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EXPLICIT EXAMPLES OF NONSOLVABLE WEAKLY HYPERBOLIC OPERATORS WITH REAL COEFFICIENTS

KAREL PRAVDA-STAROV

University of California, Berkeley

Abstract. We give in this paper two explicit examples of nonsolvable weakly hyperbolic operators with real coefficients in two-space-dimensions.

Keywords. Solvability, doubly characteristic operators, weakly hyperbolic operators.

MSC. 35A07.

1 Introduction

We provide here some explicit examples of nonsolvable weakly hyperbolic operators with real coefficients. These are, with $(t, x, y) \in \mathbb{R}^3$,

$$L_1 = \partial_t(\partial_t + y\partial_x) + \partial_y,$$

$$L_2 = \partial_t^2 - H(-y)|y|^k\partial_x^2 + \partial_y, \quad k \in \mathbb{N}^*, \quad H = \mathbb{1}_{\mathbb{R}_+},$$

where the notation $\mathbb{1}_{\mathbb{R}_+}$ stands for the characteristic function of the set $\mathbb{R}_+$. Both examples are weakly hyperbolic operators in two-space-dimensions. The operator L_1 has affine coefficients and the operator L_2 has coefficients in C^{k-1} . Y.V. Egorov gave in [2] an example of a nonsolvable weakly hyperbolic operator in one-space-dimension with a quite complicated expression. Although our examples are 2-space-dimensional, we feel that their simple expression is worth noticing.

Let us begin by recalling some results about solvability for pseudo-differential operators with real principal symbols. Let L be a classical pseudo-differential operator on an open set Ω of $\mathbb{R}^n$ with a real principal symbol a_m . The double characteristic set is defined as

$$\Sigma_2 = \{(x, \xi) \in \dot{T}^*(\Omega) : a_m(x, \xi) = 0, \quad d_\xi a_m(x, \xi) = 0\},$$

where $\dot{T}^*(\Omega)$ is the cotangent bundle minus the zero section.

- If the set Σ_2 is empty, the operator L is of strong-real-principal-type and local solvability with a loss of one derivative holds according to the theorem 26.1.7 in [4].

- In the case where the operator L has a real principal symbol a_m such that its subprincipal symbol a_{m-1}^s satisfies

$$a_m(x, \xi) = 0, \quad d_\xi a_m(x, \xi) = 0 \Rightarrow \operatorname{Im} a_{m-1}^s(x, \xi) \neq 0, \quad (1.0.1)$$

if $(x, \xi) \in \dot{T}^*(\Omega)$, N. Lerner has proved in the theorem 1.1 of [5] that there is also local solvability with a loss of one derivative. For example, this is the case of most of the operators of the type

$$AB + C,$$

where A, B, C are smooth real vector fields in $\mathbb{R}^3$ such that A, B and $[A, B]$ are linearly independent, for which F. Treves has shown in [8] that they are locally solvable.

- If we now assume that the set

$$\tilde{\Sigma}_2 = \{(x, \xi) \in \dot{T}^*(\Omega) : a_m(x, \xi) = 0, \quad d_\xi a_m(x, \xi) = 0, \quad \operatorname{Im} a_{m-1}^s(x, \xi) = 0\},$$

is non-empty, different situations can occur. For example, for the class of operators $AB + C$ studied by F. Treves in [8], the set $\tilde{\Sigma}_2$ can be non-empty, but the special structure of the principal symbol which appears as a product pq with $\{p, q\} \neq 0$ at $p = q = 0$, allows this author to obtain a solvability result with a loss of derivatives. The set $\tilde{\Sigma}_2$ can also be non-empty in the cases studied by G.A. Mendoza and G.A. Uhlmann in [7], for which they introduced the additional assumption $\operatorname{Sub}(\mathcal{P})$, also with a product structure (of involutive type) for the principal symbol.

Let us mention that there is a nice example in [1] of an operator verifying (1.0.1), which is therefore locally solvable although a quasi-homogeneous version of condition (Ψ) is violated in that case. For the operators L_1 and L_2 , the set $\tilde{\Sigma}_2$ is non-empty. The nonsolvability in any neighbourhood of 0 in $\mathbb{R}^3$ of the operator L_1 is a consequence of the result of nonsolvability proved by G.A. Mendoza and G.A. Uhlmann in the theorem 1.2 of [7]. We verify in this case that the operator L_1 violates the condition $\operatorname{Sub}(\mathcal{P})$ defined in [6] and [7]. To prove the nonsolvability in any neighbourhood of 0 for the operator with C^{k-1} coefficients L_2 , we prove by building a quasimode that **no** a priori estimates of the following type could hold

$$\exists C_0 > 0, \exists N_0 \in \mathbb{N}, \exists V_0 \text{ an open neighbourhood of } 0 \text{ in } \mathbb{R}^3 \text{ such that}$$

$$\forall u \in C_0^\infty(V_0), \quad C_0 \|L_2^* u\|_{(k-3)} \geq \|u\|_{(-N_0)},$$

where the notation $\|\cdot\|_{(s)}$ stands for the $H^s(\mathbb{R}^3)$ Sobolev norm. This fact induces that there do **not** exist an integer $N_0 \in \mathbb{N}$ and an open neighbourhood V_0 of 0 in $\mathbb{R}^3$ such that for all $f \in H^{N_0}(V_0)$, there exists $u \in H^{-k+3}(\mathbb{R}^3)$ such that

$$L_2 u = f,$$

on V_0 (let us notice that the quantity $L_2 u$ is well defined for $u \in H^{-k+3}(\mathbb{R}^3)$). Indeed if it was the case, we would have using similar arguments to the ones given by L. Hörmander in the proof of Lemma 26.4.5 in [4] that for all $v \in C_0^\infty(V_0)$,

$$|(f, v)_{L^2(V_0)}| = |(L_2 u, v)| = |(u, L_2^* v)| \leq \|u\|_{(-k+3)} \|L_2^* v\|_{(k-3)}. \quad (1.0.2)$$

Let us consider

$$\begin{aligned} T_v : H^{N_0}(V_0) &\rightarrow \mathbb{C} \\ f &\mapsto (f, v)_{L^2(V_0)}, \end{aligned}$$

for v in $C_0^\infty(V_0)$. We deduce from the previous estimate that for all f in $H^{N_0}(V_0)$, there exists $u \in H^{-k+3}(\mathbb{R}^3)$ such that

$$\sup_{v \in W} |T_v(f)| \leq \|u\|_{(-k+3)} < +\infty,$$

if $W = \{v \in C_0^\infty(V_0), \|L_2^* v\|_{(k-3)} \leq 1\}$. Since T_v is a bounded linear form for v in W , we deduce from the uniform boundedness principle that there exists a positive constant C_0 such that

$$\sup_{v \in W} \|T_v\| \leq C_0 < +\infty.$$

It follows that for all $f \in H^{N_0}(V_0)$ and $v \in C_0^\infty(V_0)$, $\|L_2^* v\|_{(k-3)} \leq 1$, we have

$$|(f, v)_{L^2(V_0)}| \leq C_0 \|f\|_{(N_0)},$$

which induces by homogeneity that for all $f \in H^{N_0}(V_0)$ and $v \in C_0^\infty(V_0)$,

$$|(f, v)_{L^2(V_0)}| \leq C_0 \|f\|_{(N_0)} \|L_2^* v\|_{(k-3)}, \quad (1.0.3)$$

if $\|L_2^* v\|_{(k-3)} \neq 0$. According to (1.0.2), we notice that this estimate (1.0.3) is also fulfilled if $\|L_2^* v\|_{(k-3)} = 0$. Using now that $\|T_v\| = \|v\|_{(-N_0)}$ for all v in $C_0^\infty(V_0)$, we obtain from (1.0.3) that the following estimate

$$\forall v \in C_0^\infty(V_0), \quad C_0 \|L_2^* v\|_{(k-3)} \geq \|v\|_{(-N_0)},$$

holds, which is not possible according to our result.

2 Nonsolvability of the operator L_1

The operator L_1 is defined in standard quantization (and also in Weyl quantization) by the symbol

$$p(t, x, y; \tau, \xi, \eta) = -\tau(\tau + y\xi) + i\eta.$$

We first notice that its principal symbol, $p_2 = -\tau(\tau + y\xi)$, is real and that the doubly characteristic set

$$\Sigma_2(L_1) = \{(t, x, y; \tau, \xi, \eta) \in \dot{T}^*(\mathbb{R}^3) : p_2 = 0, d_{\tau, \xi, \eta} p_2 = 0\},$$

where $\dot{T}^*(\mathbb{R}^3)$ stands for the cotangent bundle minus the zero section, is not empty since

$$\begin{aligned} \Sigma_2(L_1) = & \{(t, x, y; \tau, \xi, \eta) \in \dot{T}^*(\mathbb{R}^3) : y = \tau = 0, (\xi, \eta) \neq (0, 0)\} \\ & \cup \{(t, x, y; \tau, \xi, \eta) \in \dot{T}^*(\mathbb{R}^3) : \tau = \xi = 0, \eta \neq 0\}. \end{aligned}$$

Let us consider the two real-valued symbols $q = -\tau$ and $s = \tau + y\xi$, we have $p_2 = qs$. The set $\Sigma_2(L_1)$ is a submanifold of codimension 2 near the point $\nu_0 = (t_0, x_0, 0; 0, 1, 0) \in \Sigma_2(L_1)$ if $t_0, x_0 \in \mathbb{R}$, which is involutive since

$$\begin{aligned} (T_\nu \Sigma_2(L_1))^\sigma &= \{(t, x, y; \tau, \xi, \eta) \in \mathbb{R}^6 : x = y = \tau = \xi = 0\} \\ &\subset T_\nu \Sigma_2(L_1) = \{(t, x, y; \tau, \xi, \eta) \in \mathbb{R}^6 : y = \tau = 0\}, \end{aligned}$$

for all ν belonging to a neighbourhood of ν_0 in $\Sigma_2(L_1)$ if $T_\nu \Sigma_2(L_1)$ stands for the tangent plane of $\Sigma_2(L_1)$ in ν . We also notice that the Hamilton vector fields H_q, H_s and the radial vector field r , which are equal to

$$H_q = -\frac{\partial}{\partial t}, \quad H_s = \frac{\partial}{\partial t} - \xi \frac{\partial}{\partial \eta}, \quad r = \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta},$$

at points in $\Sigma_2(L_1)$ near ν_0 , are independent and that the imaginary part of the subprincipal symbol, $p_1^s = i\eta$, changes sign at the first order in 0 along the following bicharacteristic of the symbol s ,

$$\begin{cases} \gamma'(t) = H_s(\gamma(t)) \\ \gamma(0) = \nu_0, \end{cases}$$

since $\text{Im } p_1^s(\nu_0) = 0$ and

$$\begin{aligned} \frac{d}{dt} [\text{Im } p_1^s(\gamma(t))] \big|_{t=0} &= d\text{Im } p_1^s(\gamma(t)) \cdot H_s(\gamma(t)) \big|_{t=0} = \\ &\sigma(H_s(\gamma(t)), H_{\text{Im } p_1^s(\gamma(t))}) \big|_{t=0} = \{s, \text{Im } p_1^s\}(\gamma(t)) \big|_{t=0} = -1 \neq 0. \end{aligned}$$

It follows that the condition $\text{Sub}(\mathcal{P})$ defined by G.A. Mendoza and G.A. Uhlmann in [7] is violated and we deduce from Theorem 1.2 in [7] that the operator L_1 is not locally solvable at $\nu_0 \in \Sigma_2(L_1)$, which induces that the operator L_1 is nonsolvable in any neighbourhood of 0 in $\mathbb{R}^3$.

