HUA OPERATORS AND POISSON TRANSFORM FOR BOUNDED SYMMETRIC DOMAINS

KHALID KOUFANY AND GENKAI ZHANG

Abstract. Let Ω be a bounded symmetric domain of non-tube type in $\mathbb{C}^n$ with rank r and S its Shilov boundary. We consider the Poisson transform $\mathcal{P}_s f(z)$ for a hyperfunction f on S defined by the Poisson kernel $P_s(z,u) = \left(h(z,z)^{\frac{n}{r}}/|h(z,u)^{\frac{n}{r}}|^2\right)^s$, $(z,u) \times \Omega \times S$, $s \in \mathbb{C}$. For all s satisfying certain non-integral condition we find a necessary and sufficient condition for the functions in the image of the Poisson transform in terms of Hua operators. When Ω is the type **I** matrix domain in $M_{n,m}(\mathbb{C})$ $(n \leq m)$, we prove that an eigenvalue equation for the second order $M_{n,n}$-valued Hua operator characterizes the image.

1. Introduction

Let $\Omega = G/K$ be a Riemannian symmetric space. Any parabolic subgroup P of G defines a boundary G/P of the symmetric space Ω. The Poisson transform is an integral operator from hyperfunctions on G/P into the space of eigenfunctions on Ω of the algebra $\mathcal{D}(\Omega)^G$ of invariant differential operators. Any such boundary G/P can be viewed as coset space of the maximal boundary G/P_{min} defined by a minimal parabolic subgroup P_{min}. In this case the most general result was obtained by Kashiwara *et al.* [9] where they proved that under certain conditions on the eigenvalues that the Poisson transform is a $G-$isomorphism between the space of hyperfunctions on G/P_{min} and the space of eigenfunctions of invariant differential operators on Ω, namely the Helgason conjecture. It thus arises the question of characterizing the image of the Poisson transform for other smaller boundaries.

Suppose Ω is a bounded symmetric domain in a complex $n-$dimensional space V. Let S be its Shilov boundary and r its rank. In this paper

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we consider the characterization of the image of the Poisson transform

$$\mathcal{P}_s\varphi(z) = \int_S P_s(z,u)\varphi(u)d\sigma(u)$$

on the Shilov boundary S when s satisfies the following condition

$$(1) \qquad -4[b+1+j\frac{a}{2}+\frac{n}{r}(s-1)] \notin \{1,2,3,\dots\}, \text{ for } j=0 \text{ and } 1$$

where a and b are some structure constants of Ω. For a specific value of s ($s=1$ in our parameterization) the kernel $P_s(z,u)$, $(z,u)\in\Omega\times S$, is the so called Poisson kernel for harmonic functions, and the corresponding Poisson transform $\mathcal{P} := \mathcal{P}_1$ maps hyperfunctions on S to harmonic functions on Ω; here harmonic functions are defined as the smooth functions that are annihilated by all invariant differential operators that annihilate the constant functions. When Ω is a tube domain Johnson and Korányi [8] proved that the image of the Poisson transform $\mathcal{P}$ is exactly the set of all Hua-harmonic functions. For non-tube domains the characterization of the image of the Poisson transform $\mathcal{P}$ was done by Berline and Vergne [1] where certain third-order differential Hua-operator was introduced to characterize the image.

In his paper [15] Shimeno considered the Poisson transform $\mathcal{P}_s$ on tube domains; it is proved that Poisson transform maps hyperfunctions on the Shilov boundary to certain solution space of the Hua operator. For general domains and for other boundaries, the image of the Poisson transform was characterized in [16]. However for the Shilov boundary of a non-tube domain the problem is still open. We will construct two Hua operators of third order and use them to give a characterization. For the matrix ball $\mathbf{I}_{r,r+b}$ of $r\times(r+b)-$matrices some eigenvalue equation for second-order Hua operator (constructed by Hua [7] and reformulated by Berline and Vergne [1]) is proved to give the characterization. We proceed to explain the content of our paper.

