# QUANTIZATION OF FORMAL CLASSICAL DYNAMICAL $r$-MATRICES: THE REDUCTIVE CASE

DAMIEN CALAQUE

Abstract. In this paper we prove the existence of a formal dynamical twist quantization for any triangular and non-modified formal classical dynamical $r$-matrix in the reductive case. The dynamical twist is constructed as the image of the dynamical $r$-matrix by a $L_\infty$-quasi-isomorphism. This quasi-isomorphism also allows us to classify formal dynamical twist quantizations up to gauge equivalence.

## Introduction

In [Fe], Felder introduced dynamical versions of both classical and quantum Yang-Baxter equations which has been generalized to the case of a nonabelian base in [EV] for the classical part and in [X3] for the quantum part. Naturally this leads to quantization problems which have been formulated in terms of twist quantization *à la* Drinfeld ([Dr1]) in [X2, X3, EE1, EE2].

Let us formulate this problem in the general context. Consider an inclusion $\mathfrak{h} \subset \mathfrak{g}$ of Lie algebras equipped with an element $Z \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$. A *(modified) classical dynamical $r$-matrix* for $(\mathfrak{g}, \mathfrak{h}, Z)$ is a regular (meaning $C^\infty$, meromorphic, formal, ... depending on the context) $\mathfrak{h}$-equivariant map $\rho : \mathfrak{h}^* \to \wedge^2\mathfrak{g}$ which satisfies the *(modified) classical dynamical Yang-Baxter equation* (CDYBE)

$$\mathrm{CYB}(\rho) - \mathrm{Alt}(d\rho) = Z \tag{1}$$

where $\mathrm{CYB}(\rho) := [\rho^{1,2}, \rho^{1,3}] + [\rho^{1,2}, \rho^{2,3}] + [\rho^{1,3}, \rho^{2,3}] = \frac{1}{2}[\rho, \rho]$ and

$$\mathrm{Alt}(d\rho) := \sum_i \big(h_i^1 \frac{\partial \rho^{2,3}}{\partial \lambda^i} - h_i^2 \frac{\partial \rho^{1,3}}{\partial \lambda^i} + h_i^3 \frac{\partial \rho^{1,2}}{\partial \lambda^i}\big)$$

Here $(h_i)$ and $(\lambda^i)$ are dual basis of $\mathfrak{h}$ and $\mathfrak{h}^*$.

Let $\Phi = 1 + O(\hbar^2) \in (U\mathfrak{g}^{\otimes 3})^{\mathfrak{g}}[[\hbar]]$ be an associator quantizing $Z$ (of which the existence was proved in [Dr2, proposition 3.10]). A *dynamical twist quantization* of a (modified) classical dynamical $r$-matrix $\rho$ associated to $\Phi$ is a regular $\mathfrak{h}$-equivariant map $J = 1 + O(\hbar) \in \mathrm{Reg}(\mathfrak{h}^*, U\mathfrak{g}^{\otimes 2})[[\hbar]]$ such that $\mathrm{Alt}\frac{J-1}{\hbar} = \rho \bmod \hbar$ and which satisfies the *(modified) dynamical twist equation* (DTE)

$$J^{12,3}(\lambda) * J^{1,2}(\lambda + \hbar h^3) = \Phi^{-1} J^{1,23}(\lambda) * J^{2,3}(\lambda) \tag{2}$$

where $*$ denotes the PBW star-product of functions on $\mathfrak{h}^*$ and

$$J^{1,2}(\lambda + \hbar h^3) := \sum_{k \geq 0} \frac{\hbar^k}{k!} \sum_{i_1, \dots, i_k} (\partial_{\lambda^{i_1}} \cdots \partial_{\lambda^{i_k}} J)(\lambda) \otimes (h_{i_1} \cdots h_{i_k})$$

Now observe that many (modified) classical dynamical $r$-matrices can be viewed as formal ones by taking their Taylor expansion at 0. In this paper we are interested in the following conjecture:

**Conjecture 0.1** ([EE1]). *Any (modified) formal classical dynamical $r$-matrix admits a dynamical twist quantization.*

Let us reformulate DTE in the formal framework. A formal (modified) dynamical twist is an element $J(\lambda) = 1 + O(\hbar) \in (U\mathfrak{g}^{\otimes 2}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ which satisfies DTE, and $J^{1,2}(\lambda + \hbar h^3) \in (U\mathfrak{g}^{\otimes 3}\hat{\otimes}\hat{S}\mathfrak{h})[[\hbar]]$ is equal to $(\mathrm{id}^{\otimes 2} \otimes \tilde{\Delta})(J)$ where $\tilde{\Delta} : \hat{S}\mathfrak{h} \to (U\mathfrak{g}\hat{\otimes}\hat{S}\mathfrak{h})[[\hbar]]$ is induced by $\mathfrak{h} \ni x \mapsto \hbar x \otimes 1 + 1 \otimes x$. Then define $K := J(\hbar\lambda) \in (U\mathfrak{g}^{\otimes 2} \otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ which we view as an element of $(U\mathfrak{g}^{\otimes 2} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ using the symmetrization map $S\mathfrak{h} \to U\mathfrak{h}$. Since $J$ is a solution of DTE $K$ satisfies the *(modified) algebraic dynamical twist equation* (ADTE)

$$K^{12,3,4}K^{1,2,34} = (\Phi^{-1})^{1,2,3}K^{1,23,4}K^{2,3,4} \tag{3}$$

Moreover and by construction, $K = 1 + \sum_{n\geq 1}\hbar^n K_n$ has the *$\hbar$-adic valuation property*. Namely, $U\mathfrak{h}$ is filtered by $(U\mathfrak{h})_{\leq n} = \ker(\mathrm{id} - \eta\circ\varepsilon)^{\otimes n+1}\circ\Delta^{(n)}$ where $\varepsilon : U\mathfrak{h} \to \mathbf{k}$ and $\eta : \mathbf{k} \to U\mathfrak{h}$ are the counit and unit maps, and $K_n \in (U\mathfrak{h})_{\leq n-1}$. Conversely, any algebraic dynamical twist having the $\hbar$-adic valuation property can be obtained from a unique formal dynamical twist by this procedure.

This paper, in which we always assume $Z = 0$ and $\Phi = 1$ (non-modified case), is organized as follow.

In section 1 we define two differential graded Lie algebras (`dgla`'s) respectively associated to classical dynamical $r$-matrices and algebraic dynamical twists. Then we formulate the main theorem of this paper which states that if $\mathfrak{h}$ admits an ad$\mathfrak{h}$-invariant complement (the *reductive* case) then these two `dgla`'s are $L_\infty$-quasi-isomorphic and we prove that it implies Conjecture 0.1 in this case, which generalizes Theorem 5.3 of [X2]:

**Theorem 0.2.** *In the reductive case, any formal classical dynamical $r$-matrix for $(\mathfrak{g},\mathfrak{h},0)$ admits a dynamical twist quantization (associated to the trivial associator).*

The second section is devoted to the proof of the main theorem of section 1: using an equivariant formality theorem for homogeneous spaces which is obtain from [Do], we construct a $L_\infty$-quasi-isomorphism which we then modify in order to obtain the desired one. We use this $L_\infty$-quasi-isomorphism to classify formal dynamical twist quantizations up to gauge equivalence for the reductive case in section 3. In section 4 we prove that if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ for $\mathfrak{h}$ abelian and $\mathfrak{m}$ a Lie subalgebra then the results of sections 1 and 2 are still true in this situation. We conclude the paper with some open questions, and recall basic results for $L_\infty$-algebras in an appendix.

