The distribution of integers with at least two divisors in a short interval

Kevin Ford* Gérald Tenenbaum

Version 7 : 12/8/2006, 1h49

1 Introduction

Whereas, in usual cases, sieving by a set of primes may be fairly well controlled, through Buchstab's identity, sieving by a set of integers is a much more complicated task. However, some fairly precise results are known in the case where the set of integers is an interval. We refer to the recent work [1] of the first author for specific statements and references.

Define

$$\begin{aligned}\tau(n;y,z) &:= |\{d|n : y < d \leqslant z\}|,\\ H(x,y,z) &:= |\{n \leqslant x : \tau(n;y,z) \geqslant 1\}|,\\ H_r(x,y,z) &:= |\{n \leqslant x : \tau(n;y,z) = r\}|,\\ H_2^*(x,y,z) &:= |\{n \leqslant x : \tau(n;y,z) \geqslant 2\}| = \sum_{r\geqslant 2} H_r(x,y,z).\end{aligned}$$

Thus, the numbers $H_r(x,y,z)$ $(r \geqslant 1)$ describe the local laws of the function $\tau(n;y,z)$. When y and z are close, it is expected that, if an integer has at least a divisor in $(y,z]$, then it usually has exactly one, in other words

$$H(x,y,z) \sim H_1(x,y,z). \tag{1.1}$$

In this paper, we address the problem of determining the exact range of validity of such behavior. In other words, we search a necessary and sufficient condition so that $H_2^*(x,y,z) = o\big(H(x,y,z)\big)$ as x and y tend to infinity.

As shown in [5], for given y, the threshold for the behavior of the function $H(x,y,z)$ lies near the critical value

$$z = z_0(y) := y\exp\{(\log y)^{1-\log 4}\} \approx y + y/(\log y)^{\log 4-1}.$$

We concentrate on the case $z_0(y) \leqslant z \leqslant 2y$. Define

$$z = \mathrm{e}^\eta y, \quad \eta = (\log y)^{-\beta}, \quad \beta = \log 4 - 1 - \Xi/\sqrt{\log_2 y}, \quad \lambda = \frac{1+\beta}{\log 2},$$

$$Q(w) = \int_1^w \log t\,\mathrm{d}t = w\log w - w + 1.$$

*First author supported by National Science Foundation grant DMS-0301083.

Here $\log_k$ denotes the kth iterate of the logarithm.

With the above notation, we have

$$\log(z/y) = \frac{\mathrm{e}^{\Xi\sqrt{\log_2 y}}}{(\log y)^{\log 4-1}}, \qquad \log\{z/z_0(y)\} = \frac{\mathrm{e}^{\Xi\sqrt{\log_2 y}}-1}{(\log y)^{\log 4-1}},$$

so

$$\begin{aligned} &0 \leqslant \Xi \leqslant (\log 4-1)\sqrt{\log_2 y} + \frac{\log_2 2}{\sqrt{\log_2 y}}, \\ &\frac{|\log_2 2|}{\log_2 y} \leqslant \beta \leqslant \log 4 - 1, \\ &\frac{1}{\log 2} + \frac{\kappa}{\log_2 y} \leqslant \lambda \leqslant 2, \end{aligned}$$

with $\kappa := |\log_2 2|/\log 2$.

From Theorem 1 of [1], we know that, uniformly in $10 \leqslant y \leqslant \sqrt{x}$, $z_0(y) \leqslant z \leqslant 2y$,

$$H(x,y,z) \asymp \frac{\beta x}{(\Xi+1)(\log y)^{Q(\lambda)}}. \tag{1.2}$$

By Theorems 5 and 6 of [1], for any $c > 0$ and uniformly in $y_0(r) \leqslant y \leqslant x^{1/2-c}$, $z_0(y) \leqslant z \leqslant 2y$, we have

$$\begin{aligned} &\frac{H_1(x,y,z)}{H(x,y,z)} \asymp_c 1, \\ \frac{\Xi+1}{\sqrt{\log_2 y}} \ll_{r,c} &\frac{H_r(x,y,z)}{H(x,y,z)} \leqslant 1 \quad (r \geqslant 2). \end{aligned} \tag{1.3}$$

When $0 \leqslant \Xi \leqslant o(\sqrt{\log_2 y})$ and $r \geqslant 2$, the upper and lower bounds above for $H_r(x,y,z)$ have different orders. We show in this paper that the lower bound represents the correct order of magnitude.

Theorem 1. *Uniformly in* $10 \leqslant y \leqslant \sqrt{x}$, $z_0(y) \leqslant z \leqslant 2y$, *we have*

$$\frac{H_2^*(x,y,z)}{H(x,y,z)} \ll \frac{\Xi+1}{\sqrt{\log_2 y}}.$$

Corollary 2. *Let* $r \geqslant 2$ *and* $c > 0$. *Uniformly in* $y_0(r,c) \leqslant y \leqslant x^{1/2-c}$, $z_0(y) \leqslant z \leqslant 2y$, *we have*

$$\frac{H_r(x,y,z)}{H(x,y,z)} \asymp_{r,c} \frac{\Xi+1}{\sqrt{\log_2 y}}.$$

Corollary 3. *Uniformly in* $10 \leqslant y \leqslant \sqrt{x}$, $y < z \leqslant y + y(\log y)^{1-\log 4+o(1)}$, *we have*

$$H_1(x,y,z) \sim H(x,y,z).$$

Since we know from (1.3) that $H_2^*(x,y,z) \gg H(x,y,z)$ when $\beta \leqslant \log 4 - 1 - \varepsilon$ for any $\varepsilon > 0$ we have therefore completely answered the question raised at the beginning of this introduction concerning the exact validity range for the asymptotic formula (1.1). This may be viewed as a complement a theorem of Hall (see [2], ch. 7; following a note mentioned by Hall in private correspondence, we slightly modify the statement) according to which

$$H(x,y,z) \sim F(-\Xi) \sum_{r \geqslant 1} rH_r(x,y,z) = F(-\Xi) \sum_{n \leqslant x} \tau(n;y,z) \tag{1.4}$$

in the range $\Xi = o(\log_2 y)^{1/6}$, $x > \exp\{\log z \log_2 z\}$ with

$$F(\xi) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\xi/\log 4} \mathrm{e}^{-u^2}\, \mathrm{d}u.$$

