

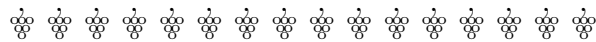
Pfaffians and Representations of the Symmetric Group

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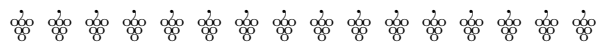
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Abstract

Pfaffians of matrices with entries $z[i, j]/(x_i + x_j)$, or determinants of matrices with entries $z[i, j]/(x_i - x_j)$, where the antisymmetrical indeterminates $z[i, j]$ satisfy the Plücker relations, can be identified with a trace in an irreducible representation of a product of two symmetric groups. Using Young's orthogonal bases, one can write explicit expressions of such Pfaffians and determinants, and recover in particular the evaluation of Pfaffians which appeared in the recent literature.



Key words: Pfaffians; Symmetric Group; Representations

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1 Introduction

Determinants or Pfaffians of order n can be written in terms of the symmetric group \mathfrak{S}_n . Determinants can be considered as generators of a 1-dimensional alternating representations. But in the case of a determinant or a Pfaffian

$$\left| \frac{a_i - a_{j+n}}{x_i - x_{j+n}} \right|_{1 \leq i, j \leq n}, \quad \mathfrak{Pfaff} \left(\frac{a_i - a_j}{x_i + x_j} \right)_{1 \leq i < j \leq n},$$

three symmetric groups occur : \mathfrak{S}_n^a acts on the indeterminates a_i , \mathfrak{S}_n^x acts on the x_j , and \mathfrak{S}_n^{ax} acts on the indices of all indeterminates simultaneously. This “diagonal action” satisfy a Cauchy-type property, each irreducible representation of \mathfrak{S}_n^a occurring in the expansion of the determinant, or of the Pfaffian, being tensored with a representation of \mathfrak{S}_n^x of conjugate type.

When $n = 2m$ is even, since the space generated by the orbit of the polynomial $(a_1 - a_2)(a_3 - a_4) \cdots (a_{n-1} - a_n)$ under \mathfrak{S}_n^a is a copy of the irreducible

representation $V_{[m,m]}^a$ of type $[m, m]$, this forces the Pfaffian to lie in the space $V_{[m,m]}^a \otimes V_{[2,\dots,2]}^x$.

An easy analysis shows that moreover the Pfaffian $\mathfrak{Pfaff} \left(\frac{a_i - a_j}{x_i + x_j} \right)$ is diagonal in Young's orthogonal basis (and thus, can be considered as a trace). In fact, the same analysis remains valid (this is our main theorem, Th.6) in the more general case

$$\mathfrak{Pfaff} (z[i, j] g[i, j])_{1 \leq i < j \leq n},$$

when taking antisymmetric indeterminates $z[i, j] = -z[j, i]$ satisfying the Plücker relations (we say *Plücker indeterminates*), instead of $(a_i - a_j)$, and symmetric indeterminates $g[i, j] = g[j, i]$ instead of $(x_i + x_j)^{-1}$. For specific $z[i, j]$ and $g[i, j]$, one may be able to write another element belonging to the same representation. Checking that two elements in the same irreducible representation coincide is very easy, and reduces to compute some specializations.

The most general case that we consider is $\mathfrak{Pfaff} (a[i, j] b[i, j] z[i, j]^{-1})$, with three families of Plücker indeterminates. In that case the Pfaffian factorizes in two factors separating the $a[i, j]$'s and $b[i, j]$'s (Th.7).

A connection with the theory of symmetric functions is provided by specializing the Plücker indeterminates into $S_\lambda(A + x_i + x_j)(x_i - x_j)$, $S_\lambda(A)$ being a fixed Schur function, to the alphabet of which one adds the letters x_i, x_j ([12]). Thus Th.7 gives the factorization of

$$\mathfrak{Pfaff} \left(\frac{S_\lambda(A + a_i + a_j) S_\mu(B + b_i + b_j) (a_i - a_j)(b_i - b_j)}{S_\nu(Z + z_i + z_j) (z_i - z_j)} \right),$$

for three Schur functions, and three families of indeterminates.

In the case of $\mathfrak{Pfaff} \left(\frac{a_i - a_j}{x_i + x_j} \right)$ first considered by Sundquist [25], and that we have taken as our generic case, it is easy to write a determinant which also lies in the space $V_{[m,m]}^a \otimes V_{[2,\dots,2]}^x$. Specializing half of the a_i 's to 1, the others to 0, one recovers the determinantal expression of Sundquist for this Pfaffian (Th.8).

Ishikawa [4], Okada [18], M. Ishikawa, S. Okada, H. Tagawa and J. Zeng [5] have given different generalizations of Sundquist's Pfaffian. We show how to connect their results to Th.6 and Th.7.

In section 8, we go back to determinants, and show how to relate

$$\left| \frac{z[i, j]}{x_i^2 - x_j^2} \right|_{1 \leq i \leq m < j \leq n} \quad \text{and} \quad \mathfrak{Pfaff} \left(\frac{z[i, j]}{x_i + x_j} \right),$$

the indeterminates $z[i, j]$ still satisfying the Plücker relations. A corollary of this analysis is that the above determinant is, up to straightforward factor, symmetrical in x_1, \dots, x_n , and not only symmetrical in x_1, \dots, x_m and

x_{m+1}, \dots, x_{2m} separately. Some determinants $\det\left(S_\lambda(A + x_i + x_j) S_\mu(B + x_i + x_j)^{-1}\right)$ present a special interest in the theory of orthogonal polynomials, or of the six-vertex model.

To be self-contained, and for lack of a reference appropriate to our needs, we first recall some properties of representations. In the last section, we give more details about the polynomial bases that one deduces from Young's orthogonal idempotents.

2 Representations of the symmetric group

2.1 Young's idempotents

The group algebra \mathcal{H} of the symmetric group \mathfrak{S}_n has by definition a linear basis consisting of all the permutations of $1, 2, \dots, n$.

Young described another basis e_{tu} , indexed by pairs of standard tableaux of the same shape with n boxes. These elements are *matrix units*, in the sense that they satisfy the relations

$$e_{t,u}e_{u,v} = e_{t,v}, \quad (1)$$

$$e_{t,u}e_{w,v} = 0 \quad \text{if } w \neq u. \quad (2)$$

In particular, the $e_{t,t}$ are idempotents: $e_{t,t}e_{t,t} = e_{t,t}$, and the identity decomposes as

$$1 = \sum_t e_{t,t},$$

where the sum is over all standard tableaux of n boxes. The subsum

$$e_\lambda = \sum_{t \in Tab(\lambda)} e_{t,t} \quad (3)$$

over standard tableaux of a given shape λ is the *central idempotent* of index λ .

2.2 Specht representations

Given any t , the right module $e_{t,t}\mathcal{H}$ is an irreducible representation of the symmetric group, with basis $\{e_{t,u} : u \text{ has the same shape as } t\}$.

There are simpler models of irreducible representations, in particular spaces of polynomials which are called *Specht representations*, though they

have been defined by Young¹.

Bases are still indexed by standard tableaux of a given shape, but now tableaux are interpreted as polynomials as follows.

A column tableau $\begin{array}{|c|} \hline k \\ \hline j \\ \hline i \\ \hline \end{array}$ is interpreted as the Vandermonde determinant in the variables x_i, x_j, \dots, x_k :

$$\begin{array}{|c|} \hline k \\ \hline j \\ \hline i \\ \hline \end{array} = (x_i - x_j)(x_i - x_k)(x_j - x_k) := \Delta^x(i, j, k),$$

and a tableau stands for the product of its columns :

$$\begin{array}{|c|} \hline 5 \\ \hline 3 & 6 \\ \hline 1 & 2 & 4 \\ \hline \end{array} = \Delta^x(1, 3, 5) \Delta^x(2, 6) \Delta^x(4).$$

We shall denote this polynomial Δ_t^x and call it *Specht polynomial*. The orbit of any Δ_t^x under the symmetric group (permuting the variables x_i) has $n!$ elements, whose linear span is of dimension the number of standard tableaux of the same shape as t .

More precisely, Young obtained, in the case of zero characteristic :

Proposition 1 *Given a partition λ , the linear span of the polynomials Δ_t^x , t varying over the set $\mathfrak{Tab}(\lambda)$ of standard tableaux of shape λ , is an irreducible representation of the symmetric group.*

Fixing a shape λ , there are two “extreme tableaux”: the one such that its columns are filled with consecutive letters, and that we shall call *top tableau* and denote ζ . For shape $[2, 3, 4]$, the top tableau is

$$\zeta = \begin{array}{|c|} \hline 3 & 6 \\ \hline 2 & 5 & 8 \\ \hline 1 & 4 & 7 & 9 \\ \hline \end{array}$$

and gives the Specht polynomial

$$\Delta_\zeta^x = \Delta^x(1, 2, 3) \Delta^x(4, 5, 6) \Delta^x(7, 8) \Delta^x(9).$$

¹Young [26, Theorem IV, p.591] uses the picturesque terminology “has the same substitutional qualities”, to state that the space $e_{i,t} \mathcal{H}$ is isomorphic to the space generated by some products of Vandermonde determinants.

