}| \le \left\|\bar{G}_\alpha - \bar{G}_1\right\|^2_{L^1_2} + C\,(1-\alpha)$$

and we also have straightforwardly $J_{2,2} = \mathcal{O}(1-\alpha)$. Gathering all these estimates, we thus obtain from (4.7)

$$\|\bar{G}_\alpha - \bar{G}_1\|_{L^1_1(m^{-1})} \le \|A^{-1}\|\,\Big[\|\bar{G}_\alpha - \bar{G}_1\|^2_{L^1_1(m^{-1})} + C\,(1-\alpha)\Big].$$

Using then the explicit result of quantification of the elastic limit in Proposition 3.6, we have that for some $\alpha_2 \in [\alpha_1, 1)$ close enough to 1:

$$\forall\,\alpha \in [\alpha_2, 1] \qquad \|A^{-1}\|\,\|\bar{G}_\alpha - \bar{G}_1\|_{L^1_1(m^{-1})} < \frac{1}{2}$$

and thus we get

$$\forall\,\alpha \in [\alpha_2, 1], \quad \|\bar{G}_\alpha - \bar{G}_1\|_{L^1_1(m^{-1})} \le 2\,C\,\|A^{-1}\|\,(1-\alpha)$$

which implies the claimed estimate.

Step 3. In order to prove the differentiability we must slightly improve the estimate established in the preceding step. On the one hand we exhibit what should be the derivative of $\bar{G}_\alpha$ at $\alpha = 1$, and denote it by R. Formally differentiating equation (1.30) at $\alpha = 1$ we have

$$Q_1'(\bar{G}_1, \bar{G}_1) + 2\tilde{Q}_1(R, \bar{G}_1) + \rho\,\nabla_v \cdot (v\,\bar{G}_1) = 0.$$

On the other hand, we may compute

$$\begin{aligned} \langle Q_\alpha'(\bar{G}_1, \bar{G}_1), |.|^2\rangle \quad &= \quad \frac{1}{4}\iiint_{\mathbb{R}^N \times \mathbb{R}^N \times S^{N-1}} b\,|u|\,\bar{G}_1\,\bar{G}_{1*}\,(|u|\,\sigma - u)\cdot\Big(|u|\,\sigma\Big)\,dv\,dv_*\,d\sigma \\ &= \quad 2\,D_{\mathcal{E}}(\bar{G}_1). \end{aligned} \tag{4.8}$$

Next, diving the equation (1.23) on the energy of G_α by $(1-\alpha)$ and formally differentiating the resulting expression we get

$$2\,\rho\,\mathcal{E}(R) - \tilde{D}_{\mathcal{E}}(\bar{G}_1, \bar{G}_1) - 4\,\tilde{D}_{\mathcal{E}}(R, \bar{G}_1) = 0.$$

We now rigorously define R in the following way

$$\bar{G}_1' = R := A^{-1}\Big(-\tilde{D}_{\mathcal{E}}(\bar{G}_1, \bar{G}_1), -F\Big), \qquad F := Q_\alpha'(\bar{G}_1, \bar{G}_1) + \rho\,\nabla_v \cdot (v\,\bar{G}_1).$$

Note that R is well-defined since $\mathcal{E}(F) = 0$ because of (4.8) and the definition of $\bar G_1$.

Step 4. We finally come back to the Step 2 and we shall construct a Taylor expansion of order 1. We want to estimate

$$\bar G_\alpha - \bar G_1 + (\alpha - 1)\, \bar G_1' = A^{-1} \left(J_1 - (\alpha-1)\, \tilde D_{\mathcal{E}}(\bar G_1, \bar G_1), J_2 - (1-\alpha)\, F \right).$$

On the one hand

$$J_1 - (\alpha - 1)\, \tilde D_{\mathcal{E}}(\bar G_1, \bar G_1) = -2\, \tilde D_{\mathcal{E}}(\bar G_1 - \bar G_\alpha, \bar G_1 - \bar G_\alpha) + (1-\alpha) \left(D(\bar G_\alpha, \bar G_\alpha) - \tilde D_{\mathcal{E}}(\bar G_1, \bar G_1) \right)$$

so that we obtain straightforwardly

$$|J_1 - (\alpha-1)\, \tilde D_{\mathcal{E}}(\bar G_1, \bar G_1)| \le C\, (1-\alpha)^2.$$

On the other hand,

$$J_2 - (1-\alpha)\, F := \Pi\, J_{2,1} + \Pi\, J_{2,2}$$

with

$$\begin{aligned} J_{2,1} \;\; = \;\; & -Q_1(\bar G_1 - \bar G_\alpha, \bar G_1 - \bar G_\alpha) + Q_1(\bar G_\alpha - \bar G_1, \bar G_\alpha) - Q_\alpha(\bar G_\alpha - \bar G_1, \bar G_\alpha) \\ & + Q_1(\bar G_1, \bar G_\alpha - \bar G_1) - Q_\alpha(\bar G_1, \bar G_\alpha - \bar G_1) + (1-\alpha)\, \nabla_v \cdot (v\, (\bar G_\alpha - \bar G_1)) \end{aligned}$$

and

$$J_{2,2} = Q_1(\bar G_1, \bar G_1) - Q_\alpha(\bar G_1, \bar G_1) - (1-\alpha)\, K.$$

It is clear from Propositions 3.1, the bound of Step 2, and some interpolation with the uniform bounds (2.1), that

$$\|J_{2,1}\|_{L^1(m^{-1})},\ \|J_{2,2}\|_{L^1(m^{-1})} \le C_k\, (1-\alpha)^k$$

for any $k \in (1,2)$. $\square$

5 Study of the spectrum and semigroup of the linearized problem

In this section we shall obtain the geometry of the spectrum of the linearized rescaled inelastic collision operator for a small inelasticity, as well as estimates on its resolvent and on the associated linear semigroup. This is based on the properties of the elastic linearized operator and some perturbation arguments again. In order to do so, one needs some common functional "ground" for the the linearized operators in the limit of vanishing inelasticity. This common functional setting is given by the study [27] in which the spectral study of the elastic linearized operator is made in L^1 spaces with exponential weights $e^{a\,|v|^s}$, $a \in (0, +\infty)$, $s \in (0,1)$.

We thus consider the operator

$$g \mapsto Q_\alpha(g,g) - \tau_\alpha\, \nabla_v \cdot (v\, g)$$

and some fluctuations h around the self-similar profile $\bar G_\alpha$: that means $g = \bar G_\alpha + h$ with $h \in L^1(m^{-1})$ where m is a fixed smooth exponential weight function, as defined in (1.28).

The corresponding linearized unbounded operator $\mathcal{L}_\alpha$ acting on $L^1(m^{-1})$ with domain $\mathrm{dom}(\mathcal{L}_\alpha) = W^{1,1}_1(m^{-1})$ if $\alpha \neq 1$ and $\mathrm{dom}(\mathcal{L}_1) = L^1_1(m^{-1})$, is defined in (1.31) (it is straightforward to check that it is closed in this space). Since the equation in self-similar variables preserves mass and the zero momentum, the correct spectral study of $\mathcal{L}_\alpha$ requires to restrict this operator to zero mean and centered distributions (which are preserved as well), that means to work in $\mathbb{L}^1(m^{-1})$. When restricted to this space, the operator $\mathcal{L}_\alpha$ is denoted by $\hat{\mathcal{L}}_\alpha$. We denote by $R(\hat{\mathcal{L}}_\alpha)$ the resolvent set of $\hat{\mathcal{L}}_\alpha$, and by $\mathcal{R}_\alpha(\xi) = (\hat{\mathcal{L}}_\alpha - \xi)^{-1}$ its resolvent operator for any $\xi \in R(\hat{\mathcal{L}}_\alpha)$.

Let us recall that for the linearized elastic hard spheres Boltzmann equation the spectrum and the asymptotic stability have been studied by many authors since the pioneering works by Hilbert [20], Carleman [12] and Grad [18], and we refer for instance to [27] for more references. The result established for $\mathcal{L}_1$ (and translated straightforwardly to $\hat{\mathcal{L}}_1$) in [27] is the following:

Theorem 5.1 *(i) There exists a decreasing sequence of real discrete eigenvalues $(\mu_n)_{n\ge1}$ (that is: eigenvalues isolated and with finite multiplicity) of $\hat{\mathcal{L}}_1$, with "energy" eigenvalue $\mu_1 = 0$ of multiplicity 1 and "energy" eigenvector ϕ_1 (defined in (1.33)), $\mu_2 < 0$ and $\lim \mu_n = \mu_\infty \in (-\infty, 0)$ such that the spectrum $\Sigma(\hat{\mathcal{L}}_1)$ of $\hat{\mathcal{L}}_1$ in $\mathbb{L}^1(m^{-1})$ writes*

$$\Sigma(\hat{\mathcal{L}}_1) = (-\infty, \mu_\infty] \cup \{\mu_n\}_{n\in\mathbb{N}}.$$

In particular, $\hat{\mathcal{L}}_1$ is onto from $\mathcal{O} \cap \mathbb{L}^1_1(m^{-1})$ onto $\mathcal{O}$.

(ii) The resolvent $\mathcal{R}_1(\xi)$ has a sectorial property for the spectrum substracted from the "energy" eigenvalue, namely there is a constructive $\mu_2 < \lambda < 0$ such that

$$\forall\, \xi \in \mathcal{A}, \quad \|\mathcal{R}_1(\xi)\|_{\mathbb{L}^1(m^{-1})} \le a + \frac{b}{|\xi+\lambda|},$$

with

$$\mathcal{A} = \left\{ \xi \in \mathbb{C}, \quad \arg(\xi+\lambda) \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right] \quad \text{and} \quad \Re e\, \xi \le \frac{\lambda}{2} \right\}.$$

(iii) The linear semigroup $S_1(t)$ associated to $\hat{\mathcal{L}}_1$ in $\mathbb{L}^1(m^{-1})$ writes

$$\forall\, t \ge 0 \qquad S_1(t) = \Pi_1 + R_1(t),$$

where Π_1 is the projection on the eigenspace associated to μ_1 and $R_1(t)$ is a semigroup which satisfies

$$\forall\, t \ge 0 \qquad \|R_1(t)\|_{\mathbb{L}^1(m^{-1})} \le C\, e^{\mu_2\, t}$$

with explicit constant C.

The main result proved in this section is a perturbation result which extends Theorem 5.1 in the following way. Let us define for any $x \in \mathbb{R}$ the half-plane Δ_x by

$$\Delta_x = \{\xi \in \mathbb{C},\ \Re e\, \xi \ge x\}.$$

Theorem 5.2 *Let us fix $\bar\mu \in (\mu_2, 0)$, $k, q \in \mathbb{N}$ and m a smooth weight exponential function with $s \in (0,1)$. Then there exists $\alpha_2 \in (\alpha_1, 1)$ such that for any $\alpha \in [\alpha_2, 1]$ the following holds:*

(i) *The spectrum* $\Sigma(\hat{\mathcal{L}}_\alpha)$ *of* $\hat{\mathcal{L}}_\alpha$ *in* $\mathbb{W}^{k,1}_q(m^{-1})$ *writes*

$$\Sigma(\hat{\mathcal{L}}_\alpha) = E_\alpha \cup \{\mu_\alpha\}, \qquad E_\alpha \subset \Delta^c_{\bar\mu},$$

where μ_α *is a* 1*-dimensional real eigenvalue which does not depend on the choice of the space* $\mathbb{W}^{k,1}_q(m^{-1})$ *and satisfies (1.32).*

(ii) *The resolvent* $\mathcal{R}_\alpha(\xi)$ *in* $\mathbb{W}^{k,1}_q(m^{-1})$ *is holomorphic on a neighborhood of* $\Delta_{\bar\mu}\backslash\{\mu_\alpha\}$ *and there are explicit constants* C_1, C_2 *such that*

$$\sup_{z\in\mathbb{C},\ \Re e\, z=\bar\mu} \|\mathcal{R}_\alpha(z)\|\|_{\mathbb{W}^{k,1}_q(m^{-1})\to\mathbb{W}^{k,1}_q(m^{-1})} \le C_1$$

and

$$\|\mathcal{R}_\alpha(\bar\mu + is)\|_{\mathbb{W}^{k+1,1}_{q+1}(m^{-1})\to\mathbb{W}^{k,1}_q(m^{-1})} \le \frac{C_2}{1+|s|}.$$

(iii) *The linear semigroup* $S_\alpha(t)$ *associated to* $\hat{\mathcal{L}}_\alpha$ *in* $\mathbb{W}^{k,1}_q(m^{-1})$ *writes*

$$S_\alpha(t) = e^{\mu_\alpha\, t}\, \Pi_\alpha + R_\alpha(t),$$

where Π_α *is the projection on the (*1*-dimensional) eigenspace associated to* μ_α *and where* $R_\alpha(t)$ *is a semigroup which satisfies*

$$\|R_\alpha(t)\|_{\mathbb{W}^{k+2,1}_{q+2}(m^{-1})\to\mathbb{W}^{k,1}_q(m^{-1})} \le C_k\, e^{\bar\mu\, t} \tag{5.1}$$

with explicit bounds.

Remark 5.3 *Note that we do not claim that the resolvent* $\mathcal{R}_\alpha$ *is sectorial for* $\alpha < 1$ *and it is likely that indeed it is not (because of the contribution of the drift term). Moreover, it is not clear how to make the spectral study in the Hilbert setting* $L^2(m^{-1})$ *with convenient weight function* m*. In particular, we are not able to prove Proposition 3.2 in an* L^2 *framework. In such a situation the spectral study and the obtaining of constructive rate of decay on the semigroup become tricky. Let us emphasize also that (as most of the results established in this paper) this result is not an easy consequence of perturbation theory of unbounded operator since the elastic limit* $\alpha \to 1$ *is strongly bad-behaved (for instance neither the relative bound nor the operator gap of [21] go to* 0*) because of the anti-drift term.*

5.1 Recalls and improvments of technical tools from [27]

Proposition 5.4 *In the statement of Theorem 5.1 one can replace everywhere* $L^1(m^{-1})$ *by* $W^{k,1}_q(m^{-1})$*,* $k, q \in \mathbb{N}$*.*

Let us first recall the key decomposition of $\hat{\mathcal{L}}_1$ in [27, Section 2] (re-written within the notation of this paper):

Let $\mathbf{1}_E$ denote the usual indicator function of the set E, let $\Theta : \mathbb{R} \to \mathbb{R}_+$ be an even C^∞ function with mass 1 and support included in $[-1, 1]$ and $\tilde\Theta : \mathbb{R}^N \to \mathbb{R}_+$ a radial C^∞

function with mass 1 and support included in $B(0,1)$. We define the following mollification functions ($\epsilon > 0$):

$$\begin{cases} \Theta_\epsilon(x) = \epsilon^{-1}\,\Theta(\epsilon^{-1}x), & (x \in \mathbb{R}) \\ \tilde\Theta_\epsilon(x) = \epsilon^{-N}\,\tilde\Theta(\epsilon^{-1}x), & (x \in \mathbb{R}^N). \end{cases}$$

Then we consider the decompositions

$$\mathcal{L}_1(g) = \mathcal{L}_1^c(g) - \mathcal{L}^\nu(g) \quad \text{ with } \quad \mathcal{L}^\nu(g) := \nu\, g$$

where $\mathcal{L}_1^c$ splits between a "gain" part $\mathcal{L}_1^+$ (denoted so because it corresponds to the linearization of Q^+) and a convolution part $\mathcal{L}^*$ (not depending on α) as

$$\mathcal{L}_1^c(g) = \mathcal{L}_1^+(g) - \mathcal{L}^*(g) \quad \text{ with } \quad \mathcal{L}^*(g) := M\,[g * \Phi],$$

(we do not write the subscript 1 when there is no dependency on α). Then for any $\delta \in (0,1)$ we set

$$\mathcal{L}_{1,\delta}^+(g) = \mathcal{I}_\delta(v) \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi(|v - v_*|)\, b_\delta(\cos\theta)\, [g' M'_* + M' g'_*]\, dv_*\, d\sigma,$$

where

$$\mathcal{I}_\delta = \tilde\Theta_\delta * \mathbf{1}_{\{|\cdot| \le \delta^{-1}\}},$$

and

$$b_\delta(z) = \left(\Theta_{\delta^2} * \mathbf{1}_{\{-1+2\delta^2 \le z \le 1-2\delta^2\}}\right)\, b(z).$$

This approximation induces $\mathcal{L}_{1,\delta} = \mathcal{L}_{1,\delta}^+ - \mathcal{L}^* - \mathcal{L}^\nu$. Then the key result is that this approximation converges (in the norm of the graph) to the original linearized operator $\mathcal{L}_1$ as $\delta \to 0$, first in the small classical linearization space $L^2(\bar G_1^{-1})$ (this technical result was in fact mostly already included in Grad's results [18]), and second most importantly in the larger space $L^1(m^{-1})$. On the basis of this approximation result the spectrum is then proved to be the same in both functional spaces, and then the norm of the resolvents within these two functional spaces are related by an explicit control.