3 Nonsolvability of the operator L_2

The second operator L_2 that we study, is defined in standard quantization (and also in Weyl quantization) by the symbol

$$p = i\eta + (\theta_k(y)\xi^2 - \tau^2) = i(\eta + i(\tau^2 - \theta_k(y)\xi^2)),$$

where θ_k is the $C^{k-1}(\mathbb{R}, \mathbb{R})$ function defined for $k \in \mathbb{N}^*$ by

$$\theta_k(y) = (-1)^k y^k H(-y) \text{ if } H = \mathbb{1}_{\mathbb{R}_+},$$

where the notation $\mathbb{1}_X$ stands for characteristic function of the set X . We notice that its principal symbol, $p_2 = \theta_k(y)\xi^2 - \tau^2$, is a real C^{k-1} symbol and that the doubly characteristic set

$$\begin{aligned} \Sigma_2(L_2) &= \{(t, x, y; \tau, \xi, \eta) \in \dot{T}^*(\mathbb{R}^3) : p_2 = 0, d_{\tau, \xi, \eta} p_2 = 0\} \\ &= \{(t, x, y; \tau, \xi, \eta) \in \dot{T}^*(\mathbb{R}^3) : \tau = 0, y \in \mathbb{R}_+\} \\ &\cup \{(t, x, y; \tau, \xi, \eta) \in \dot{T}^*(\mathbb{R}^3) : \tau = \xi = 0\}, \end{aligned}$$

is not empty. This set contains some points, $(t, x, 0; 0, \pm 1, 0) \in \Sigma_2(L_2)$, where the imaginary part of the subprincipal symbol vanishes, $p_1^s = i\eta$. Then, we notice that since the function $y \mapsto \tau^2 - \theta_k(y)\xi^2$ changes sign from $-$ to $+$ whenever $\tau\xi \neq 0$ if y increases, the symbol p violates a quasi-homogeneous version of the condition (Ψ) .

3.1 Construction of a quasimode

Let us consider $N_0 \in \mathbb{N}$,

$$\psi_1 \in C_0^\infty(\mathbb{R}^2, \mathbb{R}), \text{ supp } \psi_1 \subset [1, 4]^2, \psi_1 = 1 \text{ on } [2, 3]^2, \quad (3.1.1)$$

$$\chi_0 \in C_0^\infty(\mathbb{R}, \mathbb{R}), \text{ supp } \chi_0 \subset [-1, 1], \chi_0 = 1 \text{ on } [-1/2, 1/2], \quad (3.1.2)$$

some positive parameters α and μ such that

$$\frac{1}{k} < \alpha < \frac{2}{k} \text{ and } \frac{2}{k} < \mu < \alpha + \frac{1}{k}, \quad (3.1.3)$$

where k is the integer appearing in the definition of the operator L_2 . We set for all $\lambda \geq 1$,

$$\psi_\lambda(\tau, \xi) = \lambda^{-\frac{1}{2}-\alpha} \psi_1(\lambda^{-\alpha}\tau, \lambda^{-1-\alpha}\xi). \quad (3.1.4)$$

Let us note $\text{supp } \chi_0(\lambda^\mu(\cdot + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}))$ for the support of the function

$$y \mapsto \chi_0(\lambda^\mu(y + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}})).$$

Since using (3.1.2), we have for all $(\tau, \xi) \in [1, 4]^2$ and $\lambda \geq 1$,

$$\begin{aligned} \text{supp } \chi_0(\lambda^\mu(\cdot + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}})) &\subset \{y \in \mathbb{R} : |y + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}| \leq \lambda^{-\mu}\} \\ &\subset \{y \in \mathbb{R} : -\lambda^{-\mu} - 4^{\frac{2}{k}}\lambda^{-\frac{2}{k}} \leq y \leq \lambda^{-\mu} - 4^{-\frac{2}{k}}\lambda^{-\frac{2}{k}}\}, \end{aligned}$$

it follows from (3.1.3), $2/k < \mu$, that we can find a constant $\lambda_0 \geq 1$ and some positive constants c_1, c_2 such that $c_1 > c_2$ and for all $(\tau, \xi) \in [1, 4]^2$, $\lambda \geq \lambda_0$,

$$\text{supp } \chi_0(\lambda^\mu(\cdot + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}})) \subset \{y \in \mathbb{R} : -c_1\lambda^{-\frac{2}{k}} \leq y \leq -c_2\lambda^{-\frac{2}{k}}\}. \quad (3.1.5)$$

Let us notice that since

$$L_2^* = -\partial_y + \theta_k(y)D_x^2 - D_t^2, \quad \theta_k(y) = (-1)^k y^k H(-y), \quad H = \mathbb{1}_{\mathbb{R}_+}, \quad (3.1.6)$$

with $D_x = i^{-1}\partial_x$, $D_t = i^{-1}\partial_t$, and since the function $y \mapsto \tau^2 - \theta_k(y)\xi^2$ changes sign from $-$ to $+$ at $y = -(\tau\xi^{-1})^{2/k}$ if $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, we can find a non-negative phase function Φ_1 , which satisfies the equation

$$\begin{aligned} & (-\partial_y + \theta_k(y)\xi^2 - \tau^2)(e^{-\Phi_1(\tau, \xi, y)}) \\ &= (\partial_y \Phi_1(\tau, \xi, y) + \theta_k(y)\xi^2 - \tau^2)e^{-\Phi_1(\tau, \xi, y)} = 0, \end{aligned} \quad (3.1.7)$$

defined for all $y \in \mathbb{R}_-^*$ and $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ by

$$\Phi_1(\tau, \xi, y) = \int_{-(\tau\xi^{-1})^{\frac{2}{k}}}^y (\tau^2 - \theta_k(s)\xi^2) ds. \quad (3.1.8)$$

Indeed, since from (3.1.6) and (3.1.8),

$$\frac{\partial^2 \Phi_1}{\partial y^2}(\tau, \xi, y) = k(-y)^{k-1}\xi^2 \geq 0,$$

if $y \in \mathbb{R}_-^*$ and $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, the function $y \mapsto \Phi_1(\tau, \xi, y)$ is convex on $\mathbb{R}_-^*$ and we deduce from the fact

$$\Phi_1(\tau, \xi, -(\tau\xi^{-1})^{\frac{2}{k}}) = 0, \quad \frac{\partial \Phi_1}{\partial y}(\tau, \xi, -(\tau\xi^{-1})^{\frac{2}{k}}) = 0 \quad (3.1.9)$$

and from the Taylor formula that

$$\Phi_1(\tau, \xi, y) = (y + (\tau\xi^{-1})^{\frac{2}{k}})^2 k \xi^2 \int_0^1 (1 - \theta) ((\tau\xi^{-1})^{\frac{2}{k}}(1 - \theta) - \theta y)^{k-1} d\theta, \quad (3.1.10)$$

if $y \in \mathbb{R}_-^*$ and $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. The property of non-negativity of the function Φ_1 on $\mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_-^*$ is clear on the formula (3.1.10). We also set for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\begin{aligned} \Phi_\lambda(\tau, \xi, y) &= \Phi_1(\lambda^\alpha \tau, \lambda^{1+\alpha} \xi, y) \\ &= \int_{-(\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}}^y (\lambda^{2\alpha} \tau^2 - \theta_k(s) \lambda^{2+2\alpha} \xi^2) ds, \end{aligned} \quad (3.1.11)$$

which is also a non-negative function. A direct computation shows from (3.1.6) and (3.1.11) that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\Phi_\lambda(\tau, \xi, y) = \lambda^{2\alpha} \tau^2 y + \frac{k}{k+1} \lambda^{2\alpha - \frac{2}{k}} \tau^{2 + \frac{2}{k}} \xi^{-\frac{2}{k}} + \frac{(-1)^{k+1}}{k+1} \lambda^{2+2\alpha} \xi^2 y^{k+1}. \quad (3.1.12)$$

We can now define for all $\lambda \geq \lambda_0$ the function u_λ defined by

$$u_\lambda(t, x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x\xi + t\tau)} \psi_\lambda(\tau, \xi) \chi_0(\lambda^\mu(y + (\tau\xi^{-1})^{\frac{2}{k}})) e^{-\Phi_1(\tau, \xi, y)} d\tau d\xi. \quad (3.1.13)$$

If we note $\mathcal{F}_{t,x}$ the Fourier transform in the variables t, x , it follows from (3.1.13) that

$$\begin{aligned} U_\lambda(\tau, \xi, y) &= (\mathcal{F}_{t,x} u_\lambda)(\tau, \xi, y) \\ &= \psi_\lambda(\tau, \xi) \chi_0(\lambda^\mu(y + (\tau\xi^{-1})^{\frac{2}{k}})) e^{-\Phi_1(\tau, \xi, y)} \end{aligned} \quad (3.1.14)$$

and we can notice from (3.1.1), (3.1.4), (3.1.5) and the change of variables $(\tilde{\tau}, \tilde{\xi}) = (\lambda^{-\alpha}\tau, \lambda^{-1-\alpha}\xi)$ that for all $(\tau, \xi) \in \text{supp } \psi_\lambda$ and $\lambda \geq \lambda_0$,

$$\text{supp } U_\lambda(\tau, \xi, \cdot) \subset \mathbb{R}_-^*. \quad (3.1.15)$$

In view of (3.1.10), (3.1.14) and (3.1.15), it follows that the family $(u_\lambda)_{\lambda \geq \lambda_0}$ belongs to the space $C^\infty(\mathbb{R}_y, \mathcal{S}(\mathbb{R}_{t,x}^2))$ (because (3.1.1) and (3.1.4) imply that the function ψ_λ has a compact support in $\mathbb{R}_+^* \times \mathbb{R}_+^*$) and has its support included in the set $\mathbb{R}_{t,x}^2 \times (\mathbb{R}_y)_-$. We deduce from this fact and (3.1.6) that

$$L_2^* u_\lambda \in C^\infty(\mathbb{R}_{t,x,y}^3) \quad (3.1.16)$$

and a direct computation using (3.1.6), (3.1.7) and (3.1.14) gives that

$$\begin{aligned} \mathcal{F}_{t,x}(L_2^* u_\lambda) &= -\partial_y U_\lambda(\tau, \xi, y) + (\theta_k(y)\xi^2 - \tau^2) U_\lambda(\tau, \xi, y) \\ &= -\lambda^\mu \psi_\lambda(\tau, \xi) \chi_0'(\lambda^\mu(y + (\tau\xi^{-1})^{\frac{2}{k}})) e^{-\Phi_1(\tau, \xi, y)}. \end{aligned} \quad (3.1.17)$$

3.2 Upper bound for $\|L_2^* u_\lambda\|_{(N_0)}$

From (3.1.17) and Parseval's formula, we notice that to obtain an upper bound for the quantity $\|L_2^* u_\lambda\|_{(N_0)}$, it is enough to get an upper bound for the quantities

$$\begin{aligned} A_{j_1, j_2, j_3, j_4}(\lambda) &= \\ &= \|\lambda^{\mu(j_1+1)} \chi_0^{(j_1+1)}(\lambda^\mu(y + (\tau\xi^{-1})^{\frac{2}{k}})) \tau^{j_2} \xi^{j_3} \psi_\lambda(\tau, \xi) \partial_y^{j_4}(e^{-\Phi_1(\tau, \xi, y)})\|_{L^2(\mathbb{R}^3)}, \end{aligned} \quad (3.2.1)$$

where $(j_1, j_2, j_3, j_4) \in \mathbb{N}^4$ are some integers such that $j_1 + j_2 + j_3 + j_4 = N_0$. Using a change of variables, (3.1.4) and (3.1.11), we obtain that

$$A_{j_1, j_2, j_3, j_4}(\lambda)^2 = \lambda^{2\mu(j_1+1)+2\alpha j_2+2j_3(1+\alpha)} \int_{\mathbb{R}^3} \left[\chi_0^{(j_1+1)} \left(\lambda^\mu (y + (\tau \xi^{-1} \lambda^{-1})^{\frac{2}{k}}) \right)^2 \right. \\ \left. \times \tau^{2j_2} \xi^{2j_3} \psi_1(\tau, \xi)^2 \left| \partial_y^{j_4} (e^{-\Phi_\lambda(\tau, \xi, y)}) \right|^2 \right] dy d\tau d\xi. \quad (3.2.2)$$

Let us stress the fact that from (3.1.1), (3.1.5), (3.1.10) and (3.1.11), if $(\tau, \xi) \in \text{supp } \psi_1$ and $\lambda \geq \lambda_0$, the function $\Phi_\lambda(\tau, \xi, \cdot)$ is C^∞ on

$$\text{supp } \chi_0^{(j_1+1)} \left(\lambda^\mu (\cdot + (\tau \xi^{-1} \lambda^{-1})^{\frac{2}{k}}) \right) \subset \mathbb{R}_-^*.$$

Thus, the expression (3.2.2) is well-defined. We need now the following lemma.