Similarly, the *bottom tableau* has its rows filled with consecutive letters. We denote it by \aleph :

$$\aleph = \begin{array}{|c|c|c|c|} \hline 8 & 9 & & \\ \hline 5 & 6 & 7 & \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array}$$

with Specht polynomial

$$\Delta_{\aleph}^x = \Delta^x(1, 5, 8) \Delta^x(2, 6, 9) \Delta^x(3, 7) \Delta^x(4).$$

The standard tableaux of a given shape may be generated by using simple transpositions, starting with ζ . By the notation $\mathfrak{Tab}(\lambda)$ we mean this ranked poset, with top element ζ and bottom one, \aleph . The *distance* $\ell(t, u)$ of two tableaux is the distance in $\mathfrak{Tab}(\lambda)$.

The decomposition of any element of the Specht representation in the basis Δ_t^x is given by a so-called *straightening algorithm* (cf. [3, 2, 1]).

We shall need only one coefficient in such an expansion, the coefficient of Δ_{\aleph}^x . Given a tableau t with n boxes, and a function $f(x_1, \dots, x_n)$, let us write $f(t)$ for the specialization where each x_i is specialized to r if i lies on row r (rows are numbered from the bottom, starting with 0).

Lemma 2 *Given a partition λ , and a linear combination $f(x_1, \dots, x_n) = \sum_t c_t \Delta_t^x$, with coefficients c_t independent of x_1, \dots, x_n , then the coefficient c_{\aleph} is equal to*

$$f(\aleph) / \Delta_{\aleph}^x(\aleph).$$

Proof. All other tableaux than \aleph have in some column two entries which lie in the same row of \aleph . Q.E.D.

The Specht representation can occur in many disguises. Let us call *Plücker indeterminates* anti-symmetric indeterminates $z[i, j] = -z[j, i]$, satisfying Plücker relations[14] for all quadruples of different integers :

$$z[i, j]z[k, l] - z[i, k]z[j, l] + z[j, k]z[i, l] = 0.$$

A typical example is obtained by taking a $2 \times \infty$ generic matrix M , and defining $z[i, j]$ to be the minor on columns i, j of M . More generally, one takes an $N \times \infty$ generic matrix M , one chooses $N - 2$ columns of index α, β, \dots and define $z[i, j]$ to be the minor of maximal order of M on columns $i, j, \alpha, \beta, \dots$, with $i, j \neq \alpha, \beta, \dots$.

The following proposition gives another description of Specht representations for shape $[m, m]$.

Proposition 3 *Given an even positive number $n = 2m$, let $z[i, j]$, $1 \leq i, j \leq n$ be Plücker indeterminates. Given any numbering u of the boxes of the diagram $[m, m]$, let $z[u]$ be the product of all $z[i, j]$, where $[i, j]$ is a column of u . Let the symmetric group \mathfrak{S}_n act on the variables $z[i, j]$ by permutation of $1, 2, \dots, n$.*

Then the correspondence $z[u] = \prod z[i, j] \rightarrow \prod (x_i - x_j)$ induces an isomorphism of representations of \mathfrak{S}_n . In particular, $\{z[t]\}$, where t runs over all standard tableaux of shape $[m, m]$, is a linear basis of the span of all $z[u]$.

In short, when one has

$$z[1, 2]z[3, 4] - z[1, 3]z[2, 4] + z[1, 4]z[2, 3] = 0$$

one can as well read

$$(a_1 - a_2)(a_3 - a_4) - (a_1 - a_3)(a_2 - a_4) + (a_1 - a_4)(a_2 - a_3) = 0$$

or

$$\begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} - \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 1 & 2 \\ \hline \end{array} = 0$$

without loss of generality.

We shall use, for a rectangular shape with two columns of length $m = n/2$, three models of representations. The first one is the usual Specht representation, generated by the action of \mathfrak{S}_n on

$$\Delta^x(1, 2, \dots, m) \Delta^x(m+1, \dots, n).$$

The second model is the image of the first one under the correspondence $(x_i - x_j) \rightarrow z[i, j]$. The Specht polynomial corresponding to the top tableau of shape 2^m will now be

$$\mathfrak{Y}^{z[\cdot]}(\zeta) := \Delta^{z[\cdot]}(1, 2, \dots, m) \Delta^{z[\cdot]}(m+1, \dots, n) = \prod_{1 \leq i < j \leq m} z[i, j] \prod_{m+1 \leq i < j \leq n} z[i, j].$$

For the third one, one starts with a symmetric matrix G , with entries $g[i, j] = g[j, i]$. Let us denote the minor consisting of rows i_1, \dots, i_m and columns i_{m+1}, \dots, i_n by

$$g[i_1, \dots, i_m \mid i_{m+1}, \dots, i_n].$$

The symmetric group \mathfrak{S}_n acts formally by permuting the indices of such minors. Now, Kronecker [10, 17] has shown that such minors satisfy the

Plücker relations²

$$\sum_{i=0}^m (-)^i g[1, \dots, m-1, m+i \mid m+1, \dots, \widehat{m+i}, \dots, n] = 0.$$

This implies the following proposition :

Proposition 4 *Let G be a symmetric matrix of order $n = 2m$. Then the linear span of the minors in the orbit of*

$$g[1, \dots, m \mid m+1, \dots, n]$$

under permutation of indices, is an irreducible representation of \mathfrak{S}_n of shape 2^m .

Using the correspondence

$$g[i_1, \dots, i_m \mid i_{m+1}, \dots, i_n] \leftrightarrow \Delta^x(i_1, \dots, i_m) \Delta^x(i_{m+1}, \dots, i_n)$$

one can still speak of a *Specht basis* for these minors of a symmetric matrix.

For example, for $m = 3$, the space has basis

$$g[123 \mid 456], g[124 \mid 356], g[125 \mid 346], g[134 \mid 256], g[135 \mid 246], ,$$

and one can directly check the relation

$$g[123 \mid 456] - g[124 \mid 356] + g[125 \mid 346] - g[126 \mid 345] = 0.$$

In detail, for $m = 2$, the sum $g[12|34] - g[13|24] + g[14|23]$ expands into $g[1, 3]g[2, 4] - g[1, 4]g[2, 3] - g[1, 2]g[3, 4] + g[1, 4]g[3, 2] + g[1, 2]g[4, 3] - g[1, 3]g[4, 2]$ which is indeed zero, because $g[i, j] = g[j, i]$.

Plücker relations, hence Specht representations, also occur in the theory of symmetric functions.

Indeed, given two alphabets^[12] $A = \{a\}$, $B = \{b\}$, the *complete functions* $S_k(A - B)$ are defined by the generating function

$$\sum_k z^k S_k(A - B) = \prod_{b \in B} (1 - zb) \prod_{a \in A} (1 - za)^{-1},$$

²and, of course, all the relations obtained by permuting the rows and the columns of the original matrix, in such a way as to obtain another symmetric matrix. We could reprove directly Kronecker's relations by introducing extra variables a_1, a_2, \dots and evaluating the Pfaffian of the antisymmetric matrix $[(a_i - a_j)g[i, j]]_{i, j=1, \dots, n}$, as will become clear later.

putting $S_k = 0$ for $k < 0$. The *Schur function* $S_v(A - B)$, $v \in \mathbb{Z}^r$, has the determinantal expression $\det(S_{v_j+j-i}(A - B))$. When B is the two-letters alphabet $B = \{x, y\}$, then $(x - y)S_v(A - B)$, denoted $(x - y)S_v(A - x - y)$, is equal to the maximal minor on columns $v_1, v_2+1, \dots, v_r+r-1, x, y$ of

$$\begin{array}{c} \text{columns} \end{array} \left[\begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & x & y \\ S_0(A) & S_1(A) & S_2(A) & S_3(A) & \cdots & x^{r+1} & y^{r+1} \\ S_{-1}(A) & S_0(A) & S_1(A) & S_2(A) & \cdots & x^r & y^r \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ S_{-r-1}(A) & S_{-r}(A) & S_{-r+1}(A) & S_{-r+2}(A) & \cdots & 1 & 1 \end{array} \right].$$

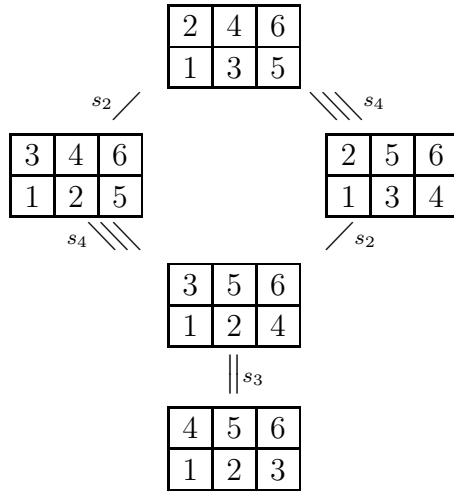
Therefore, for a given $v \in \mathbb{Z}^r$, and a given alphabet A , the $z[i, j] = (x - y)S_v(A - x - y)$ (resp. $z[i, j] = (x - y)S_v(A + x + y)$) satisfy the Plücker relations.

2.3 Orthogonal representations

Given a shape λ , then $\Delta_\zeta^x e_{\zeta, \zeta} = \Delta_\zeta^x$, and the polynomials $\Delta_\zeta^x e_{\zeta, t}$, for $t \in \mathfrak{Tab}(\lambda)$, constitute another basis of the Specht representation. Taking a linear order compatible with the poset structure, then the matrix of change of basis is lower triangular. A more precise information is given by using the Yang-Baxter relations (see the last section).