Hence the keys elements of the proof which are to be extended are, on the one hand, the approximation argument (which has to be extended from an $L^1(m^{-1})$ setting to an $W^{k,1}_q(m^{-1})$ setting), and, on the other hand the explicit control on the resolvent in the space $L^2(M^{-1})$ provided by the self-adjointness structure of the collision operator in this space and the explicit estimates on the spectral gap (see [6]), which has to be extended to an $H^k(M^{-1})$ setting. Then the rest of the proof of [27] would extend as well (up to minor technical modifications) to $W^{k,p}(m^{-1})$.

Therefore for the first point let us prove the

Proposition 5.5 *For any $k, q \in \mathbb{N}$ and $g \in W^{k,1}_{q+1}(m^{-1})$, we have*

$$\left\| \left(\mathcal{L}_1^+ - \mathcal{L}_{1,\delta}^+ \right)(g) \right\|_{W^{k,1}_q(m^{-1})} \le \varepsilon(\delta)\, \|g\|_{W^{k,1}_{q+1}(m^{-1})}$$

where $\varepsilon(\delta) > 0$ is an explicit constant going to 0 as δ goes to 0.

Proof of Proposition 5.5. The case $k = q = 0$ is provided by Proposition 3.2. Then higher-order derivatives follows by differentiation, and the incoporation of a polynomial weight is trivial. □

Concerning the second point let us prove the

Proposition 5.6 *The spectrum $\Sigma(\mathcal{L}_1)$ of $\mathcal{L}_1$ in $L^2(M^{-1})$ is the same in any $H^k(M^{-1})$, $k \in \mathbb{N}$. Moreover the control on the resolvent, which was (self-adjoint operator)*

$$\|\mathcal{R}_1(\xi)\|_{L^2(M^{-1})} \le \frac{1}{\mathrm{dist}(\xi, \Sigma(\mathcal{L}_1))}$$

in the space $L^2(M^{-1})$, extends into

$$\forall\, \xi \in \mathcal{A}, \quad \|\mathcal{R}_1(\xi)\|_{H^k(M^{-1})} \le \frac{C_k}{\mathrm{dist}(\xi, \Sigma(\mathcal{L}_1))},$$

with

$$\mathcal{A} = \left\{ \xi \in \mathbb{C}, \quad arg(\xi + \lambda) \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4} \right] \quad and \quad \Re e\, \xi \le \frac{\lambda}{2} \right\},$$

for any $k \in \mathbb{N}$ and some explicit constant $C_k > 0$,

Proof of Proposition 5.6. A quick way to prove the result for instance is the following. It is easy to prove by induction on $k \in \mathbb{N}$ the following estimate on the Dirichlet form:

$$\sum_{|s| \le k} a_s \, \langle \nabla^s \mathcal{L}_1(g), \nabla^s g \rangle_{L^2(M^{-1})} \le -\tau_k \left(\sum_{|s| \le k} \|\bar{\Pi}(\nabla^s g)\|^2_{L^2(M^{-1})} \right)$$

for some explicit $\tau_k > 0$ and $a_s > 0$, $|s| \le k$, and where $\bar{\Pi}$ denotes the orthogonal projection in $L^2(M^{-1})$ onto the functions with zero mass, momentum and energy. Therefore we deduce on $\hat{\mathcal{L}}_1$ that its semigroup satisfies

$$\forall\, k \in \mathbb{N}, \quad \|e^{t\, \hat{\mathcal{L}}_1}\|_{H^k(M^{-1})} \le C_k$$

and that obviously the same is true on the stable subspace of functions with zero energy. Then by interpolation with the rate of decay of the semigroup for functions with zero energy in $L^2(M^{-1})$, we deduce that

$$\forall\, \varepsilon > 0, \; k \in \mathbb{N}, \quad \|e^{t\, \hat{\mathcal{L}}_1} \Pi\|_{H^k(M^{-1})} \le C_{\varepsilon,k} \, e^{-(\mu_2 - \varepsilon)\, t}$$

for some explict $C_{\varepsilon,k} > 0$, and where Π is the orthogonal projection in $L^2(M^{-1})$ onto functions with zero energy. This implies on the resolvent that for any $k \in \mathbb{N}$,

$$\forall\, \xi \in \mathcal{A}, \quad \|\mathcal{R}_1(\xi)\|_{H^k(M^{-1})} \le C'_k,$$

with

$$\mathcal{A} = \left\{ \xi \in \mathbb{C}, \quad \arg(\xi + \lambda) \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4} \right] \quad \text{and} \quad \Re e\, \xi \le \frac{\lambda}{2} \right\},$$

for some explicit $C'_k > 0$. Then the result follows by straightforward interpolation with the estimates on the resolvent in $L^2(M^{-1})$. $\square$

Then we can conclude to the following extension of point (ii) of Theorem 5.1:

Proposition 5.7 *We have*

$$\forall\, \xi \in \mathcal{A}, \quad \|\mathcal{R}_1(\xi)\|_{\mathbb{W}^{k,1}_q(m^{-1})} \le a_{k,q} + \frac{b_{k,q}}{|\xi + \lambda|},$$

with

$$\mathcal{A} = \left\{ \xi \in \mathbb{C}, \quad \arg(\xi + \lambda) \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4} \right] \quad and \quad \Re e\, \xi \le \frac{\lambda}{2} \right\}.$$

for any $k, q \in \mathbb{N}$ and some explicit constant $a_{k,q}, b_{k,q} > 0$.

5.2 Decomposition of $\hat{\mathcal{L}}_\alpha$ and technical estimates

We fix once for all some $\bar\mu \in (\mu_2, 0)$ and we split the proof of Theorem 5.2 into four steps, detailed in the following four subsections.

Let us introduce the operator

$$P_\alpha = \mathcal{L}_1 - \mathcal{L}_\alpha = \mathcal{L}_1^+ - \mathcal{L}_\alpha^+ + \tau_\alpha \nabla_v \cdot (v\,\cdot).$$

Our first step in this subsection is to estimate the convergence to 0 of the first part of this operator in suitable norm. Namely we prove

Lemma 5.8 *(i) For any $k, q \in \mathbb{N}$, there exists $C = C_{k,q,m}$ such that*

$$\left\|\mathcal{L}_\alpha^+(g)\right\|_{W_q^{k,1}(m^{-1})} \le C\, \|g\|_{W_{q+1}^{k,1}(m^{-1})}, \quad \|\mathcal{L}_\alpha(g)\|_{W_q^{k,1}(m^{-1})} \le C\, \|g\|_{W_{q+1}^{k+1,1}(m^{-1})}.$$

(ii) For any $k, q \in \mathbb{N}$, there is a constructive function $\varepsilon : (0,\infty) \to (0,\infty)$ satisfying $\varepsilon(\alpha) \to 0$ as α goes to 1 and such that for any $g \in W_{q+1}^{k,1}(m^{-1})$

$$\left\|\left(\mathcal{L}_1^+ - \mathcal{L}_\alpha^+\right)(g)\right\|_{W_q^{k,1}(m^{-1})} \le \varepsilon(\alpha)\, \|g\|_{W_{q+1}^{k,1}(m^{-1})}.$$

(iii) There exists $C \in (0,\infty)$ such that for any $g \in W_3^{3,1}(m^{-1})$, we have

$$\|(\mathcal{L}_1 - \mathcal{L}_\alpha)(g)\|_{L^1(m^{-1})} \le C\,(1-\alpha)\, \|g\|_{W_3^{3,1}(m^{-1})}.$$

Proof of Lemma 5.8. The case $k = q = 0$ is proved in Proposition 3.2. Then higher-order derivatives are obtained from the $L^1(m^{-1})$ estimates by straightforward differentiation, and the incoporation of polynomial weights is trivial. □

Now let us consider some $\xi \in \mathbb{C}$ and let us define

$$A_\delta = \mathcal{L}_{1,\delta}^+ - \mathcal{L}^*$$

and

$$B_{\alpha,\delta}(\xi) = \nu + \xi + (\mathcal{L}_{1,\delta}^+ - \mathcal{L}_1^+) + P_\alpha$$

(let us recall that the approximation $\mathcal{L}_{1,\delta}^+$ was defined in the beginning of Subsection 5.1. It yields the decomposition

$$\mathcal{L}_\alpha - \xi = A_\delta - B_{\alpha,\delta}(\xi).$$

Then we have the

Lemma 5.9 *Let us consider any $k, q \in \mathbb{N}$ and ξ such that $\Re e\, \xi \ge -\min \nu$. Then*

(i) For any $\delta > 0$, the operator $A_\delta : L^1 \to W_\infty^{\infty,1}(m^{-1})$ is a bounded linear operator (more precisely it maps functions of L^1 into C^∞ functions with compact support).

(ii) For $\delta \in [0, \delta^*]$ and $\alpha \in [\alpha_2, 1]$ for some constructive $\delta^* > 0$ and $\alpha_2 \in (\alpha_1, 1)$ (depending on a lower bound on $\mathrm{dist}(\xi, \nu(\mathbb{R}^N))$), the operator

$$B_{\alpha,\delta} : W_{q+1}^{k+1,1}(m^{-1}) \to W_q^{k,1}(m^{-1})$$

is invertible

(iii) The inverse operator $B_{\alpha,\delta}(\xi)^{-1}$ satisfies for $\delta \in [0,\delta^*]$ and $\alpha \in [\alpha_3, 1]$:

$$\left\|B_{\alpha,\delta}(\xi)^{-1}\right\|_{W^{k,1}_q(m^{-1}) \to W^{k,1}_q(m^{-1})} \le \frac{C_1}{\mathrm{dist}(\Re e\,\xi, \nu(\mathbb{R}^N))}$$

and

$$\left\|B_{\alpha,\delta}(\xi)^{-1}\right\|_{W^{k+1,1}_q(m^{-1}) \to W^{k,1}_q(m^{-1})} \le \frac{C_2}{\mathrm{dist}(\xi, \nu(\mathbb{R}^N))}$$

for some explicit constants $C_1, C_2 > 0$ depending on k, q, δ^*, α_2 and a lower bound on $\mathrm{dist}(\Re e\,\xi, \nu(\mathbb{R}^N))$.

Proof of Lemma 5.9. For $\xi \in \nu(\mathbb{R}^N)^c$, it was proved in [27, Proposition 4.1, Theorem 4.2] the convergence to 0 of $(\mathcal{L}^+_{1,\delta} - \mathcal{L}^+_1)$ as $\delta \to 0$ (which was done in $L^1_1(m^{-1}) \to L^1(m^{-1})$ in [27] and is extended in any $W^{k,1}_{q+1}(m^{-1}) \to W^{k,1}_q(m^{-1})$ by Proposition 5.5), we deduce as in [27] that for δ small enough (depending on a lower bound on the coercivity norm of $\nu + \xi$, that is on a lower bound on $\mathrm{dist}(\xi, \nu(\mathbb{R}^N))$), we have

$$\|(\mathcal{L}^+_{1,\delta} - \mathcal{L}^+_1) g\|_{W^{k,1}_q(m^{-1})} \le \frac{1}{2}\, \|(\nu + \xi)\, g\|_{W^{k,1}_{q+1}}.$$

It was also proved that A_δ maps functions of L^1 into C^∞ functions with compact support (with explicit estimates).

Let now consider $B_{\alpha,\delta}(\xi)$ only in the case $k = q = 0$ (estimates for higher-order derivatives and weights are obtained by straightforward differentiation and computations). From Lemma 5.8 we have for α close enough to 1 (depending on a lower bound on $\mathrm{dist}(\xi, \nu(\mathbb{R}^N))$),

$$\left\|\left(\mathcal{L}^+_1 - \mathcal{L}^+_\alpha\right) g\right\|_{L^1(m^{-1})} \le \frac{1}{2}\, \|(\nu + \xi)\, g\|_{L^1(m^{-1})}.$$

By considering the semigroup on $L^1(m^{-1})$ of $B_{\alpha,\delta}(\xi)$ and computing the evolution of the norm in symmetric form using the formula for the differentiation of the complex modulus of a function

$$\nabla |h| = \frac{\nabla h\, \bar h + h\, \nabla \bar h}{2\,|h|},$$

it is easily seen that

$$\|e^{B_{\alpha,\delta}(\xi)\,t} g\|_{L^1(m^{-1})} \ge \|(\nu + \Re e\,\xi)\, g\|_{L^1(m^{-1})} - \frac{1}{2}\, \|(\nu + \Re e \xi)\, g\|_{L^1(m^{-1})}$$

and therefore for α close enough to 1 (depending on a lower bound on $\mathrm{dist}(\xi, \nu(\mathbb{R}^N))$), we deduce that

$$\|e^{B_{\alpha,\delta}(\xi)\,t} g\|_{L^1(m^{-1})} \ge \frac{1}{2}\, \|(\nu + \Re e\,\xi)\, g\|_{L^1(m^{-1})}$$

and thus that the operator is invertible with its inverse bounded by

$$\|B_{\alpha,\delta}(\xi)^{-1}\|_{L^1(m^{-1})} \le \frac{2}{\mathrm{dist}(\Re e\,\xi, \nu(\mathbb{R}^N))}.$$

Moreover by computing separately the evolution of the $L^1(m^{-1})$ norm in non-symmetric form (thus keeping $\nu + \xi$ but creating a term of the form $\mathcal{O}(1-\alpha)$ times a $W^{1,1}_1(m^{-1})$ norm) and the evolution of the $W^{1,1}_1(m^{-1})$ norm in symmetric form: it yields easily

$$\|e^{B_{\alpha,\delta}(\xi)\,t} g\|_{W^{1,1}(m^{-1})} \ge \frac{1}{2}\, \|(\nu + \xi)\, g\|_{L^1(m^{-1})} + \frac{1}{2}\, \|(\nu + \xi)\, \nabla_v g\|_{L^1(m^{-1})}$$

which implies the result, by droping the second term. □

5.3 Geometry of the essential spectrum and estimates on the eigenvalues

First concerning the geometry of the spectrum, following the same strategy as in [27, Subsection 3.2] we can prove the

Proposition 5.10 *Let us pick any $k, q \in \mathbb{N}$ and m a smooth exponential weight function (as defined in (1.28)). Then for any $\alpha \in [\alpha_2, 1]$, the spectrum of $\mathcal{L}_\alpha$ in $\mathbb{W}^{k,1}_q(m^{-1})$ is composed of a part included in $\Delta^c_{-\nu_0}$ (containing all possible essential spectrum), and a remaining part included in $\Delta_{-\nu_0}$ exclusively composed of discrete eigenvalues.*

Proof of Proposition 5.10. We follow the same method as in the proof of [27, Proposition 3.4]. One uses the decomposition

$$\mathcal{L}_\alpha = A_\delta - B_{\alpha,\delta}(0),$$

the compactness of the first part A_δ and the coercivity

$$\|B_{\alpha,\delta}(0)\|_{L^1(m^{-1})} \geq \|\nu\, g\|_{L^1(m^{-1})} - \varepsilon(\delta)\, \|\nu\, g\|_{L^1(m^{-1})}$$

of the second part (where $\epsilon(\delta) \to 0$ as $\delta \to 0$). Then one applies Weyl's theorem and show that (for any $\delta > 0$) $\Delta_{-\nu_0+\epsilon(\delta)}$ has to be a Fredhom set with indices $(0, 0)$ (except possibly for a countable family of points) since $[a, +\infty)$ is included in the resolvent set for a big enoug. □

Second concerning the discrete part of the spectrum, that is the isolated eigenvalues with finite multiplicity, following the same strategy as in [27, Proof of Proposition 3.5] we can prove the

Proposition 5.11 *Let us fix $\bar{\mu} \in (\mu_2, 0)$. Then for any $\alpha \in [\alpha_2, 1]$ (where α_2 is obtained from Lemma 5.9 for this choice of $\bar{\mu}$), for any $\mu \in \Delta_{\bar{\mu}}$ and $\phi \in W^{1,1}_1$ satisfying*

$$\mathcal{L}_\alpha(\phi) = \mu\, \phi$$

in L^1, we have

$$\|\phi\|_{W^{k,1}(m^{-1})} \leq C_{k,m}\, \|\phi\|_{L^1_2}$$

for any $k \in \mathbb{N}$ and $m = \exp(-a\,|v|^s)$, $a > 0$, $s \in (0, 1)$, where the constant $C_{k,m}$ depends on k, m and a lower bound on $\bar{\mu} - \mu$.

