Exact Failure Frequency Calculations for Extended Systems

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Abstract

This paper shows how the steady-state availability and failure frequency can be calculated in a single pass for very large systems, when the availability is expressed as a product of matrices. We apply the general procedure to k-out-of-n:G and linear consecutive k-out-of-n:F systems, and to a simple ladder network in which each edge and node may fail. We also give the associated generating functions when the components have identical availabilities and failure rates. For large systems, the failure rate of the whole system is asymptotically proportional to its size.

This paves the way to ready-to-use formulae for various architectures, as well as proof that the differential operator approach to failure frequency calculations is very useful and straightforward.

Index Terms

network availability, failure frequency, failure rate, k-out-of-n systems, generating function

ACRONYMS[1]

GVI	grouped variable inversion (method)
SVI	single variable inversion (method)
IE	inclusion-exclusion (principle)
OBDD	Ordered Binary Decision Diagram

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[1]The singular and plural of an acronym are always spelled the same.

NOTATION

p_i, q_i	[success, failure] probability of component i ($q_i = 1 - p_i$)
p, q	implies $p_i = p$, $q_i = q$ (for edges).
ρ	identical availability of nodes (when $\rho \neq p$).
λ_i, μ_i	[failure, repair] rate of component i
λ, μ	common [failure, repair] rate of components
A	steady-state availability of the system
U	steady-state unavailability of the system
$\overline{\nu}$	mean failure frequency of the system
$\overline{\lambda}$	mean failure rate of the system ($\overline{\nu} = A\,\overline{\lambda}$)
M'	$\left(\sum_i \lambda_i\, p_i \dfrac{\partial}{\partial p_i}\right) M$
$\mathcal{G}(z)$	generating function for the availability
$\widehat{\mathcal{G}}(z)$	generating function for the failure frequency
$A_{k,n}$	availability of a k-out-of-n:G system

I. INTRODUCTION

Steady-state system availability and failure frequency are important performance indices of a repairable system [1], [2], [3], [4], from which other key parameters such as the mean time between failures, average failure rate, Birnbaum importance, etc. may be deduced. In the steady-state regime, the frequency of system failure was first calculated by a cut-set [5] or a tie-set approach [6] in the case of statistically independent failures, which will also be considered here. These approaches are based on the inclusion-exclusion (IE) principle, where the failure or repair rates (more generally, the inverses of the mean down or up times), are adequately given for each term of the relevant expansion.

When all the terms of its IE expansion are kept, the exact availability is obtained as a function of each component availability. Several papers have provided a few simple recipes, describing how the system failure frequency and the failure rate can then be derived [7], [8], [9]. Recent refinements have been proposed when availability expressions are obtained from various instances (SVI, GVI) of sum-of-disjoint-products algorithms [10]. All these formal calculations boil down to a simple fact: the failure frequency may be derived from the availability through the application of a linear differential operator [11], [12]. This requires knowledge of the exact availability, which is hard to come by except for trivially small networks, and may have hindered the use of this method.

Unsurprisingly, several algorithms have been put forward, in which availability and failure frequency are computed side by side in a common procedure: triangle-star transformation [13], OBDD calculations [14], and another instance of differential operator calculations [15].

In this paper, we want to promote the differential operator method for the calculation of the failure frequency by showing it gives the exact result for numerous, widely used configurations, with an *arbitrary large* number of components. We take advantage of recent results establishing that the availability of recursive networks may be expressed as a product of transfer matrices that take each edge and node availabilities exactly into account [16], [17], [18].

Our paper is organized as follows. In Section II, we show how the failure frequency of a system may generally be deduced from the steady-state availability when the latter is expressed by a product of transfer matrices. We first apply this method in Section III, which is devoted to k-out-of-n systems (either k-out-of-n:G or linear consecutive k-out-of-n:F ones) with distinct components. Section IV provides a generic example for the two-terminal failure frequency of a simple ladder network, which has been solved recently for arbitrary edge and node availabilities [16]; the same procedure could easily be used for more complex networks and their all-terminal reliability too [17], [18]. In each configuration, we pay attention to the case of identical components, for which the common availability is p (for edges) and ρ (for nodes). For large systems, we show that the asymptotic failure rate has a linear dependence with size, and is given by derivatives of the largest eigenvalue of the unique transfer matrix with respect to p and ρ. We conclude by a brief outlook.