Let us call *Young's basis* the basis proportional to $\Delta_\zeta^x e_{\zeta, t}$, such that the leading term (with respect to the poset $\mathfrak{Tab}(\lambda)$) of each $\mathfrak{Y}(t)$ be Δ_t^x , and denote it $\{\mathfrak{Y}(t) : t \in \mathfrak{Tab}(\lambda)\}$.

For example, for shape $[3, 3]$, the poset of tableaux is



and the matrix expressing Young's basis in terms of the Specht basis (reading successive rows) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 & 0 \\ -1/2 & 0 & 1 & 0 & 0 \\ 1/4 & -1/2 & -1/2 & 1 & 0 \\ 2/3 & -1/3 & -1/3 & -1/3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & 0 \\ 1/4 & 1/2 & 1/2 & 1 & 0 \\ -1/4 & 1/2 & 1/2 & 1/3 & 1 \end{bmatrix}^{-1}.$$

In that special case, the two allowable linear orders on the graph give the same matrices, we did not need numbering the tableaux.

To handle other models of irreducible representations, we first need to characterize the elements corresponding to Δ_ζ^x in these models. In fact, Δ_ζ^x is the only polynomial in the Specht representation, such that $(\Delta_\zeta^x)^{s_i} = -\Delta_\zeta^x$ for all i such that $i, i+1$ are in the same column of ζ . This property still characterizes a unique element in a different copy of the Specht representation, that we shall still denote $\mathfrak{Y}(\zeta)$.

We shall still call *Young basis* the basis proportional to $\{\mathfrak{Y}(\zeta) e_{\zeta,t}\}$, with the same factors of proportionality as in the case where we start with Δ_ζ^x . More details about how to compute such a basis are given in the last section.

2.4 Cauchy Formula

Let us now take two symmetric groups $\mathfrak{S}_n^x, \mathfrak{S}_n^a$ acting respectively on the variables x_1, \dots, x_n , and a_1, \dots, a_n , with generators s_i^x, s_i^a .

We also use the group $\mathfrak{S}_n = \mathfrak{S}_n^{ax}$, which permutes simultaneously the variables a_i and x_i . We write its generators s_i , instead of s_i^{ax} . These are such that

$$s_i = s_i^a s_i^x = s_i^x s_i^a.$$

We shall give a decomposition of the two 1-dimensional idempotents $\square := (n!)^{-1} \sum \sigma$ and $\nabla := (n!)^{-1} \sum (-1)^{\ell(\sigma)} \sigma$, with respect to the groups \mathfrak{S}_n^x and \mathfrak{S}_n^a .

Indeed, by definition

$$\square^{ax} = \frac{1}{n!} \sum_{\sigma} \sigma^x \sigma^a,$$

with pairs of permutations in $\mathfrak{S}_n^a, \mathfrak{S}_n^x$ commuting with each other.

Taking the basis of Young idempotents, instead of the basis of permutations, one obtains a Cauchy-type formula :

$$\square^{ax} = \sum_{t,u} \frac{1}{d(t)} e_{t,u}^x e_{t,u}^a, \quad (4)$$

where the sum is over all pairs of standard tableaux of the same shape, and where $d(t)$ is the number of tableaux of the same shape as t .

The element ∇^{ax} is obtained from \square^{ax} under the isomorphism induced by :

$$s_i^x \rightarrow \widehat{s}_i^x := -s_i^x, \quad s_i^a \rightarrow s_i^a$$

From the expressions of \square and ∇ as sums of products of permutations, one sees that for any simple transposition,

$$\square^{ax} s_i^x = \square^{ax} s_i^a \quad \& \quad s_i^x \square^{ax} = s_i^a \square^{ax} \quad (5)$$

$$\nabla^{ax} s_i^x = -\nabla^{ax} s_i^a \quad \& \quad s_i^x \nabla^{ax} = -s_i^a \nabla^{ax}. \quad (6)$$

By taking products, one gets

$$\square^{ax} s_i^x s_j^x \cdots s_k^x = \square^{ax} s_i^a s_j^x \cdots s_k^x = \square^{ax} s_j^x \cdots s_k^x s_i^a = \cdots,$$

and therefore, for any permutation σ ,

$$\square^{ax} \sigma^x = \square^{ax} (\sigma^a)^{-1} \quad \& \quad \sigma^x \square^{ax} = (\sigma^a)^{-1} \square^{ax} \quad (7)$$

$$\nabla^{ax} \sigma^x = (-1)^{\ell(\sigma)} \nabla^{ax} (\sigma^a)^{-1} \quad \& \quad \sigma^x \nabla^{ax} = (-1)^{\ell(\sigma)} (\sigma^a)^{-1} \nabla^{ax}. \quad (8)$$

Restricting \square^{ax} or ∇^{ax} to a representation of \mathfrak{S}_n^a of type is achieved by multiplying \square^{ax} or ∇^{ax} by e_λ^a .

Since an idempotent $e_{t,t}^x$ is sent under $s_i^x \rightarrow -s_i^x$ to e_{t^\sim,t^\sim}^x , where $t \rightarrow t^\sim$ denotes the transposition of tableaux, using expression (3), we obtain the two equivalent expansions :

$$\square^{ax} e_\lambda^a = \sum_{t,u \in \mathfrak{Tab}(\lambda)} \frac{1}{d(t)} e_{t,u}^a e_{t,u}^x, \quad (9)$$

$$\nabla^{ax} e_\lambda^a = \sum_{t,u \in \mathfrak{Tab}(\lambda)} (-1)^{\ell(t,u)} \frac{1}{d(t)} e_{t,u}^a e_{t^\sim,u^\sim}^x. \quad (10)$$

For example, there are 4 standard tableaux with three boxes :

$$\alpha = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}, \quad \beta = \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}, \quad \gamma = \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}, \quad \delta = \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array},$$

and the elements \square^{ax} and ∇^{ax} decompose as :

$$\begin{aligned}\square^{ax} &= e_{\alpha,\alpha}^a e_{\alpha,\alpha}^x + \frac{1}{2} (e_{\beta,\beta}^a e_{\beta,\beta}^x + e_{\beta,\gamma}^a e_{\beta,\gamma}^x + e_{\gamma,\beta}^a e_{\gamma,\beta}^x + e_{\gamma,\gamma}^a e_{\gamma,\gamma}^x) + e_{\delta,\delta}^a e_{\delta,\delta}^x \\ \nabla^{ax} &= e_{\alpha,\alpha}^a e_{\delta,\delta}^x + \frac{1}{2} (e_{\beta,\beta}^a e_{\gamma,\gamma}^x - e_{\beta,\gamma}^a e_{\gamma,\beta}^x - e_{\gamma,\beta}^a e_{\beta,\gamma}^x + e_{\gamma,\gamma}^a e_{\beta,\beta}^x) + e_{\delta,\delta}^a e_{\alpha,\alpha}^x,\end{aligned}$$

the middle part being the component of type [1, 2].

3 Pfaffians

Let $Z = [z[i, j]]$ be an anti-symmetric matrix of even order n . Its determinant is the square of a function of the $z[i, j]$, which is called the *Pfaffian* of Z . The Pfaffian is a certain sum, with coefficients \pm , of products $z[i, j] \cdots z[k, l]$. We refer to [7] for an historical and complete presentation.

Deciding to write each monomial in $z[i, j]$ according to some lexicographic order on the variables, one can erase z , brackets and commas, and use permutations :

$$z[i, j] z[k, l] z[p, q] \rightarrow [i j k l p q]$$

The Pfaffian of Z has become an alternating sum of permutations which can be defined recursively as follows [7]. For any vector $v \in \mathbb{N}^n$ of even length n , let

$$\mathfrak{Pf}(v) = \sum_{i=1}^{n-1} (-1)^i \mathfrak{Pf}(v \setminus \{v_i, v_n\}) \cdot [v_i, v_n],$$

where the product is the concatenation product, and where $v \setminus \{v_i, v_n\}$ means suppressing the components v_i, v_n inside v .

The initial case is $\mathfrak{Pf}([]) = []$.

$$\begin{aligned}\mathfrak{Pf}([1, 2]) &= [1, 2] \quad ; \quad \mathfrak{Pf}([1, 2, 3, 4]) = [1, 2, 3, 4] - [1, 3, 2, 4] + [2, 3, 1, 4], \\ \mathfrak{Pf}([1, 2, 3, 4, 5, 6]) &= [1, 2, 3, 4, 5, 6] - [1, 2, 3, 5, 4, 6] + [1, 2, 4, 5, 3, 6] \\ &\quad - [1, 3, 2, 4, 5, 6] + [1, 3, 2, 5, 4, 6] - [1, 3, 4, 5, 2, 6] - [1, 4, 2, 5, 3, 6] \\ &\quad + [1, 4, 3, 5, 2, 6] + [2, 3, 1, 4, 5, 6] - [2, 3, 1, 5, 4, 6] + [2, 3, 4, 5, 1, 6] \\ &\quad + [2, 4, 1, 5, 3, 6] - [2, 4, 3, 5, 1, 6] - [3, 4, 1, 5, 2, 6] + [3, 4, 2, 5, 1, 6].\end{aligned}$$

Let $\mathfrak{Pf}_n := \mathfrak{Pf}([1, \dots, n])$ be the above sum of permutations (we use the notation $\mathfrak{Pfaff}(Z)$ for the Pfaffian of an antisymmetric matrix, and \mathfrak{Pf}_n for the formal sum of permutations). Our data, the $z[i, j]$, were such that $z[i, j] = -z[j, i]$, and that $z[i, j]z[k, l] = z[k, l]z[i, j]$. We shall see in the next proposition, whose proof is immediate, that the permutations appearing in \mathfrak{Pf}_n are cosets representatives, modulo the symmetries possessed by the $z[i, j]$.