Proof of Proposition 5.11. Let us sketch the idea of the proof. We use the decomposition

$$0 = \mathcal{L}_\alpha \phi - \mu\, \phi = A_\delta \phi - B_{\alpha,\delta}(\mu)\phi$$

and the fact that for the choices made for μ and α in the assumptions we have (adjusting δ as in Lemma 5.9) $B_{\alpha,\delta}(\mu)$ is invertible in any $W^{k,1}(m^{-1})$ with explicit bound, and A_δ maps L^1 into C^∞ functions with compact support. □

Remark 5.12 *An alternative proof could be to adapt the proof of Proposition 2.7.*

5.4 Estimate on the resolvent and global stability of the spectrum

Lemma 5.13 *Let us pick $k, q \in \mathbb{N}$ and m a smooth exponential weight function (as defined in (1.28)) and consider the operator $\mathcal{L}_\alpha$ in $\mathbb{W}^{k,1}_q(m^{-1})$. Then*

(i) For any $\xi \in R(\mathcal{L}_1)$, there is $\alpha_\xi \in [\alpha_2, 1)$ such that $\xi \in R(\mathcal{L}_\alpha)$ for any $\alpha \in [\alpha_\xi, 1]$.

(ii) More precisely, the resolvent $\mathcal{R}_\alpha(\xi)$ satisfies the two following estimates for $\alpha \in [\alpha_2, 1)$:

$$\|\mathcal{R}_\alpha(\xi)\|_{\mathbb{W}^{k,1}_q(m^{-1})} \le \frac{C_1 + C_2 \, \|\mathcal{R}_1(\xi)\|_{W^{k+1,1}_{q+1}(m^{-1})}}{1 - C_3\,(1-\alpha)\,\|\mathcal{R}_1(\xi)\|_{W^{k+1,1}_{q+1}(m^{-1})}}$$

$$\|\mathcal{R}_\alpha(\xi)\|_{\mathbb{W}^{k+1,1}_{q+1}(m^{-1}) \to \mathbb{W}^{k,1}_q(m^{-1})} \le \frac{1}{\delta(\xi)} \, \frac{C_1' + C_2' \, \|\mathcal{R}_1(\xi)\|_{W^{k+1,1}_{q+1}(m^{-1})}}{1 - C_3\,(1-\alpha)\,\|\mathcal{R}_1(\xi)\|_{W^{k+1,1}_{q+1}(m^{-1})}}$$

with $\delta(\xi) := \mathrm{dist}(\xi, \nu(\mathbb{R}^N))$ and where the constants C_i, C_i', $i = 1,2,3$ depend on a positive lower bound on $\mathrm{dist}(\Re e\, \xi, \nu(\mathbb{R}^N))$.

(iii) Finally, for any compact set $K \subset \rho(\mathcal{L}_1)$ there exists $\alpha_K \in [\alpha_2, 1)$, $C_K \in (0, \infty)$ such that

$$\forall\, \xi \in K,\ \alpha \in (\alpha_K, 1] \qquad \|\mathcal{R}_\alpha(\xi)\|_{\mathbb{W}^{k,1}_q(m^{-1})} \le C_K,$$
$$\forall\, \xi \in K,\ \alpha, \alpha' \in (\alpha_K, 1] \qquad \|\mathcal{R}_\alpha(\xi)\, h - \mathcal{R}_{\alpha'}(\xi)\, h\|_{\mathbb{L}^1(m^{-1})} \le C_K\,(1-\alpha)\,\|h\|_{W^{3,1}_3}.$$

Proof of Lemma 5.13. We split the proof into three steps.

Step 1. Let us consider the following operator defined from $\mathbb{W}^{k,1}_q(m^{-1})$ to $W^{k+1,1}_{q+1}(m^{-1})$ (which is seen to be well-defined at a glance)

$$I_{\alpha,\delta}(\xi) := -B_{\alpha,\delta}(\xi)^{-1} + \mathcal{R}_1(\xi)\, A_\delta\, B_{\delta,\alpha}(\xi)^{-1}.$$

Some straightforward computations show that

$$(\mathcal{L}_\alpha - \xi)\, I_{\alpha,\delta}(\xi) = -A_\delta\, B_{\alpha,\delta}(\xi)^{-1} + \mathrm{Id} + \Big[\mathrm{Id} - P_\alpha\, \mathcal{R}_1(\xi)\Big]\, A_\delta\, B_{\alpha,\delta}(\xi)^{-1}$$

which simplifies into

$$(\mathcal{L}_\alpha - \xi)\, I_{\alpha,\delta}(\xi) =: J_{\alpha,\delta}(\xi) := \mathrm{Id} - P_\alpha\, \mathcal{R}_1(\xi)\, A_\delta\, B_{\alpha,\delta}(\xi)^{-1} =: \mathrm{Id} - K_{\alpha,\delta}(\xi).$$

First using that

$$\|P_\alpha h\|_{\mathbb{W}^{k,1}_q(m^{-1})} \le C\,(1-\alpha)\,\|h\|_{W^{k+1,1}_{q+1}(m^{-1})},$$

the control of $\mathcal{R}_1(\xi)$ in $\mathbb{W}^{k+1,1}_{q+1}(m^{-1})$ and the regularization property of A_δ we deduce that

$$K_{\alpha,\delta}(\xi) = P_\alpha\, \mathcal{R}_1(\xi)\, A_\delta\, B_{\delta,\alpha}(\xi)^{-1} = \mathcal{O}(1-\alpha)$$

in the norm of bounded operators on $W^{k,1}_q(m^{-1})$, and therefore for $(1-\alpha)$ small enough (with explicit bound) we get that

$$\|K_{\alpha,\delta}(\xi)\|_{W^{k,1}_q(m^{-1})} \le C_3\,(1-\alpha)\,\|\mathcal{R}_1(\xi)\|_{\mathbb{W}^{k+1,1}_{q+1}(m^{-1})} < 1$$

and $\mathrm{Id} - K_{\alpha,\delta}(\xi)$ is invertible in $W^{k,1}_q(m^{-1})$. As a consequence

$$(\mathcal{L}_\alpha - \xi)\, I_{\alpha,\delta}(\xi)\, (\mathrm{Id} - K_{\alpha,\delta}(\xi))^{-1} = \mathrm{Id}_{\mathbb{W}^{k,1}_q(m^{-1})}$$

and we have proved that $\mathcal{L}_\alpha - \xi$ admits a right-inverse, namely so that $I_\alpha(\xi)\,(\mathrm{Id} - K_{\alpha,\delta}(\xi))^{-1}$. This proves that the operator $\mathcal{L}_\alpha - \xi$ is onto.

Step 2. In order to show that $\mathcal{L}_\alpha - \xi$ is invertible and that we have identified the resolvent it remains to prove that it is one-to-one. Let us consider the eigenvalue equation $(\mathcal{L}_\alpha - \xi)\, h = 0$ which writes

$$(\mathcal{L}_1 - \xi) h = P_\alpha h$$

from which we deduce (using Proposition 5.11 to get regularity bounds on h)

$$\begin{aligned} \|h\|_{\mathbb{W}^{k,1}_q(m^{-1})} &\le \|\mathcal{R}_1(\xi)\|_{\mathbb{W}^{k,1}_q(m^{-1})}\, \|P_\alpha h\|_{\mathbb{W}^{k,1}_q(m^{-1})} \\ &\le C\,(1-\alpha)\, \|\mathcal{R}_1(\xi)\|_{\mathbb{W}^{k,1}_q(m^{-1})}\, \|h\|_{\mathbb{W}^{k+1,1}_{q+1}(m^{-1})} \\ &\le C'\,(1-\alpha)\, \|\mathcal{R}_1(\xi)\|_{\mathbb{W}^{k,1}_q(m^{-1})}\, \|h\|_{\mathbb{W}^{k,1}_q(m^{-1})}. \end{aligned}$$

Therefore for $(1-\alpha)$ small enough (depending on the norm of $\mathcal{R}_1(\xi)$) we have that necessarily $h = 0$, and thus the operator $(\mathcal{L}_\alpha - \xi)$ is one-to-one.

For α satisfying all the previous conditions, the operator $(\mathcal{L}_\alpha - \xi)$ is bijective from $\mathbb{W}^{k+1,1}_{q+1}(m^{-1})$ to $\mathbb{W}^{k,1}_q(m^{-1})$ and its inverse is given by

$$\mathcal{R}_\alpha(\xi) = I_{\alpha,\delta}(\xi)\, J_{\alpha,\delta}(\xi)^{-1}$$

from which we get the desired bound on the resolvent thanks to the study of $B_{\alpha,\delta}(\xi)^{-1}$ in Lemma 5.9. At this point we have proved points (i), (ii) and the first estimate in (iii).

Step 3. The second estimate in point (iii) is obtained from the resolvent identity

$$\mathcal{R}_\alpha(\xi) - \mathcal{R}_1(\xi) = \mathcal{R}_\alpha(\xi)\, [\mathcal{L}_1 - \mathcal{L}_\alpha]\, \mathcal{R}_1(\xi),$$

together with the previous estimates on the resolvent and point (iii) in Lemma 5.8. □

Remark that this lemma proves the point (ii) in Theorem 5.2. Moreover, as a consequence of this estimate on the resolvent $\mathcal{R}_\alpha(\xi)$, we may go one step further in the localization of the spectrum of $\hat{\mathcal{L}}_\alpha$ around 0.

Corollary 5.14 *Let us fix $\bar\mu \in (\mu_2, 0)$. In any $\mathbb{W}^{k,1}_q(m^{-1})$ there is some constant $C \in (0,\infty)$ such that*

$$\forall\, \alpha \in [\alpha_2, 1], \quad \Sigma(\hat{\mathcal{L}}_\alpha) \cap \Delta_{\bar\mu} \subset B(0, C\,(1-\alpha)).$$

Proof of Corollary 5.14. The proof follows from the estimates in point (ii) of Lemma 5.13, together with the fact that (Proposition 4.1 of [27] in $\mathbb{L}^1(m^{-1})$ extended to $\mathbb{W}^{k,1}_q(m^{-1})$ by the previous discussion):

$$\forall\, \xi \in \Delta_{\bar\mu}, \quad \|\mathcal{R}_1(\xi)\|_{\mathbb{W}^{k,1}_q(m^{-1})} \le a + \frac{b}{|\xi|}$$

for some explicit constants $a, b > 0$. We get thus that $\|\mathcal{R}_\alpha(\xi)\|_{\mathbb{W}^{k,1}_q(m^{-1})} < \infty$ if $\xi \in \Delta_{\bar\mu}$ and $|\xi| \ge C\,(1-\alpha)$, which concludes the proof. □

5.5 Fine study of spectrum close to 0

Let us fix $r \in (0, |\bar\mu|]$ and let us choose any $\alpha_r \in [\alpha_2, 1)$ such that $C\,(1-\alpha_r) < r$ (with the notations of Corollary 5.14) in such a way that $\Sigma(\hat{\mathcal{L}}_\alpha) \cap \Delta_{\bar\lambda} \subset B(0,r)$ for any $\alpha \in [\alpha_r, 1]$. We may then define the spectral projection operator (see [21])

$$\Pi_\alpha := -\frac{1}{2\,\pi\,i}\int_{S(0,r)} \mathcal{R}_\alpha(\zeta)\,d\zeta \tag{5.2}$$

in any $\mathbb{W}^{k,1}_q(m^{-1})$, with $S(0,r) := \{\xi \in \mathbb{C},\, |\xi| = r\}$. The operator Π_α is the projection operator on the sum of eigenspaces associated to eigenvalues lying in the half plane $\{\xi \in \mathbb{C},\, \Re e\,\xi \geq -r\}$, see [21]. In particular the operator Π_1 is the projection on the energy eigenline $\mathbb{R}\,\phi_1$, where we recall that ϕ_1 is the energy eigenfunction defined by (1.33).

Lemma 5.15 *The operator Π_α satisfies*

(i) For any $k \in \mathbb{N}$ and any exponential weight function m (as defined in (1.28)), it is well-defined and bounded in $\mathbb{W}^k_q(m^{-1})$.

(ii) Moreover there is a constant $C > 0$ (depending on m) such that

$$\forall\,\alpha,\,\alpha' \in [\alpha_r, 1] \qquad \|\Pi_\alpha - \Pi_{\alpha'}\|_{W^{3,1}_3(m^{-1})\to L^1(m^{-1})} \leq C\,|\alpha' - \alpha|. \tag{5.3}$$

Proof of Lemma 5.15. It is a straightforward consequence of (5.2) and Lemma 5.13. □

Corollary 5.16 *There exists $\alpha_3 \in [\alpha_2, 1)$ such that for any $\alpha \in [\alpha_3, 1)$ there holds*

$\Sigma(\hat{\mathcal{L}}_\alpha) \cap \Delta_{\bar\mu} = \{\mu_\alpha\}$ *and the eigenspace associated to $\mu_\alpha \in \mathbb{R}$ is 1-dimensional.*

This eigenvalue is called the energy eigenvalue. We may furthermore remark that Corollary 5.14 implies

$$\forall\,\alpha \in [\alpha_3, 1) \qquad |\mu_\alpha| \leq C\,(1-\alpha). \tag{5.4}$$

Proof of Corollary 5.16. We already know that $\Sigma(\hat{\mathcal{L}}_\alpha) \cap \Delta_{\bar\mu}$ is entirely composed of discrete spectrum. Therefore we have to prove that it is of dimension 1. Indeed once this is proved, the fact that $\mu_\alpha \in \mathbb{R}$ is trivial since the operator is real, and the control (5.4) is trivial from Corollary 5.14.