The Hua operator of second-order $\mathcal{H}$ for a general symmetric domain is defined as a $\mathfrak{k}_{\mathbb{C}}$-valued operator, see section 4. For tube domains it maps the Poisson kernels into the center of $\mathfrak{k}_{\mathbb{C}}$, namely the Poisson kernels are its eigenfunctions up to an element in the center, but it is not true for non-tube domains, see section 5. However for type **I** domains of non-tube type, see section 6, there is a variant of the Hua operator, $\mathcal{H}^{(1)}$, by taking the first component of the operator, since in this case $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}^{(1)} + \mathfrak{k}_{\mathbb{C}}^{(2)}$ is a sum of two irreducible ideals. We prove that the operator $\mathcal{H}^{(1)}$ has the Poisson kernels as its eigenfunctions and we find the eigenvalues. We prove further that the eigenfunctions of the Hua operator $\mathcal{H}^{(1)}$ are also eigenfunctions of invariant differential operators on Ω. For that purpose we compute the radial part of the Hua operator

$\mathcal{H}^{(1)}$, see Proposition 6.3. We give eventually the characterization of the image of the Poisson transform in terms of the Hua operator for type $\mathbf{I}_{r,r+b}$ domains :

Theorem 1.1 (Theorem 6.1)**.** *Suppose $s \in \mathbb{C}$ satisfies the following condition*

$$-4[b+1+j+(r+b)(s-1)] \notin \{1,2,3,\cdots\}, \ \ \textit{for} \ \ j=0 \textit{ and } 1.$$

A smooth function f on $\mathbf{I}_{r,r+b}$ is the Poisson transform $\mathcal{P}_s(\varphi)$ of a hyperfunction φ on S if and only if

$$\mathcal{H}^{(1)}f = (r+b)^2 s(s-1) f I_r.$$

Our method of proving the characterization is the same as that in [10] by proving that the boundary value of the Hua eigenfunctions satisfy certain differential equations and is thus defined only on the Shilov boundary, nevertheless it requires several technically demanding computations. In section 7 we study the characterization of range of the Poisson transform for general non-tube domains. We construct two new Hua operators of third order and prove, by essentially the same method as for the previous theorem, the characterization of the image of the Poisson transform using the third-order Hua-type operators $\mathcal{U}$ and $\mathcal{W}$:

Theorem 1.2 (Theorem 7.2)**.** *Let Ω be a bounded symmetric non-tube domain of rank r in $\mathbb{C}^n$. Let $s \in \mathbb{C}$ and put $\sigma = \frac{n}{r}s$. If a smooth function f on Ω is the Poisson transform $\mathcal{P}_s$ of a hyperfunction in $\mathcal{B}(S)$, then*

$$\big(\mathcal{U} - \frac{-2\sigma^2 + 2p\sigma + c}{\sigma(2\sigma - p - b)}\mathcal{W}\big)f = 0. \tag{2}$$

Conversely, suppose s satisfies the condition

$$-4[b+1+j\frac{a}{2}+\frac{n}{r}(s-1)] \notin \{1,2,3,\cdots\}, \ \ \textit{for} \ \ j=0 \textit{ and } 1.$$

Let f be an eigenfunction $f \in \mathcal{M}(\lambda_s)$ (see (8)) with λ_s given by (11). If f satisfies (2) then it is the Poisson transform $\mathcal{P}_s(\varphi)$ of a hyperfunction φ on S.

After this paper was finished we were informed by Professor T. Oshima that he and N. Shimeno have obtained some similar results about Poisson transforms and Hua operators.