Indeed, let $\mathfrak{S}_{n/2}$ be the symmetric group which permutes the blocks $[1, 2], [3, 4], \dots, [n-1, n]$, and let Θ_n be the sum of its elements. Simple transpositions s_1, s_3, \dots, s_{n-1} commute with Θ_n .

Proposition 5 *For even n , the alternating sum of all permutations can be factorized as follows :*

$$\begin{aligned} n! \nabla &= (1 - s_1)(1 - s_3) \cdots (1 - s_{n-1}) \Theta_n \mathfrak{P}_n \\ &= \Theta_n (1 - s_1)(1 - s_3) \cdots (1 - s_{n-1}) \mathfrak{P}_n. \end{aligned}$$

As a consequence, we may write the Pfaffian of Z as

$$\mathfrak{Paff}(Z) = z[1, 2] z[3, 4] \cdots z[n-1, n] \nabla \frac{n!}{2^n (n/2)!},$$

since s_1, s_3, \dots and the permutations in Θ_n act trivially on the product $z[1, 2] z[3, 4] \cdots$.

For example, for $n = 6$,

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_6} (-1)^{\ell(\sigma)} \sigma &= (1 - s_1)(1 - s_3)(1 - s_5) ([123456] + [125634] + [341256] \\ &\quad + [345612] + [561234] + [563412]) \mathfrak{P}_6. \end{aligned}$$

An interesting approach, due to Luque and Thibon [15], to combinatorial properties of Pfaffians is through shuffle algebras. As a matter of fact, the same methods give also the *Hafnian*, i.e. the image of the Pfaffian (as an element of the group algebra) under the involution $s_i \rightarrow -s_i$, $i = 1, \dots, n-1$. We shall not need this approach, having written the Pfaffian in terms of the alternating sum of all permutations.

In [11] one finds Pfaffians and determinants associated to a family of formal series, which are needed in geometry.

4 Pfaffian, with the help of two symmetric groups

The main case that we want to treat now is the case of an antisymmetric matrix with entries

$$z[i, j] g[i, j],$$

the $z[i, j]$ satisfying the Plücker relations, and the $g[i, j]$ being symmetrical: $g[i, j] = g[j, i]$.

Of course, any antisymmetric matrix $N = [n[i, j]]$ can be written in this way, introducing extra variables a_i , and writing

$$n[i, j] = (a_i - a_j) \frac{n[i, j]}{a_i - a_j} .$$

The Pfaffian, being a sum of products of $z[i, j]$, belongs to the irreducible representation of shape $[m, m]$ (with respect to the symmetric group \mathfrak{S}_n^z acting on the indices of the indeterminates $z[i, j]$, $i = 1, \dots, n$, $m = n/2$). Hence, it can be expressed as a linear combination of Specht elements :

$$c_\zeta(g)z[\zeta] + \dots + c_{\mathfrak{N}}(g)z[\mathfrak{N}] .$$

Thanks to Prop.3, the coefficients $c_t(g)$ are the same as for the specialization $z[i, j] = (a_i - a_j)$.

The case of the Pfaffian of $(a_i - a_j)(x_i + x_j)^{-1}$ has been treated by Sundquist [25], but we need a more extensive description than his.

The coefficient $c_{\mathfrak{N}}(g)$ is obtained by specializing $a_1 = 1 = \dots = a_m$, $a_{m+1} = 0 = \dots = a_n$. In that case, the sum over all permutations in $\mathfrak{S}_n^{a, g}$

$$\sum (-)^{\ell(\sigma)} ((a_1 - a_2)g[1, 2] (a_3 - a_4)g[3, 4] \dots)^\sigma$$

reduces to

$$\sum (-)^{\ell(\sigma)} (g[1, m+1] g[2, m+2] \dots g[m-1, n])^\sigma ,$$

where the sum is now only over the subgroup $\mathfrak{S}_m \times \mathfrak{S}_m$ which permutes $1, \dots, m$, and $m+1, \dots, 2m$ separately.

This sum is equal to

$$m! g[1 \dots m \mid m+1 \dots n] . \tag{11}$$

Therefore the Pfaffian of the matrix $[(a_i - a_j)g[i, j]]$ is, up to a normalization constant, equal to the image of

$$\Omega_{a, g} := (a_1 - a_{m+1}) \dots (a_m - a_n) g[1 \dots m \mid m+1 \dots n]$$

under the anti-symmetrization $\nabla^{a, g}$.

This last element belongs to the Specht representation of \mathfrak{S}_n^a of shape $[m, m]$, and therefore, thanks to (10), belongs to the space³

$$V_{[m, m]}^a \otimes V_{[2^m]}^g .$$

³We do not need to know Kronecker's relations. Having only a component of type $[m, m]$ for \mathfrak{S}_n^a forces a component of type $[2^m]$ for \mathfrak{S}_n^g .

The action of $\nabla^{ag} = \sum \pm e_{t,t} e_{t^\sim,t^\sim}$ restricts to the tableaux t of shape $[m, m]$, and, consequently, the Pfaffian is proportional to

$$\sum_{t \in \mathfrak{Tab}([m,m])} (-1)^{\ell(\mathfrak{N},t)} \Omega_{a,g} e_{\mathfrak{N},t}^a e_{\mathfrak{N}^\sim,t^\sim}^g.$$

Eventually, taking into account normalizations, one can expand the Pfaffian in the Young basis :

$$\mathfrak{Pfaff}((a_i - a_j)g[i, j]) = \sum_{t \in \mathfrak{Tab}([m,m])} (-1)^{\ell(\zeta,t)} \mathfrak{Y}^a(t) \mathfrak{Y}^g(t^\sim). \quad (12)$$

In short, the Pfaffian may be considered as a trace in the space $V_{[m,m]}^a \otimes V_{[2m]}^g$. We summarize the preceding considerations in the following theorem.

Theorem 6 *Let $n = 2m$ be an even positive integer. Let $z[i, j]$ be Plücker indeterminates, let $g[i, j]$ be indeterminates symmetrical in i, j , for $i, j = 1 \dots n$. Then*

$$\mathfrak{Pfaff}(z[i, j]g[i, j]) = d(\mathfrak{N}) \mathfrak{Y}^z(\mathfrak{N}) \mathfrak{Y}^g(\mathfrak{N}^\sim) \nabla^{z,g} \quad (13)$$

$$= \sum_{t \in \mathfrak{Tab}([m,m])} (-1)^{\ell(\zeta,t)} \mathfrak{Y}^z(t) \mathfrak{Y}^g(t^\sim), \quad (14)$$

where $d(\mathfrak{N})$ is the number of Young tableaux of shape $[m, m]$.

This gives two ways of computing a Pfaffian. Either by a summation over all tableaux, or by antisymmetrization of the element $\mathfrak{Y}^z(\mathfrak{N}) \mathfrak{Y}^g(\mathfrak{N}^\sim)$ (one could take any other tableau than \mathfrak{N} , or take Specht polynomials instead of Young polynomials).

For example, for $n = 6$, $z[i, j] = a_i - a_j$, $g[i, j] = (x_i^4 - x_j^4)(x_i - x_j)^{-1}$, one sees that

$$g[123 | 456] = \Delta^x(1, 2, 3) \Delta^x(4, 5, 6) S_{1,1,1}(x_1, \dots, x_6).$$

In particular, $g[123 | 456]$ is the product of a Specht polynomial by a symmetric function in x_1, \dots, x_6 . Therefore, the Pfaffian is equal, up to a numerical factor, to

$$(a_1 - a_4)(a_2 - a_5)(a_3 - a_6) \Delta^x(1, 2, 3) \Delta^x(4, 5, 6) \nabla^{ax} S_{1,1,1}(x_1, \dots, x_6).$$

5 Three symmetric groups

One can use k families of Plücker indeterminates $z^1[i, j], z^2[i, j], \dots, z^k[i, j]$, together with a last family of symmetric, or antisymmetric, indeterminates $g[i, j]$, according to the parity of k . A Pfaffian

$$\mathfrak{P}\text{aff} (z^1[i, j] \cdots z^k[i, j] g[i, j])$$

still belongs to the irreducible representation

$$V_{[m,m]}^{z^1} \otimes \cdots \otimes V_{[m,m]}^{z^k}$$

of $\mathfrak{S}_n^{z^1} \times \cdots \times \mathfrak{S}_n^{z^k}$.