Let us define the space $X_\alpha := \Pi_\alpha(L^1(m^{-1})) + \Pi_1(L^1(m^{-1}))$ endowed with the norm $\|\cdot\|_{L^1(m^{-1})}$. From Proposition 5.11, there exists a constant $C_1 > 0$ such that

$$\forall\,\psi \in X_\alpha, \quad \|\psi\|_{W^{3,1}_3(m^{-1})} \leq C_1\,\|\psi\|_{L^1(m^{-1})}.$$

Thanks to the definition of Π_α and Π_1 and to Lemma 5.15, we then get

$$\begin{aligned}
\|\Pi_\alpha - \Pi_1\|_{X_\alpha\to X_\alpha} &\leq C_2 \sup_{\psi\in X_\alpha}\sup_{z\in S(0,r)} \frac{\|(\mathcal{R}_\alpha(z) - \mathcal{R}_1(z))\,\psi\|_{L^1(m^{-1})}}{\|\psi\|_{L^1(m^{-1})}} \\
&\leq C_2'\,(1-\alpha)\sup_{\psi\in X_\alpha}\frac{\|\,\psi\|_{W^{3,1}_3(m^{-1}))}}{\|\psi\|_{L^1(m^{-1})}} \\
&\leq C_2''\,(1-\alpha) < 1,
\end{aligned}$$

for $(1-\alpha)$ small enough. By classical operator theory (see for instance the arguments presented in [21, Chap 1, paragraph 4.6] in order to prove [21, Lemma 4.10]) one deduces that dimension(Π_α) = dimension(Π_1). Since dimension$(\Pi_1) = 1$ (as recalled in Theorem 5.1), this concludes the proof. □

Let us introduce for any $\psi \in L^1$ the decomposition

$$\psi = \Pi_1 \psi + \Pi_1^\perp \psi = (\pi_1 \psi)\, \phi_1 + \Pi_1^\perp \psi,$$

where $\pi_1 \psi \in \mathbb{R}$ is the coordinate of $\Pi_1 \psi$ on $\mathbb{R}\, \phi_1$ (defined thanks to the projection Π_1). For any $\alpha \in [\alpha_3, 1)$ we denote by ϕ_α the unique eigenfunction associated to μ_α such that $\|\phi_\alpha\|_{L^1_2} = 1$ and $\pi_1 \phi_\alpha \geq 0$.

We can now establish a first order approximation of the eigenfunction ϕ_α.

Lemma 5.17 *For any $k, q \in \mathbb{N}$ and any exponential weight function m (as defined in (1.28)), there exists C such that*

$$\forall\, \alpha \in [\alpha_3, 1] \qquad \|\phi_\alpha - \phi_1\|_{W^{k,1}_q(m^{-1})} \leq C\,(1-\alpha). \tag{5.5}$$

Remark 5.18 *We immediately deduce from Lemma 5.17 that $\phi_\alpha(0) < 0$ for α close enough to 1, and therefore, we get that this definition of ϕ_α coincides with the definition in Theorem 1.1.*

Proof of Lemma 5.17. On the one hand, from the normalization conditions, we have

$$\begin{aligned} \|\phi_1 - \Pi_1 \phi_\alpha\|_{L^1_2} &= |1 - \pi_1 \phi_\alpha| = \Big|\, \|\phi_\alpha\|_{L^1_2} - \|\Pi_1 \phi_\alpha\|_{L^1_2} \Big| \\ &\leq \|\phi_\alpha - \Pi_1 \phi_\alpha\|_{L^1_2} = \|\Pi_1^\perp \phi_\alpha\|_{L^1_2}. \end{aligned}$$

We then deduce

$$\|\phi_1 - \phi_\alpha\|_{L^1_2} \leq \|\Pi_1 \phi_\alpha - \phi_\alpha\|_{L^1_2} + \|\Pi_1^\perp \phi_\alpha\|_{L^1_2} \leq 2\, \|\Pi_1^\perp \phi_\alpha\|_{L^1_2}. \tag{5.6}$$

On the other hand, the eigenfunction ϕ_α satisfies

$$\hat{\mathcal{L}}_1(\phi_\alpha) = [\hat{\mathcal{L}}_1(\phi_\alpha) - \hat{\mathcal{L}}_\alpha(\phi_\alpha)] - \mu_\alpha\, \phi_\alpha.$$

Recall that from Proposition 5.11 one has uniform bounds in $W^{\infty,1}_\infty(m^{-1})$ on ϕ_α in terms of its L^1_2 norm which has been fixed to 1, so that for any $\alpha \in [\alpha_3, 1]$, $\|\phi_\alpha\|_{W^{k,1}_q(m^{-1})} \leq C$. Using Proposition 3.1 and Proposition 5.11 we get

$$\|\hat{\mathcal{L}}_1 \phi_\alpha\|_{L^1(m^{-1})} = \mathcal{O}(1-\alpha).$$

Using that $\hat{\mathcal{L}}_1$ is invertible from $\Pi_1^\perp \mathbb{L}^1_1(m^{-1})$ to $\mathbb{L}^1(m^{-1})$ we deduce that

$$\|\Pi_1^\perp \phi_\alpha\|_{L^1(m^{-1})} = \mathcal{O}(1-\alpha). \tag{5.7}$$

We conclude the proof of (5.5) holds for the L^1_2 norm gathering (5.6) and (5.7):

$$\forall\, \alpha \in [\alpha_3, 1] \qquad \|\phi_\alpha - \phi_1\|_{L^1_2} \leq C\,(1-\alpha).$$

Let now consider the eigenfunctions Φ_α associated to μ_α for $\alpha \in [\alpha_3, 1]$ such that $\pi_1 \Phi_\alpha > 0$ with the normalization condition $\|\Phi_\alpha\|_{W^{k,1}(m^{-1})} = 1$. Proceeding similarly as before (by working in the space $W^{k,1}(m^{-1})$), we can get

$$\|\Phi_\alpha - \Phi_1\|_{W^{k,1}(m^{-1})} = \mathcal{O}(1-\alpha).$$

Because the eigenspace associated to μ_α is of dimension 1, we have $\Phi_\alpha = c_\alpha \, \phi_\alpha$ for some constant $c_\alpha \in (0, \infty)$. Then

$$|c_1 - c_\alpha| \;=\; \|c_\alpha \, \phi_\alpha - c_1 \, \phi_\alpha\|_{L^1_2} \le \|\Phi_\alpha - \Phi_1\|_{L^1_2} + |c_1| \, \|\phi_1 - \phi_\alpha\|_{L^1_2} = \mathcal{O}(1-\alpha).$$

We then easily conclude that (5.5) holds for any $W^{k,1}(m^{-1})$ norm. □

We now use the linearized energy dissipation equation to get a second order expansion of the eigenvalue.

Lemma 5.19 *For $\alpha \in [\alpha_3, 1]$, the eigenvalues μ_α satisfies (with explicit bound)*

$$\mu_\alpha = -\rho \, (1-\alpha) + \mathcal{O}(1-\alpha)^2.$$

Proof of Lemma 5.19. By integrating the eigenvalue equation

$$\hat{\mathcal{L}}_\alpha \phi_\alpha = \mu_\alpha \, \phi_\alpha$$

against $|v|^2$ and dividing it by $(1-\alpha)$, we get

$$\frac{\mu_\alpha}{1-\alpha} \, \mathcal{E}(\phi_\alpha) = 2 \, \rho \, \mathcal{E}(\phi_\alpha) - 2 \, (1+\alpha) \, \tilde{D}(\bar{G}_\alpha, \phi_\alpha).$$

Using the rate of convergence of $\bar{G}_\alpha \to \bar{G}_1$ and $\phi_\alpha \to \phi_1$ established in Lemma 4.4 and Lemma 5.17 we deduce that

$$\frac{\mu_\alpha}{1-\alpha} \, \mathcal{E}(\phi_1) = 2 \, \rho \, \mathcal{E}(\phi_1) - 4 \, \tilde{D}(\bar{G}_1, \phi_1) + \mathcal{O}(1-\alpha). \tag{5.8}$$

Then we compute thanks to (A.1) and (A.2)

$$\mathcal{E}(\phi_1) = 2 \, N \, c_0 \, \rho \, \bar{\theta}_1^2, \tag{5.9}$$

where c_0 is still the normalizing constant in (1.33) such that $\|\phi_1\|_{L^1_2} = 1$. Similarly, using (A.3), (A.4) and the relation (1.27) which make a link between b_1 and $\bar{\theta}_1$, we find

$$\tilde{D}(\bar{G}_1, \phi_1) = \frac{3}{2} \, N \, c_0 \, \rho^2 \, \bar{\theta}_1^2. \tag{5.10}$$

We conclude gathering (5.8), (5.9) and (5.10). □

5.6 The map $\alpha \mapsto \bar{G}_\alpha$ is C^1

The fact that the path of self-similar profiles $\alpha \mapsto \bar{G}_\alpha$ is C^0 on $[\alpha_3, 1]$ and C^1 at $\alpha = 1$ was already proved in Lemma 4.4. Therefore we have to prove that it is C^1 for $\alpha \in [\alpha_3, 1)$.

Let us define the functional

$$(\alpha, g) \mapsto \Psi(\alpha, g) := Q_\alpha(g, g) - \tau_\alpha \nabla_v (v\, g).$$

The map Ψ is C^1 from $\mathbb{R} \times (W^{1,1}_1(m^{-1}) \cap \mathcal{C}_{\rho,0})$ into $\mathbb{L}^1(m^{-1})$ and it is such that for any $\alpha \in [\alpha_1, 1)$, the equation $\Psi(\alpha, g) = 0$ has only one solution which is the profile $\bar{G}_\alpha$. Moreover, for any $\alpha \in [\alpha_3, 1)$, the linearized operator $D_2\Psi(\alpha, \bar{G}_\alpha) = \mathcal{L}_\alpha$ is invertible from $\mathbb{W}^{1,1}(m^{-1})$ into $\mathbb{L}^1(m^{-1})$ because of the spectral properties of $\mathcal{L}_\alpha$ established in Theorem 5.2 (i) & (ii) (note that here there is no eigenvalue approaching 0 at α). Then using the same strategy as in Subsection 4.2 based on the implicit function theorem we easily conclude that $\alpha \mapsto \bar{G}_\alpha$ is C^1 from $[\alpha_3, 1)$ into $L^1(m^{-1})$. That ends the proof of Theorem 1.1 (ii).

5.7 Decay estimate on the semigroup

We start with a lemma on non sectorial semigroups in Banach spaces. This result is a tool for deriving constructive decay rate on non sectorial semigroups, from the knowkedge on the resolvent of their generator. We do not try to prove such a decay rate for the semigroup in the norm of the Banach space but instead in a weaker norm (corresponding to the norm of the graph of some power of its generator), which shall be sufficient for our study of the linearized stability of the non-linear equation (1.29).

Lemma 5.20 *Let A be a closed unbounded operator on a Banach space E with dense domain* dom(A). *We denote by $S(t)$ the associated semigroup, by $R(A)$ the associated resolvent set and by $\mathcal{R} = \mathcal{R}(\xi)$ the resolvent operator defined on $R(A)$. Assume that we have a sequence of Banach spaces $E_2 \subset E_1 \subset E_0 = E$ decreasing for inclusion (in most cases this sequence shall be provided by $E_k = dom(A^k)$ endowed with the norm of the graph of A^k). We assume on the operator that:*

(i) the resolvent set $R(A)$ contains the half plan Δ_a for some $a \in \mathbb{R}$, together with the estimates

$$\sup_{s \in \mathbb{R}} \|\mathcal{R}(a + i\, s)\|_{E_0 \to E_0} \le C_1$$

and

$$\forall\, s \in \mathbb{R}, \quad \|\mathcal{R}(a + i\, s)\|_{E_1 \to E_0},\ \|\mathcal{R}(a + i\, s)\|_{E_2 \to E_1} \le \frac{C_2}{1 + |s|}$$

for some constants $C_1, C_2 > 0$;

(ii) the semigroup $S(t)$ satisfies

$$\forall\, t \ge 0, \quad \|S(t)\|_{E_2 \to E_0} \le C_3\, e^{b\, t} \tag{5.11}$$

for some constants $C_3, b > 0$.

Then for any $a' > a$, there exists a constant C_4 depending only on a, b, a', C_1, C_2, C_3 such that

$$\forall\, t \ge 0, \quad \|S(t)\|_{E_2 \to E_0} \le C_4\, e^{a'\, t}. \tag{5.12}$$

Proof of Lemma 5.20. We split the proof into two parts.

Step 1. The first bound on the resolvent implies that for any $x \in E_0$

$$\|\mathcal{R}(a+is)x\|_{E_0} \to 0, \quad |s| \to \infty.$$

Indeed we first consider $x \in \mathrm{dom}(A)$ and then argue by density (since the domain $\mathrm{dom}(A)$ is dense). When $x \in \mathrm{dom}(A)$ the result is proved by the relation

$$\mathcal{R}(z)x = z^{-1}\left[-\mathrm{Id} + \mathcal{R}(z)\,A\right]x.$$

Step 2. Then consider the following integral of $\mathcal{R}(z)x$ on a vertical segment with real part a (for some $M > 0$)

$$I_M(x) := \int_{a-iM}^{a+iM} e^{zt}\,\mathcal{R}(z)x\,dz.$$

The function $z \to \mathcal{R}(z)$ is differentiable on this segment and we can perform an integration by part:

$$I_M(x) = \frac{e^{(a+iM)\,t}}{t}\,\mathcal{R}(a+iM)x - \frac{e^{(a-iM)\,t}}{t}\,\mathcal{R}(a-iM)x - \int_{a-iM}^{a+iM}\frac{e^{zt}}{t}\,\mathcal{R}(z)^2x\,dz$$

where we have used $R'(z) = R(z)^2$. Now we estimate the E_0 norm of this quantity:

$$\|I_M(x)\|_{E_0} \le \left\|\frac{e^{(a+iM)\,t}}{t}\,\mathcal{R}(a+iM)x\right\|_{E_0} + \left\|\frac{e^{(a+iM)\,t}}{t}\,\mathcal{R}(a+iM)x\right\|_{E_0}$$
$$+C_2^2\,\frac{e^{a\,t}}{t}\,\left(\int_{-\infty}^{+\infty}\frac{1}{(1+|s|)^2}\,ds\right)\,\|x\|_{E_2}.$$

Therefore the integral is semi-convergent and we can pass to the limit $M \to +\infty$ and use (see [32, 4]) that

$$S(t)x = \frac{1}{2i\pi}\,\lim_{M\to\infty}\int_{a-iM}^{a+iM} e^{zt}\,\mathcal{R}(z)x\,dz = \frac{1}{2i\pi}\,\lim_{M\to\infty} I_M$$

to obtain (the two boundary terms go to 0 as $M \to +\infty$ from the first step)

$$\|S(t)x\|_{E_0} \le C_2'\,\frac{e^{a\,t}}{t}\,\|x\|_{E_2}, \quad \text{with} \quad C_2' = C_2^2\,\left(\int_{-\infty}^{+\infty}\frac{1}{(1+|s|)^2}\,ds\right). \tag{5.13}$$

Using (5.11) for $t \le 1$ and (5.13) for $t \ge 1$, we conclude that (5.12) holds with $C_4 = \max(C_2', C_3\,e^{b-a'})$. □

Proof of point (iii) in Theorem 5.2. The point (ii) of Theorem 5.2 was proved in Lemma 5.13 and it shows that the operator $\bar{\mathcal{L}}_\alpha = (\mathrm{Id} - \Pi_\alpha)\,\hat{\mathcal{L}}_\alpha$ together with the sequence of Banach spaces $E_i = \mathbb{W}_i^{k+i,1}(m^{-1})$, $i = 0, 1, 2$, for any fixed $k \in \mathbb{N}$ and any exponential weight function m (as defined in (1.28)), satisfies the assumption (i) of Lemma 5.20 for any $a \in (\mu_2, 0)$. Moreover it is trivial to prove that it satisfies the assumption (ii) of Lemma 5.20 for some explicit $b > 0$ from the decomposition $\mathcal{L}_\alpha = A_\delta - B_{\alpha,\delta}(\xi)$ already introduced. □

6 Convergence to the self-similar profile

In this section, we consider the nonlinear rescaled equation (1.29) and we prove the convergence of its solutions to the self-similar profile. As a preliminary let us recall some result on propagation and appearance of moments and regularity which is picked up from [24, Proposition 3.1, Theorem 3.5, Theorem 3.6].

Lemma 6.1 *Let us consider $g_{\rm in} \in L^1_3 \cap \mathcal{C}_{\rho,0}$ and the associated solution $g \in C([0,\infty); L^1_3)$ to the rescaled equation (1.29). Then*

(i) For any exponential moment weight m (as defined in (1.28)) with exponent $s \in (0,1/2)$ and any time $t_0 \in (0,\infty)$, there exists a constant $M_1 = M_1(t_0)$ such that

$$\sup_{[t_0,\infty)} \|g(t,\cdot)\|_{L^1(m^{-1})} \le M_1. \tag{6.1}$$

Moreover, if $g_{\rm in} \in L^1(m^{-1})$ for some polynomial or exponential (with exponent $s \in (0,1)$) moment weight m then (6.1) holds (for this weight m) with $t_0 = 0$ and some constant $M_1 = M_1(\|g_{in}\|_{L^1(m^{-1})})$.