It is likely that (1.4) still holds in the range $(\log_2 y)^{1/6} \ll \Xi \leqslant o\big(\sqrt{\log_2 y}\big)$.

2 Auxiliary estimates

In the sequel, unless otherwise indicated, constants implied by $O-$ and $\ll -$ symbols are independent of any parameter.

Let m be a positive integer. We denote by $P^+(m)$ and $P^-(m)$ the smallest prime factor and largest prime factor of m, respectively, with the convention that $P^-(1) = \infty$, $P^+(1) = 1$. We write $\omega(m)$ for the number of distinct prime factors of m and $\Omega(m)$ the number of prime power divisors of m. We further define

$$\omega(m;t,u) = \sum_{\substack{p^\nu \| m \\ t < p \leqslant u}} 1, \quad \Omega(m;t,u) = \sum_{\substack{p^\nu \| m \\ t < p \leqslant u}} \nu, \quad \overline{\Omega}(m;t) = \Omega(m;2,t), \quad \overline{\Omega}(m) = \Omega(m;2,m).$$

Also, we let $\mathscr{P}(u,v)$ denote the set of integers all of whose prime factors are in $(u,v]$ and write $\mathscr{P}^*(u,v)$ for the set of squarefree members of $\mathscr{P}(u,v)$. By convention, $1 \in \mathscr{P}^*(u,v)$.

Lemma 2.1. *There is an absolute constant $C > 0$ so that for $\frac{3}{2} \leqslant u < v$, $v \geqslant \mathrm{e}^4$, $0 \leqslant \alpha \leqslant 1/\log v$, we have*

$$\sum_{\substack{m \in \mathscr{P}(u,v) \\ \omega(m) = k}} \frac{1}{m^{1-\alpha}} \leqslant \frac{(\log_2 v - \log_2 u + C)^k}{k!}.$$

Proof. For a prime $p \leqslant v$, we have $p^\alpha \leqslant 1 + 2\alpha \log p$, thus the sum in question is

$$\leqslant \frac{1}{k!} \bigg(\sum_{u < p \leqslant v} \frac{1}{p^{1-\alpha}} + \frac{1}{p^{2-2\alpha}} + \cdots \bigg)^k \leqslant \frac{\{\log_2 v - \log_2 u + O(1)\}^k}{k!}.$$

□

Lemma 2.2. *Uniformly for $u \geqslant 10$, $0 \leqslant k \leqslant 2.9 \log_2 u$, and $0 \leqslant \alpha \leqslant 1/(100 \log u)$, we have*

$$\sum_{\substack{P^+(m) \leqslant u \\ \overline{\Omega}(m) = k}} \frac{1}{m^{1-\alpha}} \ll \frac{(\log_2 u)^k}{k!}.$$

Proof. We follow the proof of Theorem 08 of [3]. Let w be a complex number with $|w| \leqslant \frac{29}{10}$. If p is prime and $3 \leqslant p \leqslant u$, then $|w/p^{1-\alpha}| \leqslant \frac{99}{100}$ and $p^\alpha \leqslant 1 + 2\alpha \log p$. Thus,

$$S(w) := \sum_{P^+(m) \leqslant u} \frac{w^{\overline{\Omega}(m)}}{m^{1-\alpha}} = \left(1 - \frac{1}{2^{1-\alpha}}\right)^{-1} \prod_{3 \leqslant p \leqslant u} \left(1 - \frac{w}{p^{1-\alpha}}\right)^{-1} \ll \mathrm{e}^{(\Re w) \log_2 u}.$$

Put $r := k/\log_2 u$. By Cauchy's formula and Stirling's formula,

$$\begin{aligned} \sum_{\substack{P^+(m) \leqslant u \\ \overline{\Omega}(m) = k}} \frac{1}{m^{1-\alpha}} &= \frac{1}{2\pi r^k} \int_{-\pi}^{\pi} \mathrm{e}^{-ik\vartheta} S(r\mathrm{e}^{i\vartheta})\, d\vartheta \\ &\ll \frac{(\log_2 u)^k}{k^k} \int_{-\pi}^{\pi} \mathrm{e}^{k \cos \vartheta}\, d\vartheta \ll \frac{(\log_2 u)^k}{k!}. \end{aligned}$$

□

Lemma 2.3. *Suppose z is large, $0 \leqslant a + b \leqslant \frac{5}{2} \log_2 z$ and*

$$\exp\{(\log x)^{9/10}\} \leqslant w \leqslant z \leqslant x, \qquad xz^{-1/(10 \log_2 z)} \leqslant Y \leqslant x.$$

The number of integers n with $x - Y < n \leqslant x$, $\overline{\Omega}(n; w) = a$ and $\omega(n; w, z) = \Omega(n; w, z) = b$, is

$$\ll \frac{Y}{\log z} \frac{\{\log_2 w\}^a}{a!} \frac{(b+1)\{\log_2 z - \log_2 w + C\}^b}{b!},$$

where C is a positive absolute constant.