Lemma 3.2.1 *For all $\nu \in \mathbb{N}^3$, there exist some functions a_l , $l = 0, \dots, |\nu|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants β_l , $l = 0, \dots, |\nu|(k+1)$ verifying*

$$\beta_l \leq 2|\nu|(\alpha + 1),$$

such that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\partial_{\tau, \xi, y}^\nu (e^{-\Phi_\lambda}) = e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) y^l \lambda^{\beta_l}. \quad (3.2.3)$$

Proof. We prove this lemma by induction on $|\nu|$. If $|\nu| = 0$, the expression (3.2.3) holds with $a_0 = 1$ and $\beta_0 = 0$. Let us assume now that for $\nu \in \mathbb{N}^3$, there exist some functions a_l , $l = 0, \dots, |\nu|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants β_l , $l = 0, \dots, |\nu|(k+1)$ verifying $\beta_l \leq 2|\nu|(\alpha + 1)$ such that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\lambda \geq 1$, the expression (3.2.3) holds. Since from (3.1.12), we have for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\partial_y \Phi_\lambda(\tau, \xi, y) = \lambda^{2\alpha} \tau^2 + (-1)^{k+1} \lambda^{2+2\alpha} \xi^2 y^k, \quad (3.2.4)$$

$$\partial_\tau \Phi_\lambda(\tau, \xi, y) = 2\lambda^{2\alpha} \tau y + 2\lambda^{2\alpha - \frac{2}{k}} \xi^{-\frac{2}{k}} \tau^{1+\frac{2}{k}}, \quad (3.2.5)$$

$$\partial_\xi \Phi_\lambda(\tau, \xi, y) = \frac{2}{k+1} \left[-\lambda^{2\alpha - \frac{2}{k}} \tau^{2+\frac{2}{k}} \xi^{-\frac{2}{k}-1} + (-1)^{k+1} \lambda^{2+2\alpha} \xi y^{k+1} \right]. \quad (3.2.6)$$

We have also for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\partial_y \left(e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) y^l \lambda^{\beta_l} \right) = \\ e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \left[-(\partial_y \Phi_\lambda) y^l + l y^{l-1} \right] \lambda^{\beta_l}, \quad (3.2.7)$$

$$\begin{aligned}
& \partial_\tau \left(e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) y^l \lambda^{\beta_l} \right) = \\
& e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} \lambda^{\beta_l} y^l \left[-(\partial_\tau \Phi_\lambda) a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) + \partial_\tau (a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}})) \right]
\end{aligned} \tag{3.2.8}$$

and

$$\begin{aligned}
& \partial_\xi \left(e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) y^l \lambda^{\beta_l} \right) = \\
& e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} \lambda^{\beta_l} y^l \left[-(\partial_\xi \Phi_\lambda) a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) + \partial_\xi (a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}})) \right].
\end{aligned} \tag{3.2.9}$$

We deduce from (3.2.4), (3.2.5), (3.2.6), (3.2.7), (3.2.8) and (3.2.9) that if

$$\tilde{\nu} \in \{\nu + (1, 0, 0), \nu + (0, 1, 0), \nu + (0, 0, 1)\},$$

there exist some functions $\tilde{a}_l$, $l = 0, \dots, |\tilde{\nu}|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants $\tilde{\beta}_l$, $l = 0, \dots, |\tilde{\nu}|(k+1)$ verifying $\tilde{\beta}_l \leq 2|\tilde{\nu}|(\alpha + 1)$ such that for all $y \in \mathbb{R}_+^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\partial_{\tau, \xi, y}^{\tilde{\nu}} (e^{-\Phi_\lambda}) = e^{-\Phi_\lambda} \sum_{l=0}^{|\tilde{\nu}|(k+1)} \tilde{a}_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) y^l \lambda^{\tilde{\beta}_l}. \tag{3.2.10}$$

Indeed, let us consider for example the case where $\tilde{\nu} = \nu + (0, 0, 1)$. We obtain from (3.2.3), (3.2.4) and (3.2.7) the expression

$$\begin{aligned}
& \partial_{\tau, \xi, y}^{\tilde{\nu}} (e^{-\Phi_\lambda}) = e^{-\Phi_\lambda} \sum_{l=0}^{|\tilde{\nu}|(k+1)-(k+2)} a_{l+1}(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (l+1) y^l \lambda^{\beta_{l+1}} \\
& - e^{-\Phi_\lambda} \sum_{l=0}^{|\tilde{\nu}|(k+1)-(k+1)} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (\tau^{\frac{1}{k}})^{2k} y^l \lambda^{\beta_l + 2\alpha} \\
& - e^{-\Phi_\lambda} \sum_{l=k}^{|\tilde{\nu}|(k+1)-1} a_{l-k}(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (-1)^{k+1} (\xi^{\frac{1}{k}})^{2k} y^l \lambda^{\beta_{l-k} + 2 + 2\alpha},
\end{aligned}$$

which can be written in the form (3.2.10). Since the power of λ is less or equal than $2 + 2\alpha$ in every term of the right-hand-side of (3.2.5) and (3.2.6), and that these terms are polynomial functions in the variables $\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}$,

$\xi^{-\frac{1}{k}}$ and y with a degree in y lower than $k+1$, we have only to use (3.2.5), (3.2.6), (3.2.8), (3.2.9) and the fact that the quantities

$$\begin{aligned} \partial_\tau(a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}})) &= \frac{1}{k} \tau^{\frac{1}{k}} (\tau^{-\frac{1}{k}})^k (\partial_1 a_l)(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \\ &\quad - \frac{1}{k} (\tau^{-\frac{1}{k}})^{k+1} (\partial_2 a_l)(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \end{aligned}$$

and

$$\begin{aligned} \partial_\xi(a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}})) &= \frac{1}{k} \xi^{\frac{1}{k}} (\xi^{-\frac{1}{k}})^k (\partial_3 a_l)(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \\ &\quad - \frac{1}{k} (\xi^{-\frac{1}{k}})^{k+1} (\partial_4 a_l)(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}), \end{aligned}$$

are some polynomial functions in the variables $\tau^{\frac{1}{k}}$, $\tau^{-\frac{1}{k}}$, $\xi^{\frac{1}{k}}$ and $\xi^{-\frac{1}{k}}$, to obtain (3.2.10) when $\tilde{\nu} = \nu + (1, 0, 0)$ or $\tilde{\nu} = \nu + (0, 1, 0)$. This proves the induction property at the rank $|\nu|+1$ and ends the proof of the lemma 3.2.1. $\square$

We deduce from (3.1.5) and the lemma 3.2.1 that there exists a positive constant C_{j_4} such that for all $(\tau, \xi) \in [1, 4]^2$, $\lambda \geq \lambda_0$ and

$$\begin{aligned} y &\in \text{supp } \chi_0(\lambda^\mu(\cdot + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})), \\ |\partial_y^{j_4}(e^{-\Phi_\lambda(\tau, \xi, y)})| &\leq C_{j_4} \lambda^{2j_4(1+\alpha)} e^{-\Phi_\lambda(\tau, \xi, y)}. \end{aligned} \quad (3.2.11)$$

Moreover, we obtain from (3.1.2) and (3.1.5) that for all $(\tau, \xi) \in [1, 4]^2$ and $\lambda \geq \lambda_0$,

$$\text{supp } \chi_0^{(j_1+1)}(\lambda^\mu(\cdot + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})) \subset \Omega_{\lambda, \tau, \xi}, \quad (3.2.12)$$

if we note

$$\Omega_{\lambda, \tau, \xi} = \{y \in \mathbb{R}^* : 2^{-1}\lambda^{-\mu} \leq |y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}}| \leq \lambda^{-\mu}\}. \quad (3.2.13)$$

Then, we deduce from (3.1.1), (3.2.2), (3.2.11), (3.2.12) and (3.2.13) that for all $\lambda \geq \lambda_0$,

$$\begin{aligned} A_{j_1, j_2, j_3, j_4}(\lambda)^2 &\leq C_{j_4}^2 \lambda^{2\mu(j_1+1)+2j_2\alpha+2j_3(1+\alpha)+4j_4(1+\alpha)} \|\chi_0^{(j_1+1)}\|_{L^\infty(\mathbb{R})}^2 \\ &\quad \times \int_{\mathbb{R}^2} \tau^{2j_2} \xi^{2j_3} \psi_1(\tau, \xi)^2 \left(\int_{\Omega_{\lambda, \tau, \xi}} e^{-2\Phi_\lambda(\tau, \xi, y)} dy \right) d\tau d\xi \\ &\leq C_{j_4}^2 \lambda^{2\mu(j_1+1)+2j_2\alpha+2j_3(1+\alpha)+4j_4(1+\alpha)-\mu} \|\chi_0^{(j_1+1)}\|_{L^\infty(\mathbb{R})}^2 \\ &\quad \times \int_{\mathbb{R}^2} \tau^{2j_2} \xi^{2j_3} \psi_1(\tau, \xi)^2 \left(\sup_{y \in \Omega_{\lambda, \tau, \xi}} e^{-2\Phi_\lambda(\tau, \xi, y)} \right) d\tau d\xi. \end{aligned} \quad (3.2.14)$$

We obtain from (3.1.10) that for all $y \in \mathbb{R}_-^*$ and $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$,

$$\begin{aligned}\Phi_1(\tau, \xi, y) &= (y + (\tau\xi^{-1})^{\frac{2}{k}})^2 k\xi^2 \int_0^1 (1-\theta) ((\tau\xi^{-1})^{\frac{2}{k}}(1-\theta) - \theta y)^{k-1} d\theta \\ &\geq (y + (\tau\xi^{-1})^{\frac{2}{k}})^2 k\xi^2 \int_0^1 (1-\theta)^k (\tau\xi^{-1})^{2-\frac{2}{k}} d\theta,\end{aligned}$$

which induces that

$$\Phi_1(\tau, \xi, y) \geq \frac{k}{k+1} \xi^2 (\tau\xi^{-1})^{2-\frac{2}{k}} (y + (\tau\xi^{-1})^{\frac{2}{k}})^2. \quad (3.2.15)$$

It follows from (3.1.11) and (3.2.15) that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in [1, 4]^2$ and $\lambda \geq 1$,

$$\begin{aligned}\Phi_\lambda(\tau, \xi, y) &\geq \frac{k}{k+1} \xi^2 (\tau\xi^{-1})^{2-\frac{2}{k}} \lambda^{2\alpha+\frac{2}{k}} (y + (\tau\lambda^{-1}\xi^{-1})^{\frac{2}{k}})^2 \\ &\geq c_3 \lambda^{2\alpha+\frac{2}{k}} (y + (\tau\lambda^{-1}\xi^{-1})^{\frac{2}{k}})^2,\end{aligned} \quad (3.2.16)$$

with $c_3 = 4^{\frac{2}{k}-2} k/(k+1) > 0$. Thus, we obtain using (3.2.13) and (3.2.16) that for all $(\tau, \xi) \in [1, 4]^2$ and $\lambda \geq \lambda_0$,

$$\sup_{y \in \Omega_{\lambda, \tau, \xi}} e^{-2\Phi_\lambda(\tau, \xi, y)} \leq e^{-\frac{c_3}{2} \lambda^{2(\alpha+\frac{1}{k}-\mu)}}. \quad (3.2.17)$$