For the two following points we now assume that for some constants $c_1, T \in (0,\infty)$ there holds

$$\inf_{[0,T]} \mathcal{E}(g(t,\cdot)) \ge c_1, \tag{6.2}$$

and we state some smoothness properties of the solution g which depend on c_1 but not on T nor α.

(ii) Assume (6.2). Then for any $k_0 \in \mathbb{N}$ there is $q_0 = q_0(k_0) \in \mathbb{N}$ such that if $\|g_{\rm in}\|_{H^{k_0}\cap L^1_{q_0}} \le C_0$ holds, then for any $c_1 \in (0,\infty)$ there exists $C_1 = C_1(C_0, c_1) \in (0,\infty)$ such that for any time $T \in (0,\infty)$, we have

$$\forall\, t \in [0,T], \quad \|g(t,\cdot)\|_{H^{k_1}} \le C_1, \tag{6.3}$$

with $k_1 = 0$ if $k_0 = 0$ and $k_1 = k_0 - 1$ if $k_0 \in \mathbb{N}^*$.

(iii) Assume (6.2) and that $g_{\rm in} \in L^2$, with $\|g_{\rm in}\|_{L^2\cap L^1_3} \le M_1 \in (0,\infty)$. Then there exists $\lambda \in (-\infty,0)$ and for any exponential weight function m with exponent $s \in (0,1/2)$ and any $k \in \mathbb{N}$, there exists a constant K (which depends on ρ, c_1, M_1, k, m) such that we may split $g = g^S + g^R$ with

$$\forall\, t \in [0,T], \quad \|g^S(t,\cdot)\|_{H^k\cap L^1(m^{-1})} \le K, \quad \|g^R(t,\cdot)\|_{L^1_3} \le K\, e^{\lambda\, t}. \tag{6.4}$$

Remark 6.2 *It is worth mentioning that these estimates are uniform with respect to the inelasticity parameter $\alpha \in (0,1)$. Indeed, one the one hand, this was already the case for the moment estimate (6.1) in [24, Proposition 3.1]. On the other hand (6.3) and (6.4) from [24, Theorem 3.5, Theorem 3.6] were (partially) based on the use of the damping effect of the anti-drift term (whose coefficient was fixed to $\tau = 1$). Here the damping effect of the anti-drift term vanishes ($\tau_\alpha \to 0$) but it is replaced (as for the elastic Boltzmann equation) by the lower bound on the energy (6.2) which allows for a control from below on the convolution term $L(g)$ appearing in the loss term of the collision operator (see Lemma 2.3), which is enough to conclude also in this case.*

6.1 Local linearized asymptotic stability

Let us first consider the nonlinear evolution equation (1.15) in $L^1(m^{-1}) \cap H^k$, and the associated equation on the fluctuation h of a solution g around the unique equilibrium $\bar G_\alpha$: $g = \bar G_\alpha + h$ and

$$\partial_t h = \mathcal{L}_\alpha h + Q_\alpha(h,h).$$

Let us start by stating an inequality that we shall need in the sequel.

Lemma 6.3 *For any exponential weight function m (as defined in (1.28)), there is a constant $C \in (0,\infty)$ such that for any $h \in W^{3,1}_3(m^{-1})$ and any $\alpha \in (0,1)$,*

$$\|\Pi_\alpha Q_\alpha(h,h)\|_{L^1(m^{-1})} \le C\,(1-\alpha)\,\|h\|^2_{W^{3,1}_3(m^{-1})}.$$

Proof of Lemma 6.3. We write

$$\Pi_\alpha Q_\alpha(h,h) = \Pi_\alpha(Q_\alpha(h,h) - Q_1(h,h)) + (\Pi_\alpha - \Pi_1)Q_1(h,h).$$

On the one hand, from Lemma 5.15 (i) and (3.4), there is $C \in (0,\infty)$ such that

$$\|\Pi_\alpha(Q_\alpha(h,h) - Q_1(h,h))\|_{L^1(m^{-1})} \le C\,(1-\alpha)\|h\|^2_{W^{3,1}_3(m^{-1})}.$$

On the other hand, from (5.3) and (3.1), we get

$$\|(\Pi_\alpha - \Pi_1)Q_1(h,h)\|_{L^1(m^{-1})} \le C\,(1-\alpha)\,\|h\|^2_{W^{3,1}_3(m^{-1})}.$$

The proof of the lemma is immediate by gathering the two previous estimates. □

We now state a first local linearized stability result.

Proposition 6.4 *For any $\alpha \in [\alpha_3, 1)$, the self-similar profile $\bar G_\alpha$ is locally asymptotically stable, with domain of stability uniform according to $\alpha \in [\alpha_3,1)$.*

More precisely, let us fix $\rho \in (0,\infty)$ and some exponential weight function m as in (1.28). There is $k_1, q_1 \in \mathbb{N}^*$ such that for any $M_0 \in (0,\infty)$ there exists $C, \varepsilon \in (0,\infty)$ such that for any $\alpha \in [\alpha_3, 1]$, for any $g_{\rm in} \in H^{k_1} \cap L^1(m^{-q_1})$ with mass ρ, momentum 0 satisfying

(6.5) $$\|g_{\rm in}\|_{H^{k_1}\cap L^1(m^{-q_1})} \le M_0, \qquad \|g_{\rm in} - \bar G_\alpha\|_{L^1(m^{-1})} \le \varepsilon,$$

the solution g to the rescaled equation (1.29) with initial datum $g_{\rm in}$ satisfies

(6.6) $$\forall\, t \ge 0, \quad \|\Pi_\alpha\,(g_t - \bar G_\alpha)\|_{L^1(m^{-1})} \le C\,\|g_{\rm in} - \bar G_\alpha\|_{L^1(m^{-1})}\, e^{\mu_\alpha\, t},$$

(6.7) $$\forall\, t \ge 0, \quad \|(\mathrm{Id} - \Pi_\alpha)\,(g_t - \bar G_\alpha)\|_{L^1(m^{-1})} \le C\,\|g_{\rm in} - \bar G_\alpha\|_{L^1(m^{-1})}\, e^{(3/2)\,\mu_\alpha\, t}.$$

Proof of Proposition 6.4. Step 1. Let us first denote by c_1 the constant given in Step 5 of Proposition 2.1 such that

$$\forall\, \alpha \in [\alpha_1, 1), \quad \mathcal{E}(\bar G_\alpha) \ge 2\, c_1.$$

We may then fix $\varepsilon_0 \in (0,\infty)$ in such a way that

(6.8) $$\|g - \bar G_\alpha\|_{L^1(m^{-1})} \le \varepsilon_0 \qquad \text{implies} \qquad \mathcal{E}(g) \ge c_1,$$

and define

$$T_* := \sup\{T,\ \mathcal{E}(g_t) \geq c_1\ \ \forall\, t \in [0,T]\} \in (0,\infty].$$

From Lemma 6.1 (i) & (ii), there exists $M \in (0,\infty)$ (depending on ρ, c_1, k_1, q_1, M_0) such that for any $T \in (0,\infty)$ there holds

$$\sup_{t\in[0,T_*]} \|g\|_{H^{k_1}\cap L^1(m^{-q_1})} \leq M. \tag{6.9}$$

Let us now consider the fluctuation $h_t = g_t - \bar{G}_\alpha$. Thanks to the mass and momentum conservations, it satisfies $h_t \in \mathcal{C}_{0,0}$ for all times, as well as the bound (6.9). We define the following decomposition on h:

$$h^1 = \Pi_\alpha h \quad \text{and} \quad h^2 = (\mathrm{Id} - \Pi_\alpha)h =: \Pi_\alpha^\perp h.$$

Since the spectral projection Π_α commutes with the linearized operator $\mathcal{L}_\alpha$, the equation on h^1 writes

$$\partial_t h^1 = \mu_\alpha\, h^1 + \Pi_\alpha Q_\alpha(h,h).$$

Multiplying that equation by $(\operatorname{sign} h)\, m^{-1}$ and integrating in the velocity variable, we deduce thanks to Lemma 6.3 and to (B.2), (6.9) that on $(0,T_*)$ the following holds

$$\begin{aligned}\frac{d}{dt}\|h^1\|_{L^1(m^{-1})} &\leq \mu_\alpha\, \|h^1\|_{L^1(m^{-1})} + C\,(1-\alpha)\, \|h\|^2_{W^{3,1}_3(m^{-1})} \\ &\leq (1-\alpha)\left[C_1\, \|h^1\|^{3/2}_{L^1_2} + C_1 \|h^2\|^{3/2}_{L^1_2} - C_2\, \|h^1\|_{L^1(m^{-1})}\right],\end{aligned} \tag{6.10}$$

for some constants C_1 depending on M and the possible choice $C_2 = \rho/2$ for C_2. For the second part h^2 we have the following equation

$$\partial_t h^2 = \Pi_\alpha^\perp\, \mathcal{L}_\alpha\, h^2 + \Pi_\alpha^\perp Q_\alpha(h,h).$$

Since the linearized operator $\mathcal{L}_\alpha$ restricted to $\Pi_\alpha^\perp$ generates the semigroup $R_\alpha(t)$ defined in point (iii) of Theorem 5.2, the Duhamel formula reads

$$h^2(t) = R_\alpha(t)\, h_{\mathrm{in}} + \int_0^t R_\alpha(t-s)\, \Pi_\alpha^\perp Q_\alpha(h,h)(s)\, ds.$$

From (5.1) and (3.1) we have

$$\|h^2(t)\|_{L^1(m^{-1})} \leq C\, e^{\bar\mu\, t}\, \|h_{\mathrm{in}}\|_{L^1(m^{-1})} + C \int_0^t e^{\bar\mu\,(t-s)}\, \|h(s)\|^2_{W^{2,1}_2(m^{-1})}\, ds.$$

We deduce

$$\|h^2_t\|_{L^1(m^{-1})} \leq C_3\, e^{\bar\mu\, t}\, \|h^2_{\mathrm{in}}\| + C_4 \int_0^t e^{\bar\mu\,(t-s)} \left(\|h^1_s\|^{3/2}_{L^1(m^{-1})} + \|h^2_s\|^{3/2}_{L^1(m^{-1})}\right) ds. \tag{6.11}$$

with C_4 depending on M thanks to (B.2) and (6.9). It is then easy to show by comparison arguments from (6.10) and (6.11) that there are $0 < \varepsilon_2 \leq \varepsilon_1 \leq \varepsilon_0$ (one can take for instance $\varepsilon_1 \le \varepsilon_0/2$ satisfying $2\,C_1\,\varepsilon_1^{1/2} < C_2$ and $2\,C_4\,\varepsilon_1^{1/2} < 1/2$ and next $\varepsilon_2 \le \varepsilon_1$ satisfying $C_3\,\varepsilon_2 < \varepsilon_1/2$) such that

$$\|h^1_{\rm in}\|_{L^1(m^{-1})} + \|h^2_{\rm in}\|_{L^1(m^{-1})} \le \varepsilon_2 \quad \text{implies} \quad \sup_{t\in[0,T_*]} \max\left\{\|h^1_t\|_{L^1(m^{-1})}, \|h^2_t\|_{L^1(m^{-1})}\right\} \le \varepsilon_1. \tag{6.12}$$

Gathering (6.8) and (6.12) we deduce that there exists $\varepsilon \in (0,\varepsilon_2)$ such that under condition (6.5) there holds $T_* = \infty$ as well as

$$\sup_{t\in(0,\infty)} \|g - \bar G_\alpha\|_{L^1(m^{-1})} \le 2\,\varepsilon_1 \le \varepsilon_0.$$

Step 2. In a second step, coming back to (6.11) and to the integral version of (6.10) and setting $y(t) = \|h^1\| + |\mu_\alpha|\,\|h^2\|$, we obtain

$$y(t) \le C_5\,e^{\mu_\alpha\,t}\,y(0) + C_6\,|\mu_\alpha| \int_0^t e^{\mu_\alpha\,(t-s)}\,y(s)^{3/2}\,ds. \tag{6.13}$$

Then we have to the following variant of the Gronwall lemma whose proof is the same that the one of [27, Lemma 4.5] and is therefore skipped:

Lemma 6.5 *Let $y = y(t)$ be a nonnegative continuous function on $\mathbb{R}_+$ such that for some constants a, b, θ, $\mu > 0$,*

$$y(t) \le a\,e^{-\mu t} X + b\left(\int_0^t e^{-\mu(t-s)} y(s)^{1+\theta}\,ds\right)$$

(as compared to [27, Lemma 4.5], X needs not necessarily be $y(0)$). Then if X and b are small enough, we have

$$y(t) \le C\,X\,e^{-\mu t}.$$

for some explicit constant $C > 0$.

Thanks to the uniform smallness estimate on $y(t)$ we can apply the lemma with $\theta = 1/4$ for instance, and we get

$$y(t) \le C_7\,y(0)\,e^{\mu_\alpha\,t}$$

from which we deduce the estimate (6.6) for the h^1 part of $g - \bar G_\alpha$. Finally, we may insert that estimate on h^1 in (6.11) and we get

$$\|h^2(t)\|_{L^1(m^{-1})} \le C_3'\,(e^{\bar\mu\,t} + e^{(3/2)\,\mu_\alpha\,t})\,\|h_{\rm in}\|_{L^1(m^{-1})} + C_4 \int_0^t e^{\bar\mu\,(t-s)}\,\|h^2(s)\|^{3/2}_{L^1(m^{-1})}\,ds.$$

The same kind of computation yields to

$$\|h^2_t\| \le C_8\,e^{(3/2)\,\mu_\alpha\,t}\,\|h(0)\|$$

from which (6.7) follows. □

6.2 Nonlinear stability estimates

In this subsection we shall prove that when the inelasticity is small, depending on the size of the initial datum (but not necessary close to the self-similar profile), the equation (1.15) is stable. This relies mainly on the fact that the entropy production timescale is of a different order (much faster) that the energy dissipation timescale as $\alpha \to 1$. This point is familiar to physicists (see for instance [9]) which separate, for granular gases with small inelasticity, the molecular timescale (the level where entropy production effects dominate) and the cooling timescale (much slower than the molecular timescale).

Proposition 6.6 *Define $k_2 := \max\{k_0, k_1\}$, $q_2 := \max\{q_0, q_1, 3\}$, where k_i and q_i are defined in Theorem 3.5 and Corollary 3.4. For any ρ, $\mathcal{E}_0$, M_0 there exists $\alpha_4 \in [\alpha_3, 1)$, $c_1 \in (0,\infty)$ and for any $\alpha \in [\alpha_4, 1]$ there exist $\varphi = \varphi(\alpha)$ with $\varphi(\alpha) \to 0$ as $\alpha \to 1$ and $T = T(\alpha)$ (possibly blowing-up as $\alpha \to 1$) such that any initial datum $0 \le g_{\rm in} \in L^1_{q_2} \cap H^{k_2} \cap \mathcal{C}_{\rho,0,\mathcal{E}_0}$ with*

$$\|g_{\rm in}\|_{L^1_{q_2} \cap H^{k_2}} \le M_0,$$

the solution g associated to the rescaled equation (1.29) satisfies

$$\forall\, t \ge 0, \quad \mathcal{E}(g_t) \ge c_1$$

and for all $\alpha' \in [\alpha_4, 1)$ and then all $\alpha \in [\alpha', 1]$

$$\forall\, t \ge T(\alpha'), \quad \left\|g_t - \bar G_\alpha\right\|_{L^1_2} \le \varphi(\alpha'). \tag{6.14}$$

Proof of Proposition 6.6. Let us consider a solution $g \in C([0,\infty); L^1_{q_2} \cap H^{k_2})$ to the rescaled equation (1.29) with given initial datum $g_{\rm in}$, whose existence has been established in [23, 24]. We split the proof of the Proposition into five steps.