Proof. There are $\ll x^{9/10}$ integers with $n \leqslant x^{9/10}$ or $2^j | n$ with $2^j \geqslant x^{1/10}$. For other n, write $n = rst$, where $P^+(r) \leqslant w$, $s \in \mathscr{P}^*(w, z)$ and $P^-(t) > z$. Here $\overline{\Omega}(r) = a$ and $\omega(s) = b$. We have either $t = 1$ or $t > z$. In the latter case $x/rs > z$, whence $Y/rs > \sqrt{z}$. We may therefore apply a standard sieve estimate to bound, for given r and s, the number of t by

$$\ll \frac{Y}{rs \log z}.$$

By Lemmas 2.1 and 2.2,

$$\sum_{r,s} \frac{1}{rs} \ll \frac{(\log_2 w)^a (\log_2 z - \log_2 w + C)^b}{a! b!}.$$

If $t = 1$, then we may assume $a + b \geqslant 1$. Set $p = P^+(n)$. If $b \geqslant 1$, then $p | s$ and we put $r_1 := r$ and $s_1 := s/p$. Otherwise, let $r_1 := r/p$ and $s_1 := s = 1$. Let $A := \overline{\Omega}(r_1)$ and $B := \omega(s_1)$, so that $A + B = a + b - 1$ in all circumstances. We have

$$p \geqslant x^{1/2\overline{\Omega}(n)} \geqslant x^{1/5 \log_2 z} \geqslant (x/Y)^2.$$

Define the non-negative integer h by $z^{\mathrm{e}^{-h-1}} < p \leqslant z^{\mathrm{e}^{-h}}$. By the Brun-Titchmarsh theorem, we see that, for each given r_1 and s_1, the number of p is $\ll Y\mathrm{e}^h/(r_1 s_1 \log z)$. Set $\alpha := 0$ if $h = 0$ and $\alpha := \mathrm{e}^h/(100 \log z)$ otherwise. For $h \geqslant 1$, we have $r_1 s_1 > x^{3/4} z^{-1/\mathrm{e}} > \sqrt{z}$. Therefore, for $h \geqslant 0$,

$$\frac{1}{r_1 s_1} \leqslant \frac{z^{-\alpha/2}}{(r_1 s_1)^{1-\alpha}} \ll \frac{\mathrm{e}^{-\mathrm{e}^h/200}}{(r_1 s_1)^{1-\alpha}}.$$

Now, Lemmas 2.1 and 2.2 imply that

$$\begin{aligned}\sum_{r_1,s_1} \frac{1}{(r_1 s_1)^{1-\alpha}} &\ll \frac{(\log_2 w)^A (\log_2 z - \log_2 w + C)^B}{A!B!} \\ &\ll (b+1)\frac{(\log_2 w)^a (\log_2 z - \log_2 w + C)^b}{a!b!},\end{aligned}$$

where we used the fact that $a \ll \log_2 w$. Summing over all h, we derive that the number of those integers $n > x^{9/10}$ satisfying the conditions of the statement is

$$\ll \frac{Y}{\log z}(b+1)\frac{(\log_2 w)^a (\log_2 z - \log_2 w + C)^b}{a!b!}.$$

Since $a!b! \leqslant (3\log_2 z)^{3\log_2 z}$, this last expression is $> x^{9/10}$. This completes the proof. $\square$

Our final lemma is a special case of a theorem of Shiu (Theorem 03 of [3]).

Lemma 2.4. *Let f be a multiplicative function such that $0 \leqslant f(n) \leqslant 1$ for all n. Then, for all x, Y with $1 < \sqrt{x} \leqslant Y \leqslant x$, we have*

$$\sum_{x-Y<n\leqslant x} f(n) \ll \frac{Y}{\log x} \exp\Big\{\sum_{p\leqslant x} \frac{f(p)}{p}\Big\}.$$

3 Decomposition and outline of the proof

Throughout, ε will denote a very small positive constant. Since Theorem 1 holds trivially for $\beta \leqslant \log 4 - 1 - \varepsilon$, we henceforth assume that

$$\log 4 - 1 - \varepsilon \leqslant \beta \leqslant \log 4 - 1. \tag{3.1}$$

Let

$$K := \lfloor \lambda \log_2 z \rfloor,$$

so that $(2 - \frac{3}{2}\varepsilon)\log_2 z \leqslant K \leqslant 2\log_2 z$. In light of (1.2), Theorem 1 reduces to

$$H_2^*(x,y,z) \ll \frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}. \tag{3.2}$$

At this stage, we notice for further reference that, by Stirling's formula, for $k \leqslant K$ we have

$$\frac{\eta(2\log_2 z)^k}{k!(\log z)^2} \leqslant \frac{\eta(2\log_2 z)^K}{K!(\log z)^2} \asymp \frac{1}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}. \tag{3.3}$$