Getting back to (3.2.14), the next proposition follows from (3.1.1), (3.2.1), (3.2.14), (3.2.17) and the fact that from (3.1.3),

$$\mu < \alpha + \frac{1}{k}.$$

Proposition 3.2.1 *We have*

$$\|L_2^* u_\lambda\|_{(N_0)} = O(e^{-\frac{c_3}{8} \lambda^{2(\alpha+\frac{1}{k}-\mu)}}) \text{ when } \lambda \rightarrow +\infty. \quad (3.2.18)$$

3.3 Lower bound for the quantity $\|u_\lambda\|_{(-N_0)}$

It follows from (3.1.4), (3.1.11), (3.1.14) and a change of variables that

$$\begin{aligned}\|u_\lambda\|_{(-N_0)}^2 &= \int_{\mathbb{R}^3} |\psi_\lambda(\tau, \xi)|^2 \left| \int_{\mathbb{R}} e^{-iy\eta} \chi_0(\lambda^\mu (y + (\xi^{-1}\tau)^{\frac{2}{k}})) e^{-\Phi_1(\tau, \xi, y)} dy \right|^2 \\ &\quad \times (1 + \eta^2 + \xi^2 + \tau^2)^{-N_0} \frac{d\eta d\tau d\xi}{(2\pi)^3} \\ &= \int_{\mathbb{R}^3} |\psi_1(\tau, \xi)|^2 \left| \int_{\mathbb{R}} e^{-iy\eta} \chi_0(\lambda^\mu (y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})) e^{-\Phi_\lambda(\tau, \xi, y)} dy \right|^2 \\ &\quad \times (1 + \eta^2 + \lambda^{2+2\alpha}\xi^2 + \lambda^{2\alpha}\tau^2)^{-N_0} \frac{d\eta d\tau d\xi}{(2\pi)^3}.\end{aligned} \quad (3.3.1)$$

By using the following estimates, for all $(\tau, \xi) \in [1, 4]^2$ and $\lambda \geq \lambda_0 \geq 1$,

$$1 + \eta^2 + \lambda^{2+2\alpha}\xi^2 + \lambda^{2\alpha}\tau^2 \leq c_4\lambda^{2+2\alpha}(1 + \eta^2),$$

$$(1 + \eta^2 + \lambda^{2+2\alpha}\xi^2 + \lambda^{2\alpha}\tau^2)^{-N_0} \geq c_4^{-N_0}\lambda^{-2(1+\alpha)N_0}(1 + \eta^2)^{-N_0},$$

where $c_4 = 33$ and from (3.1.1), $\text{supp } \psi_1 \subset [1, 4]^2$ and $\psi_1 = 1$ on $[2, 3]^2$, we deduce from (3.3.1) that for all $\lambda \geq \lambda_0$,

$$\|u_\lambda\|_{(-N_0)}^2 \geq \frac{c_4^{-N_0}}{(2\pi)^2} \lambda^{-2(1+\alpha)N_0} \int_{[2,3]^2} \|g_{\lambda,\tau,\xi}\|_{H^{-N_0}(\mathbb{R}_y)}^2 d\tau d\xi, \quad (3.3.2)$$

if we note

$$g_{\lambda,\tau,\xi}(y) = \chi_0\left(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\right) e^{-\Phi_\lambda(\tau,\xi,y)}. \quad (3.3.3)$$

Then, we use that

$$\|g_{\lambda,\tau,\xi}\|_{L^2(\mathbb{R}_y)}^2 \leq \|g_{\lambda,\tau,\xi}\|_{H^{N_0}(\mathbb{R}_y)} \|g_{\lambda,\tau,\xi}\|_{H^{-N_0}(\mathbb{R}_y)}. \quad (3.3.4)$$

The following lemma allows us to get an uniform lower bound, respectively to get an uniform upper bound with respect to the variables (τ, ξ) in $[2, 3]^2$ for the quantities $\|g_{\lambda,\tau,\xi}\|_{L^2(\mathbb{R}_y)}$ and $\|g_{\lambda,\tau,\xi}\|_{H^{N_0}(\mathbb{R}_y)}$.

Lemma 3.3.1 *We can find some positive constants c_5 and c_6 such that for all $(\tau, \xi) \in [2, 3]^2$ and $\lambda \geq \lambda_0$,*

$$\|g_{\lambda,\tau,\xi}\|_{H^{N_0}(\mathbb{R}_y)} \leq c_5 \lambda^{(\mu+2+2\alpha)N_0 - \frac{\mu}{2}} \text{ and } \|g_{\lambda,\tau,\xi}\|_{L^2(\mathbb{R}_y)} \geq c_6 \lambda^{-\frac{1+\alpha}{2}}. \quad (3.3.5)$$

Proof. To obtain the first estimate, it is enough to get a bound for $\lambda \geq \lambda_0$ of the new quantities

$$A_{j_1,j_2}(\lambda, \tau, \xi) = \|\lambda^{j_1\mu} \chi_0^{(j_1)}(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})) \partial_y^{j_2}(e^{-\Phi_\lambda(\tau,\xi,y)})\|_{L^2(\mathbb{R}_y)},$$

uniformly with respect to the variables $(\tau, \xi) \in [2, 3]^2$ where $(j_1, j_2) \in \mathbb{N}^2$ are some integers verifying $j_1 + j_2 = N_0$. We obtain using (3.2.11) and the non-negativity of the function Φ_λ that for all $(\tau, \xi) \in [2, 3]^2$ and $\lambda \geq \lambda_0$,

$$\begin{aligned} A_{j_1,j_2}(\lambda, \tau, \xi)^2 &= \lambda^{2j_1\mu} \int_{\mathbb{R}} \chi_0^{(j_1)}(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}}))^2 |\partial_y^{j_2}(e^{-\Phi_\lambda(\tau,\xi,y)})|^2 dy \\ &\leq C_{j_2}^2 \lambda^{2j_1\mu+4j_2(1+\alpha)} \int_{\mathbb{R}} \chi_0^{(j_1)}(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}}))^2 dy \\ &= C_{j_2}^2 \|\chi_0^{(j_1)}\|_{L^2(\mathbb{R})}^2 \lambda^{2j_1\mu+4j_2(1+\alpha)-\mu}. \end{aligned}$$

We deduce from this last estimate that there exists a positive constant c_5 such that for all $(\tau, \xi) \in [2, 3]^2$ and $\lambda \geq \lambda_0$,

$$\|g_{\lambda,\tau,\xi}\|_{H^{N_0}(\mathbb{R}_y)} \leq c_5 \lambda^{(\mu+2+2\alpha)N_0 - \frac{\mu}{2}},$$

which shows the first estimate of (3.3.5). We want now to get an uniform lower bound for the quantity $\|g_{\lambda,\tau,\xi}\|_{L^2(\mathbb{R}_y)}$ with respect to the variables $(\tau, \xi) \in [2, 3]^2$ for $\lambda \geq \lambda_0$. Using (3.1.10), we obtain that for all $y \in \mathbb{R}_-^*$ and $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$,

$$\begin{aligned}\Phi_1(\tau, \xi, y) &= (y + (\tau\xi^{-1})^{\frac{2}{k}})^2 k\xi^2 \int_0^1 (1-\theta) ((\tau\xi^{-1})^{\frac{2}{k}}(1-\theta) - \theta y)^{k-1} d\theta \\ &\leq k\xi^2 (y + (\tau\xi^{-1})^{\frac{2}{k}})^2 ((\tau\xi^{-1})^{\frac{2}{k}} + |y|)^{k-1}.\end{aligned}\quad (3.3.6)$$

We deduce from (3.1.11) and (3.3.6) that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in [2, 3]^2$ and $\lambda \geq 1$,

$$\Phi_\lambda(\tau, \xi, y) \leq 9k\lambda^{2+2\alpha} (y + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}})^2 (3^{\frac{2}{k}} 2^{-\frac{2}{k}} + |y|)^{k-1}. \quad (3.3.7)$$

We obtain using (3.3.7) and the change of variables, $u = y + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}$, that for all $(\tau, \xi) \in [2, 3]^2$ and $\lambda \geq 1$,

$$\begin{aligned}& \|e^{-\Phi_\lambda(\tau,\xi,y)} \mathbb{1}_{\mathbb{R}_-^*}(y)\|_{L^2(\mathbb{R}_y)}^2 \\ & \geq \int_{-\infty}^0 e^{-18k\lambda^{2+2\alpha} (y + (\tau\lambda^{-1}\xi^{-1})^{\frac{2}{k}})^2 (3^{\frac{2}{k}} 2^{-\frac{2}{k}} + |y|)^{k-1}} dy \\ & \geq \int_{-\infty}^{-(\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}} e^{-18k\lambda^{2+2\alpha} (y + (\tau\lambda^{-1}\xi^{-1})^{\frac{2}{k}})^2 (3^{\frac{2}{k}} 2^{-\frac{2}{k}} + |y|)^{k-1}} dy \\ & \geq \int_{-\infty}^0 e^{-18k\lambda^{2+2\alpha} u^2 (3^{\frac{2}{k}} 2^{-\frac{2}{k}} + |u - (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}|)^{k-1}} du.\end{aligned}\quad (3.3.8)$$

Since we can find a positive constant c_7 such that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in [2, 3]^2$ and $\lambda \geq 1$,

$$18k(3^{\frac{2}{k}} 2^{-\frac{2}{k}} + |y - (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}|)^{k-1} \leq c_7(1 + |y|^{k-1}), \quad (3.3.9)$$

it follows from (3.3.8) and (3.3.9),

$$\|e^{-\Phi_\lambda(\tau,\xi,y)} \mathbb{1}_{\mathbb{R}_-^*}(y)\|_{L^2(\mathbb{R}_y)}^2 \geq \int_{-\infty}^0 e^{-c_7\lambda^{2+2\alpha} u^2 (1 + |u|^{k-1})} du. \quad (3.3.10)$$

Then using some changes of variables, we deduce from (3.3.10) that for all $(\tau, \xi) \in [2, 3]^2$ and $\lambda \geq 1$,

$$\begin{aligned}& \|e^{-\Phi_\lambda(\tau,\xi,y)} \mathbb{1}_{\mathbb{R}_-^*}(y)\|_{L^2(\mathbb{R}_y)}^2 \\ & \geq \int_0^{+\infty} e^{-c_7\lambda^{2+2\alpha} u^2 (1 + u^{k-1})} du \\ & = \lambda^{-1-\alpha} \int_0^{+\infty} e^{-c_7 v^2 (1 + v^{k-1} \lambda^{-(1+\alpha)(k-1)})} dv \\ & \geq \lambda^{-1-\alpha} \int_0^{+\infty} e^{-c_7 v^2 (1 + v^{k-1})} dv = c_8 \lambda^{-1-\alpha}.\end{aligned}\quad (3.3.11)$$

