On the motion and collisions of rigid bodies in an ideal fluid

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Abstract

In this paper we study a coupled system of partial differential equations and ordinary differential equations. This system is a model for the 3d interactive free motion of rigid bodies immersed in an ideal fluid. Applying the least action principle of Lagrangian mechanics, we prove that the solids degrees of freedom, solve a second order system of nonlinear ordinary differential equations. Under suitable smoothness assumptions on the solids and on the fluid's domain boundaries, we prove the existence and C^∞ regularity, up to a collision between solids or between a solid with the boundary of the fluid domain, of solids motion. The case of an infinite cylinder surrounded by a fluid occupying an half space, we prove that collisions with non zero relative velocity can occur.

Keywords and Phrases: Fluid-solid interaction, ideal fluid, Lagrangian mechanics, collisions, PDE-ODE coupled system.

AMS Subject Classification: 35Q35, 35R35, 34A34, 34A12, 76B99.

1 Introduction and main results

1.1 Modelling

This paper deals with the free motion of rigid bodies in an incompressible, inviscid and irrotational fluid flow in $\mathbb{R}^3$. We denote by $S(t) = \cup_{i=1}^n S^i(t)$ the domain occupied by a collection of n solids at the instant $t \geq 0$ and by $\Omega(t)$ the domain of the surrounding fluid. We set $\Omega(0) = \Omega$ and $S^i(0) = S^i$. The boundary of $\Omega(t)$ can be decomposed into $\Gamma_1 = \partial\Omega(t) \setminus \partial S(t)$ and $\Gamma_2(t) = \cup_{i=1}^n \Gamma_2^i(t)$ with $\Gamma_2^i(t) = \partial S^i(t)$ (see figure 1). We will assume in all the paper that Γ_1 is either bounded, either an hyperplan. The center of mass of the $i-$th solid occupies the position $\mathbf{h}^i(t)$ and we denote $\mathbf{h}^i(0) = \mathbf{h}_0^i$. By the definition of a rigid motion, for any $i = 1, \ldots, n$ and for any $t > 0$, there exists an orthogonal matrix $[R^i(t)] \in SO(3)$ (the rotation group) such that the position $\mathbf{x}^i(t)$ of a point occupying the position $\mathbf{x}_0^i \in S^i$ at instant $t = 0$ is:

$$\mathbf{x}^i(t) = [R^i(t)](\mathbf{x}_0^i - \mathbf{h}_0^i) + \mathbf{h}^i(t). \tag{1.1}$$

Thus, at any time, the position of the $i-$th solid is described by a couple $\mathbf{P}^i = ([R^i], \mathbf{h}^i) \in SO(3) \times \mathbb{R}^3$ and we denote $\mathbf{P} = (\mathbf{P}^1, \ldots, \mathbf{P}^n) \in (SO(3) \times \mathbb{R}^3)^n$. The $SO(3)$ group being an indefinitely differentiable $3-$dimensional sub-manifold of

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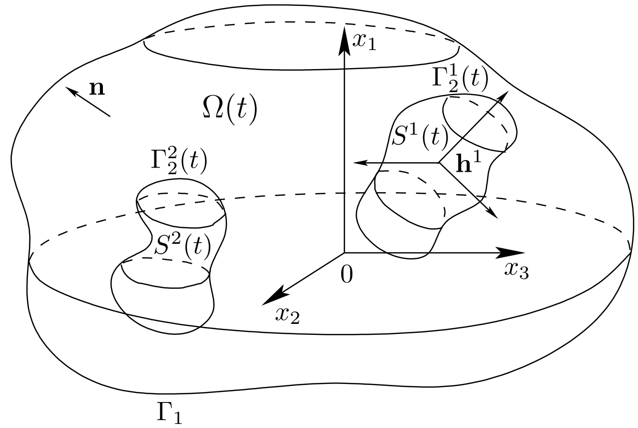

Figure 1: The rigid bodies $S^i(t)$ and the fluid domain $\Omega(t)$.

$\mathcal{M}(3)$ (the vector space of the 3×3 matrices), $(SO(3)\times\mathbb{R}^3)^n$ is a $6n-$dimensional sub-manifold of $(\mathcal{M}(3)\times\mathbb{R}^3)^n$. We denote $\mathcal{P}$ the open subset of $(SO(3)\times\mathbb{R}^3)^n$ of all the physicaly admissible positions of the solids. For any given $\widetilde{\mathbf{P}}\in\mathcal{P}$, there exists an open subset $\mathcal{Q}_{\widetilde{\mathbf{P}}}$ of $(\mathbb{R}^6)^n$ and an indefinitely differentiable diffeomorphism $\Lambda_{\widetilde{\mathbf{P}}}$ from $\mathcal{Q}_{\widetilde{\mathbf{P}}}$ onto a neighborhood of $\widetilde{\mathbf{P}}$ (i.e. $(\mathcal{Q}_{\widetilde{\mathbf{P}}},\Lambda_{\widetilde{\mathbf{P}}})$ is a local chart of $\mathcal{P}$). We call the elements $\mathbf{Q}$ of $\mathcal{Q}_{\widetilde{\mathbf{P}}}$ the local coordinates and we denote generically q_i $(1\leq i\leq 6n)$ the coordinates of $\mathbf{Q}$. The Lagrangian $\mathcal{L}(\dot{\mathbf{Q}},\mathbf{Q})$ of the fluid-solids system solves, according to the least action principle, the system of ordinary differential equations :

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{Q}}}(\dot{\mathbf{Q}},\mathbf{Q})-\frac{\partial\mathcal{L}}{\partial\mathbf{Q}}(\dot{\mathbf{Q}},\mathbf{Q})=\mathbf{0}_{6n}. \tag{1.2}$$

We shall prove that there exists a $6n\times 6n$ symmetric matrix $[K(\mathbf{Q})]$, called the virtual mass matrix (or the kinetic energy metric) such that $\mathcal{L}(\dot{\mathbf{Q}},\mathbf{Q})=(1/2)\dot{\mathbf{Q}}\cdot[K(\mathbf{Q})]\dot{\mathbf{Q}}$. Assuming this matrix to be differentiable with respect to $\mathbf{Q}$ and after straightforward computations, we deduce that the Euler-Lagrange system of ODE's (1.2) reads:

$$[K(\mathbf{Q})]\ddot{\mathbf{Q}}+\langle[\![T(\mathbf{Q})]\!],\dot{\mathbf{Q}},\dot{\mathbf{Q}}\rangle=\mathbf{0}_{6n}, \tag{1.3}$$

where $[\![T(\mathbf{Q})]\!]$ is the $6n\times 6n\times 6n$ three rank tensor defined by:

$$T_{ij_1j_2}(\mathbf{Q})=\frac{1}{2}\left(\frac{\partial K_{j_1i}}{\partial q_{j_2}}(\mathbf{Q})+\frac{\partial K_{j_2i}}{\partial q_{j_1}}(\mathbf{Q})-\frac{\partial K_{j_1j_2}}{\partial q_i}(\mathbf{Q})\right),\quad 1\leq i,j_1,j_2\leq 6n. \tag{1.4}$$

Various fluid-solids interaction models were studied during the last years. We refer for example to [15] for a precise bibliography on this topic. The first works on solids dynamics in a frictionless fluid go back to Thomson, Tait and Kirchhoff. They were also the firsts who introduced the generalized coordinates and the Euler-Lagrange equations to study this kind of problems. For the particular case of one rigid body and the system fluid-rigid filling the whole space, the analysis of the model is well understood (see Lamb, in [14]). Indeed, in this case, the derivatives with respect to $\mathbf{Q}$ and $\dot{\mathbf{Q}}$ in (1.2) can be explicitly computed and we obtain a system of ODE's satisfying the assumptions of the Cauchy-Lipschitz Theorem. The situation gets more complicated in the case of several rigid bodies or in the presence of boundaries (which is the case, for instance, if the system fluid-rigids fills a bounded domain of $\mathbb{R}^3$). Our first contribution lies in showing that, also in this case, $\mathcal{L}$ is twice differentiable with respect to $\mathbf{Q}$ and $\dot{\mathbf{Q}}$, so that we still can apply the Cauchy-Lipschitz Theorem. The proof of this fact is based on shape sensitivity analysis technics. Our second contribution is that we show that shocks (i.e. contacts with non zero velocity) can occur within this model. This situation heavily contrasts to that encountered for viscous fluids (see [16] [17], [13], [12]).

1.2 Statement of the mains results

The first Theorem concerns the existence, uniqueness and regularity of local solutions for the Euler-Lagrange system of ODE's (1.2):

Theorem 1.1 *Assume that $\partial\Omega$ is Lipschitz continuous. Then, for any time $\tilde{t} \geq 0$, any admissible position of the solids $\widetilde{\mathbf{P}} \in \mathcal{P}$ and any initial data $(\widetilde{\mathbf{Q}}, \dot{\widetilde{\mathbf{Q}}}) \in \mathcal{Q}_{\widetilde{\mathbf{P}}} \times (\mathbb{R}^6)^n$, there exists an interval $(\tilde{t}_1, \tilde{t}_2) \subset \mathbb{R}$ containing $\tilde{t}$ and a unique solution $t \in (\tilde{t}_1, \tilde{t}_2) \mapsto \mathbf{Q}^*(t, \tilde{t}, \widetilde{\mathbf{Q}}, \dot{\widetilde{\mathbf{Q}}}) \in \mathcal{Q}_{\widetilde{\mathbf{P}}}$ to the ODE (1.2) satisfying $\mathbf{Q}^*(\tilde{t}, \tilde{t}, \widetilde{\mathbf{Q}}, \dot{\widetilde{\mathbf{Q}}}) = \widetilde{\mathbf{Q}}$ and $\dot{\mathbf{Q}}^*(\tilde{t}, \tilde{t}, \widetilde{\mathbf{Q}}, \dot{\widetilde{\mathbf{Q}}}) = \dot{\widetilde{\mathbf{Q}}}$. Moreover, the function $\mathbf{Q}^*(\cdot, \tilde{t}, \widetilde{\mathbf{Q}}, \dot{\widetilde{\mathbf{Q}}})$ is indefinitely differentiable.*

Denoting $TSO(3)$ the tangent bundle to $SO(3)$, $[I_3]$ the identity 3×3 matrix and combining the Theorem above with an a priori estimate, it allows us to prove:

Corollary 1.1 *Assume that $\partial\Omega$ is Lipschitz continuous. Then, for any initial data $(([I_3], [\dot{R}_0^1]), \ldots, ([I_3], [\dot{R}_0^n])) \in (TSO(3))^n$ and $((\mathbf{h}_0^1, \dot{\mathbf{h}}_0^1), \ldots, (\mathbf{h}_0^n, \dot{\mathbf{h}}_0^n)) \in (\mathbb{R}^3 \times \mathbb{R}^3)^n$ (initial positions and initial velocities of the solids), there exists a times $T^* > 0$ and a unique indefinitely differentiable function:*

$$\mathbf{P}^* : [0, T^*) \to \mathcal{P}$$
$$t \mapsto (([R^1(t)], \mathbf{h}^1(t)), \ldots, ([R^n(t)], \mathbf{h}^n(t))),$$

for which $\mathbf{P}^*(0) = (([I_3], \mathbf{h}_0^1), \ldots, ([I_3], \mathbf{h}_0^n))$ and $\dot{\mathbf{P}}^*(0) = (([\dot{R}_0^1], \dot{\mathbf{h}}_0^1), \ldots, ([\dot{R}_0^n], \dot{\mathbf{h}}_0^n))$ and such that the trajectory of a point $\mathbf{x}^i(t)$, occupying the position $\mathbf{x}_0^i \in S^i$ at the instant $t = 0$ be given by (1.1).

The time T^* corresponds to the time of the first collision between two solids or between a solid with the fluid domain boundary. If there is no collision, then $T^* = \infty$.

A particular case of interest concerns the vertical fall of an infinite cylinder of radius 1 in an half space. For symmetry reasons, the problem reduces to be planar and the solid to be a disk. The Lagrangian $\mathcal{L}(\dot{h}, h)$ depends only on the vertical position h of the center of the disk and on its velocity $\dot{h}$. The virtual mass matrix is a scalar $k(h)$ and the Lagrangian reads $\mathcal{L}(\dot{h}, h) = (1/2)|\dot{h}|^2 k(h)$. Then the system (1.2) reduces to the ode :

$$\ddot{h} k(h) + \frac{1}{2} |\dot{h}|^2 k'(h) = 0,$$

which leads, after integrating, to $\dot{h}/\dot{h}_0 = \sqrt{k(h_0)/k(h)}$, where h_0 and $\dot{h}_0$ are the given initial position and velocity of the disk. We shall prove that $k(h)$ tends to a positive constant when $h \to 1$ and hence that

Theorem 1.2 *If we specify the solid to be an infinite cylinder and the domain of the fluid to be an half space, then there exist initial data such that the cylinder collides with the boundary in finite time with non zero relative velocity.*

Going back to the general case considered in the Theorems 1.1 and in the Corollary 1.1 and assuming additional regularity on $\partial\Omega$, we shall give an explicit expression for the tensor $[\![T(\mathbf{Q})]\!]$ involved in the equation (1.3). This is done in the Theorem 6.1 latter in.

1.3 Outline of the paper

This paper is organized as follows : the following section is devoted to the notations and the computation of the virtual mass matrix. In section 3 we deal with a generic problem of shape sensitivity analysis. The results are used in section 4 for the proofs of Theorem 1.1, Corollary 1.1 and in section 6 for Theorem 6.1. Then, in section 5, specifying the solid to be an infinite cylinder and the domain occupied by the fluid to be an half space, we shall prove that finite time collisions happen. In the same section, we will also study the behavior of the solid just before the collision time. In the last section, going back to the general case, we give the explicit form of the system of ODE's (1.2). The appendix contains the proofs of technical Propositions.

2 Notation and preliminaries

2.1 Notation

Throughout this article, we shall use bold print notations for vectors like $\mathbf{x}$, $\mathbf{h}$, $\mathbf{Q}$ whereas we keep the usual characters for their coordinates x_j, h_j, q_j and in a general way for any real valued functions like ϕ, φ. The canonical basis of the physical space $\mathbb{R}^3$ is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and the canonical basis of the $6n$ dimensional space of the local coordinates $\mathbf{Q}$ is $\{\mathbf{e}_1^1, \dots, \mathbf{e}_6^1, \mathbf{e}_1^2, \dots, \mathbf{e}_6^2, \dots, \mathbf{e}_1^n, \dots, \mathbf{e}_6^n\}$.

A matrix is denoted in square brackets $[M]$, its entries are $M_{j_1 j_2}$, $[M]^\top$ is the transposed matrix and $[I_d]$ is the identity matrix of $\mathbb{R}^d$. The Jacobian matrix of a differentiable function $\boldsymbol{\varphi}$ is $[D\boldsymbol{\varphi}]$.

We use double square brackets for three-rank tensors like $[\![T]\!]$. Its entries are $T_{j_1 j_2 j_3}$. We use the convention $([\![T]\!]\mathbf{Q})_{j_1 j_2} = \sum_{j_3} T_{j_1 j_2 j_3} q_{j_3}$ and $(\langle [\![T]\!], \mathbf{P}, \mathbf{Q}\rangle)_{j_1} = \sum_{j_2} \sum_{j_3} T_{j_1 j_2 j_3} p_{j_2} q_{j_3}$, where q_i and p_i are the coordinates of $\mathbf{Q}$ and $\mathbf{P}$.

In section 5, we will identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and any vector $\mathbf{x} = (x_1, x_2)^\top$ with the complex number $x_1 + ix_2$ where $i^2 = -1$. Due to this identification and according to the rules of notations above, we shall use bold print notations for complex numbers $\mathbf{z} = x_1 + ix_2$. In the same way, we will make no difference between a complex valued function $\mathbf{f}(x_1+ix_2) = f_1(x_1+ix_2)+if_2(x_1+ix_2)$ and the vector valued one $\mathbf{f}(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))^\top$.

2.2 Admissible positions of the solids, global and local coordinates

For any $\mathbf{P}^i = ([R^i], \mathbf{h}^i) \in SO(3)\times\mathbb{R}^3$, we denote $S^i(\mathbf{P}^i) = [R^i]S^i + \mathbf{h}^i$ and $\Gamma_2^i(\mathbf{P}^i) = \partial S^i(\mathbf{P}^i)$. We call $\mathbf{P} = (\mathbf{P}^1, \dots, \mathbf{P}^n) \in (SO(3)\times\mathbb{R}^3)^n$ the global coordinates of the solids and we introduce

$$\mathcal{P} = \{\mathbf{P} \in (SO(3)\times\mathbb{R}^3)^n \,|\, S^i(\mathbf{P}^i)\cap S^j(\mathbf{P}^j) = S^i(\mathbf{P}^i)\cap\Gamma_1 = \emptyset, \,\forall\, 1 \le i, j \le n,\, i \ne j\},$$

the set of the admissible positions of the solids. In order to build a local coordinate chart of $\mathcal{P}$, let us introduce the skew-symmetric tensors:

$$[A_1] = \mathbf{e}_3 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_3, \quad [A_2] = \mathbf{e}_1 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_1, \quad [A_3] = \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2.$$

For any given rotation matrix $[\widetilde{R}] \in SO(3)$, we set the mapping:

$$\mathcal{R}_{[\widetilde{R}]} : \mathbb{R}^3 \to SO(3), \tag{2.1a}$$

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^\top \mapsto \mathcal{R}_{[\widetilde{R}]}(\boldsymbol{\theta}) = e^{\theta_1 [A_1]} e^{\theta_2 [A_2]} e^{\theta_3 [A_3]} [\widetilde{R}]. \tag{2.1b}$$

Denoting $\mathcal{U} =]-\pi, \pi[\times]-\pi/2, \pi/2[\times]-\pi, \pi[$, the couple $(\mathcal{U}, \mathcal{R}_{[\widetilde{R}]})$ is a local C^∞ chart of $SO(3)$ and $\mathcal{R}_{[\widetilde{R}]}$ is an indefinitely differentiable diffeomorphism from $\mathcal{U}$ onto a neighborhood of $[\widetilde{R}] \in SO(3)$. The coordinates θ_k are the so-called Euler angles (with the "xyz" (pitch-roll-yaw) convention, see [9] page 603).