Let $\mathcal{H}$ denote the set of integers $n \leqslant x$ with $\tau(n; y, z) \geqslant 2$. We count separately the integers $n \in \mathcal{H}$ lying in 6 classes. In these definitions, we write $k = \overline{\Omega}(n; z) = K - b$ and for brevity we put $z_h = z^{\mathrm{e}^{-h}}$. Let

$$K_0 := (2 - 3\varepsilon)\log_2 z$$

and define

$$
\begin{aligned}
\mathcal{N}_0 &:= \{n \in \mathcal{H} : n \leqslant x/\log z \text{ or } \exists d > \log z : d^2|n\},\\
\mathcal{N}_1 &:= \{n \in \mathcal{H} \smallsetminus \mathcal{N}_0 : k \notin (K_0, K]\},\\
\mathcal{N}_2 &:= \bigcup_{1\leqslant h\leqslant 5\varepsilon\log_2 z} \mathcal{N}_{2,h},\\
\text{with } \mathcal{N}_{2,h} &:= \Big\{n \in \mathcal{H} \smallsetminus (\mathcal{N}_0 \cup \mathcal{N}_1) : \overline{\Omega}(n; z_h, z) \leqslant \tfrac{19}{10}h - \tfrac{1}{100}b\Big\}.
\end{aligned}
$$

For integers $n \in \mathcal{N}_2$, we will only use the fact that $\tau(n; y, z) \geqslant 1$. Integers in other classes do not have too many small prime factors and it is sufficient to count pairs of divisors d_1, d_2 of n in $(y, z]$. For each such pair, write $v = (d_1, d_2)$, $d_1 = vf_1$, $d_2 = vf_2$, $n = f_1f_2vu$ and assume $f_1 < f_2$. Let

$$
F_1 = \overline{\Omega}(f_1), \quad F_2 = \overline{\Omega}(f_2), \quad V = \overline{\Omega}(v), \quad U = \overline{\Omega}(u, z), \tag{3.4}
$$

and

$$
Z := \exp\{(\log z)^{1-4\varepsilon}\}. \tag{3.5}
$$

For further reference, we note that if $n \notin \mathcal{N}_0$ and $h \leqslant 5\varepsilon \log_2 z$, then

$$
\overline{\Omega}(n; z_h, z) = \omega(n; z_h, z).
$$

Now we define $\mathcal{H}^* := \mathcal{H} \smallsetminus (\mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2)$ and

$$
\begin{aligned}
\mathcal{N}_3 &:= \{n \in \mathcal{H}^* : \min(u, f_2) \leqslant Z\},\\
\mathcal{N}_4 &:= \{n \in \mathcal{H}^* : \min(u, f_2) > z^{1/10}\},\\
\mathcal{N}_5 &:= \{n \in \mathcal{H}^* : Z < \min(u, f_2) \leqslant z^{1/10}\}.
\end{aligned}
$$

In the above decomposition, the main parts are $\mathcal{N}_2$ and $\mathcal{N}_5$. We expect $\mathcal{N}_2$ to be small since, conditionally on $\overline{\Omega}(n; z) = k$, the normal value of $\overline{\Omega}(n; z_h, z)$ is $hk/\log_2 z > \frac{19}{10}h$. It is more difficult to see that $\mathcal{N}_5$ is small too. This follows from the fact that we count integers in this set according to their number of factorizations in the form $n = uvf_1f_2$ with $y < vf_1 < vf_2 \leqslant z$. Suppose for instance that $f_1, f_2 \leqslant z_j$. For $\overline{\Omega}(n; z) = k$ and $\overline{\Omega}(n; z_j, z) = G$, then, ignoring the localization of vf_1 and vf_2 in $(y, z]$, there are $4^{k-G}2^G = 4^k2^{-G}$ such factorizations. Thus, larger G means fewer factorizations. On probabilistic grounds, larger G should also mean fewer factorizations with the localization of vf_1 and vf_2.

We now briefly consider the cases of $\mathcal{N}_0$ and $\mathcal{N}_1$.

Trivially,

$$
|\mathcal{N}_0| \leqslant \frac{x}{\log z} + \sum_{d>\log z} \frac{x}{d^2} \ll \frac{x}{\log z} \ll \frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}, \tag{3.6}
$$

since $Q(\lambda) \leqslant Q(2) = \log 4 - 1$ in the range under consideration.

By the argument on pages 40–41 of [3],

$$
\sum_{\substack{n\leqslant x\\ \overline{\Omega}(n;z)>K}} 1 \ll \frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}.
$$

Setting $t := 1 - \frac{3}{2}\varepsilon$, Lemma 2.4 gives

$$\sum_{\substack{n\leqslant x\\ \tau(n;y,z)\geqslant 1\\ \overline{\Omega}(n;z)\leqslant K_0}} 1 \leqslant t^{-(2-3\varepsilon)\log_2 z} \sum_{\substack{dm\leqslant x\\ y<d\leqslant z}} t^{\overline{\Omega}(d)+\overline{\Omega}(m;z)} \ll x(\log z)^{2t-2-\beta-(2-3\varepsilon)\log t}$$

$$\ll x(\log y)^{-\beta-2\varepsilon^2} \ll x(\log y)^{-Q(\lambda)-\varepsilon^2/2}.$$

Therefore,

$$|\mathcal{N}_1| \ll \frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}. \tag{3.7}$$

In the next four sections, we show that

$$|\mathcal{N}_j| \ll \frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}} \qquad (2 \leqslant j \leqslant 5). \tag{3.8}$$

Together with (3.6) and (3.7), this will complete the proof of Theorem 1.

4 Estimation of $|\mathcal{N}_2|$

We plainly have $|\mathcal{N}_2| \leqslant \sum_h |\mathcal{N}_{2,h}|$. For $1 \leqslant h \leqslant 5\varepsilon \log_2 z$, the numbers $n \in \mathcal{N}_{2,h}$ satisfy

$$\begin{cases} x/\log z < n \leqslant x, \\ k := \overline{\Omega}(n;z) = K - b, \quad 0 \leqslant b \leqslant 3\varepsilon \log_2 z, \\ \overline{\Omega}(n; z_h, z) \leqslant \frac{19}{10}h - \frac{1}{100}b, \end{cases}$$

We note at the outset that $\mathcal{N}_{2,h}$ is empty unless $h \geqslant b/190$.