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## 1. Definitions and results

Let $\mathfrak{h} \subset \mathfrak{g}$ be an inclusion of Lie algebras.

1.1. **Algebraic structures associated to CDYBE.** Let us consider the following graded vector space

$$\underline{\mathrm{CDYB}} := \wedge^* \mathfrak{g} \otimes S\mathfrak{h} = \bigoplus_{k \geq 0} \wedge^k \mathfrak{g} \otimes S\mathfrak{h}$$

equipped with the differential d defined by

$$\mathrm{d}(x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots h_l) := -\sum_{i=1}^{l} h_i \wedge x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots h_l \hat{h}_i \tag{4}$$

With the exterior product $\wedge$ it becomes a differential graded commutative associative algebra. Moreover, one can define a graded Lie bracket of degree $-1$ on $\underline{\mathrm{CDYB}}$ which is the Lie bracket of $\mathfrak{g}$ extended to $\underline{\mathrm{CDYB}}$ in the following way:

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{(|a|-1)|b|} b \wedge [a, c] \tag{5}$$

Thus one can observe that polynomial solutions to CDYBE are exactly elements $\rho \in \underline{\mathrm{CDYB}}$ of degree 2 such that $\mathrm{d}\rho + \frac{1}{2}[\rho, \rho] = 0$. We would like to say that such a $\rho$ is a Maurer-Cartan element but $(\underline{\mathrm{CDYB}}[1], \mathrm{d}, [,])$ is not a differential graded Lie algebra (`dgla`).

Instead, remember that we are interested in $\mathfrak{h}$-equivariant solutions of CDYBE (i.e., dynamical $r$-matrices) and thus consider the subspace $\mathfrak{g}_1 = (\underline{\mathrm{CDYB}})^{\mathfrak{h}}$ of $\mathfrak{h}$-invariants with the same differential and bracket.

**Proposition 1.1.** *$(\mathfrak{g}_1[1], \mathrm{d}, [,])$ is a* `dgla`*. Moreover $(\mathfrak{g}_1, \mathrm{d}, \wedge, [,])$ is a Gerstenhaber algebra.*

*Proof.* Let $a = x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots h_s$ and $b = y_1 \wedge \cdots \wedge y_l \otimes m_1 \cdots m_t$ be $\mathfrak{h}$-invariant elements in $\mathfrak{g}_1$. We want to show that

$$\mathrm{d}[a, b] = [\mathrm{d}a, b] + (-1)^{k-1}[a, \mathrm{d}b] \tag{6}$$

The l.h.s. of (6) is equal to

$$-\Big(\sum_{i=1}^{s} h_i \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{h}_i$$
$$+\sum_{j=1}^{t} m_j \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{m}_j\Big)$$

The first term in the r.h.s. of (6) gives

$$\sum_{i=1}^{s} \big((-1)^{k-1} x_1 \wedge \cdots \wedge x_k \wedge [h_i, y_1 \wedge \cdots \wedge y_l] - h_i \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l]\big) \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{h}_i$$

and for the second term we obtain

$$\sum_{j=1}^{t} \big((-1)^{k-1} [m_j, x_1 \wedge \cdots \wedge x_k] \wedge y_1 \wedge \cdots \wedge y_l - m_j \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l]\big) \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{m}_j$$

Thus the difference between the l.h.s. and the r.h.s. of (6) is equal to

$$(-1)^k\Big(\sum_{i=1}^{k} x_1\wedge\cdots\wedge x_k\wedge[h_i,y_1\wedge\cdots\wedge y_l]\otimes h_1\cdots h_s m_1\cdots m_t\hat{h}_i$$
$$+\sum_{j=1}^{l}[m_j,x_1\wedge\cdots\wedge x_k]\wedge y_1\wedge\cdots\wedge y_l\otimes h_1\cdots h_s m_1\cdots m_t\hat{m}_j\Big)$$

Then using $\mathfrak{h}$-invariance of $a$ and $b$ one obtains

$$(-1)^{k-1}\sum_{i,j} x_1\wedge\cdots\wedge x_k\wedge y_1\wedge\cdots\wedge y_l\otimes\big(h_1\cdots h_s m_1\cdots m_t([h_i,m_j]-[m_j,h_i])\hat{h}_i\hat{m}_j\big)=0$$

The second statement of the proposition is obvious from the definition (5) of the bracket. □

Let $\rho(\lambda)\in(\wedge^2\mathfrak{g}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}$ be a formal classical dynamical $r$-matrix. Since $\rho$ satisfies CDYBE, $\alpha:=\hbar\rho(\hbar\lambda)\in\hbar\mathfrak{g}_1[[\hbar]]$ is a Maurer-Cartan element (i.e. $\mathrm{d}\alpha+\frac{1}{2}[\alpha,\alpha]=0$).

1.2. **Algebraic structures associated to ADTE.** Let us now consider the graded vector space

$$\underline{\mathrm{ADT}}:=T^*U\mathfrak{g}\otimes U\mathfrak{h}=\bigoplus_{k\geq 0}\otimes^k U\mathfrak{g}\otimes U\mathfrak{h}$$

equipped with the differential $b$ given by

$$b(P):=P^{2,\dots,k+2}+\sum_{i=1}^{k+1}(-1)^i P^{1,\dots,ii+1,\dots,k+2}\quad\text{for}\quad P\in\otimes^k U\mathfrak{g}\otimes U\mathfrak{h}\tag{7}$$

**Remark 1.2.** This is just the coboundary operator of Hochschild's cohomology with value in a comodule; and $b^2=0$ follows directly from an easy calculation.

One can define on $\underline{\mathrm{ADT}}$ an associative product $\cup$ (the *cup* product) which is given on homogeneous elements $P\in\otimes^k U\mathfrak{g}\otimes U\mathfrak{h}$ and $Q\in\otimes^l U\mathfrak{g}\otimes U\mathfrak{h}$ by

$$P\cup Q:=P^{1,\dots,k,k+1\dots k+l+1}Q^{k+1,\dots,k+l+1}$$

**Proposition 1.3.** $(\underline{\mathrm{ADT}},b,\cup)$ *is a differential graded associative algebra.*

*Proof.* The cup product is obviously associative. Thus the only thing we have to check is that

$$b(P\cup Q)=bP\cup Q+(-1)^{|P|}P\cup bQ\tag{8}$$

Let $k=|P|$ and $l=|Q|$. The l.h.s. of (8) is equal to

$$P^{2,\dots,k+1,k+2\dots k+l+2}Q^{k+2,\dots,k+l+2}+\sum_{i=1}^{k}(-1)^i P^{1,\dots,ii+1,\dots,k+1,k+2\dots k+l+2}Q^{k+2,\dots,k+l+2}$$
$$+\sum_{i=k+1}^{k+l+1}(-1)^i P^{1,\dots,k,k+1\dots k+l+2}Q^{k+1,\dots,ii+1,\dots,k+l+2}$$