With the notation above, for any given $\widetilde{\mathbf{P}} = (([\widetilde{R}^1], \widetilde{\mathbf{h}}^1), \ldots, [\widetilde{R}^n], \widetilde{\mathbf{h}}^n)) \in \mathcal{P}$ and any $\mathbf{Q} = ((\boldsymbol{\theta}^1, \mathbf{h}^1), \ldots, (\boldsymbol{\theta}^n, \mathbf{h}^n)) \in (\mathbb{R}^6)^n$, we define the function

$$\Lambda_{\widetilde{\mathbf{P}}}(\mathbf{Q}) = ((\mathcal{R}_{[\widetilde{R}^1]}(\boldsymbol{\theta}^1), \mathbf{h}^1), \ldots, (\mathcal{R}_{[\widetilde{R}^n]}(\boldsymbol{\theta}^n), \mathbf{h}^n)) \in (SO(3) \times \mathbb{R}^3)^n, \tag{2.2}$$

and the set $\mathcal{Q}_{\widetilde{\mathbf{P}}} = (\mathcal{U} \times \mathbb{R}^3)^n \cap \Lambda_{\widetilde{\mathbf{P}}}^{-1}(\mathcal{P})$. One can easily checks that $(\mathcal{Q}_{\widetilde{\mathbf{P}}}, \Lambda_{\widetilde{\mathbf{P}}})$ is a local chart of $\mathcal{P}$ and $\Lambda_{\widetilde{\mathbf{P}}}$ is a C^∞ diffeormorphism from the neighborhood $\mathcal{Q}_{\widetilde{\mathbf{P}}}$ of $\widetilde{\mathbf{Q}} = ((\mathbf{0}_3, \widetilde{\mathbf{h}}^1), \ldots, (\mathbf{0}_3, \widetilde{\mathbf{h}}^n))$ onto a neighborhood of $\widetilde{\mathbf{P}}$, satisfying $\Lambda_{\widetilde{\mathbf{P}}}(\widetilde{\mathbf{Q}}) = \widetilde{\mathbf{P}}$. Therefore:

$$\mathcal{A} = \{(\mathcal{Q}_{\mathbf{P}}, \Lambda_{\mathbf{P}}), \, \mathbf{P} \in \mathcal{P}\},$$

is an atlas of $\mathcal{P}$. The tangent space to $\mathcal{P}$ at $\widetilde{\mathbf{P}}$ is denoted $T\mathcal{P}(\widetilde{\mathbf{P}})$ and $\partial \Lambda_{\widetilde{\mathbf{P}}}/\partial \mathbf{Q}(\widetilde{\mathbf{Q}})$ is an isomorphism from $(\mathbb{R}^6)^n$ onto $T\mathcal{P}(\widetilde{\mathbf{P}})$.

For any $\mathbf{Q} \in \mathcal{Q}_{\widetilde{\mathbf{P}}}$ and any $i = 1, \ldots, n$, we denote $\mathbf{Q}^i = (\boldsymbol{\theta}^i, \mathbf{h}^i)$ and $\Lambda^i_{\widetilde{\mathbf{P}}}(\mathbf{Q}^i) = (\mathcal{R}_{[\widetilde{R}^i]}(\boldsymbol{\theta}^i), \mathbf{h}^i)$, so that $\mathbf{Q} = (\mathbf{Q}^1, \ldots, \mathbf{Q}^n)$ and $\Lambda_{\widetilde{\mathbf{P}}}(\mathbf{Q}) = (\Lambda^1_{\widetilde{\mathbf{P}}}(\mathbf{Q}^1), \ldots, \Lambda^n_{\widetilde{\mathbf{P}}}(\mathbf{Q}^n))$.

To avoid to overload notations and if no confusion is possible, we shall denote merely $\mathcal{Q}$ instead of $\mathcal{Q}_{\widetilde{\mathbf{P}}}$ and $\mathcal{R}(\boldsymbol{\theta})$ instead of $\mathcal{R}_{[\widetilde{R}]}(\boldsymbol{\theta})$. In the same way, in local coordinates, $S^i(\mathbf{Q}^i)$ will stand for $S^i(\Lambda^i_{\widetilde{\mathbf{P}}}(\mathbf{Q}^i))$, $\Gamma^i_2(\mathbf{Q}^i)$ for $\Gamma^i_2(\Lambda^i_{\widetilde{\mathbf{P}}}(\mathbf{Q}^i))$, $\Gamma_2(\mathbf{Q}) = \cup_{i=1}^n \Gamma^i_2(\mathbf{Q}^i)$ and $\Omega(\mathbf{Q})$ for $\Omega(\Lambda_{\widetilde{\mathbf{P}}}(\mathbf{Q}))$.

2.3 Computation of the virtual mass matrix

By deriving (1.1) with respect to t we obtain:

$$\dot{\mathbf{x}}(t) = \sum_{k=1}^{3} \dot{\theta}^i_k(t) \frac{\partial \mathcal{R}}{\partial \theta^i_k}(\boldsymbol{\theta}^i(t))(\mathbf{x}_0 - \mathbf{h}^i_0) + \dot{\mathbf{h}}^i(t),$$

where $\boldsymbol{\theta}^i(t) = (\theta^i_1(t), \theta^i_2(t), \theta^i_3(t))^\top = \mathcal{R}^{-1}([R^i(t)])$. Defining for all $\boldsymbol{\theta} \in \mathcal{U}$ the vectors:

$$\boldsymbol{\omega}_1(\boldsymbol{\theta}) = \mathbf{e}_1, \quad \boldsymbol{\omega}_2(\boldsymbol{\theta}) = e^{\theta_1 [A_1]} \mathbf{e}_2, \quad \boldsymbol{\omega}_3(\boldsymbol{\theta}) = e^{\theta_1 [A_1]} e^{\theta_2 [A_2]} \mathbf{e}_3,$$

and the 3×3 matrix $[\omega(\boldsymbol{\theta})]$ as the horizontal concatenation of the column vectors $\boldsymbol{\omega}_k(\boldsymbol{\theta})$ for $k = 1, 2, 3$, the following formula holds:

$$\frac{\partial \mathcal{R}}{\partial \theta_k}(\boldsymbol{\theta}) \mathcal{R}(\boldsymbol{\theta})^\top \mathbf{x} = \omega_k(\boldsymbol{\theta}) \wedge \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^3, \forall \boldsymbol{\theta} \in \mathcal{U}, \forall k = 1, 2, 3. \tag{2.3}$$

The Eulerian velocity $\mathbf{v}^i(t, \mathbf{x})$ at a point $\mathbf{x}$ of $S^i(t)$ is given by:

$$\mathbf{v}^i(t, \mathbf{x}) = [\omega(\boldsymbol{\theta}^i)] \dot{\boldsymbol{\theta}}^i \wedge (\mathbf{x} - \mathbf{h}^i) + \dot{\mathbf{h}}^i. \tag{2.4}$$

To simplify notation, we set $\boldsymbol{\omega}^i_k = \boldsymbol{\omega}_k(\boldsymbol{\theta}^i)$, $k = 1, 2, 3$ and $[\omega^i] = [\omega(\boldsymbol{\theta}^i)]$.

For $i = 1, \ldots, n$, ρ^i stands for the density of the solid S^i, $m^i > 0$ is its mass and $[J^i]$ is its inertia tensor defined by:

$$[J^i(\boldsymbol{\theta}^i)] = \mathcal{R}(\boldsymbol{\theta}^i) \left[\int_{S^i} \rho^i \left(|\mathbf{x} - \mathbf{h}^i_0|^2 [\mathrm{I}_3] - (\mathbf{x} - \mathbf{h}^i_0) \otimes (\mathbf{x} - \mathbf{h}^i_0) \right) \mathrm{d}\mathbf{x} \right] \mathcal{R}(\boldsymbol{\theta}^i)^\top.$$

The kinetic energy of the solid S^i is $\mathcal{K}^i(\mathbf{Q}^i, \dot{\mathbf{Q}}^i) = (1/2) \int_{S^i(\mathbf{Q}^i)} \rho^i |\mathbf{v}^i(t, \mathbf{x})|^2 \, \mathrm{d}\mathbf{x}$. Introducing $[\mathcal{J}^i(\mathbf{Q}^i)] = [\omega^i]^\top [J^i(\boldsymbol{\theta}^i)] \mathcal{R}[\omega^i]$, this expression can be rewritten:

$$\mathcal{K}^i(\mathbf{Q}^i, \dot{\mathbf{Q}}^i) = \frac{1}{2} \left(\dot{\boldsymbol{\theta}} \cdot [\mathcal{J}^i(\mathbf{Q}^i)] \dot{\boldsymbol{\theta}}^i + m^i |\dot{\mathbf{h}}^i|^2 \right).$$

Setting then the $6n \times 6n$ bloc-diagonal matrix

$$[K_S(\mathbf{Q})] = \operatorname{diag}([\mathcal{J}^1(\mathbf{Q}^1)], m^1[I_3], [\mathcal{J}^2(\mathbf{Q}^2)], m^2[I_3], \dots, [\mathcal{J}^n(\mathbf{Q}^n)], m^n[I_3]), \quad (2.5)$$

the total kinetic energy of the n solids reads:

$$\mathcal{K}_S(\mathbf{Q}, \dot{\mathbf{Q}}) = \sum_{i=1}^n \mathcal{K}^i(\mathbf{Q}^i, \dot{\mathbf{Q}}^i) = \frac{1}{2}\dot{\mathbf{Q}} \cdot [K_S(\mathbf{Q})]\dot{\mathbf{Q}}.$$

The Eulerian velocity $\mathbf{u}(t, \cdot)$ of the fluid in $\Omega(\mathbf{Q}(t))$ is given by:

$$\mathbf{u}(t, \cdot) = \nabla\varphi(t, \cdot),$$

where $\varphi(t, \cdot)$ is the harmonic potential. It decomposes into $\varphi(t, \cdot) = \sum_{i=1}^n \varphi^i(t, \cdot)$, each function $\varphi^i(t, \cdot)$ being harmonic in $\Omega(\mathbf{Q}(t))$ and satisfying the Neumann boundaries conditions:

$$\frac{\partial\varphi^i}{\partial\mathbf{n}}(t, \cdot) = \mathbf{v}^i(t) \cdot \mathbf{n}^i(\mathbf{Q}^i(t)) \qquad \text{on } \Gamma_2^i(\mathbf{Q}^i(t)),$$
$$\frac{\partial\varphi^i}{\partial\mathbf{n}}(t, \cdot) = 0 \qquad \text{on } \Gamma_1 \cup_{j\neq i} \Gamma_2^j(\mathbf{Q}^j(t)),$$

where the velocity $\mathbf{v}^i(t)$ is given by (2.4) and the vector $\mathbf{n}^i(\mathbf{Q}^i(t))$ stands for the unit normal to $\Gamma_2^i(\mathbf{Q}^i(t))$ directed towards the exterior of the fluid. According to (2.4), $\varphi^i(t, \cdot)$ is a linear combination of the functions $\varphi_k^i(\mathbf{Q}(t), \cdot)$, $i = 1, \dots, n$, $k = 1, \dots, 6$ defined, for any $\mathbf{Q} \in \mathcal{Q}$ as the solution (in a sense that will be made precise later in) of:

$$-\Delta\varphi_k^i(\mathbf{Q}, \cdot) = 0 \qquad \text{in } \Omega(\mathbf{Q}), \quad (2.6a)$$
$$\frac{\partial\varphi_k^i}{\partial\mathbf{n}}(\mathbf{Q}, \cdot) = g_k^i(\mathbf{Q}^i) \qquad \text{on } \Gamma_2^i(\mathbf{Q}^i), \quad (2.6b)$$
$$\frac{\partial\varphi_k^i}{\partial\mathbf{n}}(\mathbf{Q}, \cdot) = 0 \qquad \text{on } \Gamma_1 \cup_{j\neq i} \Gamma_2^j(\mathbf{Q}^j), \quad (2.6c)$$

where $g_k^i(\mathbf{Q}^i, \cdot) = \big(\boldsymbol{\omega}_k^i \wedge (\mathbf{x} - \mathbf{h}^i)\big) \cdot \mathbf{n}^i(\mathbf{Q}^i, \cdot)$ for $k = 1, 2, 3$ and $g_{3+k}^i(\mathbf{Q}^i, \cdot) = \mathbf{e}_k \cdot \mathbf{n}^i(\mathbf{Q}^i, \cdot)$ for $k = 1, 2, 3$.

Thus, denoting $\boldsymbol{\varphi}^i(\mathbf{Q}, \cdot) = (\varphi_1^i(\mathbf{Q}, \cdot), \dots, \varphi_6^i(\mathbf{Q}, \cdot))^\top$, and $\boldsymbol{\varphi}(\mathbf{Q}, \cdot)$ the vertical concatenation of $\boldsymbol{\varphi}^i(\mathbf{Q}, \cdot)$, $i = 1, \dots, n$, we obtain the formula:

$$\varphi^i(t, \cdot) = \dot{\mathbf{Q}}^i(t) \cdot \boldsymbol{\varphi}^i(\mathbf{Q}(t), \cdot), \quad \varphi(t, \cdot) = \dot{\mathbf{Q}}(t) \cdot \boldsymbol{\varphi}(\mathbf{Q}(t), \cdot). \quad (2.7)$$

The kinetic energy of the fluid which is assumed to be of constant density ρ_F is $\mathcal{K}_F(\mathbf{Q}, \dot{\mathbf{Q}}) = (1/2)\rho_F \int_{\Omega(\mathbf{Q})} |\mathbf{u}(t)|^2 \mathrm{d}\mathbf{x} = (1/2)\rho_F \int_{\Omega(\mathbf{Q})} |\nabla\varphi(t, \mathbf{x})|^2 \mathrm{d}\mathbf{x}$. Taking into account (2.7), we define for all $\mathbf{Q} \in \mathcal{Q}$ the $6n \times 6n$ matrix:

$$[K_F(\mathbf{Q})] = \rho_F \int_{\Omega(\mathbf{Q})} [D\boldsymbol{\varphi}(\mathbf{Q}, \mathbf{x})][D\boldsymbol{\varphi}(\mathbf{Q}, \mathbf{x})]^\top \mathrm{d}\mathbf{x}.$$

We obtain that:

$$\mathcal{K}_F(\mathbf{Q}, \dot{\mathbf{Q}}) = \frac{1}{2}\dot{\mathbf{Q}} \cdot [K_F(\mathbf{Q})]\dot{\mathbf{Q}}.$$

Following Lamb [14], we will derive the governing equations for the solids motion by using Lagrangian mechanics. In the absence of body forces, the Lagrangian of the system is the sum of the kinetic energy of the fluid and the kinetic energy of the solids:

$$\mathcal{L}(\mathbf{Q}, \dot{\mathbf{Q}}) = \frac{1}{2}\dot{\mathbf{Q}} \cdot ([K_S(\mathbf{Q})] + [K_F(\mathbf{Q})])\dot{\mathbf{Q}}. \quad (2.8)$$

The matrix $[K(\mathbf{Q})] = [K_S(\mathbf{Q})] + [K_F(\mathbf{Q})]$ is called the virtual mass matrix of the system fluid-solids. With the above notation, the equations of the motion are given by (1.2) (see [3] page 53).

3 Shape sensitivity analysis

3.1 The context

Let Ω be an open connected subset of $\mathbb{R}^3$. We denote $\Gamma = \partial\Omega$ and we assume that

$\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, Γ_2 is bounded and Γ_1 is either bounded either an hyperplan. (H$_1$)

Following A. Henrot and M. Pierre in [11], we introduce for all $m \geq 1$ the Banach spaces $C^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3) = C^m(\mathbb{R}^3, \mathbb{R}^3) \cap W^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ endowed with the norm:

$$\|\Theta\|_m = \sum_{|\alpha| \leq m} \|D^\alpha \Theta\|_{L^\infty(\mathbb{R}^3)}, \quad \forall \Theta \in C^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3),$$

$\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. We consider also $C_0^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ the subset of the functions compactly supported in $C^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3)$. In this section, $\mathcal{U}$ stands for a neighborhood of $[I_3]$ in $C^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ such that the mapping $\Theta \in \mathcal{U} \mapsto [D\Theta]^{-1} \in L^\infty(\mathbb{R}^3, \mathcal{M}(3))$ is well defined (and indefinitely differentiable).

Let now $\mathcal{Q}$ be an open subset of $\mathbb{R}^p$ ($p \geq 1$) containing $\mathbf{0}_p$ and consider a C^α mapping ($\alpha \geq 1$):

$$\mathbf{Q} = (q_1, \ldots, q_p)^\top \in \mathcal{Q} \mapsto \phi(\mathbf{Q}, \cdot) \in C^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3), \tag{3.1}$$

such that the following hypothesis be fulfilled:

- For all $\mathbf{Q} \in \mathcal{Q}$, $\phi(\mathbf{Q}, \cdot) \in \mathcal{U}$, $\phi(\mathbf{Q}, \cdot) - [I_3] \in C_0^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ and $\phi(\mathbf{0}_p, \cdot) = [I_3]$. (H$_2$)
- For all $\mathbf{Q} \in \mathcal{Q}$, $\phi(\mathbf{Q}, \cdot)$ is the identity in a neighborhood of Γ_1 and $\phi(\mathbf{Q}, \cdot)$ is a rigid motion in a neighborhood of Γ_2. (H$_3$)

We denote $\Omega(\mathbf{Q}) = \phi(\mathbf{Q}, \Omega)$, $\Gamma(\mathbf{Q}) = \phi(\mathbf{Q}, \Gamma)$ and $\Gamma_2(\mathbf{Q}) = \phi(\mathbf{Q}, \Gamma_2)$ and we define for all $\mathbf{Q} \in \mathcal{Q}$:

$$[A(\mathbf{Q}, \cdot)] = [D\phi(\mathbf{Q}, \cdot)]^{-1} [D\phi(\mathbf{Q}, \cdot)]^{-1\top} |\det[D\phi(\mathbf{Q}, \cdot)]| \in L^\infty(\mathbb{R}^3, \mathcal{M}(3)).$$

The hypothesis (H$_2$) leads to $[A(\mathbf{0}_p, \cdot)] = [I_3]$ and $[A(\mathbf{Q}, \cdot)] - [I_3] \in C_0^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ and (H$_3$) ensures that $[A(\mathbf{Q})] = [I_3]$ in a neighborhood of Γ. Let $0 < \delta < 1$ and choose $\mathcal{Q}$ small enough such that:

$$\|[A(\mathbf{Q}, \cdot)] - [I_3]\|_{L^\infty(\mathbb{R}^3, \mathcal{M}(3))} \leq \delta, \quad \forall\, \mathbf{Q} \in \mathcal{Q}. \tag{H$_4$}$$

For any $\mathbf{Q} \in \mathcal{Q}$, we consider the solutions (in a sense that will be made precise later in) $u_1(\mathbf{Q}, \cdot)$ and $u_2(\mathbf{Q}, \cdot)$, of the following Neumann problems:

$$-\Delta u_k(\mathbf{Q}, \cdot) = 0 \qquad \text{in } \Omega(\mathbf{Q}), \tag{3.2a}$$

$$\frac{\partial u_k}{\partial \mathbf{n}}(\mathbf{Q}, \cdot) = g_k(\mathbf{Q}, \cdot) \text{ on } \Gamma(\mathbf{Q}), \quad k = 1, 2, \tag{3.2b}$$

where $g_1(\mathbf{Q}, \cdot)$ and $g_2(\mathbf{Q}, \cdot)$ are given functions on $\Gamma(\mathbf{Q})$. We are interested in studying the sensitivity with respect to $\mathbf{Q}$ of the functional:

$$\Upsilon(\mathbf{Q}) = \int_{\Omega(\mathbf{Q})} \nabla u_1(\mathbf{Q}, \mathbf{x}) \cdot \nabla u_2(\mathbf{Q}, \mathbf{x})\, \mathrm{d}\mathbf{x}. \tag{3.3}$$