2. Preliminaries and notation

2.1. **General setting.** We recall some basic facts about the Jordan triple characterization of bounded symmetric domains and fix notations. Our presentation is mainly based on [11]. Let Ω be an irreducible bounded symmetric domain in a complex $n-$dimensional space V. Let G be the identity component of the group of biholomorphic automorphisms of Ω, and K be the isotropy subgroup of G at the point $0 \in \Omega$. Then K is a maximal compact subgroup of G and as a Hermitian symmetric space, $\Omega = G/K$. Let $\mathfrak{g}$ be the Lie algebra of G, and

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be its Cartan decomposition. The Lie algebra $\mathfrak{k}$ of K has one dimensional center $\mathfrak{z}$. Then there exists an element $Z_0 \in \mathfrak{z}$ such that $\mathrm{ad}Z_0$ defines the complex structure of $\mathfrak{p}$. Let

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^{+} \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^{-} \tag{3}$$

be the corresponding eigenspace decomposition of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$. We will use the Jordan theoretic characterization of Ω; the corresponding Lie theoretic characterization will be then more transparent and which we will also use.

There exists a quadratic form $Q : V \to End(\bar{V}, V)$ (here $\bar{V}$ is the complex conjugate of V), such that

$$\mathfrak{p} = \{\xi_v; v \in V\},$$

where $\xi_v(z) = v - Q(z)\bar{v}$. We will hereafter identify $\mathfrak{p}^{+}$ with V via the natural mapping

$$\frac{1}{2}(\xi_v - i\xi_{iv}) = v \mapsto v,$$

and $\mathfrak{p}^{-}$ with $\bar{V}$ via the mapping

$$-\frac{1}{2}(\xi_v + i\xi_{iv}) = Q(z)\bar{v} \mapsto \bar{v} \in \bar{V};$$

we will write $\bar{v} = Q(z)\bar{v}$ when viewed as element in the Lie algebra and when no ambiguity would arise.

Let $\{z\bar{v}w\}$ the polarization of $Q(z)\bar{v}$, i.e.,

$$\{z\bar{v}w\} = Q(z+w)\bar{v} - Q(z)\bar{v} - Q(w)\bar{v}.$$

This defines a *triple product* $V \times \bar{V} \times V \to V$, with respect to which V is a $JB^{\star}-$*triple*, see [20]. We define $D(z, \bar{v}) \in End(V)$ by

$$D(z, \bar{v})w = \{z\bar{v}w\}.$$

The space V carries a $K-$invariant inner product

$$\langle z, w\rangle = \frac{1}{p}\,\mathbf{tr}\, D(z, \bar{w}), \tag{4}$$

where "**tr**" is the trace functional on $End(V)$, and $p = p(\Omega)$ is the genus of Ω (see (6) below). Beside the Euclidean norm, V carries also the *spectral norm*,

$$\|z\| = \|\frac{1}{2}D(z, \bar{z})\|^{1/2},$$

where the norm of an operator in $End(V)$ is taken with respect to the Hilbert norm $\langle\cdot,\cdot\rangle^{\frac{1}{2}}$ on V. The domain Ω can now be realized as the open unit ball of V with respect to the spectral norm,

$$\Omega = \{z \in V\ ;\ \|z\| < 1\}.$$

An element $c \in V$ is a *tripotent* if $\{c\bar{c}c\} = c$. In the matrix Cartan domains (of type **I**, **II**, and **III**, see below) the tripotents are exactly the partial isometries. Each tripotent $c \in V$ gives rise to a *Peirce decomposition* of V,

$$V = V_0(c) \oplus V_1(c) \oplus V_2(c)$$

where

$$V_j(c) = \{v \in V\ :\ D(c, \bar{c})v = jv\}.$$

Two tripotents c_1 and c_1 are *orthogonal* if $D(c_1, \bar{c}_2) = 0$. Orthogonality is a symmetric relation. A tripotent c is *minimal* if it can not be written as a sum of two non-zero orthogonal tripotents. A tripotent c is *maximal* if $V_0(c) = \{0\}$. A *Jordan frame* is a maximal family of pairwise orthogonal, minimal tripotents. It is known that the group K acts transitively on Jordan frames. In particular, the cardinality of all Jordan frames is the same, and is equal to the rank r of Ω. Every $z \in V$ admits a (unique) *spectral decomposition* $z = \sum_{j=1}^{r} s_j v_j$, where $\{v_j\}$ is a Jordan frame and $s_1 \geq s_2 \geq \cdots \geq s_r \geq 0$ are the *spectral values* of z. The spectral norm of z is equal to the largest spectral value s_1.