Therefore, it can be expanded into the Young basis :

$$\mathfrak{P}\text{aff} (z^1[i, j] \cdots z^k[i, j] g[i, j]) = \sum_{t_1, \dots, t_k} \mathfrak{Y}^{z^1}(t_1) \cdots \mathfrak{Y}^{z^k}(t_k) f(t_1, \dots, t_k; g),$$

sum over k -tuples of standard tableaux of shape $[m, m]$. To find the coefficients $f(t_1, \dots, t_k; g)$, one may take indeterminates $a_i^r : i = 1 \dots n, r = 1 \dots k$, and put $z^1[i, j] = a_i^1 - a_j^1, \dots, z^k[i, j] = a_i^k - a_j^k$.

The main difference with the case $k = 1$ is that a single specialization of the a_i^r is not enough to determine the Pfaffian. However, since any element of $V_{[m,m]}^a$ is characterized by the set of all its specializations

$$a_{\sigma_1} = 1 = \cdots = a_{\sigma_m}; a_{\sigma_{m+1}} = 0 = \cdots = a_{\sigma_n}, \sigma \in \mathfrak{S}_n.$$

It is in fact sufficient to take the specializations corresponding to the permutations obtained by reading the standard tableaux as permutations (reading rows from bottom to top). In that way, the number of specializations is equal to the number of indeterminate coefficients, and representation theory tells us that this system is solvable.

I do not see anything more to say for general k , but shall restrict to $k = 2$.

Theorem 7 *Let $a[i, j], b[i, j], z[i, j], 1 \leq i < j \leq n = 2m$ be three families of Plücker indeterminates. Then*

$$\mathfrak{P}\text{aff} \left(\frac{a[i, j]b[i, j]}{z[i, j]} \right) = (\mathfrak{Y}^a(\aleph) \mathfrak{Y}^z(\zeta) \nabla^{az}) (\mathfrak{Y}^b(\aleph) \mathfrak{Y}^z(\zeta) \nabla^{bz}) \frac{d(\aleph)^2}{\prod z[i, j]}, \quad (15)$$

where \aleph is the bottom tableau of shape $[m, m]$, and ζ , the top tableau of shape 2^m , $d(\aleph)$ still being the number of tableaux of the same shape as \aleph .

Proof. As we already used, we take $a[i, j] = a_i - a_j$, $b[i, j] = b_i - b_j$. For $z[i, j]$, we can assume that we are given a generic $N \times (N+n-1)$ matrix, N sufficiently big.

$$M = \begin{bmatrix} x_1 & x_2 & \cdots & x_n & \cdots \\ y_1 & y_2 & \cdots & y_n & \cdots \\ \cdots & \cdots & \cdots & \cdots & \boxed{K} \end{bmatrix},$$

with K a submatrix of order $N-2$. One then takes $z[i, j]$ to be the maximal minor containing x_i, y_j and X_i (resp Y_j) to be the minor of order $N-1$ containing K and x_i (resp. y_j). Sylvester's relation [14] states that $\det(K) z[i, j] = X_i Y_j - X_j Y_i$. In all, the value of the Pfaffian is determined by the case

$$\mathfrak{Pfaff} \left(\frac{(a_i - a_j)(b_i - b_j)}{X_i Y_j - X_j Y_i} \right).$$

The first step is still to specialize $a_1 = 1 = \cdots = a_m$, $a_{m+1} = 0 = \cdots = a_n$, as for Th.6. The Pfaffian becomes

$$F := \det \left((b_i - b_j)(X_i Y_j - X_j Y_i)^{-1} \right)_{1 \leq i < m < j \leq n}.$$

One has now to specialize b_1, \dots, b_n . Instead of taking all permutations $\sigma \in \mathfrak{S}_n$, by reordering rows and columns, one can suppose that there exist α, β, γ : $\sigma = [1, \dots, \alpha, \alpha+\beta, \dots, \alpha+\gamma, \alpha+1, \dots, \alpha+b-1, \alpha+\gamma+1, \dots, n]$. In that case, the specialization $b_{\sigma_1} = 1, \dots, b_{\sigma_m} = 1, b_{\sigma_{m+1}} = 0, \dots, b_{\sigma_n} = 0$ of F factorizes into two blocks, each of which is of Cauchy type $\det((X_i Y_j - Y_i X_j)^{-1})_{i,j=1 \dots k}$. For example, for $m = 4$, the specialization $b_1 = b_2 = b_5 = b_6 = 1, b_3 = b_4 = b_7 = b_8 = 0$ is

$$\begin{vmatrix} 0 & 0 & (X_1 Y_7 - X_7 Y_1)^{-1} & (X_1 Y_8 - X_8 Y_1)^{-1} \\ 0 & 0 & (X_2 Y_7 - X_7 Y_2)^{-1} & (X_2 Y_8 - X_8 Y_2)^{-1} \\ -(X_3 Y_5 - X_5 Y_3)^{-1} & -(X_3 Y_6 - X_6 Y_3)^{-1} & 0 & 0 \\ -(X_4 Y_5 - X_5 Y_4)^{-1} & -(X_4 Y_6 - X_6 Y_4)^{-1} & 0 & 0 \end{vmatrix}.$$

The evaluation of a Cauchy determinant is, of course, immediate (since 1812), and in final, for any σ , the specialization $b_{\sigma_1} = 1, \dots, b_{\sigma_m} = 1, b_{\sigma_{m+1}} = 0, \dots, b_{\sigma_n} = 0$ of F is equal to

$$(-1)^{\ell(\sigma)} \Delta^z(\sigma_1, \dots, \sigma_m) \Delta^z(\sigma_{m+1}, \dots, \sigma_n) \prod_{1 \leq i < m < j \leq n} (z_i - z_j)^{-1}.$$

Therefore, F coincides, up to a numerical factor, with

$$\begin{aligned} & b_1 \cdots b_m \Delta^z(1, \dots, m) \Delta^z(m+1, \dots, n) \nabla^{b,z} \prod_{1 \leq i < m < j \leq n} (z_i - z_j)^{-1} \\ &= (b_1 - b_{m+1}) \cdots (b_m - b_n) \Delta^z(m+1, \dots, n) \nabla^{b,z} \prod_{1 \leq i < m < j \leq n} (z_i - z_j)^{-1}. \end{aligned}$$

From this specialization, one writes the Pfaffian as

$$\mathfrak{Y}^b(\aleph)\mathfrak{Y}^z(\zeta)\nabla^{bz}\frac{\Delta^z(1,\dots,m)\Delta^z(m+1,\dots,n)}{\Delta^z(1,\dots,n)}\nabla^{abz}.$$

Since any permutation in \mathfrak{S}^{abz} commutes with $\nabla^{bz}\Delta^z(1,\dots,n)^{-1}$, the theorem follows Q.E.D.

Okada [18, Th.3.4] (see also [5, Formula 1.8]) has already computed $\mathfrak{Pfaff}((a_i-a_j)(b_i-b_j)(x_i-x_j)^{-1})$. His formula can be written

$$\begin{aligned} \mathfrak{Pfaff}\left(\frac{(a_i-a_j)(b_i-b_j)}{x_i^2-x_j^2}\right) \\ = \prod_{i<j} \frac{x_i+x_j}{x_i-x_j} \mathfrak{Pfaff}\left(\frac{a_i-a_j}{x_i+x_j}\right) \mathfrak{Pfaff}\left(\frac{b_i-b_j}{x_i+x_j}\right). \end{aligned} \quad (16)$$

In [18] and [5], one finds many evaluations of Pfaffians and determinants, with entries which are specializations of Plücker indeterminates. For example, Okada [18, Th.3.4] takes

$$a[i,j] = \begin{vmatrix} 1+a_ix_i & x_i+a_i \\ 1+a_jx_j & x_j+a_j \end{vmatrix} \text{ or } a[i,j] = \begin{vmatrix} 1+a_ix_i^2 & x_i+a_ix_i & x_i^2+a_i \\ 1+a_jx_j^2 & x_j+a_jx_j & x_j^2+a_j \\ 1+cz^2 & z+cz & z^2+c \end{vmatrix}.$$

All these families satisfy the Plücker relations, and one could take determinants of higher order of the type below, as indeterminates $z[i,j]$. On the other hand, Pfaffians involving elliptic functions as in [19] do not fall in this category, the Riemann relations replacing in that case Plücker relations.