3.2 Some function spaces

We denote $\mathcal{D}'(\Omega)$ the distribution space and:

$$\widetilde{L}^2(\Gamma) = \{G \in L^2(\Gamma) \,|\, G|_{\Gamma_1} \equiv 0,\ \int_{\Gamma_2} G(\mathbf{x})\,\mathrm{d}\sigma_x = 0\},$$
$$\widetilde{H}^{1/2}(\Gamma) = \widetilde{L}^2(\Gamma) \cap H^{1/2}(\Gamma),$$

and the weighted Lebesgue spaces:

$$\mathcal{L}^2(\Omega) = \{u \in \mathcal{D}'(\Omega) \,|\, (1+|\mathbf{x}|^2)^{1/2} u \in L^2(\Omega)\},\ \text{when } \Omega \text{ is not bounded},$$
$$\mathcal{L}^2(\Omega) = \{u \in L^2(\Omega) \,|\, \int_\Omega u(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0\},\ \text{when } \Omega \text{ is bounded}.$$

We define also:

$$\widetilde{\mathcal{V}}^1 = \left\{ u \in \mathcal{D}'(\mathbb{R}^3) \,|\, u/(1+|\mathbf{x}|^2)^{1/2} \in L^2(\mathbb{R}^3), \nabla u \in L^2(\mathbb{R}^3, \mathbb{R}^3) \right\},$$

and, if Ω is bounded, we set for all $\mathbf{Q} \in \mathcal{Q}$ the quotient space:

$$\mathcal{V}^1(\mathbf{Q}) = \left\{ u \in \mathcal{D}'(\Omega(\mathbf{Q})) \,|\, u \in L^2(\Omega(\mathbf{Q})), \nabla u \in L^2(\Omega(\mathbf{Q}), \mathbb{R}^3) \right\} / \mathbb{R}.$$

If Ω is not bounded, the definition above turns into:

$$\mathcal{V}^1(\mathbf{Q}) = \left\{ u \in \mathcal{D}'(\Omega(\mathbf{Q})) \,|\, u/(1+|\mathbf{x}|^2)^{1/2} \in L^2(\Omega(\mathbf{Q})), \nabla u \in L^2(\Omega(\mathbf{Q}), \mathbb{R}^3) \right\}.$$

At last, we will need:

$$\widetilde{\mathcal{V}}^2 = \left\{ u \in \widetilde{\mathcal{V}}^1 \,|\, (1+|\mathbf{x}|^2)^{1/2} [D^2 u] \in L^2(\mathbb{R}^3, \mathcal{M}(3)) \right\},$$

and

$$\mathcal{V}^2(\mathbf{Q}) = \Big\{ u \in \mathcal{V}^1(\mathbf{Q}) \,|\, (1+|\mathbf{x}|^2)^{1/2} [D^2 u] \in L^2(\Omega(\mathbf{Q}), \mathcal{M}(3)),$$
$$\nabla u \cdot \mathbf{n} \equiv 0 \text{ on } \Gamma_1,\ \int_{\Gamma_2(\mathbf{Q})} \nabla u \cdot \mathbf{n}\,\mathrm{d}\sigma_x = 0 \Big\},$$

where $[D^2 u]$ stands for the Hessian matrix of u. We denote merely $\mathcal{V}^1 = \mathcal{V}^1(\mathbf{0}_p)$ and $\mathcal{V}^2 = \mathcal{V}^2(\mathbf{0}_p)$. The spaces $\mathcal{V}^1(\mathbf{Q})$ and $\mathcal{V}^2(\mathbf{Q})$ are endowed respectively with the scalar products:

$$(u, v)_{\mathcal{V}^1(\mathbf{Q})} = \int_{\Omega(\mathbf{Q})} \nabla u \cdot \nabla v\,\mathrm{d}\mathbf{x},$$
$$(u, v)_{\mathcal{V}^2(\mathbf{Q})} = (u, v)_{\mathcal{V}^1(\mathbf{Q})} + \int_{\Omega(\mathbf{Q})} [D^2 u] : [D^2 v](1+|\mathbf{x}|^2)\,\mathrm{d}\mathbf{x}.$$

We refer to [4], [1], [2] for details about these spaces and for the proof of the following Lemma when Ω is not bounded. The case Ω bounded is classical (see [5]).

Lemma 3.1 *Assume that Γ is lipschitz continuous and that $m = 1$ in* (3.1)*. Then for any $g_k(\mathbf{Q}, \cdot)$ ($k = 1, 2$) on $\Gamma(\mathbf{Q})$ such that $g_k(\mathbf{Q}, \phi(\mathbf{Q}, \cdot)) \in \widetilde{L}^2(\Gamma)$, the system* (3.2) *admits in $\mathcal{V}^1(\mathbf{Q})$ a unique weak solution $u_k(\mathbf{Q}, \cdot)$ defined by:*

$$\int_{\Omega(\mathbf{Q})} \nabla u_k(\mathbf{Q}, \mathbf{x}) \cdot \nabla \varphi(\mathbf{x})\, d\mathbf{x} = \int_{\Gamma(\mathbf{Q})} g_k(\mathbf{Q}, \mathbf{x})\, \varphi(\mathbf{x})\, d\sigma_x, \quad \forall \varphi \in \mathcal{V}^1(\mathbf{Q}). \tag{3.4}$$

If Γ is of class $C^{1,1}$, $m = 2$ and $g_k(\mathbf{Q}, \phi(\mathbf{Q}, \cdot)) \in \widetilde{H}^{1/2}(\Gamma)$ the solution $u_k(\mathbf{Q}, \cdot)$ is in $\mathcal{V}^2(\mathbf{Q})$ ($k = 1, 2$).

3.3 Regularity results

From now on we shall denote $G_k(\mathbf{Q},\cdot) = g_k(\mathbf{Q},\phi(\mathbf{Q},\cdot))$ and $U_k(\mathbf{Q},\cdot) = u_k(\mathbf{Q},\phi(\mathbf{Q},\cdot))$ for all $\mathbf{Q} \in \mathcal{Q}$, $k = 1,2$.

Proposition 3.1 *Assume that Γ is Lipschitz continuous, $m = 1$ in (3.1) and that $\mathbf{Q} \in \mathcal{Q} \mapsto G_k(\mathbf{Q},\cdot) \in \widetilde{L}^2(\Gamma)$ is of class C^β, $\beta \geq 1$. Then, the mappings:*

$$\mathbf{Q} \in \mathcal{Q} \mapsto U_k(\mathbf{Q},\cdot) \in \mathcal{V}^1, \quad \mathbf{Q} \in \mathcal{Q} \mapsto \Upsilon(\mathbf{Q}) \in \mathbb{R},$$

are of class $C^{\min\{\alpha,\beta\}}$ in a neighborhood of $\mathbf{Q} = \mathbf{0}_p$.

If Γ is of class $C^{1,1}$, $m = 2$ in (3.1) and $\mathbf{Q} \in \mathcal{Q} \mapsto G_k(\mathbf{Q},\cdot) \in \widetilde{H}^{1/2}(\Gamma)$ is of class C^1, then:

$$\mathbf{Q} \in \mathcal{Q} \mapsto U_k(\mathbf{Q},\cdot) \in \mathcal{V}^2,$$

is of class C^1 in a neighborhood of $\mathbf{Q} = \mathbf{0}_p$. Furthermore, for all compact set $K \subset \Omega$, there exists a neighborhood $\mathcal{Q}_K$ of $\mathbf{Q} = \mathbf{0}_p$ in $\mathcal{Q}$ such that $K \subset \Omega(\mathbf{Q})$, for all $\mathbf{Q} \in \mathcal{Q}_K$ and the mappings:

$$\mathbf{Q} \in \mathcal{Q}_K \mapsto u_k(\mathbf{Q},\cdot)\,|_K \in L^2(K), \quad \mathbf{Q} \in \mathcal{Q}_K \mapsto \nabla u_k(\mathbf{Q},\cdot)\,|_K \in L^2(K,\mathbb{R}^3),$$

are also of class C^1, with the following regularity for the derivatives:

$$\frac{\partial u_k}{\partial q_i}(\mathbf{0}_p,\cdot) \in \mathcal{V}^1, \quad \frac{\partial(\nabla u_k)}{\partial q_i}(\mathbf{0}_p,\cdot) \in L^2(\Omega,\mathbb{R}^3) \quad \forall\, i = 1,\ldots,p.$$

The local results of this Proposition which are available only in a neighborhood of $\mathbf{Q} = \mathbf{0}_p$ can easily be extended to the whole set $\mathcal{Q}$. Indeed, denoting $\widetilde{\Omega} = \phi(\widetilde{\mathbf{Q}},\Omega)$ for any $\widetilde{\mathbf{Q}} \in \mathcal{Q}$ and defining, for all $\mathbf{Q}$ in a neighborhood of $\mathbf{Q} = \mathbf{0}_p$ the function

$$\widetilde{\phi}(\mathbf{Q},\cdot) = \phi(\widetilde{\mathbf{Q}} + \mathbf{Q}, \phi^{-1}(\widetilde{\mathbf{Q}},\cdot)),$$

there is only to apply the Proposition with the function $\widetilde{\phi}$ to obtain the same results in a neighborhood of $\mathbf{Q} = \widetilde{\mathbf{Q}}$. Therefore, it makes sense to define, for all $\mathbf{Q} = (q_1,\ldots,q_n)^\top \in \mathcal{Q}$:

$$\frac{\partial u_k}{\partial q_i}(\mathbf{Q},\cdot) \in \mathcal{V}^1(\mathbf{Q}) \quad \text{and} \quad \frac{\partial(\nabla u_k)}{\partial q_i}(\mathbf{Q},\cdot) \in L^2(\Omega(\mathbf{Q}),\mathbb{R}^3), \quad i = 1,\ldots p.$$

It is possible to compute explicitly the expressions of the partial derivatives of Υ with respect to q_i:

Proposition 3.2 *Assume that Γ is of class $C^{1,1}$, $m = 2$ in (3.1) and $\mathbf{Q} \in \mathcal{Q} \mapsto G_k(\mathbf{Q},\cdot) \in \widetilde{H}^{1/2}(\Gamma)$ is of class C^1. Then, we have:*

$$\begin{aligned}
\frac{\partial \Upsilon}{\partial q_i}(\mathbf{Q}) = &-\int_{\Gamma_2(\mathbf{Q})} \nabla u_1(\mathbf{Q}) \cdot \nabla u_2(\mathbf{Q}) (\mathbf{V}^i(\mathbf{Q}) \cdot \mathbf{n})\, d\sigma_x \\
&+ \int_{\Gamma_2(\mathbf{Q})} \left(\frac{\partial G_1}{\partial q_i}(\phi^{-1}(\mathbf{Q})) - \nabla_\tau g_1(\mathbf{Q}) \cdot \mathbf{V}^i(\mathbf{Q}) \right) u_2(\mathbf{Q})\, d\sigma_x \\
&+ \int_{\Gamma_2(\mathbf{Q})} \left(\frac{\partial G_2}{\partial q_i}(\phi^{-1}(\mathbf{Q})) - \nabla_\tau g_2(\mathbf{Q}) \cdot \mathbf{V}^i(\mathbf{Q}) \right) u_1(\mathbf{Q})\, d\sigma_x \\
&+ 2 \int_{\Gamma_2(\mathbf{Q})} g_1(\mathbf{Q}) g_2(\mathbf{Q}) (\mathbf{V}^i(\mathbf{Q}) \cdot \mathbf{n})\, d\sigma_x \\
&- \int_{\Gamma_2(\mathbf{Q})} (\kappa_1(\mathbf{Q}) + \kappa_2(\mathbf{Q}))(g_1(\mathbf{Q}) u_2(\mathbf{Q}) + g_2(\mathbf{Q}) u_1(\mathbf{Q}))(\mathbf{V}^i(\mathbf{Q}) \cdot \mathbf{n})\, d\sigma_x \\
&+ \int_{\Gamma_1} \frac{\partial g_1}{\partial q_i}(\mathbf{Q}) u_2(\mathbf{Q}) + \frac{\partial g_2}{\partial q_i}(\mathbf{Q}) u_1(\mathbf{Q})\, d\sigma_x, \quad (3.5)
\end{aligned}$$

where $\kappa_1(\mathbf{Q})$ and $\kappa_2(\mathbf{Q})$ stand for the principal curvatures of the surface $\Gamma_2(\mathbf{Q})$ (we refer to [7], page 129 for a definition), ∇_τ is the tangential gradient and

$$\mathbf{V}^i(\mathbf{Q},\cdot) = \frac{\partial \phi}{\partial q_i}(\mathbf{Q},\phi^{-1}(\mathbf{Q},\cdot)), \quad i=1,\dots,p. \tag{3.6}$$

The proofs of the Propositions are given in the Appendix.

4 Existence, uniqueness and regularity of the solids trajectories

4.1 The problem in local coordinates

Considering the expression (2.8) of the Lagrangian and taking into account the symmetry of the matrices, we deduce easily that $\mathcal{L}$ is analytic with respect to $\dot{\mathbf{Q}}$ and that

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{Q}}}(\mathbf{Q}) = ([K_S(\mathbf{Q})] + [K_F(\mathbf{Q})])\dot{\mathbf{Q}}.$$

Assuming for a while the matrix $[K(\mathbf{Q})] = [K_S(\mathbf{Q})] + [K_F(\mathbf{Q})]$ to be derivable with respect to $\mathbf{Q}$ and applying the chain rule

$$\frac{d}{dt}[K(\mathbf{Q})] = \left\langle \frac{\partial [K(\mathbf{Q})]}{\partial \mathbf{Q}}, \dot{\mathbf{Q}} \right\rangle,$$

we obtain that:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{Q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{Q}} = [K(\mathbf{Q})]\ddot{\mathbf{Q}} + \left\langle \frac{\partial [K(\mathbf{Q})]}{\partial \mathbf{Q}}, \dot{\mathbf{Q}} \right\rangle \dot{\mathbf{Q}} - \frac{1}{2}\dot{\mathbf{Q}}^\top \left\langle \frac{\partial [K(\mathbf{Q})]}{\partial \mathbf{Q}}, \cdot \right\rangle \dot{\mathbf{Q}}. \tag{4.1}$$

The Euler-Lagrange system of ODE's (1.2) can formally be rewritten under the normal form:

$$\frac{d}{dt}\begin{bmatrix} \dot{\mathbf{Q}} \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} [K(\mathbf{Q})]^{-1} & [\mathbf{0}_3] \\ [\mathbf{0}_3] & [I_3] \end{bmatrix} \begin{bmatrix} -\left\langle \frac{\partial [K(\mathbf{Q})]}{\partial \mathbf{Q}}, \dot{\mathbf{Q}} \right\rangle \dot{\mathbf{Q}} + \frac{1}{2}\dot{\mathbf{Q}}^\top \left\langle \frac{\partial [K(\mathbf{Q})]}{\partial \mathbf{Q}}, \cdot \right\rangle \dot{\mathbf{Q}} \\ \dot{\mathbf{Q}} \end{bmatrix}. \tag{4.2}$$

To study this system, we begin with the:

Lemma 4.1 *Denote $\lambda_i > 0$ the smallest eigenvalue of the inertia tensor $[J^i]$ (see [9] page 195) and*

$$\alpha = \min\{\lambda_i, m_i, i = 1,\dots,n\},$$

where we recall that m_i is the mass of the $i-$th solid. Then $\alpha > 0$ and

$$\mathbf{X}^\top [K(\mathbf{Q})]\mathbf{X} = |[K(\mathbf{Q})]\mathbf{X}|^2 \geq \alpha |\mathbf{X}|^2, \quad \forall \mathbf{X} \in (\mathbb{R}^6)^n, \forall \mathbf{Q} \in \mathcal{Q}. \tag{4.3}$$

In particular, the matrix $[K(\mathbf{Q})]$ is invertible for all $\mathbf{Q} \in \mathcal{Q}$.

Proof : The matrix $[K(\mathbf{Q})]$ is the sum of $[K_F(\mathbf{Q})]$ which is positive and $[K_S(\mathbf{Q})]$ which is definite positive for all $\mathbf{Q} \in \mathcal{Q}$. Indeed, $[K_S(\mathbf{Q})]$ is a bloc diagonal matrix, each bloc being either a 3×3 diagonal matrix $\mathrm{diag}(m^i, m^i, m^i)$ with $m^i > 0$ for all $i = 1,\dots,n$, either a matrix $[\omega^i]^\top [J^i][\omega^i]$. The matrix $[J^i]$ is definite positive and simple computations yield $\det[\omega^i] = \cos(\theta_2^i) \neq 0$ because $\boldsymbol{\theta}^i = (\theta_1^i, \theta_2^i, \theta_3^1)^\top \in \mathcal{U} =]-\pi,\pi[\times]-\pi/2,\pi/2[\times]-\pi,\pi[$. ■

In order to apply the Cauchy-Lipschitz Theorem to the ODE (4.2), the key point consists in proving that $\partial[K(\mathbf{Q})]/\partial\mathbf{Q}$ is well defined and to determine its regularity with respect to $\mathbf{Q}$. This task will be achieved using arguments of shape sensitivity analysis developed in section 3.