The first line of this expression is equal to

$$bP\cup Q-(-1)^{k+1}P^{1,\dots,k,k+1\dots k+l+2}Q^{k+2,\dots,k+l+2}$$

and the last term of the same expression gives

$$(-1)^k\big(P\cup bQ-P^{1,\dots,k,k+1\dots k+l+2}Q^{k+2,\dots,k+l+2}\big)$$

The proposition is proved. □

Recall that in the case $\mathfrak{h} = \{0\}$ one can define a brace algebra structure on $(T^*U\mathfrak{g})[1]$ (see [Ge]). Unfortunately we are not able to extend this structure to $\underline{\mathrm{ADT}}$ in general. Since we deal with $\mathfrak{h}$-equivariant solutions of ADTE we can consider the subspace $\mathfrak{g}_2 = (\underline{\mathrm{ADT}})^{\mathfrak{h}}$ of $\mathfrak{h}$-invariants. Let us now define a collection of linear homogeneous maps of degree zero $\{-|-,\dots,-\} : \mathfrak{g}_2[1] \otimes \mathfrak{g}_2[1]^{\otimes m} \to \mathfrak{g}_2[1]$ indexed by $m \geq 0$, and $\{P|Q_1,\dots,Q_m\}$ is given by

$$\sum_{\substack{0\leq i_1, i_m+k_m\leq n\\ i_l+k_l\leq i_{l+1}}} (-1)^\epsilon P^{1,\dots,i_1+1\dots i_1+k_1,\dots,i_m+1\dots i_m+k_m,\dots,n+1} \prod_{s=i}^{m} Q_s^{i_s+1,\dots,i_s+k_s,i_s+k_s+1\dots n+1}$$

where $k_s = |Q_s|$, $n = |P| + \sum_s k_s - m$ and $\epsilon = \sum_s (k_s-1)i_s$.

**Proposition 1.4.** $(\mathfrak{g}_2[1], \{-|-,\dots,-\})$ *is a brace algebra.*

*Proof.* Since we work with $\mathfrak{h}$-invariant elements one can remark that if $i_s + k_s \leq i_t$ then $Q_s^{i_s+1,\dots,i_s+k_s,i_s+k_s+1\dots n+1}$ and $Q_t^{i_t+1,\dots,i_t+k_t,i_t+k_t+1\dots n+1}$ commute. Using this the proof becomes identical to the case when $\mathfrak{h} = 0$ (see [Ge] for example). □

Now observe that since $m = 1^{\otimes 3} \in (\otimes^2 U\mathfrak{g} \otimes U\mathfrak{h})^{\mathfrak{h}}$ is such that $\{m|m\} = 0$ one obtains a $B_\infty$-algebra structure ([Ba]) on $\mathfrak{g}_2$ (see [Kh]). More precisely, we have a differential graded bialgebra structure on the cofree tensorial coalgebra $T(\mathfrak{g}_2[1])$ of which structure maps $a^n, a^{p,q}$ are given by

- $a^1(P) = bP = (-1)^{|P|-1}[m,P]_G$, where
$$[P,Q]_G := \{P|Q\} - (-1)^{(|P|-1)(|Q|-1)}\{Q|P\}$$
- $a^2(P,Q) = \{m|P,Q\} = P \cup Q$
- $a^{0,1} = a^{1,0} = \mathrm{id}$
- $a^{1,n}(P;Q_1,\dots,Q_n) = \{P|Q_1,\dots,Q_n\}$ for $n \geq 1$
- all other maps are zero

In particular, we have

**Proposition 1.5.** $(\mathfrak{g}_2[1], b, [,]_G)$ *is a* `dgla`.

**Remark 1.6.** Since that for any graded vector space $V$, `dg` bialgebra structures on the cofree coassociative coalgebra $T^cV$ are in one-to-one correspondence with `dg` Lie bialgebra structures on the cofree Lie coalgebra $L^cV$ (see [Ta], section 5), then $L^c(\mathfrak{g}_2[1])$ becomes a `dg` Lie bialgebra with differential and Lie bracket given by maps $l^n, l^{p,q}$ such that $l^1 = b$ and $l^{1,1} = [,]_G$. Therefore $d_2 := \sum_{i\geq 0} l^i + \sum_{p,q\geq 0} l^{p,q} : C^c(L^c(\mathfrak{g}_2[1])) \to C^c(L^c(\mathfrak{g}_2[1]))$ defines a $G_\infty$-algebra structure on $\mathfrak{g}_2$ ($d_2 \circ d_2 = 0$ since $d_2$ is just the Chevalley-Eilenberg differential on the `dg` Lie algebra $L^c(\mathfrak{g}_2[1])$).

1.3. **Main result and proof of theorem 0.2.** First of all, observe that $\underline{\mathrm{CDYB}}$, $\mathfrak{g}_1$ and $\mathcal{G}_1 := C^c(\mathfrak{g}_1[2])$ have a natural grading induced by the one of $S\mathfrak{h}$. In the same way $\underline{\mathrm{ADT}}$, $\mathfrak{g}_2$ and $\mathcal{G}_2 := C^c(\mathfrak{g}_2[2])$ have a natural filtration induced by the one of $U\mathfrak{h}$. Our main goal is to prove the following theorem, which is sufficient to obtain algebraic dynamical twists from formal dynamical $r$-matrices.

**Theorem 1.7.** *In the reductive case, there exists a $L_\infty$-quasi-isomorphism*

$$\Psi : (\mathcal{G}_1, \mathrm{d} + [,]) \to (\mathcal{G}_2, b + [,]_G)$$

*with the following two filtration properties:*

(F1) $\forall X\in(\mathfrak{g}_1)_k,\ \Psi^1(X)=(\mathrm{alt}\otimes\mathrm{sym})(X) \bmod (\mathfrak{g}_2)_{\leq k-1}$
(F2) $\forall X\in(\Lambda^n\mathfrak{g}_1)_k,\ \Psi^n(X)\in(\mathfrak{g}_2)_{\leq n+k-1}$

*Proof of Theorem 0.2.* Now consider a formal solution $\rho(\lambda)\in(\wedge^2\mathfrak{g}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}$ to CDYBE. Let us define $\alpha:=\hbar\rho(\hbar\lambda)\in\hbar\mathfrak{g}_1[[\hbar]]$ which is a Maurer-Cartan element in $\hbar\mathfrak{g}_1[[\hbar]]$. The $L_\infty$-morphism property implies that $\widetilde{\alpha}:=\sum_{n=1}^\infty\frac{1}{n!}\Psi^n(\Lambda^n\alpha)$ is a Maurer-Cartan element in $\hbar\mathfrak{g}_2[[\hbar]]$; this exactly means that $K:=1+\widetilde{\alpha}\in(\otimes^2U\mathfrak{g}\otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ satisfies ADTE. Moreover, due to (F2) the coefficient $K_n$ of $\hbar^n$ in $K$ lies in $(\mathfrak{g}_2)_{\leq n-1}$. It means that there exists $J\in(U\mathfrak{g}^{\otimes 2}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ satisfying DTE and such that $K=(\mathrm{id}^{\otimes 2}\otimes\mathrm{sym})(J(\hbar\lambda))$. Finally, property (F1) obviously implies that the semi-classical limit condition $\frac{J-J^{\mathrm{op}}}{\hbar}=\rho \bmod \hbar$ is satisfied. □

## 2. Proof of theorem 1.7

In this section we assume that $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ with $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$. Let us denote by $\mathrm{p}:\mathfrak{g}\to\mathfrak{m}$ the projection on $\mathfrak{m}$ along $\mathfrak{h}$; it is $\mathfrak{h}$-equivariant.