Next, if we note

$$\tilde{\Omega}_{\lambda,\tau,\xi} = \{y \in \mathbb{R}_-^* : |y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}}| \geq 2^{-1}\lambda^{-\mu}\}, \quad (3.3.12)$$

using from (3.1.2) that $\chi_0 = 1$ on $[-1/2, 1/2]$ and (3.2.16), we obtain that for all $(\tau, \xi) \in [2, 3]^2$ and $\lambda \geq 1$,

$$\begin{aligned} & \left\| [1 - \chi_0(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}}))] e^{-\Phi_\lambda(\tau,\xi,y)} \mathbb{1}_{\mathbb{R}_-^*}(y) \right\|_{L^2(\mathbb{R}_y)}^2 \\ & \leq \|1 - \chi_0\|_{L^\infty(\mathbb{R})}^2 \int_{\tilde{\Omega}_{\lambda,\tau,\xi}} e^{-2\Phi_\lambda(\tau,\xi,y)} dy \\ & \leq \|1 - \chi_0\|_{L^\infty(\mathbb{R})}^2 \int_{\tilde{\Omega}_{\lambda,\tau,\xi}} e^{-2c_3\lambda^{2\alpha+\frac{2}{k}}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})^2} dy \\ & \leq e^{-\frac{c_3}{4}\lambda^{2\alpha+\frac{2}{k}-2\mu}} \|1 - \chi_0\|_{L^\infty(\mathbb{R})}^2 \int_{\mathbb{R}} e^{-c_3\lambda^{2\alpha+\frac{2}{k}}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})^2} dy \\ & \leq \pi^{\frac{1}{2}} c_3^{-\frac{1}{2}} \|1 - \chi_0\|_{L^\infty(\mathbb{R})}^2 \lambda^{-\alpha-\frac{1}{k}} e^{-\frac{c_3}{4}\lambda^{2(\alpha+\frac{1}{k}-\mu)}}, \end{aligned} \quad (3.3.13)$$

since if $y \in \tilde{\Omega}_{\lambda,\tau,\xi}$, we have

$$e^{-2c_3\lambda^{2\alpha+\frac{2}{k}}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})^2} \leq e^{-\frac{c_3}{4}\lambda^{2\alpha+\frac{2}{k}-2\mu}} e^{-c_3\lambda^{2\alpha+\frac{2}{k}}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})^2}$$

and since a change of variables gives that

$$\int_{\mathbb{R}} e^{-c_3\lambda^{2\alpha+\frac{2}{k}}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})^2} dy = \int_{\mathbb{R}} e^{-c_3\lambda^{2\alpha+\frac{2}{k}}y^2} dy = \pi^{\frac{1}{2}} c_3^{-\frac{1}{2}} \lambda^{-\alpha-\frac{1}{k}}.$$

In view of (3.1.5) and (3.3.3), the use of the triangular inequality for all $(\tau, \xi) \in [2, 3]^2$ and $\lambda \geq \lambda_0$,

$$\begin{aligned} \|g_{\lambda,\tau,\xi}\|_{L^2(\mathbb{R}_y)} &= \left\| \chi_0(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})) e^{-\Phi_\lambda(\tau,\xi,y)} \mathbb{1}_{\mathbb{R}_-^*}(y) \right\|_{L^2(\mathbb{R}_y)} \\ &\geq \|e^{-\Phi_\lambda(\tau,\xi,y)} \mathbb{1}_{\mathbb{R}_-^*}(y)\|_{L^2(\mathbb{R}_y)} \\ &\quad - \left\| [1 - \chi_0(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}}))] e^{-\Phi_\lambda(\tau,\xi,y)} \mathbb{1}_{\mathbb{R}_-^*}(y) \right\|_{L^2(\mathbb{R}_y)}, \end{aligned}$$

with the estimates (3.3.11) and (3.3.13), shows that there exists a positive constant c_6 such that for all $(\tau, \xi) \in [2, 3]^2$ and $\lambda \geq \lambda_0$,

$$\|g_{\lambda,\tau,\xi}\|_{L^2(\mathbb{R}_y)} \geq c_6 \lambda^{-\frac{1+\alpha}{2}},$$

because from (3.1.3),

$$\alpha + \frac{1}{k} - \mu > 0.$$

This ends the proof of Lemma 3.3.1. $\square$

The previous lemma permits us to obtain from the estimate (3.3.4) that for all $(\tau, \xi) \in [2, 3]^2$ and $\lambda \geq \lambda_0$,

$$\|g_{\lambda, \tau, \xi}\|_{H^{-N_0}(\mathbb{R}_y)} \geq c_6^2 c_5^{-1} \lambda^{-\alpha-1+\frac{\mu}{2}-(\mu+2+2\alpha)N_0}.$$

Then using (3.3.2), we obtain the following proposition.

Proposition 3.3.1 *There exists a positive constant c_9 such that for all $\lambda \geq \lambda_0$,*

$$\|u_\lambda\|_{(-N_0)} \geq c_9 \lambda^{-\alpha-1+\frac{\mu}{2}-(\mu+3+3\alpha)N_0}. \quad (3.3.14)$$

We need now to cutoff in the variables t, x to obtain a quasimode localized in an arbitrary neighbourhood of 0 in $\mathbb{R}^3$.

3.4 Cutoff in variables t and x

We need first to make the result of the lemma 3.2.1 more precise when there is no differentiation in the variable y .

Lemma 3.4.1 *For all $\rho \in \mathbb{N}^2$, there exist some functions a_l , $l = 0, \dots, |\rho|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants β_l , $l = 0, \dots, |\rho|(k+1)$ verifying*

$$\beta_l \leq 2|\rho| \left(\alpha - \frac{1}{k} \right),$$

such that for all $y \in \mathbb{R}_-^$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,*

$$\partial_{\tau, \xi}^\rho (e^{-\Phi_\lambda}) = e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (\lambda^{\frac{2}{k}} y)^l \lambda^{\beta_l}. \quad (3.4.1)$$

Proof. We prove again this lemma by induction on $|\rho|$. If $|\rho| = 0$, the expression (3.4.1) holds with $a_0 = 1$ and $\beta_0 = 0$. Let us assume now that for $\rho \in \mathbb{N}^2$, there exist some functions a_l , $l = 0, \dots, |\rho|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants β_l , $l = 0, \dots, |\rho|(k+1)$ verifying $\beta_l \leq 2|\rho|(\alpha - 1/k)$ such that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\lambda \geq 1$, the expression (3.4.1) holds. Since we can write (3.2.5) and (3.2.6) in the following way,

$$\partial_\tau \Phi_\lambda(\tau, \xi, y) = \lambda^{2\alpha-\frac{2}{k}} (2\tau(\lambda^{\frac{2}{k}} y) + 2\xi^{-\frac{2}{k}} \tau^{\frac{k+2}{k}}), \quad (3.4.2)$$

$$\partial_\xi \Phi_\lambda(\tau, \xi, y) = \frac{2}{k+1} \lambda^{2\alpha-\frac{2}{k}} \left(-\tau^{\frac{2k+2}{k}} \xi^{-\frac{k+2}{k}} + (-1)^{k+1} \xi(\lambda^{\frac{2}{k}} y)^{k+1} \right) \quad (3.4.3)$$

and

$$\begin{aligned}
& \partial_\tau \left(e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (\lambda^{\frac{2}{k}} y)^l \lambda^{\beta_l} \right) \\
&= e^{-\Phi_\lambda} \left(\sum_{l=0}^{|\rho|(k+1)} \left[-(\partial_\tau \Phi_\lambda) a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \right. \right. \\
&\quad \left. \left. + \partial_\tau (a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}})) \right] (\lambda^{\frac{2}{k}} y)^l \lambda^{\beta_l} \right), \quad (3.4.4)
\end{aligned}$$

$$\begin{aligned}
& \partial_\xi \left(e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (\lambda^{\frac{2}{k}} y)^l \lambda^{\beta_l} \right) \\
&= e^{-\Phi_\lambda} \left(\sum_{l=0}^{|\rho|(k+1)} \left[-(\partial_\xi \Phi_\lambda) a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \right. \right. \\
&\quad \left. \left. + \partial_\xi (a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}})) \right] (\lambda^{\frac{2}{k}} y)^l \lambda^{\beta_l} \right), \quad (3.4.5)
\end{aligned}$$

we obtain that

$$\begin{aligned}
& \partial_\tau \left(e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (\lambda^{\frac{2}{k}} y)^l \lambda^{\beta_l} \right) = \\
& - e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} 2(\tau^{\frac{1}{k}})^k a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (\lambda^{\frac{2}{k}} y)^{l+1} \lambda^{\beta_l+2(\alpha-\frac{1}{k})} \\
& - e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} 2(\xi^{-\frac{1}{k}})^2 (\tau^{\frac{1}{k}})^{k+2} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (\lambda^{\frac{2}{k}} y)^l \lambda^{\beta_l+2(\alpha-\frac{1}{k})} \\
& + e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} \left(\frac{1}{k} \left[\tau^{\frac{1}{k}} (\tau^{-\frac{1}{k}})^k (\partial_1 a_l)(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \right. \right. \\
& \quad \left. \left. - (\tau^{-\frac{1}{k}})^{k+1} (\partial_2 a_l)(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \right] (\lambda^{\frac{2}{k}} y)^l \lambda^{\beta_l} \right) \quad (3.4.6)
\end{aligned}$$

and

$$\begin{aligned}
& \partial_\xi \left(e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (\lambda^{\frac{2}{k}} y)^l \lambda^{\beta_l} \right) = \\
& e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} \frac{2}{k+1} (\tau^{\frac{1}{k}})^{2k+2} (\xi^{-\frac{1}{k}})^{k+2} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (\lambda^{\frac{2}{k}} y)^l \lambda^{\beta_l+2(\alpha-\frac{1}{k})} \\
& - e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} \frac{2}{k+1} (-1)^{k+1} (\xi^{\frac{1}{k}})^k a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (\lambda^{\frac{2}{k}} y)^{l+k+1} \lambda^{\beta_l+2(\alpha-\frac{1}{k})} \\
& + e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} \left(\frac{1}{k} [\xi^{\frac{1}{k}} (\xi^{-\frac{1}{k}})^k (\partial_3 a_l)(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \right. \\
& \quad \left. - (\xi^{-\frac{1}{k}})^{k+1} (\partial_4 a_l)(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}})] (\lambda^{\frac{2}{k}} y)^l \lambda^{\beta_l} \right). \quad (3.4.7)
\end{aligned}$$

Since from (3.1.3), $\alpha > 1/k$, we deduce from (3.4.6) and (3.4.7) that if $\tilde{\rho} \in \{\rho + (1, 0), \rho + (0, 1)\}$, there exist some functions $\tilde{a}_l$, $l = 0, \dots, |\tilde{\rho}|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants $\tilde{\beta}_l$, $l = 0, \dots, |\tilde{\rho}|(k+1)$ verifying $\tilde{\beta}_l \leq 2|\tilde{\rho}|(\alpha - 1/k)$ such that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\partial_{\tau, \xi}^{\tilde{\rho}} (e^{-\Phi_\lambda}) = e^{-\Phi_\lambda} \sum_{l=0}^{|\tilde{\rho}|(k+1)} \tilde{a}_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (\lambda^{\frac{2}{k}} y)^l \lambda^{\tilde{\beta}_l}.$$

This proves the induction property at the rank $|\rho| + 1$ and ends the proof of the lemma 3.4.1. $\square$

We can now prove the following lemma.