Step 1. From the propagation and appearance of uniform moment bounds [24, Proposition 3.1, (iii)], which it is worth noticing have been obtained uniformly with respect to the elastic coefficient (see also [8]), there exists $C_1 \in (0,\infty)$ such that

$$\sup_{t\ge 0} \|g\|_{L^1_{q_2}} \le C_1. \tag{6.15}$$

Let us define $c_1 := \min\{\mathcal{E}(\bar G_1), \mathcal{E}_0\}/4$, and

$$T_* := \sup\{T \;;\; \forall\, t \in [0,T],\ \mathcal{E}(g(t,\cdot)) \ge c_1\}. \tag{6.16}$$

Next from the equation on the evolution of energy

$$\mathcal{E}'(t) = -(1-\alpha^2)\, b_1\, D_{\mathcal{E}}(g) + (1-\alpha)\, 2\, \rho\, \mathcal{E} \tag{6.17}$$

and (6.15) there holds

$$|\mathcal{E}'(t)| \le C_2\,(1-\alpha) \qquad \forall\, t \ge 0$$

(take for instance $C_2 = 2\, b_1\, C_1^2 + C_1$), from which we deduce that we necessarily have

$$T_* \ge C_3\,(1-\alpha)^{-1}$$

(take for instance $C_3 = (3/4)\,\mathcal{E}_0/C_2$).

Step 2. From point (ii) of Lemma 6.1, we have for some constant $C_5 \in (0,\infty)$

$$\forall\, t \in [0,T_*] \qquad \|g_t\|_{H^{k_2}} \le C_4. \tag{6.18}$$

Moreover from Lemma 2.6, for any time $t_1 \in (0,T_*)$, there exists some constant $C_5 = C_5(\rho, C_4, t_1)$ such that

$$\forall\, t \in [t_1,T_*] \qquad g(t,v) \ge C_5^{-1}\, e^{-C_5\,|v|^8}. \tag{6.19}$$

Step 3. With the notations of Theorem 3.5, we compute the evolution of the relative entropy of $g(t,\cdot)$ with respect to the associated Maxwellian $M[g(t,\cdot)]$, and we obtain

$$\begin{aligned}\frac{d}{dt} H(g|M[g]) &= \frac{d}{dt} H(g) - \frac{d}{dt}\int_{\mathbb{R}^N} g \,\ln\, M[g] = \frac{d}{dt} H(g) - \frac{\rho\,N}{2\,\mathcal{E}}\frac{d}{dt}\mathcal{E} \\ &= -D_{H,\alpha}(g) - \frac{\rho\,N}{2\,\mathcal{E}}\,(1-\alpha)\,D_{\mathcal{E}}(g).\end{aligned}$$

Next from Lemma 3.4 and the estimates (6.15), (6.16), (6.18) and (6.19) we have

$$\frac{d}{dt} H(g|M[g]) \quad = \quad -D_{H,1}(g,g) + \mathcal{O}(1-\alpha) \quad \text{on} \quad (t_1,T_*).$$

Then from (3.11), we are then led to the following differential inequation on the relative entropy

$$\frac{d}{dt} H(g|M[g]) \le -C_6\, H(g|M[g])^2 + C_7\,(1-\alpha) \quad \text{on} \quad (t_1,T_*).$$

By straightforward computations we deduce that *independently of the value of* $H(g_{t_1}|M[g_{t_1}])$ (this “loss of memory” effect is typical of differential equations with overlinear damping terms), we have

$$\forall\, t \in [t_1,T_*], \quad H(g_t|M[g_t]) \le C_8\,(1-\alpha)^{1/2}\, \frac{1+e^{-C_9\,(1-\alpha)^{1/2}\,(t-t_1)}}{1-e^{-C_9\,(1-\alpha)^{1/2}\,(t-t_1)}}$$

for some explicit constants. As a conclusion, defining $t_2 := t_1 + C_9^{-1}\,(1-\alpha)^{-1/2}$ and choosing $\bar\alpha \in [\alpha_3,1)$ in such a way that $t_2 < T_*$ we have for $\alpha \in [\alpha',1)$

$$\forall\, t \in [t_2,T_*] \quad H(g(t)|M[g]) \le C_{10}\,(1-\alpha)^{1/2}.$$

Finally, using Csiszár-Kullback-Pinsker inequality (3.10), as well as Hölder inequality, we obtain under the same conditions on α and the time variable:

$$\|g - M[g]\|_{L^1_3} \le C\,\|g-M[g]\|_{L^1}^{1/2}\,\|g\|_{L^1_6}^{1/2} \le C\,H(g|M[g])^{1/4} \le C\,(1-\alpha)^{1/8}. \tag{6.20}$$

Step 4. Now let us go back to the energy equation (6.17). First, with the help of the moment bound (6.15), one may write

$$\mathcal{E}'(t) = 2\,(1-\alpha)\,[\rho\,\mathcal{E} - b_1\,D_{\mathcal{E}}(g) + \mathcal{O}(1-\alpha)].$$

Thanks to (6.20) we deduce

$$\mathcal{E}'(t) = 2\,(1-\alpha)\,(\rho\,\mathcal{E} - b_1\,D_{\mathcal{E}}(M[g]) + \mathcal{O}((1-\alpha)^{1/8})) \quad \text{on} \quad (t_2,T_*).$$

Finally, thanks to (3.16), (3.17) and the relation $\mathcal{E}(g) = \rho\, N\, \theta(g)$, we get on (t_2, T_*)

$$\mathcal{E}'(t) := \Sigma(\mathcal{E}(t), \alpha) = (1-\alpha)\,[k_3\, \mathcal{E}\, (\bar{\mathcal{E}}_1^{1/2} - \mathcal{E}^{1/2}) + \mathcal{O}((1-\alpha)^{1/8})], \tag{6.21}$$

where $\bar{\mathcal{E}}_1 = \rho\, N\, \bar\theta_1$ with $\bar\theta_1$ is the quasi-elastic self-similar temperature defined in (1.27). We may then choose $\alpha'' \in [\alpha', 1)$ such that $\Sigma(c_1, \alpha) > 0$ for any $\alpha \in [\alpha'', 1)$. We conclude by maximum principle that $T_* = \infty$ for $\alpha \in [\alpha'', 1)$. In particular, all the previous estimates on g are uniform on (t_2, ∞).

Step 5. Thanks to (6.21) we easily get

$$\frac{d}{dt} (\mathcal{E} - \bar{\mathcal{E}}_1)^2 \le -(1-\alpha)\,[k_5\, (\mathcal{E} - \bar{\mathcal{E}}_1)^2 + \mathcal{O}((1-\alpha)^{1/8})],$$

so that (for some constants $a, b > 0$)

$$\forall\, t \ge t_2, \quad |\mathcal{E}(t) - \bar{\mathcal{E}}_1| \le |\mathcal{E}(t_2) - \bar{\mathcal{E}}_1|\, e^{-a\,(1-\alpha)\,(t-t_2)} + b\,(1-\alpha)^{1/8}.$$

Setting $T(\alpha) = \max\{t_2, c\,(1-\alpha)^{-1}\}$ for some suitable constant $c > 0$, we then obtain

$$|\mathcal{E} - \bar{\mathcal{E}}_1| = \mathcal{O}((1-\alpha)^{1/8}) \quad \text{on} \quad [T(\alpha), \infty). \tag{6.22}$$

In order to conclude that (6.14) holds, we write

$$g(t) - \bar G_\alpha = (g(t) - M[g(t)]) + (M[g(t)] - \bar G_1) + (\bar G_1 - \bar G_\alpha),$$

and we estimate the first term thanks to (6.20), the second term thanks to (6.22) and the third term by (3.12). □

6.3 Decomposition and Liapunov functional for smooth initial datum

The proof of the gobal convergence (point (v) of Theorem 1.1) for smooth initial data only amounts to connect the two previous results of Propositions 6.4 and 6.6 by choosing α such that $\varphi(\alpha) \le \varepsilon$ where ε is the size of the attraction domain in Proposition 6.4 and $\varphi(\alpha)$ is defined in Propositions 6.6. More precisely, we state without proof the straightforward combination of Propositions 6.6 and Proposition 6.4.

Corollary 6.7 *Let us fix an exponential weight function m as in (1.28), with exponent $s \in (0,1)$. Then for any ρ, $\mathcal{E}_0$, M_0 there exists C and $\alpha_5 \in [\alpha_4, 1)$ (depending on ρ, $\mathcal{E}_0$, M_0, m) such that for any $\alpha \in [\alpha_5, 1)$ and any initial datum $0 \le g_{\rm in} \in L^1(m^{-q_2}) \cap H^{k_2}$ satisfying*

$$g_{\rm in} \in \mathcal{C}_{\rho,0,\mathcal{E}_0}, \qquad \|g_{\rm in}\|_{L^1(m^{-q_2}) \cap H^{k_2}} \le M_0,$$

the solution g associated to the rescaled equation (1.29) satisfies

$$\forall\, t \ge 0, \quad \|\Pi_\alpha\, (g_t - \bar G_\alpha)\|_{L^1(m^{-1})} \le C\, e^{\mu_\alpha\, t},$$

$$\forall\, t \ge 0, \quad \|(\mathrm{Id} - \Pi_\alpha)\, (g_t - \bar G_\alpha)\|_{L^1(m^{-1})} \le C\, e^{(3/2)\, \mu_\alpha\, t}.$$

Remark 6.8 *Note that the constant C in the rate of decay does not depend on α. This comes from the fact the size of the linearized stability domain is uniform as α goes to 1 in Proposition 6.4, which allows in Propositions 6.6 to pick a fixed α' such that in the estimate (6.14) $\varphi(\alpha')$ is less than this size, and therefore that the time $T(\alpha')$ required to enter this neighborhood does not blow-up as α goes to 1.*

As a by-product of the previous propositions, we state and prove a result which provides a partial answer to the question (important from the physical viewpoint) of finding Liapunov functionals for this particles system. Let us define the required objects. We consider a fixed mass ρ and some restitution coefficient α whose range will be specified below. At initial times, non-linear effects dominate and therefore we define

$$\mathcal{H}_1(g) := H(g|M[g]) + (\mathcal{E} - \bar{\mathcal{E}}_\alpha)^2$$

where $\bar{\mathcal{E}}_\alpha = \mathcal{E}(\bar{G}_\alpha)$ is the energy of the self-similar profile corresponding to α and the mass ρ. At eventual times, linearized effects dominate. Therefore we define a quite natural candidate from the spectral study:

$$\mathcal{H}_2(g) := \|h^1\|^2_{L^1(m^{-1})} + (1-\alpha) \int_0^{+\infty} \left\|R_\alpha(s)\, h^2\right\|^2_{L^2}\, ds,$$

with $h^1 = \Pi_\alpha h$, $h^2 = \Pi^\perp_\alpha h$ and $h = g - \bar{G}_\alpha$.

Proposition 6.9 *There is $k_4 \in \mathbb{N}$ big enough (this value is specified in the proof) such that for any exponential weight function m as defined in (1.28), any time $t_0 \in (0,\infty)$ and any $\rho, \mathcal{E}_0, M_0 \in (0,\infty)$, there exists $\kappa_* \in (0,\infty)$ and $\alpha_6 \in [\alpha_5, 1)$ such that for any $\alpha \in [\alpha_6, 1]$ and initial datum $g_{\rm in} \in H^{k_4} \cap L^1(m^{-1})$ satisfying*

$$g_{\rm in} \in \mathcal{C}_{\rho,0,\mathcal{E}_0}, \qquad \|g_{\rm in}\|_{H^{k_4}\cap L^1(m^{-1})} \le M_0, \qquad g_{\rm in}(v) \ge M_0^{-1}\, e^{-M_0\,|v|^8},$$

the solution g to the rescaled equation (1.29) with initial datum $g_{\rm in}$ is such that the functional

$$\mathcal{H}(g_t) = \mathcal{H}_1(g_t)\, \mathbf{1}_{\{\mathcal{H}_1(g_t)\ge\kappa_*\}} + \mathcal{H}_2(g_t)\, \mathbf{1}_{\{\mathcal{H}_1(g_t)\le\kappa_*\}}$$

is decreasing for all times $t \in [0,+\infty)$. Moreover, $\mathcal{H}(g(t,\cdot))$ is strictly decreasing as long as $g(t,\cdot)$ has not reached the self-similar state $\bar{G}_\alpha$.

Proof of Proposition 6.9. We split the proof into three steps.

Step 1: Initial times. Taking $k_4 \ge k_2$ and $\alpha \in [\alpha_4, 1)$, we know from the proof of Proposition 6.6 that the solution g satisfies that

$$\forall\, t \in [t_0,\infty), \quad \|g(t,\cdot)\|_{H^{k_4}\cap L^1(m^{-1})} \le M_1, \quad g(t,v) \ge M_1^{-1}\, e^{-M_1\,|v|^8},$$

for some constant $M_1 \in (0,\infty)$ (recall that α_4 was adjusted in terms of $\rho, \mathcal{E}_0, M_0$). Coming back then to Steps 3 and 4 in the proof of Proposition 6.6, we obtain the two following differential equation on (t_0,∞)

$$\frac{d}{dt} H(g|M[g]) \le -K_1\, H(g|M[g])^2 + \mathcal{O}(1-\alpha)$$

and

$$\frac{d}{dt}\mathcal{E} = 2\,\rho\,(1-\alpha)\, \left[K_2\, \mathcal{E}\, (\bar{\mathcal{E}}_\alpha^{1/2} - \mathcal{E}^{1/2})\, (\mathcal{E} - \bar{\mathcal{E}}_\alpha) + \mathcal{O}((1-\alpha)^{1/8})\right],$$

for some constants $K_i \in (0,\infty)$. We easily deduce that for any $\kappa \in (0,\infty)$ there exists $\alpha_\kappa \in [\alpha_5, \infty)$ such that

$$\frac{d}{dt}\mathcal{H}_1(g_t) < 0 \;\text{ for any }\; t \in (0,\infty) \text{ such that } \mathcal{H}_1(g_t) \ge \kappa. \tag{6.23}$$

Step 2: Eventual times. Let us first remark that from point (iii) in Theorem 5.2 (iii) and the interpolation inequality (B.2), for any $q \in \mathbb{N}^*$ there exists $k, k' \in \mathbb{N}$ and $C_i \in (0,\infty)$ such that

$$\begin{aligned} \left\|R_\alpha\, h^2\right\|_{L^2} &\le C_1 \left\|R_\alpha\, h^2\right\|_{W^{k,1}(m^{-q/2})} \\ &\le C_2\, e^{\bar\mu\, s} \left\|h^2\right\|_{W_2^{k+2,1}(m^{-q/2})} \le C_3\, e^{\bar\mu\, s} \left\|h\right\|_{H^{k'}\cap L^1(m^{-q})}, \end{aligned}$$

so that, taking k_4 big enough, the functional $\mathcal{H}_2(g(t,.))$ is well-defined for any times $t \in (0,\infty)$. First observe that from (6.10) there holds

$$\frac{d}{dt}\|h^1\|^2_{L^1(m^{-1})} \le (1-\alpha)\left[K_1\,\|h\|^{5/2}_{L^1(m^{-1})} - K_2\,\|h^1\|^2_{L^1(m^{-1})}\right]. \tag{6.24}$$

Second, we compute (with the notation of Subsection 5.7)

$$\frac{d}{dt}\int_0^{+\infty} \left\|R_\alpha(s)\, h_t^2\right\|^2_{L^2}\, ds = 2\int_0^{+\infty}\int_{\mathbb{R}^N} (e^{s\,\bar{\mathcal{L}}_\alpha}\, h^2)\,[e^{s\,\bar{\mathcal{L}}_\alpha}\,(\bar{\mathcal{L}}_\alpha h^2 + \Pi^\perp_\alpha Q_\alpha(h,h))]\, ds\, dv.$$