2.1. **Resolutions.** Let us first observe that the bilinear map $[,]_{\mathfrak{m}}:=(\wedge^{\cdot}\mathrm{p})\circ[,]$ defines a graded Lie bracket of degree $-1$ on $(\wedge^*\mathfrak{m})^{\mathfrak{h}}$. Then we prove

**Proposition 2.1.** *The natural map $p_1:(\mathfrak{g}_1[1],\mathrm{d},[,])\to((\wedge^*\mathfrak{m})^{\mathfrak{h}}[1],0,[,]_{\mathfrak{m}})$ is a morphism of* `dgla`*'s. Moreover, there exists an operator $\delta:\mathfrak{g}_1^*\to\mathfrak{g}_1^{*-1}$ such that $\delta\mathrm{d}+\mathrm{d}\delta=\mathrm{id}-p_1$, $\delta\circ\delta=0$ and $\delta\big((\mathfrak{g}_1)_k\big)\subset(\mathfrak{g}_1)_{k+1}$. In particular, $p_1$ induces an isomorphism in cohomology.*

*Proof.* The projection $p_1:=(\wedge^{\cdot}\mathrm{p})\otimes\varepsilon:(\underline{\mathrm{CDYB}},\mathrm{d})\to(\wedge^*\mathfrak{m},0)$ is a $\mathfrak{h}$-equivariant morphism of complexes, and it obviously restricts to a morphism of (differential) graded Lie algebras at the level of $\mathfrak{h}$-invariants.

Moreover, $\wedge^n\mathfrak{g}\otimes S\mathfrak{h}\cong\bigoplus_{p+q=n}\wedge^p\mathfrak{m}\otimes\wedge^q\mathfrak{h}\otimes S\mathfrak{h}$ as a $\mathfrak{h}$-module; and under this identification d becomes $-\mathrm{id}\otimes d_K$, where $d_K:\wedge^*\mathfrak{h}\otimes S\mathfrak{h}\to\wedge^{*+1}\mathfrak{h}\otimes S\mathfrak{h}$ is Koszul's coboundary operator, and $p_1$ corresponds to the projection on the part of zero antisymmetric and symmetric degrees in $\mathfrak{h}$. Let us define $\delta=\mathrm{id}\otimes\delta_K$ with $\delta_K:\wedge^*\mathfrak{h}\otimes S^*\mathfrak{h}\to\wedge^{*-1}\mathfrak{h}\otimes S^{*+1}\mathfrak{h}$ defined by

$$\delta_K(x_1\wedge\cdots\wedge x_n\otimes h_1\cdots h_m)=\begin{cases}\frac{1}{m+n}\sum_i(-1)^ix_1\wedge\cdots\hat{x_i}\cdots\wedge x_n\otimes h_1\cdots h_mx_i & \text{if } m+n\neq 0\\ 0 & \text{otherwise}\end{cases}$$

Finally remark $\delta$ is a $\mathfrak{h}$-equivariant homotopy operator: $\delta\mathrm{d}+\mathrm{d}\delta=\mathrm{id}-p_1$ and $\delta\circ\delta=0$. The proposition is proved. □

Now we prove a similar result for $\mathfrak{g}_2$. Let us first define $U\mathfrak{m}:=\mathrm{sym}(S\mathfrak{m})\subset U\mathfrak{g}$; this is a sub-coalgebra of $U\mathfrak{g}$ and thus $T^*U\mathfrak{m}$ equipped with its Hochschild's coboundary operator $b_{\mathfrak{m}}$ becomes a cochain subcomplex of the Hochschild complex $(T^*U\mathfrak{g},b_{\mathfrak{g}})$ of $U\mathfrak{g}$. We also have the following

**Lemma 2.2.** *$U\mathfrak{g}=U\mathfrak{g}\cdot\mathfrak{h}\oplus U\mathfrak{m}$ as a filtered $\mathfrak{h}$-module. Moreover $[,]_{G,\mathfrak{m}}:=(\otimes^{\cdot}\mathrm{p})\circ[,]$ defines a graded Lie bracket of degree $-1$ on $(T^*U\mathfrak{m})^{\mathfrak{h}}$*

*Proof.* See [He, Ch.II §4.2] for the first statement. The second statement follows from a direct computation. □

Then we prove the

**Proposition 2.3.** *The natural map* $p_2 : (\mathfrak{g}_2[1], b, [,]_G) \to ((T^*U\mathfrak{m})^{\mathfrak{h}}[1], b_{\mathfrak{m}}, [,]_{G,\mathfrak{m}})$ *is a morphism of* `dgla`*'s. Moreover, there exists an operator* $\kappa : \mathfrak{g}_2^* \to \mathfrak{g}_2^{*-1}$ *such that* $\kappa b + b\kappa = \mathrm{id} - p_2$*,* $\kappa \circ \kappa = 0$ *and* $\kappa\big((\mathfrak{g}_2)_{\leq k}\big) \subset (\mathfrak{g}_2)_{\leq k+1}$*. In particular,* $p_2$ *induces an isomorphism in cohomology.*

*Proof.* The projection $p_2 := (\otimes^{\cdot}\mathrm{p}) \otimes \varepsilon : (\underline{\mathrm{ADT}}, b) \to (T^*U\mathfrak{m}, b_{\mathfrak{m}})$ is a $\mathfrak{h}$-equivariant morphism of complexes, and it obviously restricts to a morphism of `dgla`'s at the level of $\mathfrak{h}$-invariants (by lemma 2.2).

Remember that $\mathfrak{g}_2$ has a natural filtration induced by the one of $U\mathfrak{h}$. Then one obtains a spectral sequence of which we compute the first terms:

$$\begin{array}{ll} E_0^{*,*} = (T^*U\mathfrak{g} \otimes S^*\mathfrak{h})^{\mathfrak{h}} & d_0 = b_{\mathfrak{g}} \otimes \mathrm{id} \\ E_1^{*,*} = (\wedge^*\mathfrak{g} \otimes S^*\mathfrak{h})^{\mathfrak{h}} & d_1 = \mathrm{d} \\ E_2^{*,*} = E_2^{*,0} = (\wedge^*\mathfrak{m})^{\mathfrak{h}} & d_2 = 0 \end{array}$$

Then the proposition follows from proposition 2.1. □

2.2. **Inverting $p_2$.** In this subsection, taking our inspiration from [Mo, appendix], we prove the following

**Proposition 2.4.** *There exists a* $L_\infty$*-quasi-isomorphism*

$$\mathcal{Q}_2 : (C^c((T^*U\mathfrak{m})^{\mathfrak{h}}[2]), b_{\mathfrak{m}} + [,]_{G,\mathfrak{m}}) \to (C^c(\mathfrak{g}_2[2]), b + [,]_G)$$

*such that* $\mathcal{Q}_2^1$ *is the natural inclusion and* $\mathcal{Q}_2^n$ *takes values in* $(\mathfrak{g}_2)_{\leq n-1}$*.*

*Proof.* Let $(N, b_N) \subset (\mathfrak{g}_2, b)$ be the kernel of the surjective morphism of complexes $p_2 : (\mathfrak{g}_2, b) \to ((T^*U\mathfrak{m})^{\mathfrak{h}}, b_{\mathfrak{m}})$. It follows from the proofs of propositions 2.1 and 2.3 that there exists an operator $H : N^* \to N^{*-1}$ such that $H \circ H = 0$, $b_N H + H b_N = \mathrm{id}$ and $H(N_{\leq n}) \subset N_{\leq n+1}$.