Lemma 3.4.2 *For all $\rho \in \mathbb{N}^2$, there exists a positive constant M_ρ such that for all $(\tau, \xi) \in [1, 4]^2$, $\lambda \geq \lambda_0$ and $y \in \text{supp } \chi_0(\lambda^\mu(\cdot + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}}))$,*

$$|\partial_{\tau, \xi}^\rho (e^{-\Phi_\lambda(\tau, \xi, y)})| \leq M_\rho \lambda^{2|\rho|(\alpha - \frac{1}{k})}. \quad (3.4.8)$$

Proof. We recall that the above notation $\text{supp } \chi_0(\lambda^\mu(\cdot + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}}))$ stands for the support of the function

$$y \mapsto \chi_0(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})).$$

It follows from the previous lemma that there exist some functions a_l , $l = 0, \dots, |\rho|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants β_l , $l = 0, \dots, |\rho|(k+1)$ verifying

$$\beta_l \leq 2|\rho| \left(\alpha - \frac{1}{k} \right), \quad (3.4.9)$$

such that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\partial_{\tau, \xi}^\rho (e^{-\Phi_\lambda}) = e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) (\lambda^{\frac{2}{k}} y)^l \lambda^{\beta_l}. \quad (3.4.10)$$

Using the non-negativity of the phase function Φ_λ (see (3.1.10) and (3.1.11)), we deduce from (3.1.5) and (3.4.9) that for $l = 0, \dots, |\rho|(k+1)$, there exists a positive constant $c_{10,l}$ such that for all $(\tau, \xi) \in [1, 4]^2$, $\lambda \geq \lambda_0$ and

$$y \in \text{supp } \chi_0(\lambda^\mu(\cdot + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})),$$

$$|e^{-\Phi_\lambda} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \lambda^{\beta_l}| \leq c_{10,l} \lambda^{2|\rho|(\alpha - \frac{1}{k})}, \quad (3.4.11)$$

since from (3.1.3), $\alpha > 1/k$. Finally, in view of (3.1.5), (3.4.10) and (3.4.11), we deduce that there exists a positive constant M_ρ such that the estimate (3.4.8) holds. This ends the proof of the lemma 3.4.2. $\square$

Let us now consider the function v_λ defined by

$$v_\lambda(t, x, y) = \chi_1(\lambda^\gamma t, \lambda^\gamma x) u_\lambda(t, x, y), \quad (3.4.12)$$

where γ is a parameter verifying

$$0 < \gamma < \frac{1}{k} \text{ and } \gamma + \alpha < \frac{2}{k}. \quad (3.4.13)$$

This choice is possible in view of (3.1.3). The function χ_1 is taken in the space $C_0^\infty(\mathbb{R}^2, \mathbb{R})$ such that

$$\text{supp } \chi_1 \subset B(0, 1) \text{ and } \chi_1 = 1 \text{ on } B(0, 1/2), \quad (3.4.14)$$

where the notation $B(0, r)$ stands for the closed Euclidean ball centered in 0 with a radius r . We start by getting a lower bound for the quantity $\|v_\lambda\|_{(-N_0)}$. To do this, we prove the following lemma.

Lemma 3.4.3 *For all $M \in \mathbb{N}$, there exists a positive constant K_M such that for all $\lambda \geq \lambda_0$,*

$$\|(1 - \chi_1(\lambda^\gamma t, \lambda^\gamma x)) u_\lambda(t, x, y)\|_{(-N_0)} \leq K_M \lambda^{-M}. \quad (3.4.15)$$

Proof. Since from (3.1.4), (3.1.11), (3.1.13) and a change of variables

$$u_\lambda(t, x, y) = \frac{\lambda^{\frac{1}{2} + \alpha}}{(2\pi)^2}$$

$$\times \int_{\mathbb{R}^2} e^{i(x\xi\lambda^{1+\alpha} + t\tau\lambda^\alpha)} \psi_1(\tau, \xi) \chi_0(\lambda^\mu(y + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}})) e^{-\Phi_\lambda(\tau, \xi, y)} d\tau d\xi, \quad (3.4.16)$$

we deduce from (3.1.1), (3.1.5) and (3.4.14) that for all $\lambda \geq \lambda_0$,

$$\begin{aligned}
& \left\| (1 - \chi_1(\lambda^\gamma t, \lambda^\gamma x)) u_\lambda(t, x, y) \right\|_{(-N_0)} \\
& \leq \left\| (1 - \chi_1(\lambda^\gamma t, \lambda^\gamma x)) u_\lambda(t, x, y) \right\|_{L^2(\mathbb{R}^3)} \\
& \leq \|1 - \chi_1\|_{L^\infty(\mathbb{R}^2)} \\
& \quad \times \left(\int_{-c_1 \lambda^{-\frac{2}{k}}}^{-c_2 \lambda^{-\frac{2}{k}}} \left[\int_{\{(t,x) \in \mathbb{R}^2: t^2+x^2 \geq 4^{-1} \lambda^{-2\gamma}\}} |u_\lambda(t, x, y)|^2 dt dx \right] dy \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.4.17}$$

Now, some integrations by parts on (3.4.16) show that for all $q \in \mathbb{N}$, $(t, x) \neq (0, 0)$, $y \in \mathbb{R}$ and $\lambda \geq \lambda_0$,

$$\begin{aligned}
u_\lambda(t, x, y) &= \frac{\lambda^{\alpha+\frac{1}{2}} (i\lambda^{1+\alpha}x + \lambda^\alpha t)^{-q}}{(2\pi)^2} \int_{\mathbb{R}^2} (\partial_\xi - i\partial_\tau)^q (e^{i(\lambda^{1+\alpha}x\xi + \lambda^\alpha t\tau)}) \\
&\quad \times \psi_1(\tau, \xi) \chi_0(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})) e^{-\Phi_\lambda(\tau, \xi, y)} d\tau d\xi \\
&= \frac{\lambda^{\alpha+\frac{1}{2}} (i\lambda^{1+\alpha}x + \lambda^\alpha t)^{-q}}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(\lambda^{1+\alpha}x\xi + \lambda^\alpha t\tau)} (i\partial_\tau - \partial_\xi)^q \\
&\quad \left[\chi_0(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})) \psi_1(\tau, \xi) e^{-\Phi_\lambda(\tau, \xi, y)} \right] d\tau d\xi.
\end{aligned} \tag{3.4.18}$$

We need the following lemma.

Lemma 3.4.4 *For all $\rho \in \mathbb{N}^2$ and $l \in \mathbb{N}$, there exists a positive constant $c_{11, \rho, l}$ such that for all $y \in \mathbb{R}$, $(\tau, \xi) \in [1, 4]^2$ and $\lambda \geq 1$,*

$$\left| \partial_{\tau, \xi}^\rho \left[\chi_0^{(l)}(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})) \right] \right| \leq c_{11, \rho, l} \lambda^{|\rho|(\mu - \frac{2}{k})}. \tag{3.4.19}$$

Proof of the lemma 3.4.4. We start by proving that for all $\rho \in \mathbb{N}^2$ and $l \in \mathbb{N}$, there exist some polynomial functions $P_{\rho, l, j}$ in $\mathbb{R}^4$, $j = 0, \dots, |\rho|$, such that for all $y \in \mathbb{R}$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\begin{aligned}
& \partial_{\tau, \xi}^\rho \left[\chi_0^{(l)}(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})) \right] = \\
& \sum_{j=0}^{|\rho|} P_{\rho, l, j}(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \chi_0^{(l+j)}(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})) \lambda^{j(\mu - \frac{2}{k})}.
\end{aligned} \tag{3.4.20}$$

Let us consider $l \in \mathbb{N}$. We prove (3.4.20) by induction on $|\rho|$. If $|\rho| = 0$, the identity (3.4.20) holds with $P_{\rho, l, 0} = 1$. Let us assume now that the identity

(3.4.20) holds for $\rho \in \mathbb{N}^2$. The two direct computations using (3.4.20),

$$\begin{aligned}
& \partial_{\tau, \xi}^{\tilde{\rho}} \left[\chi_0^{(l)} \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) \right] \\
&= \sum_{j=0}^{|\tilde{\rho}|-1} \left[\frac{2}{k} (\xi^{-\frac{1}{k}})^2 (\tau^{\frac{1}{k}})^{2-k} P_{\rho, l, j}(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \right. \\
&\quad \left. \times \chi_0^{(l+j+1)} \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) \lambda^{(j+1)(\mu - \frac{2}{k})} \right] \\
&+ \sum_{j=0}^{|\tilde{\rho}|-1} \left[\frac{1}{k} (\tau^{\frac{1}{k}})^{1-k} (\partial_1 P_{\rho, l, j})(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \right. \\
&\quad \left. \times \chi_0^{(l+j)} \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) \lambda^{j(\mu - \frac{2}{k})} \right] \\
&- \sum_{j=0}^{|\tilde{\rho}|-1} \left[\frac{1}{k} (\tau^{-\frac{1}{k}})^{k+1} (\partial_2 P_{\rho, l, j})(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \right. \\
&\quad \left. \times \chi_0^{(l+j)} \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) \lambda^{j(\mu - \frac{2}{k})} \right],
\end{aligned}$$

if $\tilde{\rho} = \rho + (1, 0)$ and

$$\begin{aligned}
& \partial_{\tau, \xi}^{\tilde{\rho}} \left[\chi_0^{(l)} \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) \right] = \\
&- \sum_{j=0}^{|\tilde{\rho}|-1} \left[\frac{2}{k} (\xi^{-\frac{1}{k}})^{k+2} (\tau^{\frac{1}{k}})^2 P_{\rho, l, j}(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \right. \\
&\quad \left. \times \chi_0^{(l+j+1)} \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) \lambda^{(j+1)(\mu - \frac{2}{k})} \right] \\
&+ \sum_{j=0}^{|\tilde{\rho}|-1} \left[\frac{1}{k} (\xi^{\frac{1}{k}})^{1-k} (\partial_3 P_{\rho, l, j})(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \right. \\
&\quad \left. \times \chi_0^{(l+j)} \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) \lambda^{j(\mu - \frac{2}{k})} \right] \\
&- \sum_{j=0}^{|\tilde{\rho}|-1} \left[\frac{1}{k} (\xi^{-\frac{1}{k}})^{k+1} (\partial_4 P_{\rho, l, j})(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}) \right. \\
&\quad \left. \times \chi_0^{(l+j)} \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) \lambda^{j(\mu - \frac{2}{k})} \right],
\end{aligned}$$

if $\tilde{\rho} = \rho + (0, 1)$, prove that the induction property holds at the rank $|\rho| + 1$. This proves (3.4.20). Since from (3.1.2), $\chi_0 \in C_0^\infty(\mathbb{R}, \mathbb{R})$ and from (3.1.3), $\mu > 2/k$, we deduce from (3.4.20) that there exists a positive constant $c_{11, \rho, l}$ such that for all $y \in \mathbb{R}$, $(\tau, \xi) \in [1, 4]^2$ and $\lambda \geq 1$,

$$\left| \partial_{\tau, \xi}^{\tilde{\rho}} \left[\chi_0^{(l)} \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) \right] \right| \leq c_{11, \rho, l} \lambda^{|\rho|(\mu - \frac{2}{k})},$$

which ends the proof of the lemma 3.4.4. $\square$

Then, we obtain using the lemma 3.4.2, (3.1.1), (3.4.19) and the Leibniz formula on the expression

$$(i\partial_\tau - \partial_\xi)^q \left[\chi_0 \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) \psi_1(\tau, \xi) e^{-\Phi_\lambda(\tau, \xi, y)} \right],$$

where $q \in \mathbb{N}$, that there exist some positive constants $c_{12,j}$, $j = 0, \dots, q$ and c_{13} such that for all $y \in \mathbb{R}$, $(\tau, \xi) \in \mathbb{R}^2$ and $\lambda \geq \lambda_0$,

$$\begin{aligned} & \left| (i\partial_\tau - \partial_\xi)^q \left[\chi_0 \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) \psi_1(\tau, \xi) e^{-\Phi_\lambda(\tau, \xi, y)} \right] \right| \\ & \leq \sum_{j=0}^q c_{12,j} \lambda^{(\mu - \frac{2}{k})j} \lambda^{2(q-j)(\alpha - \frac{1}{k})} \\ & \leq c_{13} \lambda^{q \max(\mu - \frac{2}{k}, 2\alpha - \frac{2}{k})}, \end{aligned} \quad (3.4.21)$$

since from (3.1.3),

$$\mu > \frac{2}{k} \text{ and } \alpha > \frac{1}{k}.$$

Since from (3.1.1), $\text{supp } \psi_1 \subset [1, 4]^2$, it follows from (3.4.18) and (3.4.21) that for all $q \in \mathbb{N}$, there exists a positive constant $c_{14,q}$ such that for all $(t, x) \neq (0, 0)$, $y \in \mathbb{R}$ and $\lambda \geq \lambda_0$,

$$|u_\lambda(t, x, y)| \leq c_{14,q} \lambda^{q \max(\mu - \frac{2}{k}, 2\alpha - \frac{2}{k}) - q\alpha + \alpha + \frac{1}{2}} |x^2 + t^2|^{-\frac{q}{2}}. \quad (3.4.22)$$