6 Special Pfaffians

There are cases where $\mathfrak{Y}^z(\aleph)\mathfrak{Y}^g(\aleph^{\sim})\nabla^{z,g}$ can be written as a determinant. Indeed, let us take $z[i,j] = (a_i-a_j)$, and $g[i,j] = (x_i+x_j)^{-1}$ as Sundquist[25]. For any integer k , let $U(a,x^k)$ be the determinant of order n

$$U(a,x^k) := \left| a_ix_i^0, a_ix_i^k, \dots, a_ix_i^{k(m-1)}, x_i^0, x_i^k, \dots, x_i^{k(m-1)} \right|_{i=1\dots n}. \quad (17)$$

The Laplace expansion of $U(a,x)$ along its first m columns shows that it belongs to the Specht representation of \mathfrak{S}_n^x of shape $[2^m]$, and therefore, that $U(a,x)$ is an element of $V_{[m,m]}^a \otimes V_{[2^m]}^x$, which is identified by the specialization $a_1 = 1 = \dots = a_m$, $a_{m+1} = 0 = \dots = a_n$. Needless to add that this specialization is

$$\Delta(x_1, \dots, x_m) \Delta(x_{m+1}, \dots, x_n).$$

On the other hand, the Pfaffian $\mathfrak{P}\text{faff}((a_i - a_j)(x_i + x_j)^{-1})$ belongs to the same space, and the same specialization sends it, according to (11), and thanks to the Cauchy identity relative to the determinant $\left| (x_i + y_j)^{-1} \right|_{i,j=1\dots m}$, to

$$c_{\mathbb{N}}(x) = \frac{\Delta^x(1, \dots, m) \Delta^x(m+1, \dots, n)}{\prod_{i \leq m < j} x_i + x_j}.$$

Taking a symmetrical denominator, one rather writes

$$c_{\mathbb{N}}(x) = \frac{\Delta^{xx}(1, \dots, m) \Delta^{xx}(m+1, \dots, n)}{\prod_{1 \leq i < j \leq n} x_i + x_j}.$$

Therefore, this specialization coincides with the one of $U(a, x^2)$, and one recovers the following theorem.

Theorem 8 (Sundquist) *Let $n = 2m$, a_1, \dots, a_n , x_1, \dots, x_n be indeterminates. Let $U(a, x)$ be the determinant (17).*

Then

$$\begin{aligned} \mathfrak{P}\text{faff} \left(\frac{a_i - a_j}{x_i + x_j} \right) &= \frac{d}{n!} \sum_{\sigma} (-1)^{\ell(\sigma)} \left((a_1 - a_{m+1}) \cdots (a_{m-1} - a_n) \times \right. \\ &\quad \left. \Delta^{xx}(1, \dots, m) \Delta^{xx}(m+1, \dots, n) \right)^{\sigma} \prod_{1 \leq i < j \leq n} (x_i + x_j)^{-1} \\ &= \det(U(a, x^2)) \prod_{1 \leq i < j \leq n} (x_i + x_j)^{-1}, \end{aligned}$$

where the sum is over all permutations $\sigma \in \mathfrak{S}_n^{a,x}$, and d is the number of Young tableaux of shape $[m, m]$.

We have given another expression in Th6, using the Young basis. To stay nearer the expansion of $U(a, x)$, we can also use the Specht basis, but there will be extra terms corresponding to pairs of non-orthogonal tableaux (as soon as $n = 6$).

Indeed, for $n = 4$, we have

$$\begin{aligned} \prod_{i < j} (x_i + x_j) \mathfrak{P}\text{faff} \left(\frac{a_i - a_j}{x_i + x_j} \right) &= (a_1 - a_2)(a_3 - a_4)(x_1^2 - x_3^2)(x_2^2 - x_4^2) \\ &\quad - (a_1 - a_3)(a_2 - a_4)(x_1^2 - x_2^2)(x_3^2 - x_4^2). \end{aligned}$$

For $n = 6$, the expansion in the Specht basis is

$$\begin{aligned} \prod_{i < j} (x_i + x_j) \mathfrak{Pfaff} \left(\frac{a_i - a_j}{x_i + x_j} \right) &= (a_1 - a_2)(a_3 - a_4)(a_5 - a_6) \times \\ &\Delta(x_1^2, x_3^2, x_5^2) \Delta(x_2^2, x_4^2, x_6^2) (1 - s_2^{ax} - s_4^{ax} + s_2^{ax} s_4^{ax} - s_2^{ax} s_4^{ax} s_3^{ax}) \\ &\quad - (a_1 - a_2)(a_3 - a_4)(a_5 - a_6) \Delta(x_1^2, x_2^2, x_3^2) \Delta(x_4^2, x_5^2, x_6^2) \\ &= \begin{vmatrix} a_1 & a_1 x_1^2 & a_1 x_1^4 & 1 & x_1^2 & x_1^4 \\ a_2 & a_2 x_2^2 & a_2 x_2^4 & 1 & x_2^2 & x_2^4 \\ a_3 & a_3 x_3^2 & a_3 x_3^4 & 1 & x_3^2 & x_3^4 \\ a_4 & a_4 x_4^2 & a_4 x_4^4 & 1 & x_4^2 & x_4^4 \\ a_5 & a_5 x_5^2 & a_5 x_5^4 & 1 & x_5^2 & x_5^4 \\ a_6 & a_6 x_6^2 & a_6 x_6^4 & 1 & x_6^2 & x_6^4 \end{vmatrix}. \end{aligned}$$

The first five terms, written as images of the first one, are the Specht polynomials for pairs of tableaux transposed of each other, but there is a sixth term corresponding to the only non-zero entry outside the diagonal in the matrix of scalar products.

In the case where the a_i 's are fixed powers of the indeterminates x_i 's, then the determinant $U(a, x^2)$ is a determinant of powers of x_i 's, proportional to a Schur function. Thus, Th.8 implies

Corollary 9 *Let r, k be positive integers, $n = 2m$ be an even integer. Let $q = 2(k-1)$, and λ be the (increasing) partition $[0, r, q, q+r, \dots, (m-2)q, (m-2)q+r, (m-1)q, m-1)q+r]$ Then*

$$\mathfrak{Pfaff} \left(\frac{x_i^{r+1} - x_j^{r+1}}{x_i^k + x_j^k} \right) = \frac{\Delta^x(1, \dots, n)}{\prod_{1 \leq i < j \leq n} x_i^k + x_j^k} S_\lambda(x_1, \dots, x_n).$$

We can give more interesting examples of Pfaffians $\mathfrak{Pfaff}(z[i, j]g[i, j])$, with $g[i, j]$ a symmetric function in x_i, x_j , which admits a symmetric function in x_1, \dots, x_n as a factor.

For example, take a partition λ , an alphabet B , and variables x_1, \dots, x_n . Chosing a positive k , we want to evaluate

$$\mathfrak{Pfaff} \left((a_i - a_j) S_\lambda(B + x_i + x_j) \frac{x_i - x_j}{x_i^k - x_j^k} \right).$$

According to Th.6, we need only compute the specialization $a_1 = 1, \dots, a_m = 1, a_{m+1} = 0, \dots, a_n = 0$, which is equal to

$$\det \left(S_\lambda(B + x_i + x_j) \frac{x_i - x_j}{x_i^k - x_j^k} \right)_{1 \leq i < m < j \leq n}.$$

To proceed further, one needs to evaluate such determinants. We shall do that in the next section. For the moment, let us only use the fact that the determinant in question is *the product of a symmetric function* $f(x_1, \dots, x_n)$ by $\Delta^x(1 \dots m)\Delta^x(m+1 \dots n) \prod_{1 \leq i \leq m < j \leq n} (x_i - x_j)(x_i^k - x_j^k)^{-1}$.

The symmetric function can then be factored out, so that the Pfaffian $\mathfrak{P}\text{faff}((a_i - a_j)S_\lambda(B + x_i + x_j)(x_i - x_j)(x_i^k - x_j^k)^{-1})$ is finally equal to

$$f(x_1, \dots, x_n) \mathfrak{P}\text{faff} \left((a_i - a_j) \frac{x_i - x_j}{x_i^k - x_j^k} \right). \quad (18)$$

Thus, the evaluation of the Pfaffian of order $2m$ has been reduced to the evaluation of a determinant of order m .

Ishikawa [4], Okada [18], and Ishikawa-Okada-Tagawa-Zeng [5] have given many generalizations of Sundquist's Pfaffian. Instead of using Plücker coordinates, they use specific determinants (which, of course, satisfy built-in Plücker relations).

7 Determinants and two symmetric groups

The fundamental surveys of Krattenthaler [8, 9] describe many methods to evaluate determinants. We would like to add to them one more method, using two symmetric groups.

In the course of proving Th.7, we have met a determinant which happened to possess an unsuspected global symmetry, and that we record now (notice that ∇^{az} is given by a summation on the full symmetric group \mathfrak{S}_n).

Corollary 10 *Let $\{a[i, j]\}$, $\{z[i, j]\}$, $1 \leq i < j \leq n = 2m$ be two families of Plücker indeterminates. Then*

$$\det \left(\frac{a[i, j]}{z[i, j]} \right)_{1=1 \dots m, j=m+1 \dots n} = d(\aleph) \mathfrak{Y}^a(\aleph) \mathfrak{Y}^z(\zeta) \nabla^{az} \prod_{1 \leq i < j \leq n} z[i, j]^{-1}, \quad (19)$$

where \aleph is the bottom tableau of shape $[m, m]$, ζ , the top tableau of shape 2^m .

The special case where $z[i, j] = x_i - x_j$, or $z[i, j] = x_i^k - x_j^k$ is worth commenting, since it reveals a symmetry in x_1, \dots, x_n that we emphasize in the next theorem theorem.

To evaluate $\det(a[i, j](x_i - x_j)^{-1})_{1=1 \dots m, j=m+1 \dots n}$, we already used that we need only take $a[i, j] = a_i - a_j$, and specialize $[a_1, \dots, a_n]$ in all permutations of $[1^m, 0^m]$. However, writing

$$R^x(1 \dots m | m+1 \dots n) := \prod_{1 \leq i \leq m} \prod_{m+1 \leq j \leq n} (x_i - x_j),$$

it is clear that

$$R^x(1 \dots m \mid m+1 \dots n) \left| \frac{a_i - a_j}{x_i - x_j} \right|_{1 \leq i \leq m < j \leq 2m}$$

has the same specializations as $U(a, x)$.