Now let us construct a $L_\infty$-isomorphism

$$\mathcal{F} : \big(C^c(\mathfrak{g}_2[2]), b + [,]_G\big) \xrightarrow{\sim} \big(C^c((T^*U\mathfrak{m})^{\mathfrak{h}}[2] \oplus N[2]), b_{\mathfrak{m}} + b_N + [,]_{G,\mathfrak{m}}\big)$$

with structure maps $\mathcal{F}^n : \Lambda^n\mathfrak{g}_2 \to ((T^*U\mathfrak{m})^{\mathfrak{h}} \oplus N)[1-n]$ such that

- $\mathcal{F}^1$ is the sum of $p_2$ with the projection on $N$ along $(T^*U\mathfrak{m})^{\mathfrak{h}}$ (in some sense $\mathcal{F}^1$ is the identity),
- for any $n > 1$ and $X \in (\Lambda^n\mathfrak{g}_2)_{\leq k}$, $\mathcal{F}^n(X) \in N_{\leq n+k-1}$.

Let us prove it by induction on $n$. First $\mathcal{F}^1$ is a morphism of complexes by definition. Then let us define $\mathcal{K}_2 : \Lambda^2\mathfrak{g}_2 \to ((T^*U\mathfrak{m})^{\mathfrak{h}} \oplus N)[1]$ by

$$\mathcal{K}_2(x \Lambda y) = [\mathcal{F}^1(x), \mathcal{F}^1(y)]_{G,\mathfrak{m}} - \mathcal{F}^1([x,y]_G)$$

It takes values in $N[1]$ and is such that $b_N\mathcal{K}_2(x,y) + \mathcal{K}_2(bx,y) + \mathcal{K}_2(x,by) = 0$. Consequently $\mathcal{F}^2 := H \circ \mathcal{K}_2 : \Lambda^2\mathfrak{g}_2 \to N$ is such that

$$b_N\mathcal{F}^2(x,y) - \mathcal{F}^2(bx,y) - \mathcal{F}^2(x,by) = \mathcal{K}_2(x,y) \quad (L_\infty\text{-condition for } \mathcal{F}^2)$$

and for any $X \in (\Lambda^2\mathfrak{g}_2)_{\leq k}$, $\mathcal{F}^2(X) \in N_{\leq k+1}$. After this, suppose we have constructed $\mathcal{F}^1, \ldots, \mathcal{F}^n$ and let us define

$$\mathcal{K}_{n+1} := [,]_{G,\mathfrak{m}} \circ \mathcal{F}^{\leq n} - \mathcal{F}^{\leq n} \circ [,]_G : \Lambda^2\mathfrak{g}_2 \to ((T^*U\mathfrak{m})^{\mathfrak{h}} \oplus N)[1]$$

It obviously takes values in $N[1]$ and is such that $b_N K_{n+1} + K_{n+1} b = 0$. Consequently $\mathcal{F}^{n+1} := H \circ K_{n+1}$ satisfies the $L_\infty$-condition

$$b_N\mathcal{F}^{n+1} - \mathcal{F}^{n+1}b = b_N H K_{n+1} - H K_{n+1} b = (b_N H + H b_N) K_{n+1} = K_{n+1}$$

and for any $X \in (\Lambda^n\mathfrak{g}_2)_{\leq n+1}$, $\mathcal{F}^{n+1}(X) \in N_{\leq n+k}$ (since $\mathcal{K}_{n+1}(X) \in N_{\leq n+k-1}$).

Now let $\mathcal{H}$ be the inverse of the isomorphism $\mathcal{F}$, it is such that for any $n \geq 1$ and $X \in (\Lambda^n \mathfrak{g}_2)_{\leq k}$, $\mathcal{H}^n(X) \in N_{\leq n+k-1}$. Finally we obtain $\mathcal{Q}_2$ by composing $\mathcal{H}$ with the inclusion of $\mathtt{dgla}$'s $(T^*U\mathfrak{m})^{\mathfrak{h}}[1] \hookrightarrow ((T^*U\mathfrak{m})^{\mathfrak{h}} \oplus N)[1]$. □

2.3. **End of the proof.** Recall from [He, Ch.II §4.2] that $(T^*U\mathfrak{m})^{\mathfrak{h}} = \mathrm{Diff}^*(G/H)^G$ and $(\wedge^*\mathfrak{m})^{\mathfrak{h}} = \Gamma(G/H, \wedge^*T(G/H))^G$ as $\mathtt{dgla}$'s. Remember also from [No, Ch.II §8] that $G$-invariant connections on $G/H$ are in one-to-one correspondence with $\mathfrak{h}$-equivariant linear maps $\alpha : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$, and that the torsion tensor is given by $\alpha - \alpha^{21} - \mathrm{p} \circ [,]$. Thus $G/H$ is equipped with a $G$-invariant torsion free connection $\nabla$, corresponding to the map $\alpha := \frac{1}{2}\mathrm{p} \circ [,]$. Then using a theorem of Dolgushev, see [Do, theorem 5], we obtain a $G$-equivariant $L_\infty$-quasi-isomorphism $\phi : \Gamma(G/H, \wedge^*T(G/H)) \to \mathrm{Diff}^*(G/H)$ with first structure map $\phi^1 = \mathrm{alt}$, which restricts to a $L_\infty$-quasi-isomorphism at the level of $G$-invariants. Let us define $\psi := \mathcal{Q}_2 \circ \phi \circ p_1 : (C^c(\mathfrak{g}_1[2]), \mathrm{d} + [,]) \to (C^c(\mathfrak{g}_2[2]), b + [,]_G)$; it is a $L_\infty$-quasi-isomorphism with first structure map $\psi^1 = (\mathrm{alt} \otimes 1) \circ (\wedge^{\cdot}\mathrm{p} \otimes \varepsilon)$.

Finally define $V := (\mathrm{alt} \otimes \mathrm{sym}) \circ \delta : \mathfrak{g}_1 \to \mathfrak{g}_2[-1]$ and use lemma A.3 to construct a $L_\infty$-quasi-morphism $\Psi : (C^c(\mathfrak{g}_1[2]), \mathrm{d} + [,]) \to (C^c(\mathfrak{g}_2[2]), b + [,]_G)$ with first structure map $\Psi^1 = \psi^1 + b \circ V + V \circ \mathrm{d}$. Since for any $X \in (\mathcal{G}_1)_k$, then

$$b \circ (\mathrm{alt} \otimes \mathrm{sym})(X) = (\mathrm{alt} \otimes \mathrm{sym}) \circ \mathrm{d}(X) \mod (\mathfrak{g}_2)_{\leq k-1}$$

$$\text{and thus} \quad \begin{aligned} \Psi^1(X) &= \psi^1(X) + bV(X) + V(\mathrm{d}X) \\ &= (\mathrm{alt} \otimes \mathrm{sym}) \circ (p_1 + \mathrm{d}\delta + \delta\mathrm{d})(X) \mod (\mathfrak{g}_2)_{\leq k-1} \\ &= (\mathrm{alt} \otimes \mathrm{sym})(X) \mod (\mathfrak{g}_2)_{\leq k-1} \end{aligned}$$

Consequently $\Psi$ satisfies (F1). Moreover, it follows from remark A.4 that $\Psi$ also satisfies (F2). □

## 3. Classification

Theorem 1.7 implies a stronger result than just the existence of the twist quantization. Namely, since $\Psi$ is a $L_\infty$-quasi-isomorphism there is a bijection between the moduli spaces of Maurer-Cartan elements of $\mathtt{dgla}$'s $(\mathfrak{g}_1[1])[[\hbar]]$ and $(\mathfrak{g}_2[1])[[\hbar]]$.