On the one hand,

$$\begin{aligned} I_1 &= 2\int_0^{+\infty}\int_{\mathbb{R}^N} (e^{s\,\bar{\mathcal{L}}_\alpha}\, h^2)\,[e^{s\,\bar{\mathcal{L}}_\alpha}\,\bar{\mathcal{L}}_\alpha h^2]\, ds dv \\ &= \int_0^{+\infty} \frac{d}{ds}\|e^{s\,\bar{\mathcal{L}}_\alpha}\, h^2\|^2_{L^2}\, ds = -\|h^2\|^2_{L^2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} I_2 &= 2\int_0^{+\infty}\int_{\mathbb{R}^N} (R_\alpha(s)\, h^2)\,[R_\alpha(s)\,\Pi^\perp_\alpha Q_\alpha(h,h))]\, ds dv \\ &\le 2\,C_1^2 \int_0^{+\infty} \|R_\alpha(s)\, h^2\|_{W^{k_1,1}(m^{-q/2})}\, \|R_\alpha(s)\,\Pi^\perp_\alpha Q_\alpha(h,h)\|_{W^{k_1,1}(m^{-q/2})}\, ds \\ &\le C_2' \left(\int_0^{+\infty} e^{2\bar\mu\, s}\, ds\right) \|h^2\|_{W_2^{k_1+1,1}(m^{-q/2})}\, \|Q_\alpha(h,h)\|_{W_2^{k_1+1,1}(m^{-q/2})} \\ &\le C_3'\, \|h^2\|^{3/4}_{L^2}\, \|h^2\|^{1/4}_{H^{k_3}\cap L^1(m^{-1})}\, \|h\|^{3/2}_{L^2}\, \|h\|^{1/2}_{H^{k_3}\cap L^1(m^{-1})}, \end{aligned}$$

for some $k_3 \in \mathbb{N}$ given by Proposition B.1. Taking $k_4 \ge k_3$, we then obtain

$$\frac{d}{dt}\int_0^{+\infty} \left\|R_\alpha(s)\, h_t^2\right\|^2_{L^2}\, ds \le K_3\,\|h\|^{9/4}_{L^2} - \|h^2\|^2_{L^2}. \tag{6.25}$$

Gathering (6.24) and (6.25) and using some interpolation again, we deduce that there exists $\kappa' \in (0,\infty)$ such that

$$\frac{d}{dt}\mathcal{H}_2(g_t) < 0 \quad \text{for any } t \in (0,\infty) \text{ such that } \|h_t\|_{L^1} \le \kappa'. \tag{6.26}$$

Step 3. We conclude putting together (6.23) and (6.26), and using (3.10), (3.12) in order to prove that

$$\mathcal{H}_1(g) \le \kappa \quad \text{implies} \quad \|h_t\|_{L^1} \le \kappa',$$

for $\alpha \in [\alpha_6, 1]$ for some $\alpha_6 \in [\alpha_5, 1)$. □

6.4 Global stability for general initial datum

We first state and prove a regularity result on the iterated gain term which is the inelastic collision operator version of the same result proved for the elastic collision operator in [26, 1].

Lemma 6.10 *There exists a constant C such that for any $f, g, h \in L^1_2(\mathbb{R}^N)$ and any $\alpha \in (0,1]$ there holds*

$$\|Q^+_\alpha(f, Q^+_\alpha(g,h))\|_{L^3} \le C\, \|f\|_{L^1_2}\, \|g\|_{L^1_2}\, \|h\|_{L^1_2}. \tag{6.27}$$

Proof of Lemma 6.10. We follow [26, lemma 2.1] and [1, lemma 2.1] and we make use of the Carleman representation introduced in [24, Proposition 1.5]. Let us consider $f, g, h \in L^1_2(\mathbb{R}^N)$ and $\phi \in L^\infty(\mathbb{R}^N)$. We apply twice the weak formulation of the gain term

$$\begin{aligned}&\int_{\mathbb{R}^N} Q^+(f, Q^+(g,h))(v)\,\phi(v)\,dv \\ &= \int_{\mathbb{R}^N} Q^+(g,h)(v)\,\left[\int_{\mathbb{R}^N} f(v_2)\,|v-v_2| \int_{S^2} \phi(w_2')\,d\sigma_2 dv_2\right] dv \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g(v)\,h(v_1)\,f(v_2)\,\left[|v-v_1| \int_{\mathbb{S}^2} |v_1'-v_2| \int_{\mathbb{S}^2} \phi(v_2'')\,d\sigma_2\,d\sigma_1\right] dv\,dv_1\,dv_2\end{aligned}$$

with $w_2' = V'(v_2, v, \sigma_2)$, $v_1' = V'(v, v_1, \sigma_1)$ and therefore $v_2'' = V'(v_2, v_1', \sigma_2)$. Recall that for any given $v, v_*, \sigma \in \mathbb{R}^N$, we define

$$w = \frac{v+v_*}{2}, \quad u = v - v_*, \quad \gamma = \frac{1+e}{2}, \quad u' = (1-\gamma)\,u + \gamma\,|u|\,\sigma$$

and then

$$V' = V'(v, v_*, \sigma) = \frac{w}{2} + \frac{u'}{2} = v + \frac{\gamma}{2}\,(|u|\,\sigma - u)$$

$$V_*' = V_*'(v, v_*, \sigma) = \frac{w}{2} - \frac{u'}{2} = v_* - \frac{\gamma}{2}\,(|u|\,\sigma - u).$$

We denote by $\Phi = \Phi(v, v_1, v_2)$ the term between brackets in the last integral. Introducing the point w_1 and the set $S_{v,v_1,\varepsilon}$ defined by

$$w_1 := (1-\gamma/2)\,v + (\gamma/2)\,v_1, \qquad S_{v,v_1,\varepsilon} := \left\{z \in \mathbb{R}^N;\ \Big||z-w_1| - (\gamma/2)\,|v-v_1|\Big| \le \varepsilon/2\right\},$$

we get

$$\Phi = \frac{(2/\gamma)^2}{|v-v_1|} \lim_{\varepsilon \to 0} \frac{\Psi_\varepsilon}{\varepsilon}, \quad \Psi_\varepsilon = \int_{\mathbb{R}^N}\int_{\mathbb{S}^2} \mathbf{1}_{S_{v,v_1,\varepsilon}}(v_1')\,|v_1'-v_2|\,\phi(v_2'')\,d\sigma_2\,dv_1'. \tag{6.28}$$

Remarking that $v_2'' = v_2 + (\gamma/2)\,(|u_2|\,\sigma_2 - u_2)$ with $u_2 = v_1' - v_2$, we observe that the integral term Ψ_ε is very similar to the collision term Q^+ (here v_2 (resp. v_1, σ_2, γ, v_2'') plays the role of v (resp. v_1, σ, β, $'v$) in the gain term) and therefore we may give a Carleman representation of Ψ_ε. The same computations as performed in [24, Proposition 1.5] yield

$$\Psi_\varepsilon = \frac{4}{\gamma^2} \int_{\mathbb{R}^N}\int_{E_{v_2,v_2''}} \mathbf{1}_{S_{v,v_1,\varepsilon}}(v_1')\,|v_2''-v_2|^{-1}\,\phi(v_2'')\,dE(v_3'')\,dv_2''$$

where $E_{v_2,v_2''}$ is the hyperplan orthogonal to the vector $v_2 - v_2''$ and passing through the point $\Omega_{v_2,v_2''} = v_2 + (1-\gamma^{-1})\,(v_2 - v_2'')$. Here v_3'' stands for the post collision velocity issued from v_1', that is $v_3'' = V_*'(v_2, v_1', \sigma_2)$, and then, thanks to the momentum conservation, $v_1' := v_2'' + v_3'' - v_2$. We finally define $\Pi_{v_2,v_2''}$ the hyperplan orthogonal to the vector $v_2 - v_2''$ and passing through the point $\Omega'_{v_2,v_2''} = v_2'' + (1-\gamma^{-1})\,(v_2 - v_2'')$ and we get

$$\Psi_\varepsilon = \frac{4}{\gamma^2} \int_{\mathbb{R}^N} \int_{\Pi_{v_2,v_2''}} \mathbf{1}_{S_{v,v_1,\varepsilon}}(v_1')\, |v_2'' - v_2|^{-1}\, \phi(v_2'')\, dE(v_1') dv_2''. \tag{6.29}$$

Now, arguing as in [1, lemma 2.1], we see that the measure of the intersection Γ_ε between the plane $\Pi_{v_2,v_2''}$ and the thickened sphere $S_{v,v_1,\varepsilon}$ is bounded by $\pi\,\varepsilon\,\gamma\,|v - v_1|$ and that $v_1'' \in \Gamma_\varepsilon$ implies that $v_2'' \in B^\varepsilon$ with

$$B^\varepsilon := \Big\{ z \in \mathbb{R}^N; |z|^2 \le |v|^2 + |v_1|^2 + 2\,\varepsilon\,(|v| + |v_1|) + \varepsilon^2\,|v_2|^2 \Big\}.$$

Gathering these estimates with (6.28) and (6.29) we get

$$\begin{aligned}
\Phi &= \frac{(2/\gamma)^4}{|v - v_1|} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^N} \frac{\phi(v_2'')}{|v_2'' - v_2|}\, \mathrm{mes}(\Gamma_\varepsilon)\, dv_2'' \\
&\le \frac{2^4\,\pi}{\gamma^3} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \frac{\phi(v_2'')}{|v_2'' - v_2|}\, \mathbf{1}_{B^\varepsilon}(v_2'')\, dv_2'' = \frac{2^4\,\pi}{\gamma^3} \int_{\mathbb{R}^N} \frac{\phi(v_2'')}{|v_2'' - v_2|}\, \mathbf{1}_{B^0}(v_2'')\, dv_2''
\end{aligned}$$

where we have defined $B^0 := \{z \in \mathbb{R}^N; |z|^2 \le |v|^2 + |v_1|^2\}$. Using [1, lemma 2.2] we may conclude as in the end of [1, lemma 2.1] and therefore (6.27) follows. □

We second establish that the solution g of the rescaled equation (1.29) decomposes between a regular part and a small remaining part as it has been proved for the elastic Boltzmann equation in [28], and then partially extended to the inelastic Boltzmann equation in [24]. As compared to this last paper, this result relaxes the assumption on the initial datum to $g_{\rm in} \in L^1_3$, but at the price of the hypothesis of a lower bound on the energy.

Lemma 6.11 *Consider $g_{\rm in} \in L^1_3$ and the associated solution $g \in C([0,\infty); L^1_3)$ to the rescaled equation (1.29). Assume that for some constant $\rho, c_1, M_1, T \in (0,\infty)$ there holds*

$$g_{\rm in} \in \mathcal{C}_{\rho,0}, \qquad \|g_{\rm in}\|_{L^1_3} \le M_1, \qquad \forall\, t \in [0,T], \quad \mathcal{E}(g(t,\cdot)) \ge c_1. \tag{6.30}$$

Then, there are $\alpha_7 \in [\alpha_6, 1)$ and $\lambda \in (-\infty, 0)$, and for any exponential weight function m (as defined in (1.28) and any $k \in \mathbb{N}$, there exists a constant K (which depends on ρ, c_1, M_1, k, m) such that for any $\alpha \in [\alpha_7, 1]$, we may split $g = g^S + g^R$ with

$$\forall\, t \in [0,T], \quad \|g^S(t,\cdot)\|_{H^k \cap L^1(m^{-1})} \le K, \qquad \|g^R(t,\cdot)\|_{L^1_3} \le K\, e^{\lambda\, t}. \tag{6.31}$$

Proof of Lemma 6.11 The starting point is to write the rescaled equation (1.29) in the following way

$$\frac{\partial g}{\partial t} + \tau_\alpha\, v \cdot \nabla_v g + \ell\, g = Q_\alpha^+(g,g),$$

with $\ell(t,v) := \tau_\alpha N + L(g(t,\cdot))(v)$. Introducing the linear semigroup

$$(U_t\,h)(v) = h(v\,e^{-\tau_\alpha\,t})\,\exp\left(-\int_0^t \ell(s,v)\,ds\right)$$

and using the Duhamel formula, we have

$$g_t = U_t g_{\rm in} + \int_0^t U_{t-s} Q^+_\alpha(g_s,g_s)\,ds.$$

We iterate that last identity and we obtain $g = g_1^R + g_1^S$ with

$$g_1^R = U_t g_{\rm in} + \int_0^t U_{t-s} Q^+_\alpha(g_s, U_s g_{\rm in})\,ds, \quad g_1^S = \int_0^t\int_0^s U_{t-s} Q^+_\alpha(g_s, U_{s-u} Q^+_\alpha(g_u,g_u))\,du\,ds.$$

On the one hand, the energy lower bound (6.30) and Lemma 2.3 imply that there exists a constant $c_2 \in (0,\infty)$ such that

$$(U_t\,h)(v) \le e^{-c_2\,t}\,(V_{\xi_t} h)(v) \quad \text{with} \quad (V_\xi h)(v) = h(\xi\,v) \quad \text{and} \quad \xi_t = e^{-\tau_\alpha\,t}$$

On the other hand, straightforward homogeneity arguments leads to

$$Q^+_\alpha(g, V_\xi h) = \xi^{-N-1}\,V_{\xi^{-1}}\,Q^+_\alpha(V_{\xi^{-1}} g, h)$$

and $\|h_\xi\,|.|^q\|_{L^p} = \xi^{-q-N/p}\,\|h\,|.|^q\|_{L^p}$ for any functions g,h and positive real ξ. From these considerations we deduce that

$$\|g_1^R(t)\|_{L^1} \le e^{(N\,\tau_\alpha - c_2)\,t}\,\|g_{\rm in}\|_{L^1} + e^{((N+1)\,\tau_\alpha - c_2)\,t}\,\|g_{\rm in}\|_{L^1_1}\,\sup_{s\ge 0}\,\|g_s\|_{L^1_1} \le C\,e^{-(c_2/2)\,t},$$

for some constant C and for any $(1-\alpha)$ small enough. In the same way, we have

$$\|g_1^S(t)\|_{L^3} \quad \le \quad \int_0^t\int_0^s e^{[(2N/3+1)\,\tau_\alpha - c_2]\,(t-\sigma)}\,\|Q^+_\alpha(V_{\xi^{-1}_{s-\sigma}}\,g_s, Q^+_\alpha(g_\sigma,g_\sigma))\|_{L^3}\,d\sigma ds.$$

Taking $(1-\alpha)$ smaller if necessary and using Lemma 6.10, we obtain

$$\|g_1^S(t)\|_{L^3} \quad \le \quad \left(\int_0^t\int_0^s e^{-(c_2/2)\,(t-\sigma)}\,d\sigma\,ds\right)\,\sup_{s\ge 0}\,\|g_s\|^3_{L^1_2},$$

which ends the proof of (6.31) in the case $k=0$, with the help of point (i) of Lemma 6.1. The general case $k \in \mathbb{N}^*$ is then treated by following the strategy introduced in [28] and using the result of appearance of regularity proved in [24] (and recalled in point (iii) of Lemma 6.1). □

We third recall a classical L^1 stability result for the elastic Boltzmann equation which has been established in [24, Proposition 3.2] for the rescaled equation (1.29).