II. General procedure

In many systems, as will be explicitly shown in the following sections, the availability A (or the unavailability U) is given by an expression of the form

$$A = \mathbf{v}_L \, M_n \, \cdots \, M_1 \, \mathbf{v}_R \, , \tag{1}$$

where M_k $(1 \leq k \leq n)$ is a transfer matrix, the elements of which are multilinear polynomials of individual component availabilities, and where $\mathbf{v}_L$ and $\mathbf{v}_R$ are two vectors in which these availabilities do not appear. The mean failure frequency $\overline{\nu}$ is obtained from [11], [12]

$$\overline{\nu} = \sum_i \lambda_i \, p_i \, \frac{\partial A}{\partial p_i} = \sum_i \mu_i \, q_i \, \frac{\partial U}{\partial q_i}. \tag{2}$$

In order to avoid unnecessarily heavy notation, we call M'_k the matrix obtained by applying the linear differential operator $\sum_i \lambda_i \, p_i \, \frac{\partial}{\partial p_i}$ to M_k. Therefore,

$$\begin{aligned}\overline{\nu} &= \mathbf{v}_L \, M'_n \, M_{n-1} \, \cdots \, M_1 \, \mathbf{v}_R \\ &\quad + \mathbf{v}_L \, M_n \, M'_{n-1} \, \cdots \, M_1 \, \mathbf{v}_R \\ &\quad + \cdots \\ &\quad + \mathbf{v}_L \, M_n \, M_{n-1} \, \cdots \, M'_1 \, \mathbf{v}_R \, . \end{aligned} \tag{3}$$

Since M_k's elements are at most linear functions of each p_i, the derivation of M'_k is straightforward. For instance, a matrix element $p_1 + p_2 \, p_3 - p_1 \, p_2 \, p_3$ in M_k would give rise to $\lambda_1 \, p_1 + (\lambda_2 + \lambda_3) \, p_2 \, p_3 - (\lambda_1 + \lambda_2 + \lambda_3) \, p_1 \, p_2 \, p_3$; the recipes given in [7], [8], [11] fully apply.

Both availability and failure frequency may be obtained in a single pass in the following way. Let us initialize the procedure by setting

$$\mathcal{A}_1 = M_1 \, \mathbf{v}_R \, , \tag{4}$$

$$\mathcal{V}_1 = M'_1 \, \mathbf{v}_R \, . \tag{5}$$

The recursion equations are

$$\mathcal{A}_k = M_k \, \mathcal{A}_{k-1} \, , \tag{6}$$

$$\mathcal{V}_k = M_k \, \mathcal{V}_{k-1} + M'_k \, \mathcal{A}_{k-1} \, , \tag{7}$$

from which we deduce the final results

$$A = \mathbf{v}_L \, \mathcal{A}_n \, , \tag{8}$$

$$\overline{\nu} = \mathbf{v}_L \, \mathcal{V}_n \, . \tag{9}$$

We can now turn to a few ‘real-life’ applications.

III. k-OUT-OF-n SYSTEMS

k-out-of-n systems are widely used, in various configurations; they have therefore contributed to a huge body of literature (see [4], [19], [20] and references therein). We start our discussion with these systems because each transfer matrix actually refers to a single equipment only.