3.1. **Classification of algebraic and formal dynamical twists.** Following [EE1], two dynamical twists $J(\lambda)$ and $J'(\lambda)$ are said to be *gauge equivalent* if there exists a regular $\mathfrak{h}$-equivariant map $T(\lambda) = \exp(q) + O(\hbar) \in \mathrm{Reg}(\mathfrak{h}^*, U\mathfrak{g})^{\mathfrak{h}}[[\hbar]]$, with $q \in \mathrm{Reg}(\mathfrak{h}^*, \mathfrak{g})^{\mathfrak{h}}$ such that $q(0) = 0$, and satisfying

$$J'(\lambda) = T^{12}(\lambda) * J(\lambda) * T^2(\lambda)^{-1} * T^1(\lambda + \hbar h^2)^{-1} \tag{9}$$

Dealing with formal functions one can easily derive an equivalence relation for the corresponding algebraic dynamical twists $K = J(\hbar\lambda)$ and $K' = J'(\hbar\lambda)$:

$$K' = Q^{12,3}K(Q^{2,3})^{-1}(Q^{1,23})^{-1} \tag{10}$$

in $(U\mathfrak{g}^{\otimes 2} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$, with $Q = 1 + O(\hbar) \in (U\mathfrak{g} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ given by $T(\hbar\lambda)$.

Assume now we are in the reductive case.

Since the composition $\mathcal{Q}_2 \circ \phi : (C^c((\wedge\mathfrak{m})^{\mathfrak{h}}[2]), [,]_{\mathfrak{m}}) \to (C^c(\mathfrak{g}_2[2]), b + [,]_G)$ in the previous section is a $L_\infty$-quasi-isomorphism then we have a bijective correspondance

$$\frac{\{\pi \in \hbar(\wedge^2\mathfrak{m})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } [\pi, \pi]_{\mathfrak{m}} = 0\}}{G_0} \longleftrightarrow \frac{\{\text{algebraic dynamical twists}\}}{\text{gauge equivalence (10)}} \tag{11}$$

where $G_0$ is the prounipotent group corresponding to the Lie algebra $\hbar\mathfrak{m}^{\mathfrak{h}}[[\hbar]]$. Moreover, since the structure maps $\mathcal{Q}_2^n$ take values in $(\mathfrak{g}_2)_{\leq n-1}$ then it appears that any algebraic dynamical twist is gauge equivalent to a one with the $\hbar$-adic valuation property and thus we have a bijection

$$\frac{\{\text{algebraic dynamical twists}\}}{\text{gauge equivalence (10)}} \longleftrightarrow \frac{\{\text{formal dynamical twists}\}}{\text{gauge equivalence (9)}} \tag{12}$$

3.2. **Classical counterpart.** Assume that we are in the reductive case. Since $p_1$ is a $L_\infty$-quasi-isomorphism by propostion 2.1 then we have a bijection

$$\frac{\{\alpha\in\hbar(\wedge^2\mathfrak{g}\otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]\text{ s.t. }\mathrm{d}\alpha+\frac{1}{2}[\alpha,\alpha]=0\}}{G_1}\longleftrightarrow\frac{\{\pi\in\hbar(\wedge^2\mathfrak{m})^{\mathfrak{h}}[[\hbar]]\text{ s.t. }[\pi,\pi]_{\mathfrak{m}}=0\}}{G_0}$$

where $G_1$ is a prounipotent group and its action (by affine transformations) is given by the exponentiation of the infinitesimal action of its Lie algebra $\hbar(\mathfrak{g}\otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$:

$$q\cdot\alpha=\mathrm{d}q+[q,\alpha]\qquad\big(q\in\hbar(\mathfrak{g}\otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]\big) \tag{13}$$

Then going along the lines of subsection 2.2 one can prove the following

**Proposition 3.1.** *There exists a $L_\infty$-quasi-isomorphism*

$$\mathcal{Q}_1:(C^c((\wedge^*\mathfrak{m})^{\mathfrak{h}}[2]),[,]_{\mathfrak{m}})\to(C^c(\mathfrak{g}_1[2]),\mathrm{d}+[,])$$

*such that $\mathcal{Q}_1^1$ is the natural inclusion and $\mathcal{Q}_1^n$ takes values in $(\mathfrak{g}_1)_{\leq n-1}$.*

Consequently any Maurer-Cartan element in $(\mathfrak{g}_1[1])[[\hbar]]$ is equivalent to a one of the form $\hbar\rho_\hbar(\hbar\lambda)$, where $\rho_\hbar\in(\wedge^2\mathfrak{g}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ satisfies CDYBE. In other words $\rho_\hbar$ is $\hbar$-dependant formal dynamical $r$-matrix. On such a $\rho_\hbar$ the infinitesimal action (13) becomes

$$q\cdot\rho_\hbar=-\sum_i h_i\wedge\frac{\partial q}{\partial\lambda^i}+[q,\rho_\hbar]\qquad(q\in\mathfrak{g}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}[[\hbar]] \tag{14}$$

This action integrates in an affine action of some group $\widetilde{G_1}$ of $\mathfrak{h}$-equivariant formal maps with values in the Lie group $G$ of $\mathfrak{g}$. And then we have a bijection

$$\frac{\{\pi\in\hbar(\wedge^2\mathfrak{m})^{\mathfrak{h}}[[\hbar]]\text{ s.t. }[\pi,\pi]_{\mathfrak{m}}=0\}}{G_0}\longleftrightarrow\frac{\{\text{form. dynam. }r\text{-matrices}/\mathbb{R}[[\hbar]]\}}{\widetilde{G_1}} \tag{15}$$

**Remark 3.2.** This bijection has to be compared with Proposition 2.13 in [X2] and section 3 of [ES]

Finally, combining (15), (11) and (12) we obtain the following generalization of Theorem 6.11 in [X2] to the case of a nonabelian base:

**Theorem 3.3.** *Let $\pi\in(\wedge^2\mathfrak{m})^{\mathfrak{h}}$ such that $[\pi,\pi]_{\mathfrak{m}}=0$. Then there are bijective correspondances between*

1. *the set of $\hbar$-dependant and $G$-invariant Poisson structures $\pi_\hbar=\hbar\pi$ mod $\hbar^2$ on $G/H$, modulo the action of $G_0$,*
2. *the set of $\hbar$-dependant formal dynamical $r$-matrices $\rho_\hbar(\lambda)$ such that $\rho_\hbar(0)=\pi$ mod $\hbar$ in $\wedge^2(\mathfrak{g}/\mathfrak{h})[[\hbar]]$, modulo the action (14) of $\widetilde{G_1}$,*
3. *the set of formal dynamical twists $J(\lambda)$ satisfying $Alt\frac{J(0)-1}{\hbar}=\pi$ mod $\hbar$ in $\wedge^2(\mathfrak{g}/\mathfrak{h})[[\hbar]]$, modulo gauge equivalence (9).*

## 4. Another case when the twist quantization exists

In this section we assume that $\mathfrak{h}$ is abelian and admits a Lie subalgebra $\mathfrak{m}$ as complement.