According to the definition (2.5) of $[K_S(\mathbf{Q})]$, the mapping:

$$\mathbf{Q} \in \mathcal{Q} \mapsto [K_S(\mathbf{Q})] \in \mathcal{M}(6n),$$

is analytic. Moreover, the simple formula,

$$\frac{\partial \boldsymbol{\omega}_k}{\partial \theta_j}(\boldsymbol{\theta}) = \begin{cases} \boldsymbol{\omega}_j(\boldsymbol{\theta}) \wedge \boldsymbol{\omega}_k(\boldsymbol{\theta}), & \text{if } 1 \leq j < k \leq 3, \\ \mathbf{0}_3, & \text{if } 1 \leq k \leq j \leq 3, \end{cases} \tag{4.4}$$

leads to:

$$\frac{\partial[\omega^i]}{\partial\theta_1^i} = \mathbf{e}_2 \otimes (\boldsymbol{\omega}_1^i \wedge \boldsymbol{\omega}_2^i) + \mathbf{e}_3 \otimes (\boldsymbol{\omega}_1^i \wedge \boldsymbol{\omega}_3^i), \quad \frac{\partial[\omega^i]}{\partial\theta_2^i} = \mathbf{e}_3 \otimes (\boldsymbol{\omega}_2^i \wedge \boldsymbol{\omega}_3^i), \quad \forall\, i = 1, \ldots, n.$$

The differential of the matrix $[\omega^i]$ with respect to $\mathbf{Q}$ is therefore the three-rank tensor:

$$\frac{\partial[\omega^i]}{\partial\mathbf{Q}} = \mathbf{e}_2 \otimes (\boldsymbol{\omega}_1^i \wedge \boldsymbol{\omega}_2^i) \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes (\boldsymbol{\omega}_1^i \wedge \boldsymbol{\omega}_3^i) \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes (\boldsymbol{\omega}_2^i \wedge \boldsymbol{\omega}_3^i) \otimes \mathbf{e}_2.$$

After some algebra, one obtains that:

$$\left\langle \frac{\partial[\mathcal{J}^i(\mathbf{Q}^i)]}{\partial\mathbf{Q}^i}, \dot{\mathbf{Q}}^i \right\rangle \dot{\mathbf{Q}}^i - \frac{1}{2}\dot{\mathbf{Q}}^{i\top} \left\langle \frac{\partial[\mathcal{J}^i(\mathbf{Q}^i)]}{\partial\mathbf{Q}^i}, \cdot \right\rangle \dot{\mathbf{Q}}^i = \langle [\![T^i(\mathbf{Q}^i)]\!], \dot{\mathbf{Q}}^i, \dot{\mathbf{Q}}^i \rangle, \quad i = 1, \ldots, n,$$

where the three-rank tensor $[\![T^i(\mathbf{Q}^i)]\!]$ arising in the right hand side is defined by:

$$T^i_{kj_1j_2}(\mathbf{Q}^i) = \frac{1}{2}\big[(\boldsymbol{\omega}^i_{\min\{j_1,j_2\}} \wedge \boldsymbol{\omega}^i_{\max\{j_1,j_2\}})[J^i(\boldsymbol{\theta}^i)]\boldsymbol{\omega}^i_k + (\boldsymbol{\omega}^i_{j_1} \wedge \boldsymbol{\omega}^i_k)[J^i(\boldsymbol{\theta}^i)]\boldsymbol{\omega}^i_{j_2} + (\boldsymbol{\omega}^i_{j_2} \wedge \boldsymbol{\omega}^i_k)[J^i(\boldsymbol{\theta}^i)]\boldsymbol{\omega}^i_{j_1}\big], \tag{4.5}$$

if $1 \leq k, j_1, j_2 \leq 3$ and $T^i_{kj_1j_2}(\mathbf{Q}^i) = 0$ if $4 \leq k, j_1, j_2 \leq 6$. Thus, $\langle [\![T^i(\mathbf{Q}^i)]\!], \dot{\mathbf{Q}}^i, \dot{\mathbf{Q}}^i \rangle$ yields a $6-$length column vector. With the tensors $[\![T^i(\mathbf{Q}^i)]\!]$, we build the three-rank $6n \times 6n \times 6n$ tensor defined for all $(\mathbf{Q}, \dot{\mathbf{Q}}) \in \mathcal{Q} \times (\mathbb{R}^6)^n$ by:

$$\langle [\![T_S(\mathbf{Q})]\!], \dot{\mathbf{Q}}, \dot{\mathbf{Q}} \rangle = \begin{pmatrix} \langle [\![T^1(\mathbf{Q}^1)]\!], \dot{\mathbf{Q}}^1, \dot{\mathbf{Q}}^1 \rangle \\ \vdots \\ \langle [\![T^n(\mathbf{Q}^n)]\!], \dot{\mathbf{Q}}^n, \dot{\mathbf{Q}}^n \rangle \end{pmatrix}. \tag{4.6}$$

We get finally the equality:

$$\left\langle \frac{\partial[K_S(\mathbf{Q})]}{\partial\mathbf{Q}}, \dot{\mathbf{Q}} \right\rangle \dot{\mathbf{Q}} - \frac{1}{2}\dot{\mathbf{Q}}^\top \left\langle \frac{\partial[K_S(\mathbf{Q})]}{\partial\mathbf{Q}}, \cdot \right\rangle \dot{\mathbf{Q}} = \langle [\![T_S(\mathbf{Q})]\!], \dot{\mathbf{Q}}, \dot{\mathbf{Q}} \rangle. \tag{4.7}$$

It is clear that $[\![T_S(\mathbf{Q})]\!]$ is analytic with respect to $\mathbf{Q}$.

The use of the Proposition 3.1 will allow us to prove that $[K_F(\mathbf{Q})]$ is indefinitely differentiable with respect to $\mathbf{Q}$ if $\partial\Omega$ is Lipschitz continuous. Nevertheless, the computation of $\partial[K_F(\mathbf{Q})]/\partial\mathbf{Q}$ is only possible when $\partial\Omega$ is of class $C^{1,1}$. Each entries of the matrix $[K_F(\mathbf{Q})]$ has the form of the functional Υ defined by (3.3) in section 3. We define the function ϕ arising in (3.1) as follows: let $\mathcal{O}^0$ be an open set such that $\Gamma_1 \subset \mathcal{O}^0$. For each S^i it is possible to find an open set $\mathcal{O}^i$ such that

$\overline{S^i} \subset \mathcal{O}^i$, $\overline{\mathcal{O}^i}$ compact, $\overline{\mathcal{O}^i} \cap \overline{\mathcal{O}^0} = \emptyset$ for all $i = 1, \dots, n$ and $\overline{\mathcal{O}^i} \cap \overline{\mathcal{O}^j} = \emptyset$ if $i \neq j$. We consider then a C^∞ partition of unity $\{\zeta^0,\, \zeta^i,\, i = 1, \dots, n\}$ such that $\zeta^i \equiv 1$ in $\mathcal{O}^i$, $i = 0, 1, \dots, n$. For any $\mathbf{Q}$ in a neighborhood of $((\mathbf{0}_3, \mathbf{h}_0^1), \dots, (\mathbf{0}_3, \mathbf{h}_0^n))$ in $\mathcal{Q}$, the function

$$\phi(\mathbf{Q}, \mathbf{x}) = \zeta^0(\mathbf{x})\mathbf{x} + \sum_{i=1}^{n} \zeta^i(\mathbf{x})(\mathcal{R}(\boldsymbol{\theta}^i)(\mathbf{x} - \mathbf{h}_0^i) + \mathbf{h}^i),$$

lies in $C^{\infty,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ and satisfies the hypothesis (H_2) and (H_3) of section 3. Then we introduce, for all $i = 1, \dots, n$, the following vectorial fields on $\Gamma(\mathbf{Q})$:

$$\mathbf{V}_k^i(\mathbf{Q}^i, \mathbf{x}) = \boldsymbol{\omega}_k^i \wedge (\mathbf{x} - \mathbf{h}^i), \qquad k = 1, 2, 3 \ \text{ on } \Gamma_2^i(\mathbf{Q}^i), \tag{4.8a}$$

$$\mathbf{V}_{k+3}^i(\mathbf{Q}^i, \mathbf{x}) = \mathbf{e}_k^i, \qquad k = 1, 2, 3 \ \text{ on } \Gamma_2^i(\mathbf{Q}^i), \tag{4.8b}$$

$$\mathbf{V}_k^i(\mathbf{Q}^i, \mathbf{x}) = \mathbf{0}_3, \qquad k = 1, \dots, 6 \ \text{ on } \Gamma(\mathbf{Q}) \setminus \Gamma_2^i(\mathbf{Q}^i). \tag{4.8c}$$

The boundaries values of $\partial\varphi_k^i/\partial\mathbf{n}$ in the system (2.6) read for all $i = 1, \dots, n$ and $k = 1, \dots, 6$:

$$g_k^i(\mathbf{Q}^i, \mathbf{x}) = \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \cdot \mathbf{V}_k^i(\mathbf{Q}^i, \mathbf{x}), \qquad \text{on } \Gamma_2^i(\mathbf{Q}^i), \tag{4.9a}$$

$$g_k^i(\mathbf{Q}^i, \mathbf{x}) = 0, \qquad \text{on } \Gamma(\mathbf{Q}) \setminus \Gamma_2^i(\mathbf{Q}^i), \tag{4.9b}$$

and the functions $G_k^i(\mathbf{Q}^i, \cdot) = g_k^i(\mathbf{Q}^i, \phi(\mathbf{Q}, \cdot))$ are:

$$G_k^i(\mathbf{Q}^i, \mathbf{x}) = \mathbf{n}^i \cdot ((\mathcal{R}(\boldsymbol{\theta}^i)^\top \boldsymbol{\omega}_k^i) \wedge (\mathbf{x} - \mathbf{h}_0^i)), \tag{4.10a}$$

$$G_{k+3}^i(\mathbf{Q}^i, \mathbf{x}) = \mathbf{n}^i \cdot \mathcal{R}(\boldsymbol{\theta}^i)^\top \mathbf{e}_k, \qquad \text{on } \Gamma_2^i,\ 1 \leq k \leq 3, \tag{4.10b}$$

$$G_k^i(\mathbf{Q}^i, \mathbf{x}) = 0, \qquad \text{on } \Gamma \setminus \Gamma_2^i,\ 1 \leq k \leq 6. \tag{4.10c}$$

Since $G_k^i(\mathbf{Q}^i, \cdot) \in \widetilde{L}^2(\Gamma)$ for all $\mathbf{Q} \in \mathcal{Q}$, the Lemma 3.1 applies. It ensures the existence and uniqueness of the functions $\varphi_k^i(\mathbf{Q}, \cdot)$ in suitable weighted Sobolev spaces. The matrix $[K_F(\mathbf{Q})]$ is hence well defined for all $\mathbf{Q} \in \mathcal{Q}$. Moreover, it is clear that the functions $\mathbf{Q} \mapsto G_k^i(\mathbf{Q}^i, \cdot) \in \widetilde{L}^2(\Gamma)$ are indefinitely differentiable. According to the Proposition 3.1, the mapping $\mathbf{Q} \mapsto [K_F(\mathbf{Q})]$ is also of class C^∞ in a neighborhood of $((\mathbf{0}_3, \mathbf{h}_0^1), \dots, (\mathbf{0}_3, \mathbf{h}_0^n))$. The Cauchy Lipschitz Theorem applies to the ODE (4.2) and the solutions are indefinitely differentiable. We have thus obtained the conclusion of Theorem 1.1.

4.2 The problem in global coordinates

Let $(\mathbf{P}_0, \dot{\mathbf{P}}_0) \in \mathcal{P} \times T\mathcal{P}(\mathbf{P}_0)$ (namely, initial positions an initial velocities of the solids) and define $\mathbf{Q}_0 = ((\mathbf{0}_3, \mathbf{h}_0^1), \dots, (\mathbf{0}_3, \mathbf{h}_0^n)) \in \mathcal{Q}_{\mathbf{P}_0}$ and $\dot{\mathbf{Q}}_0 \in (\mathbb{R}^3 \times \mathbb{R}^3)^n$ such that

$$\left\langle \frac{\partial \Lambda_{\mathbf{P}_0}}{\partial \mathbf{Q}}(\mathbf{Q}_0), \dot{\mathbf{Q}}_0 \right\rangle = \dot{\mathbf{P}}_0.$$

We can apply Theorem 1.1: there exists a real interval $[0, T)$ and a unique C^∞ function $\mathbf{Q}^*(\cdot, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0)$ that solves (1.2) on $[0, T)$. We set then

$$\mathbf{P}^*(t) = \Lambda_{\mathbf{P}_0}(\mathbf{Q}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0)), \quad \forall\, t \in [0, T).$$

The principle of conservation of energy ensures that

$$\mathcal{L}(\mathbf{Q}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0), \dot{\mathbf{Q}}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0)) = \mathcal{L}(\mathbf{Q}_0, \dot{\mathbf{Q}}_0), \quad \forall\, t \in [0, T).$$

The definition (2.8) of the Lagrangian and the formula (4.3) yield:

$$|\dot{\mathbf{Q}}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0)|^2 \leq \frac{1}{\alpha} \mathcal{L}(\mathbf{Q}_0, \dot{\mathbf{Q}}_0), \quad \forall\, t \in [0, T). \tag{4.11}$$

Classical results on ODE's (see for example [6] page 13) allow to make precise the behavior of the point $(t, \mathbf{Q}^*(t,0,\mathbf{Q}_0,\dot{\mathbf{Q}}_0), \dot{\mathbf{Q}}^*(t,0,\mathbf{Q}_0,\dot{\mathbf{Q}}_0)) \in \mathbb{R} \times \mathcal{Q}(\mathbf{P}_0) \times (\mathbb{R}^6)^n$ as $t \to T$. One of the following alternatives holds:

- either $T = \infty$,
- either $|\dot{\mathbf{Q}}^*(t,0,\mathbf{Q}_0,\dot{\mathbf{Q}}_0)| \to \infty$ as $t \to T$ but this case is excluded by the estimate (4.11).
- either $\mathbf{Q}^*(t,0,\mathbf{Q}_0,\dot{\mathbf{Q}}_0) \to \widetilde{\mathbf{Q}}_1 \in \partial\mathcal{Q}_{\mathbf{P}_0}$ and $\dot{\mathbf{Q}}^*(t,0,\mathbf{Q}_0,\dot{\mathbf{Q}}_0) \to \dot{\widetilde{\mathbf{Q}}}_1 \in (\mathbb{R}^6)^n$ as $t \to T$.

In the last case, if the point $\mathbf{Q}_1$ does not correspond to a collision, we can set $\mathbf{P}_1 = \Lambda_{\mathbf{P}_0}(\widetilde{\mathbf{Q}}_1)$, $\dot{\mathbf{P}}_1 = \left\langle \partial\Lambda_{\mathbf{P}_0}/\partial\mathbf{Q}(\widetilde{\mathbf{Q}}_1), \dot{\widetilde{\mathbf{Q}}}_1 \right\rangle$, and $\mathbf{Q}^1 = \Lambda^{-1}_{\mathbf{P}_1}(\widetilde{\mathbf{Q}}_1)$,

$$\dot{\mathbf{Q}}_1 = \left\langle \frac{\partial\Lambda^{-1}_{\mathbf{P}_1}}{\partial\mathbf{Q}}(\widetilde{\mathbf{Q}}_1), \dot{\mathbf{Q}}_1 \right\rangle.$$

We can then build a new solution with initial data $(\mathbf{Q}_1, \dot{\mathbf{Q}}_1)$. This solution does continue $\mathbf{P}^*$ to the right of T. The proof of the Corollary 1.1 is then completed.

5 Motion of an infinite cylinder

The vertical motion of an infinite cylinder of radius 1 is studied in this section. For symmetry reasons, the problem reduces to the motion of a disk in the half-plane $\Omega = \{(x,y) \in \mathbb{R}^2 \text{ s.t. } y \geq 0\}$. Suppose that $(0, h(t))$ for $t > 0$ are the coordinates of the center of mass of the disk B_h and assume that the angular velocity and the first component of the linear velocity are zero. In this case the system has only one degree of freedom, h, so the virtual mass matrix reduces to the scalar $k(h)$.

$$k(h) = m + \mathcal{E}(h),$$
$$\mathcal{E}(h) = \int_{\Omega\setminus B_{h(t)}} |\nabla\varphi_h|^2(\mathbf{x})\mathrm{d}\mathbf{x},$$

where φ_h solve the following Neumann problem :

$$-\Delta\varphi_h = 0 \qquad \mathbf{x} \text{ in } \Omega \setminus B_h, \tag{5.1a}$$
$$\frac{\partial\varphi_h}{\partial\mathbf{n}} = h - x_2 \qquad \text{on } \partial B_h, \tag{5.1b}$$
$$\frac{\partial\varphi_h}{\partial\mathbf{n}} = 0 \qquad \text{on } \partial\Omega. \tag{5.1c}$$

The Euler-Lagrange system (4.1) reduces to:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{Q}}} - \frac{\partial\mathcal{L}}{\partial\mathbf{Q}} = (m + \mathcal{E}(h))\ddot{h} + \frac{1}{2}\frac{\partial\mathcal{E}(h)}{\partial h}\dot{h}^2 = 0,$$

and hence, after integrating:

$$\dot{h} = \dot{h}_0\sqrt{\frac{m + \rho_F\mathcal{E}(h_0)}{m + \rho_F\mathcal{E}(h)}}, \tag{5.2}$$

where h_0 and $\dot{h}_0$ are the initial position and velocity of the disk. The ODE (5.2) ensures that for all $t > 0$, $\dot{h}$ keeps the same signum as $\dot{h}_0$. In particular, for $\dot{h}_0 < 0$ and for any $h_0 > 1$, we have $1 \leq h \leq h_0$. To study the collision, we prove some properties on the function $\mathcal{E}$:

Lemma 5.1 *The function $\mathcal{E}$ is given by the series:*

$$\mathcal{E}(h) = 2\pi \sum_{k\geq 1} \frac{h^2-1}{[(h+\sqrt{h^2-1})^k - (h-\sqrt{h^2-1})^k]^2} - \frac{\pi}{4}. \tag{5.3}$$

This formula leads to the continuity of $\mathcal{E}$ on $[1,+\infty[$ with the following limits:

$$\lim_{h\to 1^+} \mathcal{E}(h) = \frac{\pi^3}{12} - \frac{\pi}{4}, \qquad \lim_{h\to+\infty} \mathcal{E}(h) = \frac{\pi}{4}. \tag{5.4}$$

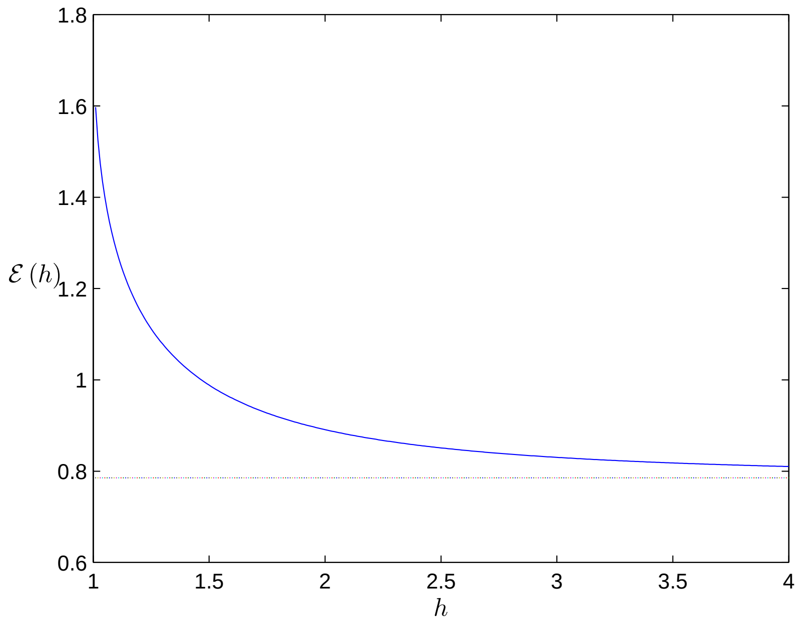

Figure 2: Graph of the function $\mathcal{E}$ with respect to h

The main result of this section is:

Proposition 5.1 *For any $h_0 > 1$ and $\dot{h}_0 < 0$, there exists $T_0 > 0$ such that the Cauchy problem (5.2) has an unique solution defined on $[0, T_0]$ and satisfying $h(T_0) = 1$ and $\dot{h}(T_0) > 0$. It means that the cylinder collides the boundary in finite time with non zero relative velocity.*

5.1 Computation of an explicit solution

We shall compute explicitly the solution of (5.1), using a conformal mapping method. In particular, we will obtain the expression of $\mathcal{E}$, involving h, and we will show the continuity and compute the finite limit of $\mathcal{E}$ when h goes to 0. To conclude, we show that $|\dot{h}(T_0)| > \alpha > 0$.
We begin by considering the following holomorphic change of variables:

Lemma 5.2 *For all $t > 0$, the conformal mapping $\mathbf{G}$ defined by:*

$$\mathbf{G}(\mathbf{z}) = \frac{\mathbf{z} - ia}{\mathbf{z} + ia},$$

where $a = \sqrt{h^2 - 1}$*, maps* B_h *onto the disk* $\mathcal{D}_\sigma$ *of center* $\mathbf{0}$ *and radius* $\sigma = h - \sqrt{h^2-1}$ *and* $\Omega \setminus B_h$ *onto* $\mathcal{A}_\sigma = \mathcal{A}(0, \sigma, 1)$*, the annulus of center* $\mathbf{0}$ *and radii* σ *and 1. For all* $\mathbf{w} \in \mathcal{A}_\sigma$*, the inverse function* $\mathbf{g} = \mathbf{G}^{-1}$ *is given by:*

$$\mathbf{g}(\mathbf{w}) = ia\frac{1+\mathbf{w}}{1-\mathbf{w}},$$

and we have $h = (\sigma^{-1} + \sigma)/2$*,* $a = (\sigma^{-1} - \sigma)/2$*.*

A proof of the Lemma below can be found in [10].