Write $n = du$ with $y < d \leqslant z$ and $u \leqslant x/y$. Let

$$\overline{\Omega}(d; z_h) = D_1, \quad \Omega(d; z_h, z) = D_2, \quad \overline{\Omega}(u; z_h) = U_1, \quad \Omega(u; z_h, z) = U_2,$$

so that $D_1 + D_2 \geqslant 1$, $D_2 + U_2 \leqslant \frac{19}{10}h - \frac{1}{100}b$ and $D_1 + D_2 + U_1 + U_2 = k$.

Fix $k = K - b$, h, D_1, D_2, U_1 and U_2. By Lemma 2.3 (with $w = z_h$, $a = U_1$, $b = U_2$), the number of u is

$$\ll \frac{x}{y\log z}\,\frac{(\log_2 z - h)^{U_1}}{U_1!}(U_2+1)\frac{(h+C)^{U_2}}{U_2!}.$$

A second application of Lemma 2.3 yields that the number of d is

$$\ll \frac{\eta y}{\log z}\frac{(\log_2 z - h)^{D_1}}{D_1!}(D_2+1)\frac{(h+C)^{D_2}}{D_2!}.$$

Since $D_2 + U_2 < 2h$, we have $(h+C)^{U_2+D_2} \leqslant \mathrm{e}^{2C} h^{U_2+D_2}$. Summing over D_1, D_2, U_1, U_2 with $G = D_2 + U_2$ fixed and using the binomial theorem, we find that the number of n in question is

$$\ll \frac{\eta x}{(\log z)^2}(\log_2 z - h)^{k-G} h^G (G+1)^2 \sum_{\substack{U_1+D_1=k-G\\ D_2+U_2=G}} \frac{1}{U_1!D_1!D_2!U_2!} \ll \frac{\eta x 2^k}{(\log z)^2} A(h,G),$$

where

$$A(h,G) = (G+1)^2 \frac{(\log_2 z - h)^{k-G} h^G}{(k-G)!G!}.$$

Since $G+1 \leqslant G_h := \lfloor \frac{19}{10}h \rfloor$, we have

$$\frac{A(h,G+1)}{A(h,G)} \geqslant \frac{h(k-G)}{(G+1)(\log_2 z - h)} \geqslant \frac{k - 10\varepsilon \log_2 z}{1.9(1-5\varepsilon)\log_2 z} > \tfrac{21}{20}$$

if ε is small enough. Next,

$$\begin{aligned} A(h,G_h) &\leqslant (G_h+1)^2 \frac{(\log_2 z - h)^{k-G_h}(hk)^{G_h}}{k!(G_h/\mathrm{e})^{G_h}} \\ &\ll (h+1)^2 \frac{(\log_2 z)^k}{k!} \big(\tfrac{20}{19}\mathrm{e}\big)^{19h/10} \mathrm{e}^{-h(k-G_h)/\log_2 z} \\ &\ll \frac{(\log_2 z)^k}{k!} \mathrm{e}^{-h/500}, \end{aligned}$$

since $(k-G_h)/\log_2 z > 2 - 13\varepsilon$ and $\frac{19}{10}\log(\frac{20}{19}\mathrm{e}) < 2 - 1/400$. Thus,

$$\sum_{b/190 \leqslant h \leqslant 5\varepsilon \log_2 z} \; \sum_{0 \leqslant G \leqslant G_h} A(h,G) \ll \sum_{b/190 \leqslant h \leqslant 5\varepsilon \log_2 z} A(h,G_h) \ll \frac{(\log_2 z)^k}{k!} \mathrm{e}^{-b/95000}$$

and so

$$\sum_{\substack{n \in \mathcal{N}_2 \\ \overline{\Omega}(n;z)=k}} 1 \ll \frac{\eta x (2\log_2 z)^k}{(\log z)^2 k!} \mathrm{e}^{-(K-k)/95000} \ll \frac{x \mathrm{e}^{-(K-k)/95000}}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}},$$

by (3.3). Summing over the range $K_0 \leqslant k \leqslant K$ furnishes the required estimate (3.8) for $j = 2$.

5 Estimation of $|\mathcal{N}_3|$

All integers $n = f_1 f_2 u v$ counted in $\mathcal{N}_3$ verify

$$\begin{cases} x/\log z < n \leqslant x, \\ \overline{\Omega}(n;z) \leqslant K, \\ y < v f_1 < v f_2 \leqslant z, \quad \min(u, f_2) \leqslant Z, \end{cases}$$

where Z is defined in (3.5). This is all we shall use in bounding $|\mathcal{N}_3|$.

Let $\mathcal{N}_{3,1}$ be the subset corresponding to the condition $f_2 \leqslant Z$ and let $\mathcal{N}_{3,2}$ comprise those $n \in \mathcal{N}_3$ such that $u \leqslant Z$.