A. k-out-of-n:G systems

We first consider the simple k-out-of-n:G system, where each component has an availability p_i ($1 \leq i \leq n$). To operate as a whole, the system needs at least k elements to function. Its availability $A_{k,n}$ may be written as (see [4], p. 244)

$$A_{k,n} = 1 - (1, 0, \cdots, 0)_k \, \mathbf{\Lambda}_n \, \mathbf{\Lambda}_{n-1} \, \mathbf{\Lambda}_1 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_k , \tag{10}$$

with

$$\mathbf{\Lambda}_i = \begin{pmatrix} q_i & p_i & 0 & \cdots & 0 \\ 0 & q_i & p_i & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & 0 & q_i & p_i \\ 0 & 0 & \cdots & 0 & q_i \end{pmatrix}_{k \times k} . \tag{11}$$

We have reduced the size of the matrix to a $k \times k$ one, instead of the original $(k+1) \times (k+1)$, because of the nature of $\mathbf{v}_L = (1, 0, \ldots, 0)_k$ and $\mathbf{v}_R$ in eq. (10).

The 'derivative' of $\mathbf{\Lambda}_i$ is

$$\mathbf{\Lambda}'_i = \begin{pmatrix} -\lambda_i \, p_i & \lambda_i \, p_i & 0 & \cdots & 0 \\ 0 & -\lambda_i \, p_i & \lambda_i \, p_i & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & 0 & -\lambda_i \, p_i & \lambda_i \, p_i \\ 0 & 0 & \cdots & 0 & -\lambda_i \, p_i \end{pmatrix}_{k \times k} , \tag{12}$$

so that the computation of the failure frequency following the method given in section II is straightforward (care should of course be taken of the minus sign in eq. (10)).

Let us revisit Example 7.2 of [4] (see p. 245) for the 5-out-of-8:G system with p_i = 0.90, 0.89, ..., 0.83. Assuming a unique repair rate for all components, namely μ, the failure rates λ_i are such that $\lambda_i \, p_i = \mu \, (1 - p_i)$. From the procedure detailed in Section II, we deduce $A_{5,8} = \frac{615925280183}{625000000000} \approx 0.98548045$ and a failure frequency $\overline{\nu}_{5,8} = \frac{80129143 59}{156250000000} \, \mu \approx 0.051283 \, \mu$. The failure rate $\overline{\lambda}_{5,8} = \overline{\nu}_{5,8} / A_{5,8}$ is then equal to $0.0520382 \, \mu$.

When all components are identical ($p_i \equiv p$ and $\lambda_i \equiv \lambda$), only one transfer matrix appears. Admittedly, $A_{k,n}$ is so simple that a matrix formulation is hardly necessary. Nonetheless, we can give a compact expression for the generating function $\mathcal{G}_k(z) = \sum_{n=0}^{\infty} A_{k,n}\, z^n$ (the derivation is given in the appendix):

$$\mathcal{G}_k(k\text{-out-of-}n\text{:G}; z) = \frac{p^k\, z^k}{(1-z)\,(1-(1-p)\,z)^k}\,. \tag{13}$$

Since the generating function is a formal power-series expansion, we can apply the linear differential operator $\lambda\, p\, \frac{\partial}{\partial p}$ to eq. (13) so that $\widehat{\mathcal{G}}_k(z) = \sum_{n=0}^{\infty} \overline{\nu}_{k,n}\, z^n$ is easily found to be

$$\widehat{\mathcal{G}}(k\text{-out-of-}n\text{:G}; z) = \frac{\lambda\, k\, p^k\, z^k}{(1-(1-p)\,z)^{k+1}}\,, \tag{14}$$

which is another formulation of the well-known result $\overline{\nu}_{k,n} = \lambda\, k \begin{pmatrix} n \\ l \end{pmatrix} p^k\,(1-p)^{n-k}$ (eq. (7.10) of [4], p. 234).