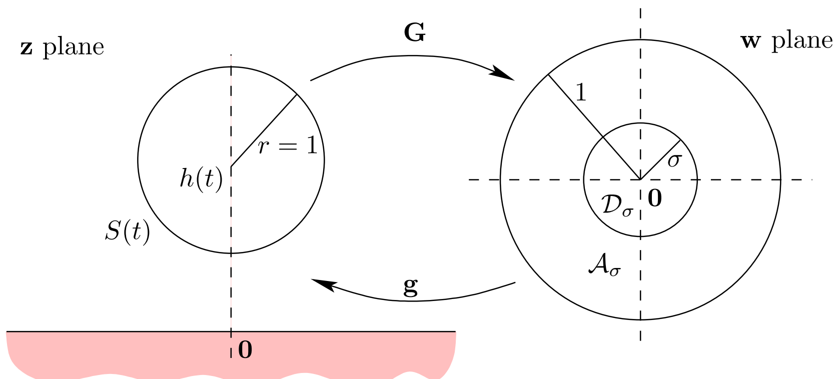

Figure 3: The conformal mappings $\mathbf{G}$ and $\mathbf{g}$.

Let us recall some useful classical properties of an holomorphic change of variables:

Lemma 5.3 *Let* $\mathcal{O}$ *and* $\widetilde{\mathcal{O}}$ *be two open subsets of* $\mathbb{R}^2$ *and* $\mathbf{P} = P_1 + iP_2 : \mathcal{O} \to \widetilde{\mathcal{O}}$ *be a conformal mapping.*

1. *Let* $f \in C^2(\widetilde{\mathcal{O}})$ *and* $F = f \circ \mathbf{P}$*. Then* $F \in C^2(\mathcal{O})$ *and we have the relation:*

$$\Delta F(u_1, u_2) = \left[\left(\frac{\partial P_1}{\partial u_1}\right)^2 + \left(\frac{\partial P_2}{\partial u_1}\right)^2\right] \Delta f(\mathbf{P}(u_1, u_2)),$$

for all $(u_1, u_2)^\top \in \mathcal{O}$*.*

2. *Let* Γ *be a* C^1 *curve,* $\Gamma \subset \mathcal{O}$ *and* $\mathbf{n}$ *be the unit normal to* Γ*. Denote also* $\widetilde{\Gamma} = \mathbf{P}(\Gamma) \subset \widetilde{\mathcal{O}}$ *and* $\widetilde{\mathbf{n}}$ *the unit normal to* $\widetilde{\Gamma}$*. Then we have:*

$$\frac{\partial F}{\partial \mathbf{n}}(u_1, u_2) = \left|\frac{\partial \mathbf{P}}{\partial u_1}(u_1, u_2)\right| \frac{\partial f}{\partial \widetilde{\mathbf{n}}}(\mathbf{P}(u_1, u_2)). \tag{5.5}$$

Applying Lemma 5.3, the new function $\phi_\sigma(u_1, u_2) = \varphi_h(\mathbf{g}(u_1, u_2))$ solves the Neumann problem:

$$\Delta\phi_\sigma(u_1, u_2) = 0 \qquad \text{for } (u_1, u_2) \in \mathcal{A}_\sigma, \tag{5.6a}$$

$$\frac{\partial \phi_\sigma}{\partial \mathbf{n}}(u_1, u_2) = 0 \qquad \text{for } u_1^2 + u_2^2 = 1, \tag{5.6b}$$

$$\frac{\partial \phi_\sigma}{\partial \mathbf{n}}(u_1, u_2) = \frac{a[2\sigma^2 - (1+\sigma^2)u_1]}{\sigma(1+\sigma^2 - 2u_1)^2} \quad \text{for } u_1^2 + u_2^2 = \sigma^2. \tag{5.6c}$$

For $t > 0$, according to [8, Theorem 3.1], the harmonicity of ϕ_σ ensures the existence of an harmonic conjugate function ψ and of an holomorphic function $\mathbf{F}$ defined in $\mathcal{A}_\sigma$ by $\mathbf{F} = \phi_\sigma + i\psi$. Remark that, since $\mathcal{A}_\sigma$ is not a simple connected set, this existence result is not obvious. The function $\mathbf{F}$ is defined up to an additive complex constant because ϕ_σ is the solution of a Neumann problem. From the boundaries conditions (5.6b) and (5.6c) for ϕ_σ, we derive the boundaries conditions for $\mathbf{F}$. On the boundary $|\mathbf{w}| = 1$ of $\mathcal{A}_\sigma$, the exterior unit normal is $\mathbf{n} = \mathbf{w}$, then $\partial\mathbf{F}/\partial\mathbf{n}(\mathbf{w}) = \mathbf{F}'(\mathbf{w})\mathbf{w}$. On the boundary $|\mathbf{w}| = \sigma$, the normal vector is $\mathbf{n} = -\mathbf{w}/\sigma$, therefore we get $-\sigma\partial\mathbf{F}/\partial\mathbf{n}(\mathbf{w}) = \mathbf{F}'(\mathbf{w})\mathbf{w}$. Finally, with $\mathbf{w} = u_1 + iu_2$, (5.6b) and (5.6c) yield for $\mathbf{F}$ the conditions:

$$\mathcal{R}(\mathbf{F}'(\mathbf{w})\mathbf{w}) = 0 \qquad \text{for } |\mathbf{w}| = 1, \tag{5.7a}$$

$$\mathcal{R}(\mathbf{F}'(\mathbf{w})\mathbf{w}) = \frac{-a[2\sigma^2 - (1+\sigma^2)u_1]}{(1+\sigma^2-2u_1)^2} \qquad \text{for } |\mathbf{w}| = \sigma. \tag{5.7b}$$

The function $\mathbf{F}$ being holomorphic in the annulus $\mathcal{A}_\sigma$, it admits an expansion under the form of a Laurent series.

Lemma 5.4 *The Laurent series of the function* $\mathbf{F}$*, holomorphic in* $\mathcal{A}_\sigma$ *and satisfying* (5.7) *is:*

$$\mathbf{F}(\mathbf{w}) = \boldsymbol{\gamma} - a\sum_{n\geq 1}\frac{\sigma^{2n}}{1-\sigma^{2n}}(\mathbf{w}^n + \mathbf{w}^{-n}), \quad \forall\, \mathbf{w} \in \mathcal{A}_\sigma, \tag{5.8}$$

where $\boldsymbol{\gamma} \in \mathbb{C}$ *is an arbitrary constant.*

Remark 5.1 *The above series converges for* $\sigma^2 < |\mathbf{w}| < \sigma^{-2}$*. This implies in particular that the series converges in* $\mathcal{A}_\sigma$*.*

Proof : Seeking the coefficients of the Laurent series (5.8), we write formally that:

$$\mathbf{F}(\mathbf{w}) = \boldsymbol{\gamma} + \sum_{n\geq 1}(a_n + ib_n)\mathbf{w}^n + (c_n + id_n)\mathbf{w}^{-n} \quad \text{for all } \mathbf{w} \in \mathcal{A}_\sigma, \tag{5.9a}$$

and then

$$\mathcal{R}(\mathbf{F}'(\mathbf{w})\mathbf{w}) = \sum_{n\geq 1} n[r^n a_n - r^{-n}c_n]\cos(n\theta) - n[r^n b_n + r^{-n}d_n]\sin(n\theta) \quad \text{for all } \mathbf{w} = re^{i\theta}. \tag{5.9b}$$

For $r = 1$, according to (5.7a), we obtain that $a_n = c_n$ and $b_n = -d_n$. For $r = \sigma$, the identity (5.7b) allows us to deduce the coefficients a_n and b_n. Thus, we get:

$$a_n + ib_n = \frac{-\sigma^n}{n(1-\sigma^{2n})}\frac{1}{\pi}\int_0^{2\pi}\alpha(\theta)e^{-in\theta}\,\mathrm{d}\theta = \frac{-\sigma^n}{n(1-\sigma^{2n})}\widehat{\alpha}(n),$$

$$c_n + id_n = \frac{-\sigma^n}{n(1-\sigma^{2n})}\frac{1}{\pi}\int_0^{2\pi}\alpha(\theta)e^{in\theta}\,\mathrm{d}\theta = \frac{-\sigma^{-n}}{-n(1-\sigma^{-2n})}\widehat{\alpha}(-n),$$

where $\widehat{\alpha}(n)$ is the n-th Fourier coefficient of α, the function defined for all $\theta \in \mathbb{R}$ by:

$$\alpha(\theta) = \frac{-a\sigma[2\sigma - (1+\sigma^2)\cos(\theta)]}{[1+\sigma^2-2\sigma\cos(\theta)]^2}.$$

Plugging the expressions of a_n, b_n, c_n and d_n in the formal expansion (5.9a), we get:

$$\mathbf{F}(\mathbf{w}) = \boldsymbol{\gamma} + \sum_{n\in\mathbb{Z}^*}\frac{-\sigma^n}{n(1-\sigma^{2n})}\widehat{\alpha}(n)\mathbf{w}^n, \tag{5.10}$$

where $\boldsymbol{\gamma} \in \mathbb{C}$ is an arbitrary constant. We next compute explicitly the Fourier coefficients of α. Set $\mathbf{z} = e^{i\theta}$, it follows that:

$$\widehat{\alpha}(n) = \frac{1}{2i\pi} \int_{\partial \mathcal{D}} \frac{-a\sigma[4\sigma\mathbf{z} - (1+\sigma^2)(\mathbf{z}^2+1)]}{[(1+\sigma^2)\mathbf{z} - \sigma(\mathbf{z}^2+1)]^2} \frac{\mathrm{d}\mathbf{z}}{\mathbf{z}^n}. \tag{5.11}$$

Since:

$$\frac{-a\sigma[4\sigma\mathbf{z} - (1+\sigma^2)(\mathbf{z}^2+1)]}{[(1+\sigma^2)\mathbf{z} - \sigma(\mathbf{z}^2+1)]^2} = a\sigma \left[\frac{1}{(\mathbf{z}-\sigma)^2} + \frac{1}{\sigma^2} \frac{1}{(\mathbf{z}-\sigma^{-1})^2} \right],$$

the relation (5.11), combined with (5.10) gives, according to the residue Theorem:

$$\begin{aligned} \widehat{\alpha}(n) &= na\sigma^n \qquad \text{for all } n \geq 1, \\ \widehat{\alpha}(n) &= -na\sigma^{-n} \text{ for all } n \leq -1. \end{aligned}$$

Plugging these identities into (5.10), we get (5.8) and the proof is then completed.
■

We give now proof of the formula (5.3) and compute the limits of the function $\mathcal{E}$.

Proof : The expression of $\mathcal{E}(h) = \int_{\Omega \setminus B_h} |\nabla \varphi_h|^2 \mathrm{d}\mathbf{x}$ is:

$$\mathcal{E}(h) = \tilde{\mathcal{E}}(h - \sqrt{h^2-1}) = \tilde{\mathcal{E}}(\sigma) = \int_{\mathcal{A}_\sigma} |\nabla \phi_\sigma|^2 \mathrm{d}\mathbf{w}, \tag{5.12}$$

where ϕ_σ is the solution of the system (5.6). The above formula comes from the holomorphic properties of $\mathbf{G}$ and $\mathbf{g}$. Hence, taking into account the Cauchy Riemann conditions, their Jacobian matrices can be written as:

$$\begin{aligned} [J_G(x_1,x_2)] &= \left| \frac{\partial \mathbf{G}}{\partial x_1}(x_1,x_2) \right| [Q_G(x_1,x_2)] \text{ for all } (x_1,x_2) \in \Omega \setminus B_h, \\ [J_g(u_1,u_2)] &= \left| \frac{\partial \mathbf{g}}{\partial u_1}(u_1,u_2) \right| [Q_g(u_1,u_2)] \qquad \text{for all } (u_1,u_2) \in \mathcal{A}_\sigma, \end{aligned}$$

where $[Q_G]$ and $[Q_g]$ are orthogonal matrices. We deduce the expression of $\nabla\varphi_h$ in terms of $\nabla\phi_\sigma$:

$$\begin{aligned} \nabla\varphi_h(x_1,x_2) &= \left| \frac{\partial \mathbf{G}}{\partial x_1}(x_1,x_2) \right| [Q_G(x_1,x_2)] \nabla\phi_\sigma(\mathbf{G}(x_1,x_2)) \\ &= \left| \frac{\partial \mathbf{g}}{\partial u_1}(\mathbf{G}(x_1,x_2)) \right|^{-1} [Q_G(x_1,x_2)] \nabla\phi_\sigma(\mathbf{G}(x_1,x_2)). \end{aligned}$$

Since $|\nabla\phi_\sigma(u_1,u_2)|^2 = |\mathbf{F}'(\mathbf{w})|^2$, where $\mathbf{F}$ is defined by (5.8),we have in cylindrical coordinates:

$$\tilde{\mathcal{E}}(\sigma) = 2\pi \int_\sigma^1 \left[\frac{1}{2\pi} \int_0^{2\pi} |\mathbf{F}'(\rho\cos(\theta), \rho\sin(\theta))|^2 \, \mathrm{d}\theta \right] \rho \, \mathrm{d}\rho.$$

In order to apply the Parseval Theorem, we need to prove that for all $\rho \in [\sigma, 1]$ the function $\theta \mapsto \mathbf{F}'(\rho\cos(\theta), \rho\sin(\theta))$ is in $L^2(0, 2\pi)$. We have the estimates:

$$\begin{aligned} |\mathbf{F}'(\rho\cos(\theta), \rho\sin(\theta))| &\leq \frac{a}{\sigma} \sum_{n\geq 1} \frac{n\sigma^{2n}}{1-\sigma^{2n}} (1+\sigma^{-n}) \\ &\leq \frac{a}{\sigma(1-\sigma)} \sum_{n\geq 1} n\sigma^n = \frac{a}{(1-\sigma)^3}. \end{aligned}$$

So we can apply the Parseval equality to $|\mathbf{F}'|$ and we obtain for all $\sigma \in]0,1[$:

$$\tilde{\mathcal{E}}(\sigma) = 2\pi \int_\sigma^1 \rho \,\mathrm{d}\rho \sum_{n\geq 1} \left[\frac{an\sigma^{2n}}{1-\sigma^{2n}}\right]^2 [\rho^{2n-2} + \rho^{-2n-2}] \tag{5.13a}$$

$$= \pi a^2 \sum_{n\geq 1} n\sigma^{2n} \frac{1+\sigma^{2n}}{1-\sigma^{2n}}. \tag{5.13b}$$

The well known identities $x/(1-x) = \sum_{n\geq 1} x^n$ and $x/(1-x)^2 = \sum_{n\geq 1} nx^n$, for all $|x| < 1$ together with (5.13b), allow us to turn the expression of $\mathcal{E}$ into:

$$\tilde{\mathcal{E}}(\sigma) = \pi a^2 \sum_{n\geq 1} n(1+\sigma^{2n}) \sum_{k\geq 1} \sigma^{2nk} \quad \text{for all } 0 < \sigma < 1. \tag{5.14}$$

We can invert the order of both sums because all the terms are non-negative, to get

$$\begin{aligned}\tilde{\mathcal{E}}(\sigma) &= \pi a^2 \sum_{k\geq 1}\sum_{n\geq 1} n\sigma^{2nk} + n\sigma^{2n(k+1)} \\ &= \pi a^2 \sum_{k\geq 1} \frac{\sigma^{2k}}{(1-\sigma^{2k})^2} + \frac{\sigma^{2(k+1)}}{(1-\sigma^{2(k+1)})^2} \quad \text{for all } 0 < \sigma < 1.\end{aligned}$$

We split the series and change the index in the the second one to get the final expression for $\tilde{\mathcal{E}}$:

$$\tilde{\mathcal{E}}(\sigma) = \frac{\pi}{2}\left[\sum_{k\geq 1} \sigma^{2k-2}\left[\frac{1-\sigma^2}{1-\sigma^{2k}}\right]^2\right] - \frac{\pi}{4} \quad \text{for all } 0 < \sigma < 1.$$

Introducing $\sigma = h - \sqrt{h^2-1}$ in the above expression, $\mathcal{E}$ reduces to (5.3). In the sequel, we prove the continuity of the function $\mathcal{E}$. We apply Lebesgue's Theorem to establish the continuity of $\tilde{\mathcal{E}}$ defined by the series (5.14) on $]0,1]$. Let $f_k(\sigma) = \sigma^{2k-2}\left[1-\sigma^2/1-\sigma^{2k}\right]^2$ for $\sigma \in]0,1[$. These functions are continuous on $]0,1[$ and can be extended by $1/k^2$ for $\sigma = 1$. We have the following inequality for all $k \geq 2$:

$$\frac{1-\sigma^k}{1-\sigma} = \sum_{p=0}^{k-1} \sigma^p \geq k\sigma^{k-1} \qquad \text{for all } 0 < \sigma \leq 1,$$

and then

$$f_k(\sigma) = \sigma^{2k-2}\left[\frac{1+\sigma}{1+\sigma^k}\frac{1-\sigma}{1-\sigma^k}\right]^2 \leq \frac{4}{k^2} \qquad \text{for all } 0 \leq \sigma \leq 1.$$

Finally the terms of $\tilde{\mathcal{E}}$ are uniformly bounded by a convergent series for all $0 \leq \sigma \leq 1$. Then $\tilde{\mathcal{E}}$ is a continuous function of σ. We can pass to the limit in the series as σ goes to 1 and as σ goes to 0 and we get the values for the energy given by (5.4). ■

We give the proof of the proposition (5.1).
Proof : The ODE for $\dot{h}$ is:

$$\dot{h} = \dot{h_0}\sqrt{\frac{m + \rho_F \mathcal{E}(h_0)}{m + \rho_F \mathcal{E}(h)}}.$$

We recall that $\dot{h}$ keeps the same signum as $\dot{h}_0 < 0$. By the Lemma (5.1), the function $\mathcal{E}$ is continuous on $[1, h_0]$. Let $k = \sup\{\mathcal{E}(h), h \in [1,h_0]\}$ and $\alpha =$

$\dot{h}_0\sqrt{(m+\rho_F\mathcal{E}(h_0))/(m+\rho_F k)}$. We have

$$\dot{h}(t) \le \alpha < 0 \text{ for } t > 0,$$
$$h(t) \le \alpha t + h_0 \text{ for } t > 0.$$

When t tends to $+\infty$, h goes to $-\infty$. We conclude that there exists $T_0 > 0$ such that $h(T_0) = 1$ and $|\dot{h}(T_0)| > 0$. ■

5.2 Velocity damping at the collision point

We have seen that our model of fluid-solid interaction, dealing with an ideal fluid flow, allows the rigid body to collide the boundary of the fluid domain. Nevertheless, the fluid has a damping effect on the velocity of the solid. Considering again the configuration of the preceding section, we can rewrite (5.2):

$$\frac{\dot{h}}{\dot{h}_0} = \sqrt{\frac{\pi + \rho_F/\rho_S\mathcal{E}(h_0)}{\pi + \rho_F/\rho_S\mathcal{E}(h)}}. \tag{5.16}$$

We have drawn in the figure 5.2 the disk velocity damping percentage $100(1-\dot{h}/\dot{h}_0)$ with respect to h for some values of ρ_F/ρ_S. Note that the d'Alembert's paradox, holding for a disk in the whole space $\mathbb{R}^2$, does not apply any more in our case. The resulting hydrodynamic force on the disk is non-null and the disk slows down when approaching the fluid boundary.