We deduce by getting back to (3.4.17), using (3.4.22) and a change of variables that for all $q \in \mathbb{N} \setminus \{0, 1\}$ and $\lambda \geq \lambda_0$,

$$\begin{aligned} & \left\| (1 - \chi_1(\lambda^\gamma t, \lambda^\gamma x)) u_\lambda \right\|_{(-N_0)} \\ & \leq c_{14,q} (c_1 - c_2)^{\frac{1}{2}} \lambda^{q \max(\mu - \alpha - \frac{2}{k}, \alpha - \frac{2}{k}) + \alpha + \frac{1}{2} - \frac{1}{k}} \|1 - \chi_1\|_{L^\infty(\mathbb{R}^2)} \\ & \quad \times \left(\int_{\{(t,x) \in \mathbb{R}^2: t^2 + x^2 \geq 4^{-1} \lambda^{-2\gamma}\}} |t^2 + x^2|^{-q} dt dx \right)^{\frac{1}{2}} \\ & \leq c_{14,q} (c_1 - c_2)^{\frac{1}{2}} \lambda^{q \max(\mu - \alpha - \frac{2}{k} + \gamma, \alpha + \gamma - \frac{2}{k}) + \alpha + \frac{1}{2} - \frac{1}{k} - \gamma} \|1 - \chi_1\|_{L^\infty(\mathbb{R}^2)} \\ & \quad \times \left(\int_{\{(t,x) \in \mathbb{R}^2: t^2 + x^2 \geq 4^{-1}\}} |t^2 + x^2|^{-q} dt dx \right)^{\frac{1}{2}}. \end{aligned} \quad (3.4.23)$$

We recall in view of (3.1.5) that $c_1 > c_2$. Let us notice that

$$\max \left(\mu - \alpha - \frac{2}{k} + \gamma, \alpha + \gamma - \frac{2}{k} \right) < 0, \quad (3.4.24)$$

because from (3.4.13),

$$\alpha + \gamma < \frac{2}{k}$$

and from (3.1.3) and (3.4.13),

$$\mu - \alpha - \frac{2}{k} + \gamma < \gamma - \frac{1}{k} < 0.$$

Finally, we obtain using (3.4.23) and (3.4.24) the estimate (3.4.15). This ends the proof of the lemma 3.4.3. $\square$

We can now use (3.4.12) and the triangular inequality for all $\lambda \geq \lambda_0$,

$$\|u_\lambda\|_{(-N_0)} - \|(1 - \chi_1(\lambda^\gamma t, \lambda^\gamma x))u_\lambda\|_{(-N_0)} \leq \|v_\lambda\|_{(-N_0)},$$

with the estimates (3.3.14) and (3.4.15) to prove the following result.

Proposition 3.4.1 *There exists a positive constant c_{15} such that for all $\lambda \geq \lambda_0$,*

$$\|v_\lambda\|_{(-N_0)} \geq c_{15} \lambda^{-\alpha-1+\frac{\mu}{2}-(\mu+3+3\alpha)N_0}. \quad (3.4.25)$$

We now need to get an upper bound for the quantity $\|L_2^* v_\lambda\|_{(N_0)}$, $N_0 \in \mathbb{N}$, with respect to the parameter λ . It follows from (3.1.6) and (3.4.12) that for all $\lambda \geq \lambda_0$,

$$\begin{aligned} L_2^* v_\lambda &= \chi_1(\lambda^\gamma t, \lambda^\gamma x) L_2^* u_\lambda - \lambda^{2\gamma} \theta_k(y) (\partial_x^2 \chi_1)(\lambda^\gamma t, \lambda^\gamma x) u_\lambda \\ &\quad - 2\lambda^\gamma \theta_k(y) (\partial_x \chi_1)(\lambda^\gamma t, \lambda^\gamma x) \partial_x u_\lambda + \lambda^{2\gamma} (\partial_t^2 \chi_1)(\lambda^\gamma t, \lambda^\gamma x) u_\lambda \\ &\quad + 2\lambda^\gamma (\partial_t \chi_1)(\lambda^\gamma t, \lambda^\gamma x) \partial_t u_\lambda. \end{aligned} \quad (3.4.26)$$

We note respectively A_λ , B_λ , C_λ , D_λ and E_λ the terms appearing in the right-hand-side of the last expression. Let us first notice that these five terms are C^∞ on $\mathbb{R}^3$. Indeed, we have already proved after (3.1.15) that $u_\lambda \in C^\infty(\mathbb{R}_y, \mathcal{S}(\mathbb{R}_{t,x}^2))$ and, it follows from (3.1.1), (3.1.5) and (3.4.16) that for all $(t, x) \in \mathbb{R}^2$, $\lambda \geq \lambda_0$,

$$\text{supp } u_\lambda(t, x, \cdot) \subset [-c_1 \lambda^{-\frac{2}{k}}, -c_2 \lambda^{-\frac{2}{k}}] \quad (3.4.27)$$

and from (3.1.6), $\theta_k(y) = (-y)^k$ if $y \in \mathbb{R}_-$. Moreover, we have already proved in (3.1.16) that $L_2^* u_\lambda$ is C^∞ on $\mathbb{R}^3$. Then, we want to get an upper bound for the $H^{N_0}(\mathbb{R}^3)$ norm of the term A_λ . To do this, it is enough to get an upper bound for the quantity

$$\|\partial_{t,x}^{M_1} (\chi_1(\lambda^\gamma t, \lambda^\gamma x)) \partial_{t,x,y}^{M_2} (L_2^* u_\lambda)\|_{L^2(\mathbb{R}^3)},$$

where $M_1 \in \mathbb{N}^2$ and $M_2 \in \mathbb{N}^3$ verify $|M_1| + |M_2| = N_0$. Since

$$\|\partial_{t,x}^{M_1} (\chi_1(\lambda^\gamma t, \lambda^\gamma x)) \partial_{t,x,y}^{M_2} (L_2^* u_\lambda)\|_{L^2(\mathbb{R}^3)} \leq \lambda^{\gamma|M_1|} \|\partial_{t,x}^{M_1} \chi_1\|_{L^\infty(\mathbb{R}^2)} \|L_2^* u_\lambda\|_{(|M_2|)},$$

it follows that there exists a positive constant c_{16} such that for all $\lambda \geq \lambda_0$,

$$\|A_\lambda\|_{(N_0)} \leq c_{16} \lambda^{\gamma N_0} \left(\sup_{|j| \leq N_0} \|\partial_{t,x}^j \chi_1\|_{L^\infty(\mathbb{R}^2)} \right) \|L_2^* u_\lambda\|_{(N_0)}. \quad (3.4.28)$$

Thus, we deduce from (3.1.3), (3.2.18) and (3.4.28) that

$$\|A_\lambda\|_{(N_0)} = O\left(e^{-\frac{c_3}{16}\lambda^{2(\alpha+\frac{1}{k}-\mu)}}\right) \text{ when } \lambda \rightarrow +\infty. \quad (3.4.29)$$

Let us now consider $(j_1, j_2, j_3) \in \mathbb{N}^3$. It follows from (3.4.16) that for all $\lambda \geq \lambda_0$,

$$\begin{aligned} \partial_t^{j_1} \partial_x^{j_2} \partial_y^{j_3} u_\lambda(t, x, y) &= \frac{i^{j_1+j_2}}{(2\pi)^2} \lambda^{\alpha+\frac{1}{2}+j_1\alpha+j_2(1+\alpha)} \int_{\mathbb{R}^2} e^{i(\lambda^{1+\alpha}x\xi+\lambda^\alpha t\tau)} \\ &\times \tau^{j_1} \xi^{j_2} \psi_1(\tau, \xi) \partial_y^{j_3} \left[\chi_0\left(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\right) e^{-\Phi_\lambda(\tau, \xi, y)} \right] d\tau d\xi. \end{aligned} \quad (3.4.30)$$

We can make again some integrations by parts in (3.4.30) as in (3.4.18). Thus, we obtain that for all $q \in \mathbb{N}$, $(t, x) \neq (0, 0)$, $y \in \mathbb{R}$ and $\lambda \geq \lambda_0$,

$$\begin{aligned} &\partial_t^{j_1} \partial_x^{j_2} \partial_y^{j_3} u_\lambda(t, x, y) \\ &= \frac{i^{j_1+j_2}}{(2\pi)^2} \lambda^{\alpha+\frac{1}{2}+j_1\alpha+j_2(1+\alpha)} (i\lambda^{1+\alpha}x + \lambda^\alpha t)^{-q} \int_{\mathbb{R}^2} e^{i(\lambda^{1+\alpha}x\xi+\lambda^\alpha t\tau)} (i\partial_\tau - \partial_\xi)^q \\ &\left(\tau^{j_1} \xi^{j_2} \psi_1(\tau, \xi) \partial_y^{j_3} \left[\chi_0\left(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\right) e^{-\Phi_\lambda(\tau, \xi, y)} \right] \right) d\tau d\xi. \end{aligned} \quad (3.4.31)$$

Let us prove the following lemma.

Lemma 3.4.5 *For all $\rho \in \mathbb{N}^2$, $j_3 \in \mathbb{N}$, there exists a positive constant c_{17, ρ, j_3} such that for all $(\tau, \xi) \in [1, 4]^2$, $\lambda \geq \lambda_0$ and $y \in \text{supp } \chi_0\left(\lambda^\mu(\cdot + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\right)$,*

$$\begin{aligned} &\left| \partial_{\tau, \xi}^\rho \partial_y^{j_3} \left[\chi_0\left(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\right) e^{-\Phi_\lambda(\tau, \xi, y)} \right] \right| \\ &\leq c_{17, \rho, j_3} \lambda^{(2+2\alpha+\mu)j_3+|\rho|\max(2\alpha-\frac{2}{k}, \mu-\frac{2}{k})}. \end{aligned} \quad (3.4.32)$$

Proof. The Leibniz formula first proves that

$$\begin{aligned} &\partial_y^{j_3} \left[\chi_0\left(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\right) e^{-\Phi_\lambda} \right] = \\ &\sum_{l=0}^{j_3} C_{j_3}^l \lambda^{l\mu} \chi_0^{(l)}\left(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\right) \partial_y^{j_3-l} (e^{-\Phi_\lambda}). \end{aligned} \quad (3.4.33)$$

We deduce from the lemma 3.2.1 and (3.4.33) that there exist some functions $a_{r, l}$, $r = 0, \dots, j_3(k+1)$, $l = 0, \dots, j_3$, which are polynomial in $\mathbb{R}^4$ and some constants $\beta_{r, l}$, $r = 0, \dots, j_3(k+1)$, $l = 0, \dots, j_3$, verifying

$$\beta_{r, l} \leq 2j_3(\alpha + 1), \quad (3.4.34)$$

such that for all $y \in \mathbb{R}_+^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\begin{aligned} \partial_y^{j_3} \left[\chi_0 \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) e^{-\Phi_\lambda} \right] = \\ \sum_{\substack{0 \leq l \leq j_3 \\ 0 \leq r \leq j_3(k+1)}} \chi_0^{(l)} \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) y^r a_{r,l} \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}} \right) \lambda^{l\mu + \beta_{r,l}} e^{-\Phi_\lambda}. \end{aligned} \quad (3.4.35)$$