B. Linear consecutive k-out-of-n:F systems

These systems have been studied in many papers [19], [20] and a recent textbook [4]. The reliability $\widetilde{A}_{k,n}$ — the probability of operation of a system of n components, which fails if at least k *consecutive* elements fail — of such a system is given by (see also eq. (9.48) of [4], p. 344)

$$\widetilde{A}_{k,n} = (1,0,\ldots,0)_k\; \widetilde{\mathbf{\Lambda}}_n\, \widetilde{\mathbf{\Lambda}}_{n-1}\, \widetilde{\mathbf{\Lambda}}_1 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_k\,, \tag{15}$$

with

$$\widetilde{\mathbf{\Lambda}}_i = \begin{pmatrix} p_i & q_i & 0 & \cdots & 0 \\ p_i & 0 & q_i & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ p_i & 0 & \cdots & 0 & q_i \\ p_i & 0 & 0 & \cdots & 0 \end{pmatrix}_{k\times k}\,. \tag{16}$$

Here again, we have reduced the size of the matrix and the vectors with respect to their original formulation. Consequently,

$$\widetilde{\mathbf{\Lambda}}'_i = \begin{pmatrix} \lambda_i\, p_i & -\lambda_i\, p_i & 0 & \cdots & 0 \\ \lambda_i\, p_i & 0 & -\lambda_i\, p_i & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ \lambda_i\, p_i & 0 & \cdots & 0 & -\lambda_i\, p_i \\ \lambda_i\, p_i & 0 & 0 & \cdots & 0 \end{pmatrix}_{k\times k}\,, \tag{17}$$

leading once again to a straightforward calculation of the failure frequency.

A numerical application may be found in the Lin/Con/4/11:F model, as in Example 9.6 of [4], where the p_i's range from 0.7 to 0.9 by steps of 0.02. Assuming again that the repair rate for each equipment is μ, we get $A_{4,11} = \frac{30105385968617}{30517578125000} \approx 0.98649329$ and a failure frequency $\overline{\nu}_{4,11} = \frac{155495836041}{30517578125000}\,\mu \approx 0.050953\,\mu$. The corresponding failure rate is then equal to $0.0516505\,\mu$.

For the sake of completeness, we give the generating function for identical components is (see eq. (2.2) of [21], [22])

$$\mathcal{G}_k(\text{Lin/Con}/k/n\text{:F}; z) = \frac{1-(1-p)^k\, z^k}{1 - z + p\,(1-p)^k\, z^{k+1}}\,. \tag{18}$$

Use of eq. (18) to obtain the failure frequency generating function is straightforward, and will not be repeated here.

IV. Simple ladder

We consider in this section the two-terminal availability of a simple ladder network, displayed in Fig. 1, where successive nodes are labelled S_i or T_j, and where the larger black dots mark the source s and terminal t. This network is a simplified description of a standard architecture for long-hail communication networks: it consists in primary and backup paths, plus additional connections between transit nodes enabling the so-called "local protection" policy by bypassing faulty intermediate nodes or edges. Such an architecture of "absolutely reliable nodes and unreliable edges," with up to 25 edges, was chosen as Example 5 in [23] for a comparison of different "sum of disjoint products" minimizing algorithms, or by Rauzy [24] as well as Kuo and collaborators in OBDD test calculations [25], [26], [27]. We showed [16] that the two-terminal availability has a beautiful algebraic structure [28], since its exact expression is given by a product of 3×3 transfer matrices (see eqs. (19–21) below). Consequently, it can also be determined for a network of arbitrary size.

Using the notation $\mathcal{R}_{S_n}$ (resp. $\mathcal{R}_{T_n}$) for the two-terminal availability between S_0 and S_n

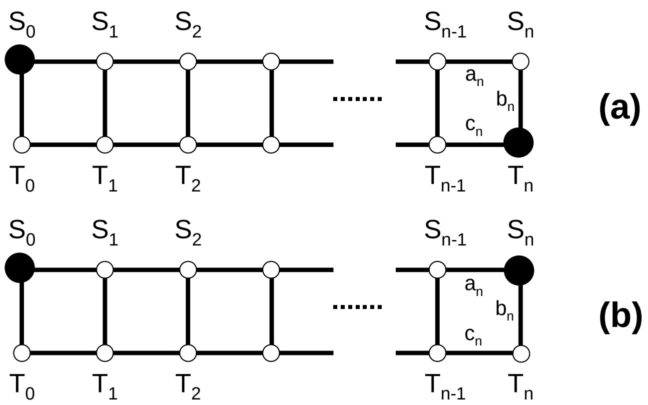

Fig. 1. Different source-terminal connections for a simple ladder. Edges and nodes are indexed by their availabilities: (a_i, b_i, c_i) and (S_i, T_i).