If $f_2 \leqslant Z$, then $v > z^{1/2}$ and $u > x/\{vZ^2 \log z\} > x^{1/3}$. For $\frac{1}{2} \leqslant t \leqslant 1$ we have

$$\begin{aligned} |\mathcal{N}_{3,1}| &\leqslant \sum_{f_1, f_2, v, u} t^{\overline{\Omega}(f_1 f_2 u v; z) - K} \\ &= t^{-K} \sum_{f_1 \leqslant Z} t^{\overline{\Omega}(f_1)} \sum_{f_1 < f_2 \leqslant \mathrm{e}^{\eta} f_1} t^{\overline{\Omega}(f_2)} \sum_{y/f_1 < v \leqslant z/f_1} t^{\overline{\Omega}(v)} \sum_{u \leqslant x/f_1 f_2 v} t^{\overline{\Omega}(u;z)}. \end{aligned}$$

Apply Lemma 2.4 to the three innermost sums. The u-sum is

$$\ll \frac{x}{f_1 f_2 v}(\log z)^{t-1} \leqslant \frac{x}{f_1 y}(\log z)^{t-1}.$$

and the v-sum is

$$\ll \frac{\eta y}{f_1}(\log z)^{t-1}.$$

The f_2-sum is $\ll \eta f_1 (\log f_1)^{t-1}$ if $f_1 > \eta^{-3}$ and otherwise is $\ll \eta f_1$ trivially (note that $\eta f_1 \gg 1$ follows from the fact that $(f_1+1)/f_1 \leqslant f_2/f_1 \leqslant \mathrm{e}^{\eta}$). Next

$$\begin{aligned}\sum_{f_1 \leqslant \eta^{-3}} \frac{1}{f_1} + \sum_{2 \leqslant f_1 \leqslant Z} \frac{t^{\overline{\Omega}(f_1)}}{f_1}(\log f_1)^{t-1} &\ll \log_2 z + (\log_2 z) \max_{j \leqslant \log_2 Z} \mathrm{e}^{j(t-1)} \sum_{f_1 \leqslant \exp\{\mathrm{e}^j\}} \frac{t^{\overline{\Omega}(f_1)}}{f_1} \\ &\ll (\log_2 z)(\log Z)^{2t-1}.\end{aligned}$$

Thus,

$$|\mathcal{N}_{3,1}| \ll x(\log_2 x)(\log x)^E$$

with $E = -2\beta - \lambda \log t + 2t - 2 + (2t-1)(1-4\varepsilon)$. We select optimally $t := \frac{1}{4}\lambda/(1-2\varepsilon)$, and check that $t \geqslant \frac{1}{2}$ since $\lambda \geqslant 2 - \varepsilon/\log 2$. Then

$$\begin{aligned}E &= -Q(\lambda) + \lambda \log(1-2\varepsilon) + 4\varepsilon \leqslant -Q(\lambda) + (2 - \varepsilon/\log 2)(-2\varepsilon - 2\varepsilon^2) + 4\varepsilon \\ &< -Q(\lambda) - \varepsilon^2.\end{aligned}$$

Next, we consider the case when $u \leqslant Z$. We observe that this implies

$$\tfrac{1}{4} v z^2 \leqslant v x \leqslant v n \log z = u f_1 v f_2 v \log z \leqslant Z z^2 \log z$$

hence $v \leqslant 4Z \log z \leqslant Z^2$, and therefore

$$\min(f_1, f_2) > z^{1/2}.$$

Also, $z > x^{1/3}$ since $x/\log z < n = uvf_1f_2 \leqslant Zz^2$. Thus, for $\frac{1}{2} \leqslant t \leqslant 1$, we have

$$\begin{aligned}|\mathcal{N}_{3,2}| &\leqslant \sum_{f_1, f_2, v, u} t^{\overline{\Omega}(f_1 f_2 u v; z) - K} \\ &= t^{-K} \sum_{v \leqslant Z^2} t^{\overline{\Omega}(v)} \sum_{u \leqslant xv/y^2} t^{\overline{\Omega}(u)} \sum_{y/v < f_1 \leqslant z/v} t^{\overline{\Omega}(f_1)} \sum_{y/v < f_2 \leqslant z/v} t^{\overline{\Omega}(f_2)}.\end{aligned}$$

The sums upon f_1 and f_2 are each

$$\ll \frac{\eta y}{v}(\log z)^{t-1}$$

and the u-sum is

$$\ll \frac{xv}{y^2}(\log 2xv/y^2)^{t-1} \leqslant \frac{xv}{y^2}(\log 2v)^{t-1}.$$

Thus, selecting the same value $t := \frac{1}{4}\lambda/(1-2\varepsilon)$, we obtain

$$\begin{aligned}|N_{3,2}| &\ll t^{-K} x \eta^2 (\log z)^{2t-1} \sum_{v \leqslant Z^2} \frac{t^{\overline{\Omega}(v)}(\log 2v)^{t-1}}{v} \\ &\ll x(\log_2 z)(\log z)^E \leqslant x(\log_2 z)(\log z)^{-Q(\lambda) - \varepsilon^2}.\end{aligned}$$

This completes the proof of (3.8) with $j = 3$.

6 Estimation of $|\mathcal{N}_4|$

We now consider those integers $n = f_1 f_2 uv$ such that

$$\begin{cases} x/\log z < n \leqslant x, \\ k := \overline{\Omega}(n; z) = K - b, \quad 0 \leqslant b \leqslant 3\varepsilon \log_2 z, \\ y < vf_1 < vf_2 \leqslant z, \quad \min(u, f_2) > z^{1/10}. \end{cases}$$

With the notation (3.4), fix k, F_1, F_2, U and V. Here u, f_1 and f_2 are all $> \frac{1}{2}z^{1/10}$. By Lemma 2.3 (with $w = z$), for each triple f_1, f_2, v the number of u is

$$\ll \frac{x}{f_1 f_2 v \log z} \frac{(\log_2 z)^U}{U!}.$$

Using Lemma 2.3 two more times, we obtain, for each v,

$$\sum_{y/v < f_1 \leqslant z/v} \frac{1}{f_1} \sum_{y/v < f_2 \leqslant z/v} \frac{1}{f_2} \ll \frac{\eta^2}{(\log z)^2} \frac{(\log_2 z)^{F_1+F_2}}{F_1! F_2!}.$$