Since using the Leibniz formula on (3.4.35), we can write

$$\begin{aligned} \partial_{\tau,\xi}^\rho \partial_y^{j_3} \left[\chi_0 \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) e^{-\Phi_\lambda} \right] = \\ \sum_{l,r,\rho_1,\rho_2,\rho_3} \left[c_{18,l,r,\rho_1,\rho_2,\rho_3} \partial_{\tau,\xi}^{\rho_1} \left(\chi_0^{(l)} \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) \right) \right. \\ \left. \times y^r \partial_{\tau,\xi}^{\rho_2} \left(a_{r,l} \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}} \right) \right) \partial_{\tau,\xi}^{\rho_3} \left(e^{-\Phi_\lambda} \right) \lambda^{l\mu + \beta_{r,l}} \right], \end{aligned}$$

where the above sum is taken on $0 \leq l \leq j_3$, $0 \leq r \leq j_3(k+1)$, $(\rho_1, \rho_2, \rho_3) \in (\mathbb{N}^2)^3$, $\rho_1 + \rho_2 + \rho_3 = \rho$ and where $c_{18,l,r,\rho_1,\rho_2,\rho_3}$ are some constants, we deduce from (3.1.5), (3.4.8), (3.4.19) and (3.4.34), that there exist some positive constant $c_{19,l,r,\rho_1,\rho_2,\rho_3}$ and c_{20} such that for all $(\tau, \xi) \in [1, 4]^2$, $\lambda \geq \lambda_0$ and $y \in \text{supp } \chi_0(\lambda^\mu(\cdot + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}))$,

$$\begin{aligned} & \left| \partial_{\tau,\xi}^\rho \partial_y^{j_3} \left[\chi_0 \left(\lambda^\mu (y + (\lambda^{-1} \xi^{-1} \tau)^{\frac{2}{k}}) \right) e^{-\Phi_\lambda} \right] \right| \\ & \leq \sum_{l,r,\rho_1,\rho_2,\rho_3} c_{19,l,r,\rho_1,\rho_2,\rho_3} \lambda^{|\rho_1|(\mu - \frac{2}{k})} \lambda^{2|\rho_3|(\alpha - \frac{1}{k})} \lambda^{l\mu + 2j_3(\alpha+1)} \\ & \leq c_{20} \lambda^{j_3(2+2\alpha+\mu) + |\rho| \max(2\alpha - \frac{2}{k}, \mu - \frac{2}{k})}, \end{aligned}$$

where the sum of the previous expression is taken on $0 \leq l \leq j_3$, $0 \leq r \leq j_3(k+1)$, $(\rho_1, \rho_2, \rho_3) \in (\mathbb{N}^2)^3$, $\rho_1 + \rho_2 + \rho_3 = \rho$. This proves (3.4.32) and ends the proof of the lemma 3.4.5. $\square$

Thus, since from (3.1.1) and (3.1.3),

$$\text{supp } \psi_1 \subset [1, 4]^2 \text{ and } \max \left(2\alpha - \frac{2}{k}, \mu - \frac{2}{k} \right) > 0,$$

we can deduce from (3.4.31) and the previous lemma that for all $q \in \mathbb{N}$, there exists a positive constant $c_{21,q}$ such that for all $(t, x) \neq (0, 0)$, $y \in \mathbb{R}$ and $\lambda \geq \lambda_0$,

$$\begin{aligned} & |\partial_t^{j_1} \partial_x^{j_2} \partial_y^{j_3} u_\lambda(t, x, y)| \leq \\ & c_{21,q} \lambda^{\alpha + \frac{1}{2} + j_1\alpha + j_2(1+\alpha) + j_3(\mu + 2 + 2\alpha) + q \max(2\alpha - \frac{2}{k}, \mu - \frac{2}{k}) - q\alpha} |t^2 + x^2|^{-\frac{q}{2}}. \end{aligned} \quad (3.4.36)$$

It follows from (3.4.27) and from (3.1.6), $\theta_k(y) = (-y)^k$ for all $y \in \mathbb{R}_-$ that there exists a positive constant c_{22} such that we have the following estimate of the $H^{N_0}(\mathbb{R}^3)$ norm of the terms B_λ , C_λ , D_λ and E_λ defined in (3.4.26),

$$\begin{aligned} & \max(\|B_\lambda\|_{(N_0)}, \|C_\lambda\|_{(N_0)}, \|D_\lambda\|_{(N_0)}, \|E_\lambda\|_{(N_0)}) \\ & \leq c_{22} \sum_{\beta_1, \beta_2} \|\partial_{t,x}^{\beta_1}(\chi_1(\lambda^\gamma t, \lambda^\gamma x)) \partial_{t,x,y}^{\beta_2} u_\lambda\|_{L^2(\mathbb{R}^3)}. \end{aligned} \quad (3.4.37)$$

where the sum is taken on $\beta_1 = (l_1, l_2)$, $\beta_2 = (l_3, l_4, l_5)$ with l_j , $j = 1, \dots, 5$ some integers verifying $0 \leq l_j \leq N_0 + 2$ and $l_1 + l_2 \geq 1$. Using these notations, let us consider some integers l_j , $j = 1, \dots, 5$ verifying

$$0 \leq l_j \leq N_0 + 2 \text{ and } l_1 + l_2 \geq 1. \quad (3.4.38)$$

We set $\beta_1 = (l_1, l_2)$ and $\beta_2 = (l_3, l_4, l_5)$. Since from (3.4.14) and (3.4.38),

$$\text{supp } \partial_{t,x}^{\beta_1} \chi_1 \subset \{(t, x) \in \mathbb{R}^2 : t^2 + x^2 \geq 1/4\},$$

we deduce from (3.4.27) and (3.4.36) that for all $q \geq 2$, there exists a positive constant $c_{23,q}$ such that for all $\lambda \geq \lambda_0$,

$$\begin{aligned} & \|\partial_{t,x}^{\beta_1}(\chi_1(\lambda^\gamma t, \lambda^\gamma x)) \partial_{t,x,y}^{\beta_2} u_\lambda\|_{L^2(\mathbb{R}^3)} \\ & \leq c_{23,q} \lambda^{\alpha + \frac{1}{2} + \gamma(l_1 + l_2) + \alpha l_3 + (1 + \alpha)l_4 + (\mu + 2 + 2\alpha)l_5 + q \max(\alpha - \frac{2}{k}, \mu - \alpha - \frac{2}{k})} \|\partial_{t,x}^{\beta_1} \chi_1\|_{L^\infty(\mathbb{R}^2)} \\ & \quad \times \left(\int_{-c_1 \lambda^{-\frac{2}{k}}}^{-c_2 \lambda^{-\frac{2}{k}}} \left(\int_{\{(t,x) \in \mathbb{R}^2 : t^2 + x^2 \geq 4^{-1} \lambda^{-2\gamma}\}} |t^2 + x^2|^{-q} dt dx \right) dy \right)^{\frac{1}{2}}. \end{aligned} \quad (3.4.39)$$

Since with our choice of the integers l_j in (3.4.38), we have

$$\gamma(l_1 + l_2) + \alpha l_3 + (1 + \alpha)l_4 + (\mu + 2 + 2\alpha)l_5 \leq (N_0 + 2)(3 + 2\gamma + 4\alpha + \mu). \quad (3.4.40)$$

We deduce from (3.4.38), (3.4.39), (3.4.40) and a change of variables that

$$\begin{aligned} & \|\partial_{t,x}^{\beta_1}(\chi_1(\lambda^\gamma t, \lambda^\gamma x)) \partial_{t,x,y}^{\beta_2} u_\lambda\|_{L^2(\mathbb{R}^3)} \\ & \leq c_{23,q} (c_1 - c_2)^{\frac{1}{2}} \lambda^{\alpha + \frac{1}{2} - \frac{1}{k} - \gamma + (N_0 + 2)(3 + 2\gamma + 4\alpha + \mu) + q \max(\alpha - \frac{2}{k}, \mu - \alpha - \frac{2}{k}) + q\gamma} \\ & \quad \left(\sup_{|\beta_1| \leq 2N_0 + 4} \|\partial_{t,x}^{\beta_1} \chi_1\|_{L^\infty(\mathbb{R}^2)} \right) \left(\int_{\{(T,X) \in \mathbb{R}^2 : T^2 + X^2 \geq 4^{-1}\}} |T^2 + X^2|^{-q} dT dX \right)^{\frac{1}{2}}. \end{aligned} \quad (3.4.41)$$

We recall that in view of (3.1.5), we have $c_1 > c_2$. Then, it follows from (3.4.37) and (3.4.41) that for all $q \geq 2$, there exists a positive constant $c_{24,q}$ such that

$$\begin{aligned} & \max(\|B_\lambda\|_{(N_0)}, \|C_\lambda\|_{(N_0)}, \|D_\lambda\|_{(N_0)}, \|E_\lambda\|_{(N_0)}) \\ & \leq c_{24,q} \lambda^{\alpha + \frac{1}{2} - \frac{1}{k} - \gamma + (N_0 + 2)(3 + 2\gamma + 4\alpha + \mu) + q \max(\alpha + \gamma - \frac{2}{k}, \mu + \gamma - \alpha - \frac{2}{k})}. \end{aligned} \quad (3.4.42)$$

Since from (3.1.3) and (3.4.13),

$$\max\left(\alpha + \gamma - \frac{2}{k}, \mu + \gamma - \alpha - \frac{2}{k}\right) < 0, \quad (3.4.43)$$

because

$$\mu - \alpha - \frac{1}{k} < 0 \text{ and } \gamma - \frac{1}{k} < 0,$$

we obtain from (3.4.26), (3.4.29), (3.4.42) and (3.4.43) that for all $M \in \mathbb{N}$, there exists a positive constant C_M such that for all $\lambda \geq \lambda_0$,

$$\|L_2^* v_\lambda\|_{(N_0)} \leq C_M \lambda^{-M}. \quad (3.4.44)$$

To sum up, we have built a family $(v_\lambda(t, x, y))_{\lambda \geq \lambda_0}$ in (3.4.12), which is C^∞ on $\mathbb{R}^3$ and has according to (3.4.14) and (3.4.27), its support in the compact set

$$B(0, \lambda^{-\gamma}) \times [-c_1 \lambda^{-\frac{2}{k}}, -c_2 \lambda^{-\frac{2}{k}}]. \quad (3.4.45)$$

The estimates obtained in (3.4.25) and (3.4.44), for all $\lambda \geq \lambda_0$,

$$\|v_\lambda\|_{(-N_0)} \geq c_{15} \lambda^{-\alpha-1+\frac{\mu}{2}-(\mu+3+3\alpha)N_0}, \quad (3.4.46)$$

$$\forall M \in \mathbb{N}, \exists C_M > 0, \|L_2^* v_\lambda\|_{(N_0)} \leq C_M \lambda^{-M}, \quad (3.4.47)$$

allow us to prove that **no** a priori estimates of the following type can hold

$$\begin{aligned} &\exists C_0 > 0, \exists N_0 \in \mathbb{N}, \exists V_0 \text{ an open neighbourhood of } 0 \text{ in } \mathbb{R}^3 \text{ such that} \\ &\forall u \in C_0^\infty(V_0), \quad C_0 \|L_2^* u\|_{(k-3)} \geq \|u\|_{(-N_0)}. \end{aligned}$$

This proves that the operator L_2 is nonsolvable in any neighbourhood of 0 in $\mathbb{R}^3$ in the sense where there do **not** exist an integer $N_0 \in \mathbb{N}$ and an open neighbourhood V_0 of 0 in $\mathbb{R}^3$ such that for all $f \in H^{N_0}(V_0)$, there exists $u \in H^{-k+3}(\mathbb{R}^3)$ such that

$$L_2 u = f \text{ on } V_0.$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, EVANS
HALL, BERKELEY, CA 94720, USA
E-mail address: **karel@math.berkeley.edu**