Note that since $\mathfrak{h}$ is abelian and $\mathfrak{m}$ a Lie subalgebra, the projection $p : \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{m}$ along $\mathfrak{h}$ extends to a morphsim of graded Lie algebras $\wedge^{\cdot} p : (\wedge\mathfrak{g})^{\mathfrak{h}} \to (\wedge\mathfrak{g})^{\mathfrak{h}}$ at the level of $\mathfrak{h}$-invariants. And thus $\wedge^{\cdot} p \otimes \varepsilon : (\mathfrak{g}_1[1], \mathrm{d}, [,]) \to ((\wedge\mathfrak{g})^{\mathfrak{h}}[1], 0, [,])$ is a morphism of dgla's. Then the natural inclusion $\mathrm{id}\otimes 1 : (T^*U\mathfrak{g})^{\mathfrak{h}} \to \mathfrak{g}_2$ obviously allows one to consider $(T^*U\mathfrak{g})^{\mathfrak{h}}[1]$ as a sub-dgla of $\mathfrak{g}_2[1]$. Finally recall from [Ca, section 3.3] that there exists a $L_\infty$-quasi-isomorphism $\mathcal{F} : C^c((\wedge^*\mathfrak{g})^{\mathfrak{h}}[2]) \to C^c((T^*U\mathfrak{g})^{\mathfrak{h}}[2])$ with $\mathcal{F}^1 = \mathrm{alt}$. By composing these maps one obtains a $L_\infty$-morphism

$$\widetilde{\mathcal{F}} : (\mathcal{G}_1, \mathrm{d} + [,]) \to (\mathcal{G}_2, b + [,]_G)$$

with values in $(\mathcal{G}_2)_{\leq 0}$ and first structure map $\widetilde{\mathcal{F}}^1 = (\mathrm{alt}\otimes 1)\circ(\wedge^{\cdot} p \otimes \varepsilon)$.

**Theorem 4.1.** *There exists a $L_\infty$-quasi-iomorphism*

$$\Psi : (\mathcal{G}_1, \mathrm{d} + [,]) \to (\mathcal{G}_2, b + [,]_G)$$

*with properties* (F1) *and* (F2) *of Theorem 1.7.*

*Proof.* First observe that since $\mathfrak{h}$ is abelian then $\mathfrak{g}_1 \cong ((\wedge\mathfrak{g})^{\mathfrak{h}} \cap \wedge\mathfrak{m}) \otimes \wedge\mathfrak{h} \otimes S\mathfrak{h}$ as a vector space. Thus if $\delta_K$ is as in the proof of proposition 2.1 then $\delta := \mathrm{id}\otimes\delta_K$ is a homotopy operator: $\delta\mathrm{d} + \mathrm{d}\delta = \mathrm{id} - \wedge^{\cdot} p\otimes\varepsilon$ and $\delta\circ\delta = 0$.

Now we proceed like in subsection 2.3: use lemma A.3 to construct a $L_\infty$-morphism $\Psi$ with first structure map $\Psi^1 = \widetilde{\mathcal{F}}^1 + b\circ V + V\circ\mathrm{d}$, where $V := (\mathrm{alt}\otimes\mathrm{sym})\circ\delta : \mathfrak{g}_1 \to \mathfrak{g}_2[-1]$.

It remains to prove that $\Psi$ is a quasi-isomorphism. It follows from the first observation in this proof that $H^*(\mathfrak{g}_1, \mathrm{d}) = (\wedge\mathfrak{g})^{\mathfrak{h}} \cap \wedge\mathfrak{m}$, which also equals $H^*(\mathfrak{g}_2, b)$ due to the spectral sequence argument. Consequently $\widetilde{\mathcal{F}}^1$ is a quasi-isomorphism of complexes, and so is $\Psi^1$. □

Finally using the same argumentation as in the proof of theorem 0.2 (subsection 1.3) one obtains the

**Theorem 4.2.** *If $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ with a Lie subalgebra as a complement, then any formal classical dynamical $r$-matrix for $(\mathfrak{g}, \mathfrak{h}, 0)$ admits a dynamical twist quantization (associated to the trivial associator).*

**Example 4.3.** In particular, this allows us to quantize dynamical $r$-matrices arizing from semi-direct products $\mathfrak{g} = \mathfrak{m} \ltimes \mathbb{C}^n$ like in [EN, example 3.7].

## Concluding remarks

Let us first observe that if $\mathfrak{h}$ is abelian then $(\wedge^*\mathfrak{g})^{\mathfrak{h}} \cap \wedge^*\mathfrak{m}[1]$ (resp. $(T^*U\mathfrak{g})^{\mathfrak{h}} \cap T^*\mathrm{sym}(S\mathfrak{m})[1]$) inherits a dgla structure from the one of $\mathfrak{g}_1[1]$ (resp. $\mathfrak{g}_2[1]$) and $H^*(\mathfrak{g}_1, \mathrm{d}) = (\wedge^*\mathfrak{g})^{\mathfrak{h}} \cap \wedge^*\mathfrak{m} = H^*(\mathfrak{g}_2, b)$, for any complement $\mathfrak{m}$ of $\mathfrak{h}$. Thus I conjecture that there exists a $L_\infty$-quasi-isomorphism between $(\wedge^*\mathfrak{g})^{\mathfrak{h}} \cap \wedge^*\mathfrak{m}[1]$ and $(T^*U\mathfrak{g})^{\mathfrak{h}} \cap T^*\mathrm{sym}(S\mathfrak{m})[1]$ which generalizes together $\phi$ of subsection 2.3 and $\mathcal{F}$ of section 4. In particular this would imply conjecture 0.1 in the abelian (and non-modified) case.

Let us then mention that one can consider a non-triangular (i.e., non-antisymmetric) version of non-modified classical dynamical $r$-matrices. Namely, $\mathfrak{h}$-equivariant maps

$r \in \mathrm{Reg}(\mathfrak{h}^*, \mathfrak{g}\otimes\mathfrak{g})$ such that $\mathrm{CYB}(r) - \mathrm{Alt}(dr) = 0$. According to [X3], a quantization of such a $r$ is a $\mathfrak{h}$-equivariant map $R = 1 + \hbar r + O(\hbar^2) \in \mathrm{Reg}(\mathfrak{h}^*, U\mathfrak{g}^{\otimes 2})[[\hbar]]$ that satisfies the *quantum dynamical Yang-Baxter equation* (QDYBE)

$$(16) \quad R^{1,2}(\lambda) * R^{1,3}(\lambda + \hbar h^2) * R^{2,3}(\lambda) = R^{2,3}(\lambda + \hbar h^1) * R^{1,3}(\lambda) * R^{1,2}(\lambda + \hbar h^3)$$

**Question 4.4.** *Does such a quantization always exist?*

The most famous example of non-triangular dynamical $r$-matrices was found in [AM] by Alekseev and Meinrenken, then extended successively to a more general context in [EV, ES, EE1], and quantized in [EE1].