Now, Lemma 2.2 gives

$$\sum_v \frac{1}{v} \ll \frac{(\log_2 z)^V}{V!}.$$

Gathering these estimates and using (3.3) yields

$$\begin{aligned} |\mathcal{N}_4| &\ll \frac{x\eta^2}{(\log z)^3} \sum_{(2-3\varepsilon)\log_2 z \leqslant k \leqslant K} \sum_{F_1+F_2+U+V=k} \frac{(\log_2 z)^k}{F_1! F_2! U! V!} \\ &= \frac{x\eta^2}{(\log z)^3} \sum_{(2-3\varepsilon)\log_2 z \leqslant k \leqslant K} \frac{(2\log_2 z)^k}{k!} 2^k \\ &\ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}} \frac{2^K \eta}{\log z} \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \end{aligned}$$

Thus (3.8) holds for $j = 4$.

7 Estimation of $|\mathcal{N}_5|$

It is plainly sufficient to bound the number of those $n = f_1 f_2 uv$ satisfying the following conditions

$$\begin{cases} x/\log z < n \leqslant x, \\ k := \overline{\Omega}(n; z) = K - b, \quad 0 \leqslant b \leqslant 3\varepsilon \log_2 z, \\ \overline{\Omega}(n; z_h, z) > \frac{19}{10}h - \frac{1}{100}b \quad (1 \leqslant h \leqslant 5\varepsilon \log_2 z) \\ y < vf_1 < vf_2 \leqslant z, \quad Z < \min(u, f_2) \leqslant z^{1/10}. \end{cases}$$

Define j by $z_{j+2} < \min(u, f_2) \leqslant z_{j+1}$. We have $1 \leqslant j \leqslant 5\varepsilon \log_2 z$. Let $\mathcal{N}_{5,1}$ be the set of those n satisfying the above conditions with $u \leqslant z_{j+1}$ and let $\mathcal{N}_{5,2}$ be the complementary set, for which $f_2 \leqslant z_{j+1}$.

If $u \leqslant z_{j+1}$, then $v \leqslant (z^2 u \log z)/x \leqslant 4u \log z \leqslant z_j$ and $f_2 > f_1 > z^{1/2}$. Recall notation (3.4) and write

$$F_{11} := \overline{\Omega}(f_1; z_j), \quad F_{12} := \Omega(f_1; z_j, z), \quad F_{21} := \overline{\Omega}(f_2; z_j), \quad F_{22} := \Omega(f_2; z_j, z),$$

so that the initial condition upon $\overline{\Omega}(n; z_h, z)$ with $h = j$ may be rewritten as

$$F_{12} + F_{22} \geqslant G_j := \max(0, \lfloor \tfrac{19}{10} j - b/100 \rfloor).$$

We count those n in a dyadic interval $(X, 2X]$, where $x/(2\log z) \leqslant X \leqslant x$. Fix k, j, X, U, V, F_{rs} and apply Lemma 2.3 to sums over u, f_1, f_2. The number of n is question is

$$\begin{aligned} &\leqslant \sum_{v \leqslant z_j} \sum_{vX/z^2 \leqslant u \leqslant 2vX/y^2} \sum_{y/v < f_1 \leqslant z/v} \sum_{y/v < f_2 \leqslant z/v} 1 \\ &\ll \frac{\eta^2 X \mathrm{e}^j}{(\log z)^3} \frac{(\log_2 z - j)^{U + F_{11} + F_{21}}}{U! F_{11}! F_{21}!} (F_{12}+1)(F_{22}+1) \frac{(j+C)^{F_{12}+F_{22}}}{F_{12}! F_{22}!} \sum_{v \leqslant z_j} \frac{1}{v}. \end{aligned}$$

Bounding the v-sum by Lemma 2.2, and summing over X, U, V, F_{rs} with $F_{12} + F_{22} = G$ yields

$$|\mathcal{N}_{5,1}| \ll \frac{\eta^2 x}{(\log z)^3} \sum_{(2-3\varepsilon)\log_2 z \leqslant k \leqslant K} 4^k \sum_{1 \leqslant j \leqslant 5\varepsilon \log_2 z} \sum_{G_j \leqslant G \leqslant k} M(j, G),$$

where

$$M(j, G) := \mathrm{e}^j (G+1)^2 \frac{(\log_2 z - j)^{k-G} (j+C)^G}{2^G (k-G)! G!}.$$

Let $j_b = \lfloor \tfrac{1}{2} b + 100C + 100 \rfloor$. If $j \leqslant j_b$, then $j + C \leqslant \frac{99}{100}(j + C_b)$ with $C_b := 3C + 2 + \frac{b}{100}$ and, introducing $R := \max_{G \geqslant 0} \{(G+1)^2 (\frac{99}{100})^G\}$, we have

$$\begin{aligned} \sum_{1 \leqslant j \leqslant j_b} \sum_{G_j \leqslant G \leqslant k} M(j, G) &\leqslant R \sum_{1 \leqslant j \leqslant j_b} \mathrm{e}^j \sum_{0 \leqslant G \leqslant k} \frac{(\log_2 z - j)^{k-G} (j + C_b)^G}{2^G G! (k-G)!} \\ &\ll \frac{1}{k!} \sum_{1 \leqslant j \leqslant j_b} \mathrm{e}^j \left(\log_2 z - \tfrac{1}{2} j + \tfrac{1}{2} C_b\right)^k \\ &\ll \frac{(\log_2 z)^k}{k!} \sum_{1 \leqslant j \leqslant j_b} \mathrm{e}^{j + (b/200 - j/2)k/\log_2 z} \\ &\ll \frac{(\log_2 z)^k}{k!} \mathrm{e}^{b/100 + 2\varepsilon j_b} \ll \frac{(\log_2 z)^k}{k!} \mathrm{e}^{b/50}. \end{aligned}$$