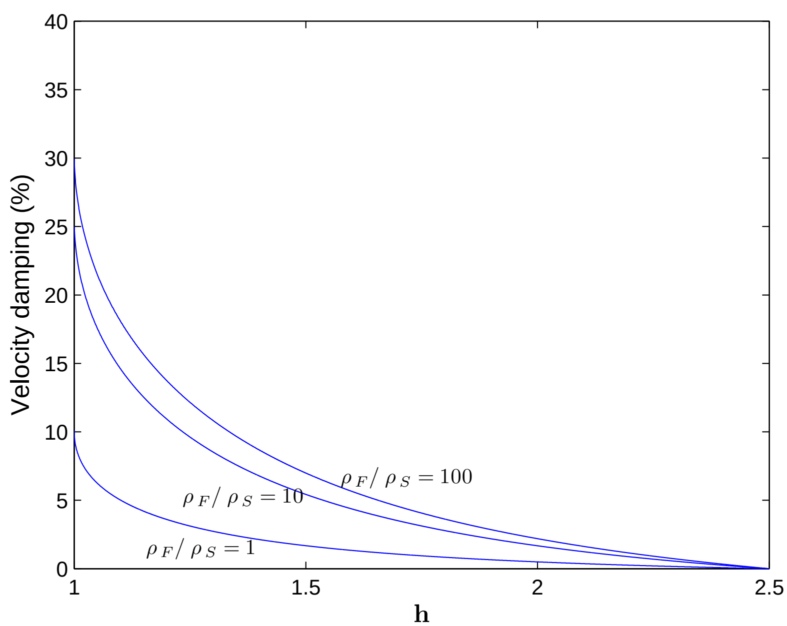

Figure 4: Ball velocity damping for $h_0 = 2.5$

No surprisingly, the damping effect increases as h decreases and is maximum at the collision point, that is to say for $h = 1$. Since $\mathcal{E}(1) = \pi^3/12 - \pi/4$ and $\mathcal{E}(\infty) = \pi/4$, we can compute that a disk coming from infinity has lost $100(1 -$

$\sqrt{(\pi+\rho_F/\rho_S\pi/4)/(\pi+\rho_F/\rho_S(\pi^3/12-\pi/4))}$ percents of its initial velocity at the moment of the impact. This quantity, represented in figure 5.2, increases with ρ_F/ρ_S and tends to $100(1-\sqrt{1/(\pi^2/3-1)}) \sim 34\%$ as ρ_F/ρ_S goes to infinity.

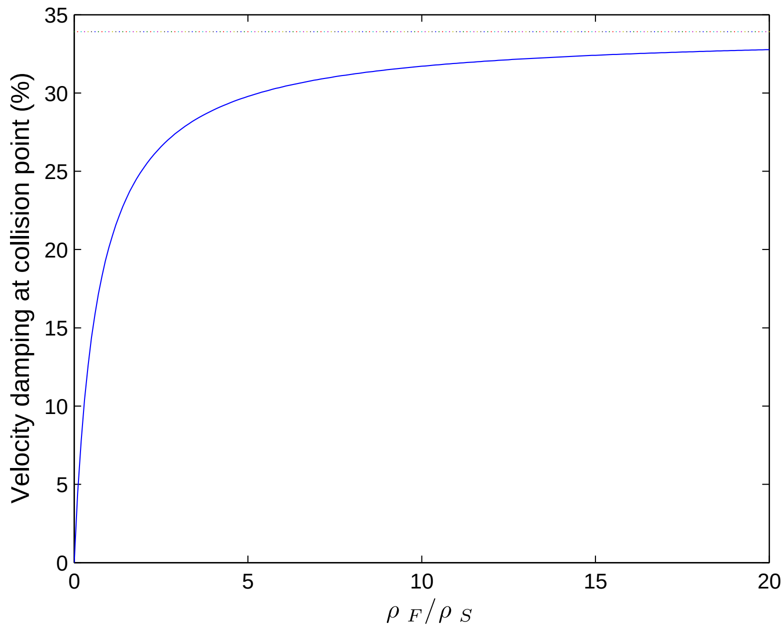

Figure 5: Velocity damping at the collision point with respect to ρ_F/ρ_S

In particular, even with an infinite density fluid, the disk does collide the boundary.

6 Explicit form of the Lagrangian system of ODE's in the general case

Assuming additional regularity for $\partial\Omega$, it is possible to make explicit the Euler-Lagrange equations (1.2). The expression shall involve $\boldsymbol{\varphi}(\mathbf{Q},\cdot)$ and $[D\boldsymbol{\varphi}(\mathbf{Q},\cdot)]$ on $\Gamma_2(\mathbf{Q})$ and geometric data of the solids. Thus, for any $\mathbf{x}\in\Gamma_2^i(\mathbf{Q}^i)$, $\boldsymbol{\tau}_1^i(\mathbf{Q}^i,\mathbf{x})$ and $\boldsymbol{\tau}_2^i(\mathbf{Q}^i,\mathbf{x})$ stand for the principal directions on $\Gamma_2^i(\mathbf{Q})$ at the point $\mathbf{x}$ and $\kappa_1^i(\mathbf{Q}^i,\mathbf{x})$ and $\kappa_2^i(\mathbf{Q}^i,\mathbf{x})$ are the principal curvatures. The set $\{\boldsymbol{\tau}_1^i(\mathbf{Q}^i,\mathbf{x}),\boldsymbol{\tau}_2^i(\mathbf{Q}^i,\mathbf{x}),\mathbf{n}^i(\mathbf{Q}^i,\mathbf{x})\}$ is a direct orthonormal basis of $\mathbb{R}^3$ and $\Pi^i(\mathbf{Q}^i,\mathbf{x})=\kappa_1^i(\mathbf{Q}^i,\mathbf{x})\boldsymbol{\tau}_1^i(\mathbf{Q}^i,\mathbf{x})\otimes\boldsymbol{\tau}_1^i(\mathbf{Q}^i,\mathbf{x})+\kappa_2^i(\mathbf{Q}^i,\mathbf{x})\boldsymbol{\tau}_2^i(\mathbf{Q}^i,\mathbf{x})\otimes\boldsymbol{\tau}_2^i(\mathbf{Q}^i,\mathbf{x})$ is the second fundamental form on $\Gamma_2^i(\mathbf{Q})$ at the point $\mathbf{x}$. We refer to [7] for details and precisions on this topic. We introduce the $6-$length column vectors defined on $\Gamma_2(\mathbf{Q})$ by:

$$\mathbf{W}_{\mathbf{n}}^i(\mathbf{Q}^i,\mathbf{x})=\begin{bmatrix}[\omega^i]((\mathbf{x}-\mathbf{h}^i)\wedge\mathbf{n}^i(\mathbf{Q}^i,\mathbf{x}))\\ \mathbf{n}^i(\mathbf{Q}^i,\mathbf{x}))\end{bmatrix}, \tag{6.1a}$$

$$\mathbf{W}_{\tau_k}^i(\mathbf{Q}^i,\mathbf{x})=\begin{bmatrix}[\omega^i]((\mathbf{x}-\mathbf{h}^i)\wedge\boldsymbol{\tau}_k^i(\mathbf{Q}^i,\mathbf{x}))\\ \boldsymbol{\tau}_k^i(\mathbf{Q}^i,\mathbf{x})\end{bmatrix}, \quad k=1,2,\ \forall\mathbf{x}\in\Gamma_2^i(\mathbf{Q}^i), \tag{6.1b}$$

and $\mathbf{W}_{\mathbf{n}}^i(\mathbf{Q}^i,\mathbf{x})=\mathbf{W}_{\tau_k}^i(\mathbf{Q}^i,\mathbf{x})=\mathbf{0}_6$ if $\mathbf{x}\in\Gamma_2^j(\mathbf{Q}^j)$, $j\neq i$. The $6n-$length column vectors $\mathbf{W}_{\mathbf{n}}(\mathbf{Q},\mathbf{x})$ and $\mathbf{W}_{\tau_k}(\mathbf{Q},\mathbf{x})$ are obtained by concatenating respectively the n vectors $\mathbf{W}_{\tau_k}^i(\mathbf{Q}^i,\mathbf{x})$ and the n vectors $\mathbf{W}_{\mathbf{n}}^i(\mathbf{Q}^i,\mathbf{x})$. For all $\mathbf{x}\in\Gamma_2(\mathbf{Q})$ and $k=1,2$ we set $\kappa_k(\mathbf{Q},\mathbf{x}) = \kappa_k^i(\mathbf{Q}^i,\mathbf{x})$ if $\mathbf{x} \in \Gamma_2^i(\mathbf{Q}^i)$. The $6n \times 6n$ symmetric matrix $[B(\mathbf{Q},\mathbf{x})]$ is bloc-diagonal $[B(\mathbf{Q},\mathbf{x})] = \mathrm{diag}([B^1(\mathbf{Q}^1,\mathbf{x})],\ldots,[B^n(\mathbf{Q}^n,\mathbf{x})])$. Each 6×6 symmetric sub-matrix $[B^i(\mathbf{Q}^i,\mathbf{x})]$ is defined on $\Gamma_2^i(\mathbf{Q}^i)$ by:

$$[B^i(\mathbf{Q}_i,\mathbf{x})] = \begin{bmatrix} [\omega^i]^\top & [0_3] \\ [0_3] & [\mathrm{I}_3] \end{bmatrix} [\widetilde{B}^i(\mathbf{Q}^i,\mathbf{x})] \begin{bmatrix} [\omega^i] & [0_3] \\ [0_3] & [\mathrm{I}_3] \end{bmatrix}, \tag{6.2}$$

with

$$\begin{aligned}
\widetilde{B}^i_{kl}(\mathbf{Q}_i,\mathbf{x}) &= -n^i_k(x_l - h^i_l), && 1 \le k < l \le 3,\\
\widetilde{B}^i_{kk}(\mathbf{Q}_i,\mathbf{x}) &= (\mathbf{x}-\mathbf{h}^i)\cdot\mathbf{n}^i - (x_k - h^i_k)n^i_k, && k = 1,2,3,\\
\widetilde{B}^i_{kl}(\mathbf{Q}_i,\mathbf{x}) &= -n^i_l(x_k - h^i_k), && 1 \le l < k \le 3,\\
\widetilde{B}^i_{kl}(\mathbf{Q}_i,\mathbf{x}) &= 0, && k > 3 \text{ or } l > 3.
\end{aligned}$$

On $\Gamma_2^j(\mathbf{Q}^j)$ for $j \neq i$, we have $[B^i(\mathbf{Q}^i,\mathbf{x})] = [0_{6\times 6}]$. At last, the matrix $[C(\mathbf{Q},\mathbf{x})]$ is also bloc-diagonal $[C(\mathbf{Q},\mathbf{x})] = \mathrm{diag}([C^1(\mathbf{Q}^1,\mathbf{x})],\ldots,[C^n(\mathbf{Q}^n,\mathbf{x})])$. Each sub-matrix $[C^i(\mathbf{Q}^i,\mathbf{x})]$ is defined on $\Gamma_2^i(\mathbf{Q}^i)$ by:

$$[C^i(\mathbf{Q}_i,\mathbf{x})] = \begin{bmatrix} [\omega^i]^\top & [0_3] \\ [0_3] & [\mathrm{I}_3] \end{bmatrix} [\widetilde{C}^i(\mathbf{Q}^i,\mathbf{x})] \begin{bmatrix} [\omega^i] & [0_3] \\ [0_3] & [\mathrm{I}_3] \end{bmatrix}, \tag{6.3}$$

with $[\widetilde{C}^i(\mathbf{Q}^i,\mathbf{x})] = \sum_{k=3}^{6} \mathbf{e}^i_k \otimes (\mathbf{n}^i \wedge \mathbf{e}_k)$ and on $\Gamma_2^j(\mathbf{Q}^j)$ for $j \neq i$ by $[C^i(\mathbf{Q}^i,\mathbf{x})] = [0_{6\times 6}]$. We set then the following three-rank $6n \times 6n \times 6n$ tensors:

$$\begin{aligned}
[\![T_{F_\mathbf{n}}(\mathbf{Q})]\!] = {} & -\rho_F \int_{\Gamma_2(\mathbf{Q})} [D\boldsymbol{\varphi}(\mathbf{Q})][D\boldsymbol{\varphi}(\mathbf{Q})]^\top \otimes \mathbf{W}_\mathbf{n}(\mathbf{Q})\,\mathrm{d}\sigma_x \\
& + \rho_F \frac{1}{2} \int_{\Gamma_2(\mathbf{Q})} \mathbf{W}_\mathbf{n}(\mathbf{Q}) \otimes [D\boldsymbol{\varphi}(\mathbf{Q})][D\boldsymbol{\varphi}(\mathbf{Q})]^\top\,\mathrm{d}\sigma_x,
\end{aligned} \tag{6.4}$$

and for $k = 1,2$:

$$\begin{aligned}
&[\![T_{F_\tau}(\mathbf{Q})]\!] = \\
&\quad \sum_{k=1}^{2} \rho_F \int_{\Gamma_2(\mathbf{Q})} \kappa_k(\mathbf{Q})\boldsymbol{\varphi}(\mathbf{Q}) \otimes [\mathbf{W}_{\tau_k}(\mathbf{Q}) \otimes \mathbf{W}_{\tau_k}(\mathbf{Q}) - \mathbf{W}_\mathbf{n}(\mathbf{Q}) \otimes \mathbf{W}_\mathbf{n}(\mathbf{Q})]\,\mathrm{d}\sigma_x,
\end{aligned} \tag{6.5}$$

and:

$$[\![T_R(\mathbf{Q})]\!] = \rho_F \int_{\Gamma_2(\mathbf{Q})} \boldsymbol{\varphi}(\mathbf{Q}) \otimes ([B(\mathbf{Q})] + [C(\mathbf{Q})]) + \frac{1}{2}([C(\mathbf{Q})] - [C(\mathbf{Q})]^\top) \otimes \boldsymbol{\varphi}(\mathbf{Q})\,\mathrm{d}\sigma_x. \tag{6.6}$$

In the following Theorem, the tensor $[\![T_S(\mathbf{Q})]\!]$ is defined by (4.6).

Theorem 6.1 *Assume that $\partial\Omega$ is of class $C^{1,1}$. Then, the Euler-Lagrange system of ODE's* (1.2) *reads:*

$$\begin{aligned}
&([K_S(\mathbf{Q})] + [K_F(\mathbf{Q})])\ddot{\mathbf{Q}} + \langle [\![T_S(\mathbf{Q})]\!], \dot{\mathbf{Q}}, \dot{\mathbf{Q}} \rangle \\
&\qquad + \langle [\![T_{F_\mathbf{n}}(\mathbf{Q})]\!], \dot{\mathbf{Q}}, \dot{\mathbf{Q}} \rangle + \langle [\![T_{F_\tau}(\mathbf{Q})]\!], \dot{\mathbf{Q}}, \dot{\mathbf{Q}} \rangle + \langle [\![T_R(\mathbf{Q})]\!], \dot{\mathbf{Q}}, \dot{\mathbf{Q}} \rangle = \mathbf{0}_{6n}.
\end{aligned}$$

Furthermore, all the matrices and tensors involved in this equation are well defined and indefinitely differentiable with respect to $\mathbf{Q}$ for all $\mathbf{Q} \in \mathcal{Q}$.

Proof : We recall that the vectors $\mathbf{V}_k^i$ and the functions g_k^i and G_k^i arising in the sequel are defined respectively by (4.8), (4.9) and (4.10). Applying the formula (2.3) and (4.4), we get:

$$\begin{aligned}
&\frac{\partial}{\partial \theta_l^i}(\mathcal{R}(\boldsymbol{\theta}^i)^\top \boldsymbol{\omega}_k^i) = -\mathcal{R}(\boldsymbol{\theta}^i)^\top(\boldsymbol{\omega}_l^i \wedge \boldsymbol{\omega}_k^i) + \mathcal{R}(\boldsymbol{\theta}^i)^\top(\boldsymbol{\omega}_l^i \wedge \boldsymbol{\omega}_k^i) = \mathbf{0}_3, \quad 1 \leq l < k \leq 3,\\
&\frac{\partial}{\partial \theta_l^i}(\mathcal{R}(\boldsymbol{\theta}^i)^\top \boldsymbol{\omega}_k^i) = -\mathcal{R}(\boldsymbol{\theta}^i)^\top(\boldsymbol{\omega}_l^i \wedge \boldsymbol{\omega}_k^i), \qquad 1 \leq k \leq l \leq 3.
\end{aligned}$$

From these identities, we deduce that, on Γ_2^i and for all $i = 1, \dots, n$:

$$\begin{aligned}
&\frac{\partial G_k^i}{\partial \theta_l^i}(\mathbf{Q}^i, \mathbf{x}) = 0, && 1 \leq l < k \leq 3,\\
&\frac{\partial G_k^i}{\partial \theta_l^i}(\mathbf{Q}^i, \mathbf{x}) = \mathbf{n}^i \cdot ((-\mathcal{R}(\boldsymbol{\theta}^i)^\top(\boldsymbol{\omega}_l^i \wedge \boldsymbol{\omega}_k^i)) \wedge (\mathbf{x} - \mathbf{h}_0^i)), && 1 \leq k \leq l \leq 3,\\
&\frac{\partial G_{k+3}^i}{\partial \theta_l^i}(\mathbf{Q}^i, \mathbf{x}) = -\mathcal{R}(\boldsymbol{\theta}^i)\mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \cdot (\boldsymbol{\omega}_l^i \wedge \mathbf{e}_k), && 1 \leq k, l \leq 3
\end{aligned}$$

and

$$\frac{\partial G_k^i}{\partial h_l^i}(\mathbf{Q}^i, \mathbf{x}) = 0, \qquad 1 \leq l \leq 3,\ 1 \leq k \leq 6.$$

Using the notation $(q_1^i, \dots, q_6^i) = (\theta_1^i, \theta_2^i, \theta_3^i, h_1^i, h_2^i, h_3^i)$, we have therefore on Γ_2^i:

$$\begin{aligned}
&\frac{\partial G_k^i}{\partial q_l^i}(\mathbf{Q}^i, \phi^{-1}(\mathbf{Q}, \mathbf{x})) = 0, && 1 \leq l < k \leq 3,\\
&\frac{\partial G_k^i}{\partial q_l^i}(\mathbf{Q}^i, \phi^{-1}(\mathbf{Q}, \mathbf{x})) = (\boldsymbol{\omega}_l^i \wedge \mathbf{n}^i(\mathbf{Q}^i)) \cdot \mathbf{V}_k^i(\mathbf{Q}^i) &&\\
&\qquad\qquad - (\boldsymbol{\omega}_k^i \wedge \mathbf{n}^i(\mathbf{Q}^i)) \cdot \mathbf{V}_l^i(\mathbf{Q}^i), && 1 \leq k \leq l \leq 3,\\
&\frac{\partial G_{k+3}^i}{\partial q_l^i}(\mathbf{Q}^i, \phi^{-1}(\mathbf{Q}, \mathbf{x})) = (\mathbf{n}^i(\mathbf{Q}^i) \wedge \mathbf{e}_k) \cdot \boldsymbol{\omega}_l^i, && 1 \leq l \leq 3, \quad 1 \leq k \leq 3,\\
&\frac{\partial G_k^i}{\partial q_l^i}(\mathbf{Q}^i, \phi^{-1}(\mathbf{Q}, \mathbf{x})) = 0, && 3 \leq l \leq 6, \quad 1 \leq k \leq 6.
\end{aligned}$$