Lemma 6.12 *Consider* $0 \le g^1_{\rm in}, g^2_{\rm in} \in L^1_3 \cap \mathcal{C}_{\rho,0}$ *and the two associated solution* $g_i \in C([0,\infty); L^1_3) \cap L^\infty(0,\infty; L^1_3)$ *to the rescaled equation (1.29). There exists* $C_{\rm stab} \in (0,\infty)$ *(only depending on* b *and* $\sup_{t\ge 0}\|g^1+g^2\|_{L^1_3}$*) such that*

$$\forall\,t \ge 0, \quad \|g^2_t - g^1_t\|_{L^1_2} \le \|g^2_{\rm in} - g^1_{\rm in}\|_{L^1_2}\,e^{C_{\rm stab} t}.$$

Proof of point (iv) of Theorem 1.1. Let us consider $g_{\rm in} \in L^1_3 \cap \mathcal{C}_{\rho,0,\mathcal{E}_{\rm in}}$ with $\|g_{\rm in}\|_{L^1_3} \le M_0$ for some fixed $\mathcal{E}_{\rm in}$, $M_0 \in (0,\infty)$ and g the associated solution to the rescaled equation (1.29) which has been built in [24]. We know that there exists $M_1 \in (0,\infty)$ such that

$$\sup_{(0,\infty)} \|g(t,\cdot)\|_{L^1_3} \le M_1. \tag{6.32}$$

Step 1. We define

$$T_* := \sup\{T \in (0,\infty),\ \mathcal{E}(g(t,\cdot)) \ge c_1\ \forall\, t \in [0,T]\}, \quad c_1 := \min\{\mathcal{E}_{\rm in}, \bar{\mathcal{E}}_1\}/2.$$

We shall prove that $T_* = +\infty$. We argue by contradiction, assuming that $T_* < \infty$. From the equation on the energy (6.17) and the uniform estimate (6.32) and from the definition of T_* we have

$$T_* \ge C_1\,(1-\alpha)^{-1} \quad \text{and} \quad \mathcal{E}'(T_*) \le 0. \tag{6.33}$$

Thanks to Lemma 6.11, we may decompose

$$g = g^S + g^R \quad \text{on} \quad (0,t_1),$$

with $t_1 \in (0,T_*)$ to be fixed. At time t_1 we initiate a new flow starting from the smooth part of g. More precisely, we decompose

$$g = \tilde g^S + \tilde g^R \quad \text{on} \quad (t_1,T_*),$$

with $\tilde g^S(t_1) = [\rho/\rho(g^S(t_1))]\, g^S(t_1)$, $\tilde g^S$ solution (with mass ρ!) to the equation (1.29) on (t_1,T_*) and $\tilde g^R := g - \tilde g^S$. On the one hand, from (6.31) and Lemma 6.12 we have

$$\|\tilde g^R(t)\|_{L^1_3} \le C\, e^{C_{\rm stab}\,(T_*-t_1)+\lambda\, t_1} \quad \text{on} \quad (t_1,T_*).$$

We choose $t_1 = \eta\, T_*$ with $\eta \in (0,1)$ in such a way that $C_{\rm stab}\,(1-\eta) + \lambda\,\eta = \lambda/2$. We have then proved

$$\|\tilde g^R(t)]\|_{L^1_3} \le C\, e^{(\lambda/2)\, C_1\, (1-\alpha)^{-1}} \quad \text{on} \quad (t_1,T_*). \tag{6.34}$$

On the other hand, following Step 3 in the proof of Proposition 6.6, we deduce a similar estimate as (6.20), namely

$$\|\tilde g^S(T_*,\cdot) - M[\tilde g^S(T_*,\cdot)]\|_{L^1_3} = \mathcal{O}((1-\alpha)^{1/8}) \tag{6.35}$$

for any $(1-\alpha)$ small enough chosen in such a way that the intermediate time t_2 defined in Step 3 of the proof of Proposition 6.6 satisfies $t_1 + t_2 \le T_*$. Gathering (6.34) and (6.35) we obtain

$$\|\tilde g(T_*,\cdot) - M[g(T_*,\cdot)]\|_{L^1_3} = \mathcal{O}((1-\alpha)^{1/8}).$$

Coming back to the equation (6.17) on the energy and proceeding like in Step 4 in the proof of Proposition 6.6, we get

$$\mathcal{E}'(T_*) \ge (1-\alpha)\left[k_3\, c_1\,(\bar{\mathcal{E}}_1^{1/2} - c_1^{1/2}) - C\,(1-\alpha)^{1/8}\right] > 0$$

for any $(1-\alpha)$ small enough. That is in contradiction with (6.33) and we conclude that $T_* = +\infty$.

Step 2. Thanks to the previous step, we have a uniform in time lower bound on the energy, and therefore we can run the decomposition theorem for all times.

By applying the decomposition theorem as in Step 1 for a given time $t \in (0,\infty)$, starting a new flow at $t_1 = \eta\, t$ taking $[\rho/\rho(g^S(t_1))]\, g^S(t_1,\cdot)$ as initial datum, and then using Corollary 6.7 on the smooth part $\tilde g^S(s,\cdot)$, $s \in [t_1,t]$, we find that at time t, the solution g_t decomposes as $\tilde g^S_t + \tilde g^R_t$, where $\tilde g^S_t$ approaches the self-similar profile with rate $C\, e^{\mu_\alpha\,(t-t_1)}$, that is $C\, e^{(1-\eta)\,\mu_\alpha\, t}$, and the remaining part $\tilde g^R$ goes to 0 with rate $C\, e^{(\lambda/2)\, t}$. Since $|\lambda/2|$ is larger than $(1-\eta)\,|\mu_\alpha|$ for $(1-\alpha)$ small enough, it concludes the proof of (1.34). □

A Appendix: Moments of Gaussians

We state here some moments of tensor product of Gaussians.

Lemma A.1 *The following identities hold*

$$\text{(A.1)}\qquad \int_{\mathbb{R}^N} M_{1,0,1}\, |v|^2\, dv = N,$$

$$\text{(A.2)}\qquad \int_{\mathbb{R}^N} M_{1,0,1}\, |v|^4\, dv = N\,(N+2),$$

$$\text{(A.3)}\qquad \int_{\mathbb{R}^N\times\mathbb{R}^N} M_{1,0,1}\, (M_{1,0,1})_*\, |u|^3\, dv\, dv_* = 2^{3/2} \int_{\mathbb{R}^N} M_{1,0,1}\, |v|^3\, dv,$$

$$\text{(A.4)}\qquad \int_{\mathbb{R}^N\times\mathbb{R}^N} M_{1,0,1}\, (M_{1,0,1})_*\, |v|^2\, |u|^3\, dv\, dv_* = \sqrt{2}\,(2N+3) \int_{\mathbb{R}^N} M_{1,0,1}(v)\, |v|^3\, dv.$$

Proof of Lemma A.1. The proof of (A.1) and (A.2) being straightforward and the proof of (A.3) being very similar to the proof of (A.4) we only prove (A.4). We first notice that

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^N} M_{1,0,1}\, (M_{1,0,1})_*\, |v|^2\, |u|^3\, dv\, dv_* \\
&\qquad = \frac{1}{2} \int_{\mathbb{R}^N\times\mathbb{R}^N} M_{1,0,1}\, (M_{1,0,1})_*\, (|v|^2 + |v_*|^2)\, |u|^3\, dv\, dv_* \\
&\qquad = \frac{1}{4} \int_{\mathbb{R}^N\times\mathbb{R}^N} M_{1,0,1}\, (M_{1,0,1})_*\, (|v+v_*|^2 + |v-v_*|^2)\, |u|^3\, dv\, dv_*.
\end{aligned}$$

Making use of the change of variable $(v,v_*) \to (x = (v+v_*)/\sqrt{2}, y = (v-v_*)/\sqrt{2})$, we then get

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^N} M_{1,0,1}\, (M_{1,0,1})_*\, |v|^2\, |u|^3\, dv\, dv_* \\
&\qquad = \sqrt{2} \int_{\mathbb{R}^N\times\mathbb{R}^N} M_{1,0,1}(x)\, M_{1,0,1}(y)\, (|x|^2 + |y|^2)\, |y|^3\, dx\, dy \\
&\qquad = \sqrt{2}\, N \int_{\mathbb{R}^N} M_{1,0,1}(v)\, |v|^3\, dv + \sqrt{2} \int_{\mathbb{R}^N} M_{1,0,1}(v)\, |v|^5\, dv \\
&\qquad = \sqrt{2}\,(2N+3) \int_{\mathbb{R}^N} M_{1,0,1}(v)\, |v|^3\, dv.
\end{aligned}$$

□

B Appendix: Interpolation inequalities

Lemma B.1 *(i) For any $k, k^*, q, q^* \in \mathbb{Z}$ with $k \geq k^*$, $q \geq q^*$ and any $\theta \in (0,1)$ there is $C \in (0,\infty)$ such that for $h \in W^{k^{**},1}_{q^{**}}(m^{-1})$*

$$\|h\|_{W^{k,1}_q(m^{-1})} \leq C\, \|h\|^{1-\theta}_{W^{k^*,1}_{q^*}(m^{-1})}\, \|h\|^{\theta}_{W^{k^{**},1}_{q^{**}}(m^{-1})}. \tag{B.1}$$

with $k^{**}, q^{**} \in \mathbb{Z}$ such that $k = (1-\theta)\, k^* + \theta\, k^{**}$, $q = (1-\theta)\, q^* + \theta\, q^{**}$.

(ii) For any $k, q \in N^*$ and any exponential weight function m as defined in (1.28), there exists $C \in (0,\infty)$ such that for any $h \in H^{k^\ddagger} \cap L^1(m^{-12})$ with $k^\ddagger := 8k + 7(1 + N/2)$

$$\|h\|_{W^{k,1}_q(m^{-1})} \leq C\, \|h\|^{1/4}_{H^{k^\ddagger}}\, \|h\|^{1/4}_{L^1(m^{-12})}\, \|h\|^{3/4}_{L^1(m^{-1})}. \tag{B.2}$$

Proof of Lemma B.1. The inequality (B.1) in point (i) is a classical result from interpolation theory. Let us focus on point (ii). We prove the inequality (B.2) for $h \in \mathcal{S}(\mathbb{R}^N)$ and then argue by density. On the one hand, we observe that for any ℓ there exists C such that

$$\|h\|^2_{H^\ell} \leq C\, \|h\|_{L^1}\, \|h\|_{H^{\ell^\dagger}}, \quad \ell^\dagger := 2\ell + 1 + N/2.$$

Iterating twice that inequality, we get (for some related exponents $k^\dagger, k^\ddagger$)

$$\|h\|_{H^{k^\dagger}} \leq C\, \|h\|^{3/4}_{L^1}\, \|h\|^{1/4}_{H^{k^\ddagger}}. \tag{B.3}$$

On the other hand, using first Cauchy-Schwartz inequality, plus the same argument as above and Hölder's inequality, we obtain

$$\begin{aligned} \|h\|_{W^{k,1}_q(m^{-1})} &\leq C\|h\|_{H^k(m^{-3/2})} \leq C\, \|h\|^{1/2}_{L^1(m^{-3})}\, \|h\|^{1/2}_{H^{k^\dagger}} \\ &\leq C\, \|h\|^{1/8}_{L^1(m^{-12})}\, \|h\|^{3/8}_{L^1}\, \|h\|^{1/2}_{H^{k^\dagger}}. \end{aligned} \tag{B.4}$$

We conclude gathering (B.4) and (B.3). □

Acknowledgments. We thank Alexander Bobylev, José Antonio Carrillo and Cédric Villani for fruitful discussions on the inelastic Boltzmann equation.

References

[1] F. Abrahamsson *Strong L^1 convergence to equilibrium without entropy conditions for the Boltzmann equation*, Comm. Partial Diff. Equations **24** (1999), 1501–1535

[2] M. Bisi, J. A. Carrillo, G. Toscani, *Contractive metrics for a Boltzmann equation for granular gases: diffusive equilibria*, J. Stat. Phys. **118** (2005), no. 1-2, 301–331.

[3] M. Bisi, J. A. Carrillo, G. Toscani, *Decay rates in probability metrics towards homogeneous cooling states for the inelastic Maxwell model*, J. Stat. Phys. (2006), in press.

[4] M. D. Blake, *A spectral bound for asymptotically norm-continuous semigroups*, J. Operator Theory **45** (2001), 111–130.

[5] A. V. Bobylev, J.A. Carillo, I. Gamba, *On some properties of kinetic and hydrodynamics equations for inelastic interactions*, J. Statist. Phys. **98** (2000), 743–773.

[6] C. Baranger, C. Mouhot, *Explicit spectral gap estimates for the linearized Boltzmann and Landau operators with hard potentials*, Rev. Matem. Iberoam. **21** (2005), 819–841

[7] A. V. Bobylev, C. Cercignani, G. Toscani, *Proof of an asymptotic property of self-similar solutions of the Boltzmann equation for granular materials*, J. Statist. Phys. **111** (2003), 403–417.

[8] A. V. Bobylev, I. Gamba, V. Panferov, *Moment inequalities and high-energy tails for the Boltzmann equations with inelastic interactions*, J. Statist. Phys. **116** (2004), 1651–1682.

[9] N. V. Brilliantov, T. Pöschel, **Kinetic theory of granular gases.** Oxford Graduate Texts. Oxford University Press, Oxford, 2004.

[10] E. Caglioti, C. Villani, *Homogeneous cooling states are not always good approximations to granular flows*, Arch. Rational Mech. Anal. **163** (2002), 329–343.

[11] T. Carleman, *Sur la théorie de l'équation intégrodifférentielle de Boltzmann*, Acta Math. **60** (1932).

[12] T. Carleman, **Problèmes mathématiques dans la théorie cinétique des gaz**, Almqvist and Wiksells Boktryckeri Ab, Uppsala 1957.

[13] C. Cercignani, *Recent developments in the mechanics of granular materials*, in Fisica matematica e ingegneria delle strutture, Pitagora Editrice, Bologna, 1995, pp. 119–132.

[14] I. Csiszár, *Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963) 85108.

[15] M. H. Ernst, R. Brito, *Driven inelastic Maxwell molecules with high energy tails*, Phys. Rev. E **65** (2002).

[16] M. H. Ernst, R. Brito, *Scaling solutions of inelastic Boltzmann equations with over-populated high energy tails*, J. Statist. Phys. **109** (2002), 407–432.

[17] I. Gamba, V. Panferov, C. Villani, *On the Boltzmann equation for diffusively excited granular media*, Comm. Math. Phys. **246** (2004), 503–541.

[18] H. Grad, *Asymptotic theory of the Boltzmann equation. II* Rarefied Gas Dynamics (Proc. 3rd Internat. Sympos., Palais de l'UNESCO, Paris, 1962), Vol. I, pp 26–59, New York, 1963.

[19] P. K. Haff, *Grain flow as a fluid-mechanical phenomenon*, J. Fluid Mech. **134** (1983).

[20] Hilbert, D. Grundzüge einer Allgemeinen Theorie der Linearen Integralgleichungen. *Math. Ann. 72*, (1912), Chelsea Publ., New York, (1953).

[21] T. Kato, **Perturbation theory for linear operators**, Springer-Verlag, Berlin, 1995.

[22] S. Kullback, **Information Theory and Statistics**, John Wiley, 1959.

[23] S. Mischler, C. Mouhot, M. Rodriguez Ricard, *Cooling process for inelastic Boltzmann equations for hard spheres, Part I: The Cauchy problem*, J. Stat. Phys **124** (2006), 655–702.

[24] S. Mischler, C. Mouhot *Cooling process for inelastic Boltzmann equations for hard spheres, Part II: Self-similar solutions and tail behavior*, J. Stat. Phys. **124**, (2006), 703–746.

[25] S. Mischler, C. Mouhot, work in progress.

[26] S. Mischler, B. Wennberg, *On the spatially homogeneous Boltzmann equation*, Ann. Inst. Henri Poincaré, Analyse non linéaire **16** (1999), 467–501.

[27] C. Mouhot *Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation*, Comm. Math. Phys. **261** (2006), 629–672.

[28] C. Mouhot, C. Villani, *Regularity theory for the spatially homogeneous Boltzmann equation with cut-off*, Arch. Rational Mech. Anal. **173** (2004), 169–212.

[29] L. Nirenberg, **Topics in nonlinear functional analysis**. With a chapter by E. Zehnder. Notes by R. A. Artino. Lecture Notes, 1973–1974. Courant Institute of Mathematical Sciences, New York University, New York, 1974.

[30] A. Pulvirenti, B. Wennberg, *A Maxwellian lower bound for solutions to the Boltzmann equation*, Comm. Math. Phys. **183** (1997), 145–160.

[31] C. Villani, *Cercignani's conjecture is sometimes true and always almost true*, Comm. Math. Phys. **234** (2003), 455–490.

[32] P. F. Yao, *On the inversion of the Laplace transform of C_0 semigroups and its applications*, SIAM J. Math. Anal. **26** (1995), 1331–1341.