When $j > j_b$, then

$$G_j \geqslant \tfrac{9}{5}(j + C) + \tfrac{1}{10}(j_b + C + 1) - \tfrac{1}{100} b - 1 \geqslant \tfrac{9}{5}(j + C) + 9 \geqslant 189.$$

Thus, for $G \geqslant G_j$ we have

$$\frac{M(j,G+1)}{M(j,G)} = \left(\frac{G+2}{G+1}\right)^2 \frac{j+C}{2(G+1)} \frac{k-G}{\log_2 z - j} \leqslant \tfrac{4}{7}.$$

Therefore,

$$\sum_{G_j \leqslant G \leqslant k} M(j,G) \ll M(j,G_j) \ll \frac{j^2 \mathrm{e}^j}{k!} \frac{(\log_2 z - j)^{k-G_j}(jk)^{G_j}}{2^{G_j} G_j!}$$
$$\leqslant \frac{j^2 \mathrm{e}^j (\log_2 z)^k}{k!} \mathrm{e}^{-j(k-G_j)/\log_2 z} \left(\frac{\mathrm{e}jk}{2G_j \log_2 z}\right)^{G_j} \ll \frac{(\log_2 z)^k}{k!} \mathrm{e}^{-j/5},$$

since $k - G_j \geqslant (2-10\varepsilon)\log_2 z$, $\mathrm{e}jk/(2G_j \log_2 z) \leqslant \frac{5}{9}\mathrm{e}$, and $-1 + \frac{19}{10}\log(\frac{5}{9}\mathrm{e}) < -\frac{1}{5}$. We conclude that

$$\sum_{1 \leqslant j \leqslant 5\varepsilon \log_2 z} \; \sum_{G_j \leqslant G \leqslant k} M(j,G) \ll \frac{(\log_2 z)^k}{k!} \mathrm{e}^{b/50} \tag{7.1}$$

and hence, by (3.3),

$$|\mathcal{N}_{5,1}| \ll \frac{\eta^2 x}{(\log z)^3} \sum_{k \leqslant K} \frac{(2\log_2 z)^k}{k!} 2^{K-b/2} \ll \frac{\eta^2 2^K x}{(\log z)^3} \frac{(2\log_2 z)^K}{K!} \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}.$$

Now assume $f_2 \leqslant z_{j+1}$. Then $\min(u,v) > \sqrt{z}$. Fix F_1, F_2 and

$$\overline{\Omega}(v; z_j) = V_1, \quad \Omega(v; z_j, z) = V_2, \quad \overline{\Omega}(u; z_j) = U_1, \quad \Omega(u; z_j, z) = U_2.$$

By Lemma 2.3, given f_1, f_2 and v, the number of u is

$$\ll \frac{x}{f_1 f_2 v \log z} \frac{(\log_2 z - j)^{U_1}(U_2+1)(j+C)^{U_2}}{U_1! U_2!}.$$

Applying Lemma 2.3 again, for each f_1 we have

$$\sum_{\substack{f_1 < f_2 \leqslant \mathrm{e}^\eta f_1 \\ y/f_1 < v \leqslant z/f_1}} \frac{1}{f_2 v} \ll \frac{\eta^2 \mathrm{e}^j}{(\log z)^2} \frac{(V_2+1)(\log_2 z - j)^{V_1+F_2}(j+C)^{V_2}}{V_1! V_2! F_2!}.$$

By Lemma 2.2,

$$\sum_{f_1 \leqslant z_j} \frac{1}{f_1} \ll \frac{(\log_2 z - j)^{F_1}}{F_1!}.$$

Combine these estimates, and sum over $F_1, F_2, U_1, U_2, V_1, V_2$ with $V_2 + U_2 = G$. As in the estimation of $|\mathcal{N}_{5,1}|$, sum over k, j, G using (3.3) and (7.1). We obtain

$$|\mathcal{N}_{5,2}| \ll \frac{\eta^2 x}{(\log z)^3} \sum_{(2-3\varepsilon)\log_2 z \leqslant k \leqslant K} 4^k \sum_{1 \leqslant j \leqslant 5\varepsilon \log_2 z} \; \sum_{G_j < G \leqslant k} M(j,G)$$
$$\ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}.$$

References

[1] K. Ford, The distribution of integers with a divisor in a given interval, (2004).

[2] R.R Hall, *Sets of multiples*, Cambridge Tracts in Mathematics 118, Cambridge University Press, Cambridge 1996.

[3] R. R. Hall and G. Tenenbaum, *Divisors*, Cambridge Tracts in Mathematics, vol. 90, Cambridge University Press, Cambridge, 1988.

[4] R. R. Hall and G. Tenenbaum, The set of multiples of a short interval, in : D.V. Chudnovsky *et al.* (eds), *Number Theory* (New York Seminar 1989–90), 119–128 (Springer Verlag 1991).

[5] G. Tenenbaum, Sur la probabilité qu'un entier possède un diviseur dans un intervalle donné, *Compositio Math.* **51** (1984), 243–263.