Thanks to Th.6 and Th.8, going back to the variables $z[i, j]$, taking variables x_i^2 instead of x_i , we have just obtained :

Theorem 11 *Let $z[i, j]$, $1 \leq i < j \leq n = 2m$ be Plücker indeterminates, and let x_1, \dots, x_n be indeterminates. Then*

$$\begin{aligned} \det \left| \frac{z[i, j]}{x_i^2 - x_j^2} \right|_{1 \leq i \leq m < j \leq n} &= \frac{1}{R^{xx}(1 \dots m \mid m+1 \dots n)} \\ &\quad \times \sum_{t \in \mathfrak{Tab}([m, m])} (-1)^{\ell(\zeta, t)} \mathfrak{Y}^z(t) \mathfrak{Y}^{xx}(t^\sim) \quad (20) \\ &= \mathfrak{Y}^z(\aleph) \mathfrak{Y}^{xx}(\aleph^\sim) \nabla^{xz} \frac{d(\aleph)}{R^{xx}(1 \dots m \mid m+1 \dots n)} \quad (21) \\ &= \frac{\prod_{1 \leq i < j \leq n} x_i + x_j}{R^{xx}(1 \dots m \mid m+1 \dots n)} \mathfrak{Pfaff} \left(\frac{z[i, j]}{x_i + x_j} \right) \quad (22) \end{aligned}$$

the Pfaffian being of order n , and the superscript xx meaning using the indeterminates x_i^2 instead of x_i .

In particular, $R^x(1 \dots m \mid m+1 \dots n) \left| \frac{z[i, j]}{x_i - x_j} \right|$ is symmetrical in x_1, \dots, x_n .

Taking $z[i, j] = S_\lambda(B + x_i + x_j)(x_i - x_j)$ and changing the powers of the variables x_i in denominator, we get the following corollary.

Corollary 12 *Let λ be a partition, B be an alphabet, m, k be two positive integers. Then*

$$\frac{1}{\Delta^x(1 \dots m) \Delta^x(m+1 \dots 2m)} \left(\prod_{1 \leq i \leq m < j \leq 2m} \frac{x_i^k - x_j^k}{x_i - x_j} \right) \left| \frac{S_\lambda(B + x_i + x_j)}{S_{k-1}(x_i + x_j)} \right|_{1 \leq i \leq m < j \leq 2m}$$

is a function symmetrical in x_1, \dots, x_{2m} .

Notice that $\left| p_2(B + x_i + x_j) \right|_{1 \leq i \leq 2 < j \leq 4}$, where p_2 is the second power sum, does not furnish a symmetric function in x_1, \dots, x_4 (this does not contradict the corollary, because, fortunately, p_2 is not a Schur function).

For $k = 1$, and $\lambda = r^p$, a rectangular partition, one gets an identity which is useful in the theory of orthogonal polynomials [12, Prop. 8.4.3] :

$$\begin{aligned} & \frac{1}{\Delta^x(1 \dots m)\Delta^x(m+1 \dots 2m)} \left| S_{r^p}(B + x_i + x_j) \right|_{1 \leq i \leq m < j \leq n} \\ &= (S_{(r+1)^{p-1}}(B))^{m-1} S_{(r-m+1)^{p+m-1}}(B + x_1 + \dots + x_n). \end{aligned} \quad (23)$$

More precisely, given *moments* $\mu_k = (-1)^k S_{1^k}(B)$, supposed to be sufficiently generic, then $\{S_{n^n}(B + x)\}$ is a family of orthogonal polynomials, with respect to this moments [12, Ch.8]. The Christoffel-Darboux kernel of order r , $K_r(x, y)$, is proportional to $S_{r^{r+1}}(B + x + y)$, and Cor.12 states that the determinant with entries $K_r(x_i, y_j)$, $1 \leq i, j \leq n$ is a symmetric function of $x_1, \dots, x_n, y_1, \dots, y_n$ and gives its precise value. This property can be directly proved, using Bazin relation on minors [13] (see also the article of Rosengren about the relations between Pfaffians and kernels [21]).

The case $k = 2$, and $\lambda = \rho := [1, \dots, r]$, has been settled by [5, 19], and reads

$$\begin{aligned} & \left| \frac{S_\rho(B + x_i + x_j)}{x_i + x_j} \right|_{1 \leq i \leq m < j \leq n} \\ &= \frac{\Delta^x(1 \dots m)\Delta^x(m+1 \dots 2m)}{\prod_{1 \leq i \leq m < j \leq 2m} x_i + x_j} (S_\rho(B))^{m-1} S_\rho(B + x_1 + \dots + x_n). \end{aligned} \quad (24)$$

In these two cases, the symmetric function has been further factorized, compared to the case of a general partition λ . This induces a factorization of Pfaffians :

$$\begin{aligned} & \mathfrak{Pfaff}(a[i, j]S_{r^p}(B + x_i + x_j)) \\ &= (S_{(r+1)^{p-1}}(B))^{m-1} S_{(r-m+1)^{p+m-1}}(B + x_1 + \dots + x_n) \\ & \quad \mathfrak{Y}^a(\mathfrak{N})\Delta^x(1 \dots m)\Delta^x(m+1 \dots 2m)\nabla^{ax}, \end{aligned} \quad (25)$$

$$\begin{aligned} & \mathfrak{Pfaff}\left(\frac{a[i, j]}{x_i + x_j} S_\rho(B + x_i + x_j)\right) \\ &= (S_\rho(B))^{m-1} S_\rho(B + x_1 + \dots + x_n) \mathfrak{Pfaff}\left(\frac{a[i, j]}{x_i + x_j}\right). \end{aligned} \quad (26)$$

Our last example will be related to the six-vertex model in physics. Stroganov [24] found that the determinant with entries

$$\sin(x_i - y_j + \eta)^{-1} \sin(x_i - y_j - \eta)^{-1},$$

with $\eta = \exp(\pi\sqrt{-1}/3)$, $i, j = 1 \dots n$, is the product of a symmetric function in $x_1, \dots, x_n, y_1, \dots, y_n$ by $\Delta^x(1, \dots, n)\Delta^y(1, \dots, n)$.

Since

$$-4 \sin(x_i - y_j + \eta) \sin(x_i - y_j - \eta) = 1 + 2 \cos(2(x-y)) = 1 + a/b + b/a,$$

with $a = \exp(2\pi x\sqrt{-1})$, $b = \exp(2\pi y\sqrt{-1})$, Stroganov's case is the evaluation of the determinant $\left| (x_i/y_j - y_j/x_i) \left((x_i/y_j)^3 - (y_j/x_i)^3 \right)^{-1} \right|$.

The following lemma gives a more general case, as a corollary of Th. 11.

Lemma 13 *Let k, r be two positive integers. Then*

$$\det \left((x_i^r - x_j^r)(x_i^k - x_j^k)^{-1} \right)_{i=1 \dots m, j=m+1 \dots 2m}$$

is equal to the product of the Schur function in x_1, \dots, x_{2m} of index

$$[0, \gamma, \beta, \beta+\gamma, 2\beta, 2\beta+\gamma, \dots, (n-1)\beta, (n-1)\beta+\gamma]$$

times $\Delta^x(1 \dots m)\Delta^x(m+1 \dots 2m)$, where $\gamma = r - 1$, $\beta = k - 2$.

For example, for $m = 3$, $r = 2$, $k = 5$, then the determinant is equal to

$$\Delta^x(1, 2, 3)\Delta^x(4, 5, 6) s_{[0,1,3,4,6,7]}(x_1, \dots, x_6).$$

8 Note: Young's basis

Young first defined *natural idempotents*, giving rise to what we have called the *Specht basis*. He then obtained orthogonal idempotents by an orthogonalization process which was later clarified by Thrall (see Rutherford [22]).

The easiest way of obtaining Young's orthogonal idempotents is to characterize them as simultaneous eigenvectors for the *Jucys-Murphy elements*

$$\xi_j := \sum_{i < j} (i, j), \quad j = 0, \dots, n,$$

where the sum is over transpositions (cf. Okounkov-Vershik [20]).

However, this approach does not provide the relations between the different idempotents for the same shape, and is inappropriate for our decomposition of Pfaffians.