(resp. T_n), we find that [16], [18]

$$\mathcal{R}_{S_n} = (1\ 0\ 0)\, M_n \cdots M_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{19}$$

$$\mathcal{R}_{T_n} = (0\ 1\ 0)\, M_n \cdots M_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{20}$$

The transfer matrix M_n is given by

$$M_n = \begin{pmatrix} a_n S_n & b_n c_n S_n T_n & a_n b_n c_n S_n T_n \\ a_n b_n S_n T_n & c_n T_n & a_n b_n c_n S_n T_n \\ -a_n b_n S_n T_n & -b_n c_n S_n T_n & a_n (1 - 2 b_n) c_n S_n T_n \end{pmatrix}. \tag{21}$$

For $n = 0$, we must set $a_0 = 1$, and may choose $c_0 = 0$ because it does not change the final result. It is worth noting that all five availabilities of the n^{th} “cell” or building block of the network appear in a single transfer matrix M_n, which is not sparse, contrary to the matrices of Section III.

Equations (19–21) apply to the most general ladder in terms of individual availabilities. If an edge or a node is missing, its reliability p_i should be set to zero, and its failure rate may be considered arbitrary, because it will not alter the final result. Similarly, if a given edge or node is perfect, its reliability should be equal to one; its failure rate λ_i should then, of course, be set to zero.

The associated matrix M'_n is

$$M'_n = \begin{pmatrix} m'_{11} & m'_{12} & m'_{13} \\ m'_{21} & m'_{22} & m'_{23} \\ m'_{31} & m'_{32} & m'_{33} \end{pmatrix} , \tag{22}$$

with

$$\begin{aligned}
m'_{11} &= (\lambda_{a_n} + \lambda_{S_n})\, a_n\, S_n\,, \\
m'_{12} &= (\lambda_{b_n} + \lambda_{c_n} + \lambda_{S_n} + \lambda_{T_n})\, b_n\, c_n\, S_n\, T_n\,, \\
m'_{13} &= (\lambda_{a_n} + \lambda_{b_n} + \lambda_{c_n} + \lambda_{S_n} + \lambda_{T_n})\, a_n\, b_n\, c_n\, S_n\, T_n\,, \\
m'_{21} &= (\lambda_{a_n} + \lambda_{b_n} + \lambda_{S_n} + \lambda_{T_n})\, a_n\, b_n\, S_n\, T_n\,, \\
m'_{22} &= (\lambda_{c_n} + \lambda_{T_n})\, c_n\, T_n\,, \\
m'_{23} &= m'_{13}\,, \\
m'_{31} &= -m'_{21}\,, \\
m'_{32} &= -m'_{12}\,, \\
m'_{33} &= (\lambda_{a_n} + \lambda_{c_n} + \lambda_{S_n} + \lambda_{T_n})\, a_n\, c_n\, S_n\, T_n - 2\, m'_{13}\,.
\end{aligned}$$

When $a_i = b_i = c_i \equiv p$ and $S_i = T_i \equiv \rho$ the three eigenvalues ζ_0 and $\zeta_\pm$ of the transfer matrix are [16]

$$\zeta_0 = p\,\rho\,(1 - p\,\rho), \tag{23}$$

$$\zeta_\pm = \frac{p\,\rho}{2} \left(1 + 2\,p\,(1-p)\,\rho \pm \sqrt{\mathcal{B}}\right) , \tag{24}$$

with $\mathcal{B} = 1 + 4\,p^2\,\rho - 8\,p^3\,\rho^2 + 4\,p^4\,\rho^2$. The two-terminal availabilities are [16]