If $1 \leq l, k \leq 3$, we can rewrite, taking into account the definition of the vectors $\mathbf{V}_l^i$ and the properties of the wedge product:

$$\begin{aligned}
(\boldsymbol{\omega}_l^i \wedge \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x})) \cdot \mathbf{V}_k^i &= (\boldsymbol{\omega}_l^i \wedge \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x})) \cdot (\boldsymbol{\omega}_k^i \wedge (\mathbf{x} - \mathbf{h}^i))\\
&= \boldsymbol{\omega}_k^{i\top}[((\mathbf{x} - \mathbf{h}^i) \cdot \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}))[I_3] - \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \otimes (\mathbf{x} - \mathbf{h}^i)]\boldsymbol{\omega}_l^i.
\end{aligned}$$

In order to apply formula (3.5), we need also to compute the quantities $\nabla_\tau g_k^i(\mathbf{Q}^i, \mathbf{x}) \cdot \mathbf{V}_l^i(\mathbf{Q}^i, \mathbf{x})$. We will denote merely $\mathbf{V}_l^i$ instead of $\mathbf{V}_l^i(\mathbf{Q}^i, \mathbf{x})$ in the following. We have, for all $i = 1, \dots, n$ and for all $k = 1, 2, 3$:

$$\begin{aligned}
\nabla_\tau(\mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \cdot (\boldsymbol{\omega}_k^i \wedge (\mathbf{x} - \mathbf{h}^i))) \cdot \mathbf{V}_l^i &= ([\nabla_\tau \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x})]\mathbf{V}_l^i) \cdot (\boldsymbol{\omega}_k^i \wedge (\mathbf{x} - \mathbf{h}^i))\\
&\quad + \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \cdot (\boldsymbol{\omega}_k^i \wedge [\nabla_\tau(\mathbf{x} - \mathbf{h}^i)]\mathbf{V}_l^i).
\end{aligned}$$

According to the definition of the tangential gradient in [11] page 192, we get:

$$[\nabla_\tau(\mathbf{x} - \mathbf{h}^i)]\mathbf{V}_l^i = \mathbf{V}_l^i - (\mathbf{V}_l^i \cdot \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}))\mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}),$$

and therefore:

$$\nabla_\tau g_k^i(\mathbf{Q}^i,\mathbf{x})\cdot\mathbf{V}_l^i = [\nabla_\tau \mathbf{n}^i(\mathbf{Q}^i,\mathbf{x})]\mathbf{V}_l^i\cdot\mathbf{V}_k^i - (\boldsymbol{\omega}_k^i\wedge\mathbf{n}^i(\mathbf{Q}^i,\mathbf{x}))\cdot\mathbf{V}_l^i \quad 1\le k\le 3. \tag{6.7a}$$

In the same way, we can compute:

$$\nabla_\tau g_k^i(\mathbf{Q}^i,\mathbf{x})\cdot\mathbf{V}_l^i = (\nabla_\tau \mathbf{n}^i(\mathbf{Q}^i,\mathbf{x})\cdot\mathbf{V}_l^i)\cdot\mathbf{V}_k^i, \quad 4\le k\le 6. \tag{6.7b}$$

Denoting then $T\Gamma_2^i(\mathbf{Q}^i)$ the tangent bundle of $\Gamma_2^i(\mathbf{Q}^i)$, $T\mathbb{S}^2$ the tangent bundle of the 2-dimensional sphere $\mathbb{S}^2$ and $\Pi^i : T\Gamma_2^i(\mathbf{Q}^i)\times T\Gamma_2^i(\mathbf{Q}^i)\to T\mathbb{S}_2$ the second fundamental form on $\Gamma_2^i(\mathbf{Q}^i)$ (see [7] for a definition of these notions), we rewrite (6.7):

$$\begin{aligned}
\nabla_\tau g_k^i(\mathbf{Q}^i,\mathbf{x})\cdot\mathbf{V}_l^i &= -\langle \Pi^i(\mathbf{Q}^i,\mathbf{x}),\mathbf{V}_l^i,\mathbf{V}_k^i\rangle - (\boldsymbol{\omega}_k^i\wedge\mathbf{n}^i(\mathbf{Q}^i,\mathbf{x}))\cdot\mathbf{V}_l^i, && 1\le k\le 3,\\
\nabla_\tau g_k^i(\mathbf{Q}^i,\mathbf{x})\cdot\mathbf{V}_l^i &= -\langle \Pi^i(\mathbf{Q}^i,\mathbf{x}),\mathbf{V}_l^i,\mathbf{V}_k^i\rangle, && 4\le k\le 6.
\end{aligned}$$

According to the notation introduced earlier in, we have the identity:

$$\Pi^i(\mathbf{Q}^i,\mathbf{x}) = \kappa_1^i(\mathbf{Q}^i,\mathbf{x})\boldsymbol{\tau}_1^i(\mathbf{Q}^i,\mathbf{x})\otimes\boldsymbol{\tau}_1^i(\mathbf{Q}^i,\mathbf{x}) + \kappa_2^i(\mathbf{Q}^i,\mathbf{x})\boldsymbol{\tau}_2^i(\mathbf{Q}^i,\mathbf{x})\otimes\boldsymbol{\tau}_2^i(\mathbf{Q}^i,\mathbf{x}).$$

Building now the 6×6 matrix $[\Lambda^i(\mathbf{Q}^i,\mathbf{x})]$ whose entries are:

$$\begin{aligned}
\Lambda_{kl}^i(\mathbf{Q}^i,\mathbf{x}) = \frac{\partial G_k^i}{\partial q_l^i}(\mathbf{Q}^i,\mathbf{x}) - \nabla_\tau g_k^i(\mathbf{Q}^i,\mathbf{x})\cdot\mathbf{V}_l^i(\mathbf{Q}^i,\mathbf{x})\\
- (\kappa_1^i(\mathbf{Q}^i,\mathbf{x})+\kappa_2^i(\mathbf{Q}^i,\mathbf{x}))g_k^i(\mathbf{Q}^i,\mathbf{x})(\mathbf{V}_l^i(\mathbf{Q}^i,\mathbf{x})\cdot\mathbf{n}^i(\mathbf{Q}^i,\mathbf{x})),
\end{aligned}$$

we get the identity:

$$\begin{aligned}
[\Lambda^i(\mathbf{Q}^i,\mathbf{x})] &= \begin{bmatrix}[\omega^i]^\top & [0_3]\\ [0_3] & [\mathrm{I}_3]\end{bmatrix}\Big([\widetilde{A}^i(\mathbf{Q}^i)] + [\widetilde{B}^i(\mathbf{Q}^i)] + [\widetilde{C}^i(\mathbf{Q}^i)]\Big)\begin{bmatrix}[\omega^i] & [0_3]\\ [0_3] & [\mathrm{I}_3]\end{bmatrix}\\
&= [A^i(\mathbf{Q}^i)] + [B^i(\mathbf{Q}^i)] + [C^i(\mathbf{Q}^i)].
\end{aligned}$$

Let us drop $\mathbf{Q}^i$ and $\mathbf{x}$ in the sequel to simplify the notations. The matrix $[A^i]$ decomposes into four 3×3 sub-matrices $[A_{kl}]$, $1\le k,l\le 2$ defined by:

$$\begin{aligned}
[\widetilde{A}_{11}^i] &= \sum_{k=1}^{2}\kappa_k^i((\mathbf{x}-\mathbf{h}^i)\wedge\boldsymbol{\tau}_k^i\otimes(\mathbf{x}-\mathbf{h}^i)\wedge\boldsymbol{\tau}_k^i - (\mathbf{x}-\mathbf{h}^i)\wedge\mathbf{n}^i\otimes(\mathbf{x}-\mathbf{h}^i)\wedge\mathbf{n}^i),\\
[\widetilde{A}_{21}^i] &= [\widetilde{A}_{12}^i]^\top = \sum_{k=1}^{2}\kappa_k^i(\boldsymbol{\tau}_k^i\otimes(\mathbf{x}-\mathbf{h}^i)\wedge\boldsymbol{\tau}_k^i - \mathbf{n}^i\otimes(\mathbf{x}-\mathbf{h}^i)\wedge\mathbf{n}^i),\\
[\widetilde{A}_{22}^i] &= \sum_{k=1}^{2}\kappa_k^i(\boldsymbol{\tau}_k^i\otimes\boldsymbol{\tau}_k^i - \mathbf{n}^i\otimes\mathbf{n}^i).
\end{aligned}$$

The matrices $[\widetilde{B}^i]$ and $[\widetilde{C}^i]$ are defined by (6.2) and (6.3). Applying formula (3.5), we get:

$$\begin{aligned}
\left\langle\frac{\partial[K_F(\mathbf{Q})]}{\partial\mathbf{Q}},\dot{\mathbf{Q}}\right\rangle\dot{\mathbf{Q}} &= -\rho_F\int_{\Gamma_2(\mathbf{Q})}[D\boldsymbol{\varphi}][D\boldsymbol{\varphi}]^\top\dot{\mathbf{Q}}(\mathbf{W_n}\cdot\dot{\mathbf{Q}})\,\mathrm{d}\sigma_x\\
&+\sum_{k=1}^{2}\rho_F\int_{\Gamma_2(\mathbf{Q})}\kappa_k(\boldsymbol{\varphi}(\mathbf{W}_{\tau_k}\cdot\dot{\mathbf{Q}}) + \mathbf{W}_{\tau_k}(\boldsymbol{\varphi}\cdot\dot{\mathbf{Q}}))(\mathbf{W}_{\tau_k}\cdot\dot{\mathbf{Q}})\,\mathrm{d}\sigma_x\\
&+\rho_F\int_{\Gamma_2(\mathbf{Q})}\boldsymbol{\varphi}(\dot{\mathbf{Q}}^\top([B]+[C])\dot{\mathbf{Q}}) + ([B]+[C])\dot{\mathbf{Q}}(\boldsymbol{\varphi}\cdot\dot{\mathbf{Q}})\,\mathrm{d}\sigma_x\\
&-\rho_F\int_{\Gamma_2(\mathbf{Q})}(\kappa_1+\kappa_2)(\boldsymbol{\varphi}(\mathbf{W_n}\cdot\dot{\mathbf{Q}}) + \mathbf{W_n}(\boldsymbol{\varphi}\cdot\dot{\mathbf{Q}}))(\mathbf{W_n}\cdot\dot{\mathbf{Q}})\,\mathrm{d}\sigma_x,
\end{aligned}$$

where the vectors $\mathbf{W}_{\tau_1}(\mathbf{Q},\cdot)$, $\mathbf{W}_{\tau_2}(\mathbf{Q},\cdot)$ and $\mathbf{W}_{\mathbf{n}}(\mathbf{Q},\cdot)$ are defined on $\Gamma_2(\mathbf{Q})$ by (6.1). Proceeding as well we get:

$$\begin{aligned}\dot{\mathbf{Q}}^\top\left\langle\frac{\partial[K_F(\mathbf{Q})]}{\partial\mathbf{Q}},\cdot\right\rangle\dot{\mathbf{Q}} &= -\rho_F\int_{\Gamma_2(\mathbf{Q})}(\dot{\mathbf{Q}}^\top[D\boldsymbol{\varphi}][D\boldsymbol{\varphi}]^\top\dot{\mathbf{Q}})\mathbf{W}_{\mathbf{n}}\,\mathrm{d}\sigma_x\\ &+2\sum_{k=1}^{2}\rho_F\int_{\Gamma_2(\mathbf{Q})}\kappa_k(\boldsymbol{\varphi}\cdot\dot{\mathbf{Q}})(\mathbf{W}_{\tau_k}\cdot\dot{\mathbf{Q}})\mathbf{W}_{\tau_k}\,\mathrm{d}\sigma_x+2\int_{\Gamma_2(\mathbf{Q})}(\boldsymbol{\varphi}\cdot\dot{\mathbf{Q}})[B]\dot{\mathbf{Q}}\,\mathrm{d}\sigma_x\\ &+\int_{\Gamma_2(\mathbf{Q})}(\boldsymbol{\varphi}\cdot\dot{\mathbf{Q}})([C]+[C]^\top)\dot{\mathbf{Q}}\,\mathrm{d}\sigma_x-2\rho_F\int_{\Gamma_2(\mathbf{Q})}(\kappa_1+\kappa_2)(\boldsymbol{\varphi}\cdot\dot{\mathbf{Q}})(\mathbf{W}_n\cdot\dot{\mathbf{Q}})\,\mathrm{d}\sigma_x.\end{aligned}$$

After some simplifications, we deduce that:

$$\begin{aligned}&\left\langle\frac{\partial[K_F(\mathbf{Q})]}{\partial\mathbf{Q}},\dot{\mathbf{Q}}\right\rangle\dot{\mathbf{Q}}-\frac{1}{2}\dot{\mathbf{Q}}^\top\left\langle\frac{\partial[K_F(\mathbf{Q})]}{\partial\mathbf{Q}},\cdot\right\rangle\dot{\mathbf{Q}}=\\ &-\rho_F\int_{\Gamma_2(\mathbf{Q})}[D\boldsymbol{\varphi}][D\boldsymbol{\varphi}]^\top\dot{\mathbf{Q}}(\mathbf{W}_{\mathbf{n}}\cdot\dot{\mathbf{Q}})\,\mathrm{d}\sigma_x+\frac{1}{2}\rho_F\int_{\Gamma_2(\mathbf{Q})}|[D\boldsymbol{\varphi}]^\top\dot{\mathbf{Q}}|^2\mathbf{W}_{\mathbf{n}}\,\mathrm{d}\sigma_x\\ &+\sum_{k=1}^{2}\rho_F\int_{\Gamma_2(\mathbf{Q})}\kappa_k\boldsymbol{\varphi}[(\mathbf{W}_{\tau_k}\cdot\dot{\mathbf{Q}})^2-(\mathbf{W}_{\mathbf{n}}\cdot\dot{\mathbf{Q}})^2]\,\mathrm{d}\sigma_x\\ &+\rho_F\int_{\Gamma_2(\mathbf{Q})}\boldsymbol{\varphi}(\dot{\mathbf{Q}}^\top([B]+[C])\dot{\mathbf{Q}})+\frac{1}{2}([C]-[C]^\top)\dot{\mathbf{Q}}(\boldsymbol{\varphi}\cdot\dot{\mathbf{Q}})\,\mathrm{d}\sigma_x.\end{aligned}\quad(6.8)$$

Introducing the tensors $[\![T_{F_{\mathbf{n}}}(\mathbf{Q})]\!]$, $[\![T_{F_\tau}(\mathbf{Q})]\!]$ and $[\![T_R(\mathbf{Q})]\!]$ defined respectively by (6.4), (6.5) and (6.6) and plugging the expressions (4.7) and (6.8) into (4.1), we obtain the formula of Theorem 6.1. ■

A Proofs of the Propositions of section 3

Proof of Proposition 3.1: Let φ be a function of $\mathcal{V}^1$ and define $\widetilde{\varphi}(\mathbf{Q},\cdot)=\varphi(\phi^{-1}(\mathbf{Q},\cdot))$. Simple computations yield:

$$\nabla\widetilde{\varphi}(\mathbf{Q},\cdot)=\nabla\varphi(\phi^{-1}(\mathbf{Q},\cdot))[D\phi(\mathbf{Q},\phi^{-1}(\mathbf{Q},\cdot))]^{-1},$$
$$\int_{\Omega(\mathbf{Q})}|\nabla\widetilde{\varphi}(\mathbf{Q},\mathbf{x})|^2\,\mathrm{d}\mathbf{x}=\int_\Omega\nabla\varphi(\mathbf{x})[A(\mathbf{Q},\mathbf{x})]\nabla\varphi^\top(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

and according to the hypothesis (H_4), $\varphi\circ\phi^{-1}(\mathbf{Q},\cdot)$ lies in $\mathcal{V}^1(\mathbf{Q})$. The variational formulation (3.4) can be rewritten with the test function $\varphi\circ\phi^{-1}(\mathbf{Q},\cdot)$ and yields, upon a change of variables:

$$\int_\Omega\nabla U_k(\mathbf{Q},\mathbf{x})[A(\mathbf{Q},\mathbf{x})]\nabla\varphi^\top(\mathbf{x})\,\mathrm{d}\mathbf{x}=\int_\Gamma G(\mathbf{Q},\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}\sigma_x.\quad(\mathrm{A}.1)$$

Notice that the expression of the integral on Γ results from the hypothesis (H_3). We define then the mapping:

$$\begin{aligned}F_k^1:\mathcal{Q}\times\mathcal{V}^1&\to(\mathcal{V}^1)'\\ (\mathbf{Q},U)&\mapsto\mathcal{A}(\mathbf{Q})U-\mathcal{B}_k(\mathbf{Q}),\quad k=1,2,\end{aligned}$$

where $\mathcal{A}(\mathbf{Q}):\mathcal{V}^1\to(\mathcal{V}^1)'$ and $\mathcal{B}_k(\mathbf{Q})\in(\mathcal{V}^1)'$ are defined by

$$\langle\mathcal{A}(\mathbf{Q})U,\varphi\rangle_{(\mathcal{V}^1)'\times\mathcal{V}^1}=\int_\Omega\nabla U[A(\mathbf{Q})]\nabla\varphi^\top\,\mathrm{d}\mathbf{x},\quad\forall\,U,\varphi\in\mathcal{V}^1,$$
$$\langle\mathcal{B}_k(\mathbf{Q}),\varphi\rangle_{(\mathcal{V}^1)'\times\mathcal{V}^1}=\int_\Gamma G_k(\mathbf{Q})\varphi\,\mathrm{d}\sigma_x,\quad\forall\,\varphi\in\mathcal{V}^1.$$

The mapping $\mathbf{Q} \in \mathcal{Q} \mapsto \mathcal{B}_k(\mathbf{Q}) \in (\mathcal{V}^1)'$ is of class C^β because $\mathbf{Q} \in \mathcal{Q} \mapsto G_k(\mathbf{Q}, \cdot) \in L^2(\Gamma)$ is and the mapping $\mathbf{Q} \in \mathcal{Q} \mapsto [A(\mathbf{Q})] \in L^\infty(\mathbb{R}^3, \mathcal{M}(3))$ is of class C^α, like ϕ. At last, the mapping

$$\begin{aligned} L^\infty(\mathbb{R}^3, \mathcal{M}(3)) \times \mathcal{V}^1 &\to (\mathcal{V}^1)' \\ ([A], U) &\mapsto \mathcal{A}U, \end{aligned}$$

defined by

$$\langle \mathcal{A}U, \varphi \rangle_{(\mathcal{V}^1)' \times \mathcal{V}^1} = \int_\Omega \nabla U [A] \nabla \varphi^\top \, \mathrm{d}\mathbf{x}, \quad \forall U, \varphi \in \mathcal{V}^1,$$

is of class C^∞. Therefore $(\mathbf{Q}, U) \in \mathcal{Q} \times \mathcal{V}^1 \mapsto \mathcal{A}(\mathbf{Q})U \in (\mathcal{V}^1)'$ is of class C^α and F_k^1 is of class $C^{\min\{\alpha,\beta\}}$ (remark that F_k^1 is of class C^∞ with respect to U because it is linear and continuous). Moreover, $F_k^1(\mathbf{0}_p, u_k(\mathbf{0}_p, \cdot)) = 0$ and $\partial F_k^1 / \partial U(\mathbf{0}_p, u_k(\mathbf{0}_p, \cdot))$ is an isomorphism from $\mathcal{V}^1$ onto $(\mathcal{V}^1)'$, according to the Riesz representation Theorem. Indeed, for all $U, \varphi \in \mathcal{V}^1$ we have:

$$\left\langle \frac{\partial F_k^1}{\partial U}(\mathbf{0}_p, u_k(\mathbf{0}_p, \cdot)) U, \varphi \right\rangle_{(\mathcal{V}^1)' \times \mathcal{V}^1} = \int_\Omega \nabla U \cdot \nabla \varphi \, \mathrm{d}\mathbf{x}.$$

Applying then the implicit function Theorem, we conclude that there exists a neighborhood $\mathcal{Q}_0$ of $\mathbf{Q} = \mathbf{0}_p$ in $\mathcal{Q}$ and a function $\mathbf{Q} \in \mathcal{Q}_0 \mapsto U_k(\mathbf{Q}, \cdot) \in \mathcal{V}^1$ of class $C^{\min\{\alpha,\beta\}}$ such that $F_k^1(\mathbf{Q}, U_k(\mathbf{Q}, \cdot)) = 0$ for all $\mathbf{Q} \in \mathcal{Q}_0$. By uniqueness of the solution of (3.4) in $\mathcal{V}^1(\mathbf{Q})$, we conclude that $U_k(\mathbf{Q}, \cdot) = u_k(\mathbf{Q}, \phi(\mathbf{Q}, \cdot))$.