Following [EE1], remark that for any non-triangular dynamical $r$-matrix $r$ such that $r + r^{\mathrm{op}} = t \in (S^2\mathfrak{g})^{\mathfrak{g}}$ (quasi-triangular case) one can define $\rho := r - t/2$ and $Z := \frac{1}{4}[t^{1,2}, t^{2,3}]$. Then $\rho$ is a modified dynamical $r$-matrix for $(\mathfrak{g}, \mathfrak{h}, Z)$; morever the assignment $r \longmapsto \rho$ is a bijective correspondence between quasi-triangular dynamical $r$-matrices for $(\mathfrak{g}, \mathfrak{h}, t)$ and modified dynamical $r$-matrices for $(\mathfrak{g}, \mathfrak{h}, Z)$. Now observe that if $J(\lambda)$ is a dynamical twist quantizing $\rho$, then $R(\lambda) = J^{\mathrm{op}}(\lambda)^{-1} * e^{\hbar t/2} * J(\lambda)$ is a quantum dynamical $R$-matrix quantizing $r$.

In this paper we have constructed such a dynamical twist in the triangular case $t = 0$. One can ask

**Question 4.5.** *Does such a dynamical twist exist for any quasi-triangular dynamical $r$-matrix? At least in the reductive and abelian cases?*

This question seems to be more reasonable than the previous one.
More generally one can ask if conjecture 0.1 (and its smooth and meromorphic versions) is true in general. A positive answer was given in [EE1] when $\mathfrak{h} = \mathfrak{g}$; but unfortunately it is not known in general, even for the non-dynamical case $\mathfrak{h} = \{0\}$ (which is the last problem of Drinfeld [Dr1]: quantization of coboundary Lie bialgebras).

Finally let me mention that if $r(\lambda)$ is a triangular dynamical $r$-matrix for $(\mathfrak{g}, \mathfrak{h})$, then the bivector field

$$\pi := \overrightarrow{r(\lambda)} + \sum_i \frac{\partial}{\partial \lambda^i} \wedge \overrightarrow{h_i} + \pi_{\mathfrak{h}^*}$$

is a $G\times H$-biinvariant Poisson structure on $G\times\mathfrak{h}^*$ and the projection $p : G\times\mathfrak{h}^* \to \mathfrak{h}^*$ is a momentum map. Moreover, according to [X3] any dynamical twist quantization $J(\lambda)$ of $r(\lambda)$ allows us to define a $G \times H$-biinvariant star-product $*$ quantizing $\pi$ on $G \times \mathfrak{h}^*$ as follows:

$$\begin{array}{lll} f * g = f *_{PBW} g & \text{if} & f,\, g \in C^\infty(\mathfrak{h}^*) \\ f * g = fg & \text{if} & f \in C^\infty(G),\, g \in C^\infty(\mathfrak{h}^*) \\ f * g = \exp\left(\hbar \sum_i \frac{\partial}{\partial \lambda^i} \otimes \overrightarrow{h_i}\right) \cdot (f \otimes g) & \text{if} & f \in C^\infty(\mathfrak{h}^*),\, g \in C^\infty(G) \\ f * g = \overrightarrow{J(\lambda)}(f \otimes g) & \text{if} & f,\, g \in C^\infty(G) \end{array}$$

This way the map $p^* : (\mathrm{Fct}(\mathfrak{h}^*)[[\hbar]], *_{PBW}) \to (\mathrm{Fct}(G\times\mathfrak{h}^*)[[\hbar]], *)$ becomes a quantum momentum map in the sens of [X1].
So there may be a way to see momentum maps and their quantum analogues as Maurer-Cartan elements in `dgla`'s.

## Appendix A. Homotopy Lie algebras

See [HS] for a detailed discussion of the theory.

Recall that a $L_\infty$*-algebra* structure on a graded vector space $\mathfrak{g}$ is a degree 1 coderivation $Q$ on the cofree cocommutative coalgebra $C^c(\mathfrak{g}[1])$ such that $Q\circ Q=0$. By cofreeness, such a coderivation $Q$ is uniquely determined by structure maps $Q^n:\Lambda^n\mathfrak{g}\to\mathfrak{g}[2-n]$ which satisfy an infinite collection of equations. In particular $(\mathfrak{g},Q^1)$ is a cochain complex.

**Example A.1.** Any `dgla` $(\mathfrak{g},\mathrm{d},[,])$ is canonically a $L_\infty$-algebra. Namely, $Q$ is given by structure maps $Q^1=\mathrm{d}$, $Q^2=[,]$ and $Q^n=0$ for $n>2$.

A $L_\infty$*-morphism* between two $L_\infty$-algebras $(\mathfrak{g}_1,Q_1)$ and $(\mathfrak{g}_2,Q_2)$ is a degree 0 morphism of coalgebras $F:C^c(\mathfrak{g}_1[1])\to C^c(\mathfrak{g}_2[1])$ such that $F\circ Q_1=Q_2\circ F$. Again by cofreeness, such a morphism is uniquely determined by structure maps $F^n:\Lambda^n\mathfrak{g}_1\to\mathfrak{g}_2[1-n]$ which satisfy an infinite collection of equations. In particular $F^1:\mathfrak{g}_1\to\mathfrak{g}_2$ is a morphism of complexes; when it induces an isomorphism in cohomology we say that $F$ is a $L_\infty$*-quasi-isomorphism.*

**Example A.2.** Any morphism of `dgla`'s is a $L_\infty$-morphism with all structure maps equal to zero except the first one.

In this paper we use many times the following

**Lemma A.3** ([Do])**.** *Let $F:C^c(\mathfrak{g}_1[1])\to C^c(\mathfrak{g}_2[1])$ be a $L_\infty$-morphism. For any linear map $V:\mathfrak{g}_1\to\mathfrak{g}_2[-1]$ there exists a $L_\infty$-morphism $\Psi:C^c(\mathfrak{g}_1[1])\to C^c(\mathfrak{g}_2[1])$ with first structure map $\Psi^1=F^1+Q_2^1\circ V+V\circ Q_1^1$. Moreover, if $F$ is a $L_\infty$-quasi-isomorphism then $\Psi$ is also.*

*Proof.* First remark that $V$ extends uniquely to a linear map $C^c(\mathfrak{g}_1[1])\to C^c(\mathfrak{g}_2[1])$ of degree $-1$ such that

$$\Delta_2\circ V=\big(F\otimes V+V\otimes F+\frac{1}{2}V\otimes(Q_2\circ V+V\circ Q_1)+\frac{1}{2}(Q_2\circ V+V\circ Q_1)\otimes V\big)\circ\Delta_1$$

where $\Delta_1$ and $\Delta_2$ denote comultiplications in $C^c(\mathfrak{g}_1[1])$ and $C^c(\mathfrak{g}_2[1])$, respectively. Then define $\Psi:=F+Q_2\circ V+V\circ Q_1$. □

**Remark A.4.** Assume that in the previous lemma $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are filtrated, $F$ is such that $F^n$ takes values in $(\mathfrak{g}_2)_{\leq n-1}$, and $V\big((\mathfrak{g}_1)_{\leq k}\big)\subset(\mathfrak{g}_2)_{\leq k+1}$. Then one can obviously check that for any $X\in(\Lambda^n\mathfrak{g}_1)_{\leq k}$, $F^n(X)\in(\mathfrak{g}_2)_{\leq n+k-1}$.

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IRMA, 7 rue René Descartes, F-67084 Strasbourg, France
*E-mail address*: **calaque@math.u-strasbg.fr**