$$\begin{aligned}
\mathcal{R}_{T_n}(p,\rho) &= \frac{1}{2\,p} \left(-\zeta_0^{n+1} + p\,\rho\,(1+p\,\rho)\, \frac{\zeta_+^{n+1} - \zeta_-^{n+1}}{\zeta_+ - \zeta_-} \right. \\
&\qquad \left. -(1 - 2\,p + p\,\rho)\, p^3\, \rho^3\, \frac{\zeta_+^n - \zeta_-^n}{\zeta_+ - \zeta_-} \right) , \qquad (25) \\
\mathcal{R}_{S_n}(p,\rho) &= \frac{1}{2\,p} \left(+\zeta_0^{n+1} + p\,\rho\,(1+p\,\rho)\, \frac{\zeta_+^{n+1} - \zeta_-^{n+1}}{\zeta_+ - \zeta_-} \right. \\
&\qquad \left. -(1 - 2\,p + p\,\rho)\, p^3\, \rho^3\, \frac{\zeta_+^n - \zeta_-^n}{\zeta_+ - \zeta_-} \right) . \qquad (26)
\end{aligned}$$

These expressions are identical except for the $\pm$ sign in front of the ζ_0^{n+1} term. Assuming that the common link failure rate is λ while that for the nodes is ξ, the failure frequency for the $S_0 \to T_n$ connection is

$$\overline{\nu} = \lambda\, p\, \frac{\partial \mathcal{R}_{T_n}(p,\rho)}{\partial p} + \xi\, \rho\, \frac{\partial \mathcal{R}_{T_n}(p,\rho)}{\partial \rho}\,; \tag{27}$$

a similar expression applies to $\mathcal{R}_{S_n}(p,\rho)$. When n is large, both availabilities are actually of the form $\alpha_+\,\zeta_+^n$, because the modulus of ζ_+ is larger than that of the remaining eigenvalues for $0 \le p \le 1$ [16]. When nodes are perfect, we have therefore in this limit

$$\overline{\nu} \approx \lambda\, p \left(\frac{\partial \alpha_+}{\partial p}\, \zeta_+^n + n\, \alpha_+\, \frac{\partial \zeta_+}{\partial p}\, \zeta_+^{n-1} \right)\,, \tag{28}$$

so that the failure rate is

$$\overline{\lambda} \approx \lambda \left(\frac{\partial \ln \alpha_+}{\partial \ln p} + n\, \frac{\partial \ln \zeta_+}{\partial \ln p} \right)\,, \tag{29}$$

with

$$\frac{\partial \ln \zeta_+}{\partial \ln p} = \frac{-1 + 4\,p - 6\,p^2 + 4\,p^3 + (3 - 4\,p)\,\sqrt{1 + 4\,p^2\,(1-p)^2}}{2\,(1-p)\,\sqrt{1 + 4\,p^2\,(1-p)^2}}\,, \tag{30}$$

$$\frac{\partial \ln \alpha_+}{\partial \ln p} = \frac{4 - 5\,p + 8\,p^2 - 20\,p^3 + 16\,p^4 - 4\,p^5 - (4 - 7\,p + 4\,p^2 - 2\,p^3)\,\sqrt{1 + 4\,p^2\,(1-p)^2}}{2\,(1-p)\,[1 + 4\,p^2\,(1-p)^2]} \tag{31}$$

The variations with p of $\partial \ln \zeta_+ / \partial \ln p$ and $\partial \ln \alpha_+ / \partial \ln p$ are displayed in Figs. 2 and 3. Since $|\zeta_+| < 1$ for $p < 1$, the contribution of $\partial \ln \alpha_+ / \partial \ln p$ will prevail, and $\overline{\lambda}$ will have a linear dependence with n in the large network limit (this is a general property when the eigenvalue of highest modulus is different from unity).