We need to reinterpret Young's orthogonalization in terms of the *Yang-Baxter relations* :

$$\left(s_i + \frac{1}{\alpha}\right) \left(s_{i+1} + \frac{1}{\alpha+\beta}\right) \left(s_i + \frac{1}{\beta}\right) = \left(s_{i+1} + \frac{1}{\beta}\right) \left(s_i + \frac{1}{\alpha+\beta}\right) \left(s_{i+1} + \frac{1}{\alpha}\right) \quad (27)$$

$$\left(s_i + \frac{1}{\alpha}\right) \left(s_j + \frac{1}{\beta}\right) = \left(s_j + \frac{1}{\beta}\right) \left(s_i + \frac{1}{\alpha}\right), \quad |i - j| \neq 1 \quad (28)$$

The graphical representation of these relations is easy to remember (taking $i = 1$) :

$$\begin{array}{ccc} & [123] & \\ s_1 + \frac{1}{\alpha} \begin{array}{l} // \\ \backslash \end{array} & s_2 + \frac{1}{\beta} & \\ & [213] & [132] & s_1 + \frac{1}{\alpha} \begin{array}{l} // \\ \backslash \end{array} & [1234] & s_3 + \frac{1}{\beta} \\ s_2 + \frac{1}{\alpha+\beta} \begin{array}{l} | \\ | \\ | \end{array} & \begin{array}{l} || \\ || \\ || \end{array} s_1 + \frac{1}{\alpha+\beta} & [2134] & [1243] \\ & [231] & [312] & s_3 + \frac{1}{\beta} \begin{array}{l} \backslash \\ // \end{array} & // & s_1 + \frac{1}{\alpha} \\ s_1 + \frac{1}{\beta} \begin{array}{l} \backslash \\ // \end{array} & / & s_2 + \frac{1}{\alpha} & [2143] \\ & [321] & & & & \end{array}$$

The standard Young tableaux of a given shape are the vertices of a graph obtained by generating them with simple transpositions, starting from the top ζ . We keep the same directed graph, but label an edge $t \rightarrow ts_i$ by $s_i + 1/\rho$:

$$t \xrightarrow{s_i + 1/\rho} ts_i,$$

where ρ is the diagonal distance (the difference of *contents*) between the letters i and $i+1$ in t .

A path in such a graph is interpreted as the product, in the group algebra, of the edges composing it, and the Yang-Baxter relations insure that two paths having the same end points evaluate to the same element in the group algebra.

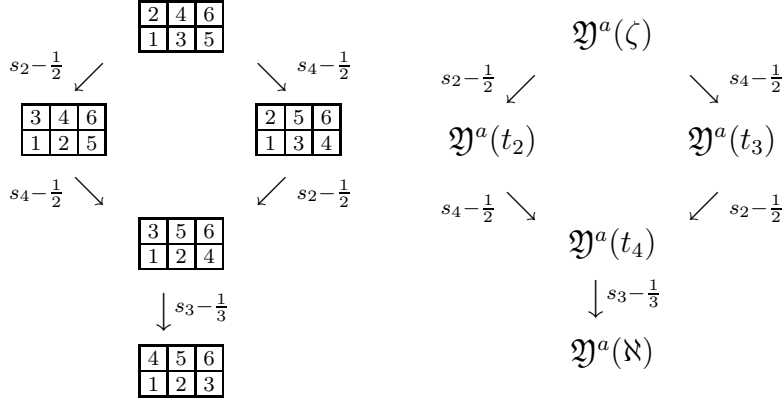
We replace now ζ by the Specht polynomial Δ_ζ^a , and define, for any other standard tableau of the same shape, the *Young polynomial* $\mathfrak{Y}^a(t)$ by :

$$\mathfrak{Y}^a(t) = \Delta_\zeta^x \left(s_i + \frac{1}{\rho}\right) \cdots \left(s_j + \frac{1}{\rho'}\right),$$

whenever

$$\zeta \xrightarrow{s_i + 1/\rho} \cdots \xrightarrow{s_j + 1/\rho'} t$$

is a path from ζ to t .



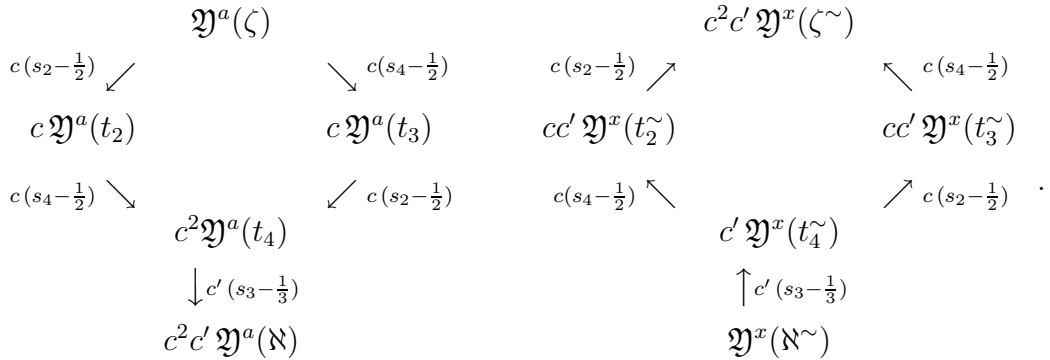
The graph on the right side describes the generation of Young's basis, starting with $\mathfrak{Y} \left(\begin{smallmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{smallmatrix} \right)$, and applying those $s_i + 1/\rho$ which label the edges.

Our graph is directed, but since $-\rho$ is the distance between $i, i+1$ in ts_i , if ρ is the distance in t , and since

$$(s_i + 1/\rho)(s_i - 1/\rho) = 1 - 1/\rho^2, \rho \neq 1,$$

one could use a double orientation by normalizing the edges, taking $(s_i + 1/\rho)/\sqrt{1 - 1/\rho^2}$ instead of $(s_i + 1/\rho)$.

Pfaffians are obtained by taking a space $V_{[m,m]}^a \otimes V_{[2m]}^x$, and using a pair of orthonormal bases that we write below in terms of the two Young bases. The graph on the left describes the orthonormal basis for shape $[3, 3]$, generated downwards, and the graph on the right, the basis for shape $[2, 2, 2]$, generated upwards. The Pfaffian is obtained by taking the sum of products of the corresponding vertices of the two graphs :



The normalization constants $\frac{1}{\sqrt{1 - 1/\rho^2}}$ are $c = \frac{1}{\sqrt{1 - 1/4}}$ and $c' = \frac{1}{\sqrt{1 - 1/9}}$, because the diagonal distances involved are ± 2 and ± 3 .

In conclusion, the sum $\sum_{t \in \mathfrak{Tab}(\{3,3\})} \pm \mathfrak{Y}^a(t) \mathfrak{Y}^x(t^\sim)$ is equal, up to the global factor $c^2 c'$, to the same sum when using the orthonormal basis rather than the Young polynomials (which are, as we chose to define them, only an orthogonal basis).

Notice that the sum of products of Young polynomials can be written

$$\begin{aligned} \mathfrak{Y}^a(\aleph) \mathfrak{Y}^x(\aleph^\sim) & \left(1 - (s_3^a + \tfrac{1}{3})(s_3^x - \tfrac{1}{3}) \frac{1}{1 - \frac{1}{9}} \times \right. \\ & \left. \left(1 - (s_2^a + \tfrac{1}{2})(s_2^x - \tfrac{1}{2}) \frac{1}{1 - \frac{1}{4}} \right) \left(1 - (s_4^a + \tfrac{1}{2})(s_4^x - \tfrac{1}{2}) \frac{1}{1 - \frac{1}{4}} \right) \right) \\ & = \mathfrak{Y}^a \left(\begin{smallmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{smallmatrix} \right) \mathfrak{Y}^x \left(\begin{smallmatrix} 3 & 6 \\ 2 & 5 \\ 1 & 4 \end{smallmatrix} \right) - \mathfrak{Y}^a \left(\begin{smallmatrix} 3 & 5 & 6 \\ 1 & 2 & 4 \end{smallmatrix} \right) \mathfrak{Y}^x \left(\begin{smallmatrix} 4 & 6 \\ 2 & 5 \\ 1 & 3 \end{smallmatrix} \right) \times \\ & \left(1 - (s_2^a + \tfrac{1}{2})(s_2^x - \tfrac{1}{2}) \frac{1}{1 - \frac{1}{4}} \right) \left(1 - (s_4^a + \tfrac{1}{2})(s_4^x - \tfrac{1}{2}) \frac{1}{1 - \frac{1}{4}} \right) \\ & = \dots = \mathfrak{Y}^a \left(\begin{smallmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{smallmatrix} \right) \mathfrak{Y}^x \left(\begin{smallmatrix} 3 & 6 \\ 2 & 5 \\ 1 & 4 \end{smallmatrix} \right) - \mathfrak{Y}^a \left(\begin{smallmatrix} 3 & 5 & 6 \\ 1 & 2 & 4 \end{smallmatrix} \right) \mathfrak{Y}^x \left(\begin{smallmatrix} 4 & 6 \\ 2 & 5 \\ 1 & 3 \end{smallmatrix} \right) \\ & + \mathfrak{Y}^a \left(\begin{smallmatrix} 3 & 4 & 6 \\ 1 & 2 & 5 \end{smallmatrix} \right) \mathfrak{Y}^x \left(\begin{smallmatrix} 5 & 6 \\ 2 & 4 \\ 1 & 3 \end{smallmatrix} \right) + \mathfrak{Y}^a \left(\begin{smallmatrix} 2 & 5 & 6 \\ 1 & 3 & 4 \end{smallmatrix} \right) \mathfrak{Y}^x \left(\begin{smallmatrix} 4 & 6 \\ 3 & 5 \\ 1 & 2 \end{smallmatrix} \right) - \mathfrak{Y}^a \left(\begin{smallmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{smallmatrix} \right) \mathfrak{Y}^x \left(\begin{smallmatrix} 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{smallmatrix} \right) \end{aligned}$$

In the preceding sections, we did not have recourse to normalization constants, but used the Young basis and checked the overall factor by computing a specialization of the Pfaffian.

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