Proceeding as in (A.1), we rewrite $\Upsilon(\mathbf{Q})$:

$$\Upsilon(\mathbf{Q}) = \int_\Omega \nabla U_1(\mathbf{Q}, \mathbf{x}) [A(\mathbf{Q}, \mathbf{x})] \nabla U_2(\mathbf{Q}, \mathbf{x})^\top \, \mathrm{d}\mathbf{x}.$$

Since $\mathbf{Q} \in \mathcal{Q}_0 \mapsto \nabla U_k(\mathbf{Q}, \cdot) \in L^2(\Omega, \mathbb{R}^3)$ is of class $C^{\min\{\alpha,\beta\}}$ and $\mathbf{Q} \in \mathcal{Q} \mapsto [A(\mathbf{Q}, \cdot)] \in L^\infty(\mathbb{R}^3, \mathcal{M}(3))$ is of class C^∞, we obtain the expected regularity result for $\mathbf{Q} \in \mathcal{Q}_0 \mapsto \Upsilon(\mathbf{Q})$, namely $C^{\min\{\alpha,\beta\}}$.

Assume now that Γ is of class $C^{1,1}$ and that $G_k(\mathbf{Q}, \cdot)$ is like in the hypothesis of the second part of the Proposition. We define

$$\begin{aligned} F_k^2 : \mathcal{Q} \times \mathcal{V}^2 &\to \mathcal{L}^2(\Omega) \times \widetilde{H}^{1/2}(\Gamma) \\ (\mathbf{Q}, U) &\mapsto \begin{pmatrix} -\mathrm{div}([A(\mathbf{Q})] \nabla U) \\ \nabla U|_\Gamma \cdot \mathbf{n} - G_k(\mathbf{Q}, \cdot) \end{pmatrix}. \end{aligned}$$

Taking into account (H$_2$), $[A(\mathbf{Q})] - [I_3]$ is compactly supported and $\mathrm{div}([A(\mathbf{Q})] \nabla U) = \mathrm{div}([A(\mathbf{Q})]) \cdot \nabla U + [A(\mathbf{Q})] : [D^2 U]$ lies in $\mathcal{L}^2(\Omega)$ when Ω is not bounded. When Ω is bounded, we have to check furthermore that $\int_\Omega \mathrm{div}([A(\mathbf{Q})] \nabla U) \, \mathrm{d}\mathbf{x} = 0$. Since $[A(\mathbf{Q})] = [I_3]$ on Γ, we have

$$\int_\Omega \mathrm{div}([A(\mathbf{Q})] \nabla U) \, \mathrm{d}\mathbf{x} = \int_\Gamma \nabla U \cdot \mathbf{n} \, \mathrm{d}\sigma_x = 0.$$

Proceeding as for F_k^1, we prove that F_k^2 is continuously differentiable. Moreover $F_k^2(\mathbf{0}_p, u_k(\mathbf{0}_p, \cdot)) = 0$ and:

$$\frac{\partial F_k^2}{\partial U}(\mathbf{0}_p, u_k(\mathbf{0}_p, \cdot)) U = \begin{pmatrix} -\Delta U \\ \nabla U|_\Gamma \cdot \mathbf{n} \end{pmatrix}.$$

Classical results for elliptic equations [5] when Ω is bounded or results of [1], [2] and [4] when Ω is not, ensure that $\partial F_k^2 / \partial U(\mathbf{0}_p, u_k(\mathbf{0}_p, \cdot))$ is an isomorphism from

$\mathcal{V}^2$ onto $\mathcal{L}^2(\Omega)$. At this point, let us stress that with our choice of functions spaces, the classical compatibility condition $\int_\Omega f\,\mathrm{d}\mathbf{x} + \int_\Gamma g\mathrm{d}\sigma_x = 0$, essential to solve the Neumann problem $-\Delta u = f$ in Ω, $\partial u/\partial \mathbf{n} = g$ on Γ when Ω is bounded, is not needed when Ω is not bounded (see [1] and [4]). We apply then the implicit function Theorem and conclude as for F_1^k.

Let K be a compact subset in Ω and consider a function $\zeta_K \in C_0^\infty(\Omega)$ such that $\zeta_K \equiv 1$ in a neighborhood of K. The function $\mathbf{Q} \mapsto \widetilde{U}_k(\mathbf{Q},\cdot) = \zeta_K(\phi(\mathbf{Q},\cdot))U_k(\mathbf{Q},\cdot) \in L^2(\mathbb{R}^3)$ is continuously differentiable and

$$\frac{\partial \widetilde{U}_k}{\partial q_i}(\mathbf{Q},\cdot) = U_k(\mathbf{Q},\cdot)\nabla\zeta_K(\phi(\mathbf{Q},\cdot))[D\phi(\mathbf{Q},\cdot)] + \zeta_k(\phi(\mathbf{Q},\cdot))\frac{\partial U_k}{\partial q_i}(\mathbf{Q},\cdot).$$

In particular:

$$\frac{\partial \widetilde{U}_k}{\partial q_i}(\mathbf{Q},\phi(\mathbf{Q},\cdot))\bigg|_K = \frac{\partial U_k}{\partial q_i}(\mathbf{Q},\phi(\mathbf{Q},\cdot))\bigg|_K.$$

Since $\partial\widetilde{U}_k/\partial q_i(\mathbf{Q},\cdot) \in H^1(\mathbb{R}^3)$, we deduce from the Lemma 5.3.3 page 181 of [11] that $\mathbf{Q} \mapsto \widetilde{u}_k(\mathbf{Q},\cdot) = \widetilde{U}_k(\mathbf{Q},\phi^{-1}(\mathbf{Q},\cdot)) = \zeta_K(\cdot)u_k(\mathbf{Q},\cdot) \in L^2(\mathbb{R}^3)$ is differentiable. Furthermore $\widetilde{U}_k$ and $\widetilde{u}_k$ are linked through the formula:

$$\frac{\partial \widetilde{u}_k}{\partial q_i}(\mathbf{Q},\cdot) = \frac{\partial \widetilde{U}_k}{\partial q_i}(\mathbf{Q},\phi^{-1}(\mathbf{Q},\cdot)) - \nabla\widetilde{U}_k(\mathbf{Q},\phi^{-1}(\mathbf{Q},\cdot))\cdot\mathbf{V}^i(\mathbf{Q},\cdot), \quad i=1,\ldots,p.$$

We can define the function $\partial u_k/\partial q_i(\mathbf{0}_p,\cdot)$ in the whole set Ω by its restriction to any compact subset of Ω :

$$\frac{\partial u_k}{\partial q_i}(\mathbf{0}_p,\cdot)\bigg|_K = \frac{\partial \widetilde{u}_k}{\partial q_i}(\mathbf{0}_p,\cdot)\bigg|_K, \quad \forall K \subset \Omega.$$

Since $\partial U_k/\partial q_i(\mathbf{0}_p,\cdot) \in \mathcal{V}^1$, $\nabla U_k(\mathbf{0}_p,\cdot) \in L^2(\Omega,\mathbb{R}^3)$ and $V^i(\mathbf{0}_p,\cdot) \equiv 0$ on Γ_1, we obtain therefore that:

$$\frac{\partial u_k}{\partial q_i}(\mathbf{0}_p,\cdot) = \frac{\partial U_k}{\partial q_i}(\mathbf{0}_p,\cdot) - \nabla U_k(\mathbf{0}_p,\cdot)\cdot\mathbf{V}^i(\mathbf{0}_p,\cdot) \in \mathcal{V}^1,$$

where $\mathbf{V}^i$ is defined by (3.6). We proceed as well to define $\partial\nabla u_k/\partial q_i(\mathbf{0}_p,\cdot) \in L^2(\Omega,\mathbb{R}^3)$, taking into account that $U_k(\mathbf{0}_p,\cdot) \in \mathcal{V}^2$. ■

Proof of Proposition 3.2: The domain Ω being of class $C^{1,1}$, we can extend any function $\varphi \in \mathcal{V}^2$ to be a function of $\widetilde{\mathcal{V}}^2$ and for all $\mathbf{Q} \in \mathcal{Q}$, $\varphi|_{\Omega(\mathbf{Q})}$ lies in $\mathcal{V}^2(\mathbf{Q})$. Choosing this test function in (3.4), we compute the derivative of the variational formulation:

$$\int_{\Omega(\mathbf{Q})} \nabla u_k(\mathbf{Q},\mathbf{x})\cdot\nabla\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Gamma G_k(\mathbf{Q},\mathbf{x})\varphi(\phi(\mathbf{Q},\mathbf{x}))\,\mathrm{d}\sigma_x,$$

at the point $\mathbf{Q} = \mathbf{0}_p$ with respect to q_i. Applying the Corollary 5.2.5 of [11] page 173, we obtain that:

$$\int_{\Omega(\mathbf{Q})} \frac{\partial\nabla u_k}{\partial q_i}(\mathbf{0}_p,\mathbf{x})\cdot\nabla\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\Gamma_2}\nabla u_k(\mathbf{0}_p,\mathbf{x})\cdot\nabla\varphi(\mathbf{x})(\mathbf{V}^i\cdot\mathbf{n})\,\mathrm{d}\sigma_x$$
$$= \int_{\Gamma_1}\frac{\partial G_k}{\partial q_i}(\mathbf{0}_p,\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}\sigma_x + \frac{d}{dq_i}\left(\int_{\Gamma_2} G_k(\mathbf{Q},\mathbf{x})\varphi(\phi(\mathbf{Q},\mathbf{x}))\,\mathrm{d}\sigma_x\right). \quad \text{(A.2)}$$

The Lemma 5.3.3 page 181 of [11] ensures that $\mathbf{Q} \in \mathcal{Q} \mapsto \varphi(\phi(\mathbf{Q},\cdot))$ is differentiable and that

$$\frac{\partial\varphi}{\partial q_i}(\phi(\mathbf{Q},\cdot)) = \nabla\varphi(\phi(\mathbf{Q},\cdot))\cdot\mathbf{V}^i(\phi(\mathbf{Q},\cdot)) \in \widetilde{\mathcal{V}}^1.$$

In particular $\partial\varphi/\partial q_i(\phi(\mathbf{Q},\cdot))|_{\Gamma_2} \in H^{1/2}(\Gamma_2)$ and we have :

$$\frac{d}{dq_i}\left(\int_{\Gamma_2} G_k(\mathbf{Q},\mathbf{x})\varphi(\phi(\mathbf{Q},\mathbf{x}))\,\mathrm{d}\sigma_x\right) = \\ \int_{\Gamma_2} \frac{\partial G_k}{\partial q_i}(\mathbf{Q},\mathbf{x})\varphi(\phi(\mathbf{Q},\mathbf{x})) + G_k(\mathbf{Q},\mathbf{x})\nabla\varphi(\phi(\mathbf{Q},\mathbf{x}))\cdot\mathbf{V}^i(\phi(\mathbf{Q},\mathbf{x}))\,\mathrm{d}\sigma_x.$$

Plugging this expression into (A.2), specifying $\mathbf{Q} = \mathbf{0}_p$ and denoting merely u_k, G_k and $\mathbf{V}^i$ for $u_k(\mathbf{0}_p,\cdot)$, $G_k(\mathbf{0}_p,\cdot)$ and $\mathbf{V}^i(\mathbf{0}_p,\cdot)$, we obtain on the one hand the variational formulation for ∇u_k:

$$\int_{\Omega(\mathbf{Q})} \frac{\partial\nabla u_k}{\partial q_i}\cdot\nabla\varphi\,\mathrm{d}\mathbf{x} + \int_{\Gamma_2}\nabla u_k\cdot\nabla\varphi(\mathbf{V}^i\cdot\mathbf{n})\,\mathrm{d}\sigma_x = \int_{\Gamma_1}\frac{\partial G_k}{\partial q_i}\varphi\,\mathrm{d}\sigma_x \\ + \int_{\Gamma_2}\frac{\partial G_k}{\partial q_i}\varphi + G_k\nabla\varphi\cdot\mathbf{V}_i\,\mathrm{d}\sigma_x, \quad \forall\varphi\in\widetilde{\mathcal{V}}^2. \quad \text{(A.3)}$$

On the other hand, computing the derivative at the point $\mathbf{Q} = \mathbf{0}_p$ of the expression of $\Upsilon(\mathbf{Q})$ and applying the Corollary 5.2.5 of [11] page 173, we get:

$$\frac{d}{dq_i}\left(\int_{\Omega(\mathbf{Q})}\nabla u_1(\mathbf{Q},\mathbf{x})\cdot\nabla u_2(\mathbf{Q},\mathbf{x})\,\mathrm{d}\mathbf{x}\right)\Bigg|_{\mathbf{Q}=\mathbf{0}_p} = \int_\Gamma \frac{\partial\nabla u_1}{\partial q_i}\cdot\nabla u_2\,\mathrm{d}\sigma_x \\ + \int_\Gamma\frac{\partial\nabla u_2}{\partial q_i}\cdot\nabla u_1\,\mathrm{d}\sigma_x + \int_{\Gamma_2}\nabla u_1\cdot\nabla u_2(\mathbf{V}^i\cdot\mathbf{n})\,\mathrm{d}\sigma_x.$$

Since u_1 and u_2 lie in $\mathcal{V}^2$, we can use the identity (A.3) in order to transform both right hand side terms. We get therefore:

$$\frac{d}{dq_i}\left(\int_{\Omega(\mathbf{Q})}\nabla u_1(\mathbf{Q},\mathbf{x})\cdot\nabla u_2(\mathbf{Q},\mathbf{x})\,\mathrm{d}\mathbf{x}\right)\Bigg|_{\mathbf{Q}=\mathbf{0}_p} = -\int_{\Gamma_2}\nabla u_1\cdot\nabla u_2(\mathbf{V}^i\cdot\mathbf{n})\,\mathrm{d}\sigma_x \\ + \int_{\Gamma_1}\frac{\partial G_1}{\partial q_i}u_2 + \frac{\partial G_2}{\partial q_i}u_1\,\mathrm{d}\sigma_x + \int_{\Gamma_2} G_1\nabla u_2\cdot\mathbf{V}^i + G_2\nabla u_1\cdot\mathbf{V}^i\,\mathrm{d}\sigma_x. \quad \text{(A.4)}$$

Then, decomposing $\mathbf{V}^i$ into its normal and tangential part on Γ_2 : $\mathbf{V}^i = (\mathbf{V}^i\cdot\mathbf{n})\mathbf{n} + \mathbf{V}^i_\tau$, one obtains that:

$$\nabla u_k\cdot\mathbf{V}^i = \frac{\partial u_k}{\partial\mathbf{n}}(\mathbf{V}^i\cdot\mathbf{n}) + \nabla_\tau u_k\cdot\mathbf{V}^i_\tau, \quad \text{(A.5)}$$

where ∇_τ stands for the tangential gradient on Γ_2. Using some results of differential geometry (see [11] pages 192-197), we have on the one hand:

$$G_1\nabla_\tau u_2\cdot\mathbf{V}^i_\tau = \mathrm{div}_\tau(G_1 u_2\mathbf{V}^i_\tau) - \nabla_\tau G_1\cdot\mathbf{V}^i_\tau u_2 - G_1 u_2\mathrm{div}_\tau(\mathbf{V}^i_\tau),$$

and on the other hand:

$$\mathrm{div}_\tau(\mathbf{V}^i_\tau) = \mathrm{div}_\tau(\mathbf{V}^i) + (\kappa_1+\kappa_2)(\mathbf{V}^i\cdot\mathbf{n}), \quad \mathrm{div}_\tau(\mathbf{V}^i) = \mathrm{div}(\mathbf{V}^i) - \mathbf{n}^\top[D\mathbf{V}^i]\mathbf{n},$$

where $\kappa_1+\kappa_2 = -\mathrm{div}_\tau(\mathbf{n})$. Taking into account ($\mathrm{H}_3$), we deduce that $\mathrm{div}(\mathbf{V}^i)\equiv 0$ and $\mathbf{n}^\top[D\mathbf{V}^i]\mathbf{n}\equiv 0$ since $[D\mathbf{V}^i]$ is a skew-symmetric matrix. We remark also, applying Lemma 5.4.10 page 194 of [11], that:

$$\int_{\Gamma_2}\mathrm{div}_\tau(G_1u_2\mathbf{V}^i_\tau)\,\mathrm{d}\sigma_x = \int_\Gamma\mathrm{div}_\tau(G_1u_2\mathbf{V}^i_\tau)\,\mathrm{d}\sigma_x = 0,$$

the first equality resulting from the hypothesis (H_3) which means in particular that $\mathbf{V}^i \equiv \mathbf{0}$ on Γ_1. Finally we get:

$$\int_{\Gamma_2} G_1 \nabla_\tau u_2 \cdot \mathbf{V}^i_\tau \, \mathrm{d}\sigma_x = - \int_{\Gamma_2} (\kappa_1 + \kappa_2) G_1 u_2 (\mathbf{V}^i \cdot \mathbf{n}) + u_2 \nabla_\tau G_1 \cdot \mathbf{V}^i_\tau \, \mathrm{d}\sigma_x. \quad (A.6)$$

Summarizing (A.5) and (A.6) and plugging the result into (A.4), we obtain the expression (3.5) of the Proposition. ■

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