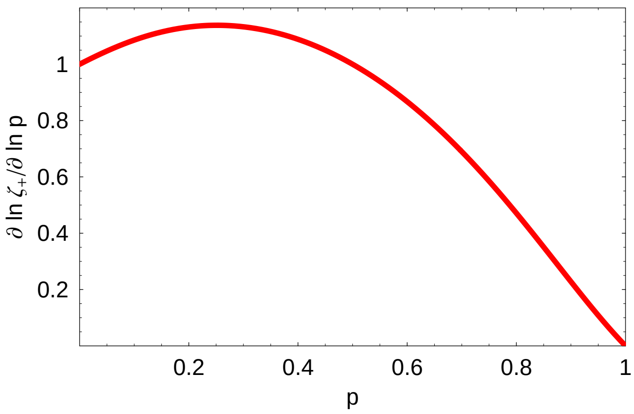

Fig. 2. Variation of $\partial \ln \zeta_+ / \partial \ln p$ with p for a simple ladder with perfect nodes (eq. (30)). The maximum, reached for $p \approx 0.251641$, is about 1.13827.

For very reliable components, eqs. (29–31) simplify. It is easy to show by a series expansion in the link unavailability q that the global failure rate is given (to first order) by $\overline{\lambda} \to (2\,n + 4)\,\lambda\,q$; this result could also have been obtained by visual inspection and enumeration of the minimal cuts.

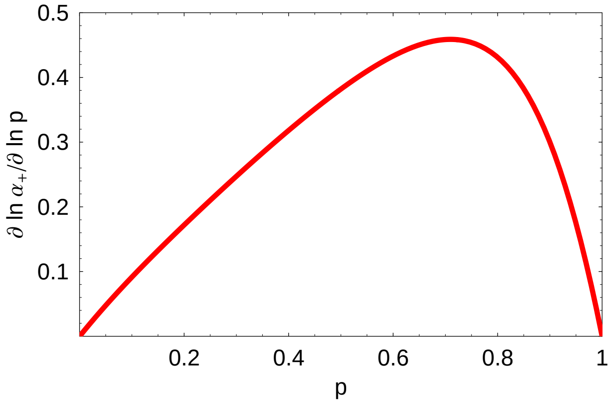

Fig. 3. Variation of $\partial \ln \alpha_+ / \partial \ln p$ with p for a simple ladder with perfect nodes (eq. (31)). The maximum, reached for $p \approx 0.709902$, is about 0.458825.

V. Conclusion and outlook

We have shown that the linear differential method for computing the failure frequency is a very simple and useful one for k-out-of-n systems as well as the two-terminal availability for recursive networks (this should hold for the all-terminal availability, too [18]). Its application is not limited to the case of extremely reliable components. Even though we restricted our discussion to expressions dealing with availabilities, a similar treatment could be performed for expressions where unavailabilities are the input data (see eq. (2)). For more complex networks, the size of the transfer matrix increases (for instance, it is a 13×13 one for the 'street $3 \times n$' of [26]) but the calculations remain straightforward. Finally, the expressions given for steady-state availabilities can also be used for time-dependent systems provided that failures and reparations are still statistically independent events, because the expressions are formally identical (the availabilities of components must be replaced by the reliabilities).

Appendix I
Proof of eq. (13)

$A_{k,n}$ is given by

$$A_{k,n} = \sum_{l=k}^{n} \binom{n}{l} p^l (1-p)^{n-l} . \tag{32}$$

The fundamental equality between binomials

$$\binom{n+1}{l} = \binom{n}{l} + \binom{n}{l-1} \tag{33}$$

leads to

$$A_{k,n+1} = (1-p)\,A_{k,n} + p\,A_{k-1,n}\,. \tag{34}$$

Setting $\mathcal{G}_k(z) = \sum_{n=0}^{\infty} A_{k,n}\,z^n$ implies

$$\frac{1}{z}\,\mathcal{G}_k(z) = (1-p)\,\mathcal{G}_k(z) + p\,\mathcal{G}_{k-1}(z)\,. \tag{35}$$

so that

$$\mathcal{G}_k(z) = \frac{p\,z}{1-(1-p)\,z}\,\mathcal{G}_{k-1}(z) = \left(\frac{p\,z}{1-(1-p)\,z}\right)^k\,\mathcal{G}_0(z)\,. \tag{36}$$

Since $A_{0,n} = 1, \forall n$, $\mathcal{G}_0(z) = 1/(1-z)$; eq. (13) follows.

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