Let us choose a Jordan frame $\{c_j\}_{j=1}^{r}$ in V. Then, by the transitivity of K on frames, each element $z \in V$ admits a *polar decomposition* $z = k\sum_{j=1}^{r} s_j c_j$, where $k \in K$ and s_j are the spectral values of z. Let $e = c_1 + c_2 + \cdots + c_r$; then e is a maximal tripotent. Let

$$V = \sum_{0\leq j\leq k\leq r} \oplus V_{j,k}$$

be the *joint Peirce decomposition* of V associated with the Jordan frame $\{c_j\}_{j=1}^{r}$, where

$$V_{j,k} = \{v \in V\ ;\ D(c_\ell, \bar{c}_\ell)v = (\delta_{\ell,j} + \delta_{\ell,k})v,\ \ 1 \leq \ell \leq r\} \tag{5}$$

for $(j,k) \neq (0,0)$ and $V_{0,0} = \{0\}$. By the minimality of c_j, $V_{j,j} = \mathbb{C}c_j$, $1 \leq j \leq r$. The transitivity of K on the frames implies that the integers

$$a := \dim V_{j,k} \ \ (1 \leq j < k \leq r); \qquad b := \dim V_{0,j} \ \ (1 \leq j \leq r)$$

are independent of the choice of the frame and of $1 \leq j < k \leq r$. The triple of integers (r,a,b) uniquely determines Ω. Since e is a maximal tripotent, the Peirce decomposition associated with e is $V = V_2 \oplus V_1$ with

$$V_2 = \sum_{1 \leq j \leq k \leq r} V_{j,k} \ \text{ and } \ V_1 = \sum_{j=1}^{r} V_{0,j}.$$

V_2 becomes a Jordan algebra for the product $xy = \{x\bar{e}y\}$ with identity element e. Let $n_1 = \dim V_1$ and $n_2 = \dim V_2$. Then we have

$$n_1 = rb, \ n_2 = r + \frac{r(r-1)}{2}a \text{ and } n = n_1 + n_2.$$

The *genus* of Ω is

$$p = p(\Omega) = \frac{1}{r}\,\mathbf{tr}\, D(e,\bar{e}) = (r-1)a + b + 2. \tag{6}$$

Thus $\langle c_j, c_j \rangle = \frac{1}{p}\,\mathbf{tr}\, D(c_j, \bar{c}_j) = \frac{1}{rp}\,\mathbf{tr}\, D(e,\bar{e}) = 1$, and this is true for every minimal tripotent in V.

The irreducible bounded symmetric domains were completely classified (up to a biholomorphic isomorphism) by Élie Cartan [2]. We give here a list of all irreducible bounded symmetric domains and the corresponding Jordan triples, for more details see [11] .

Ω	V	(r,a,b)
$\mathbf{I}_{n,m}$ $(n \leq m)$	$M_{n,m}(\mathbb{C})$	$(n, 2, m-n)$
$\mathbf{II}_n$	$\{z \in M_{n,n}(\mathbb{C}) \,:\, z^t = -z\}$	$(n/2, 4, 0)$ (n even)
		$((n-1)/2, 4, 2)$ (n odd)
$\mathbf{III}_n$	$\{z \in M_{n,n}(\mathbb{C}) \,:\, z^t = z\}$	$(n, 1, 0)$
$\mathbf{IV_n}$	$\mathbb{C}^n$	$(2, n-2, 0)$
$\mathbf{V}$	$M_{1,2}(\mathbb{O})$	$(2, 6, 4)$
$\mathbf{VI}$	$\{z \in M_{3,3}(\mathbb{O}) \,:\, \bar{z}^t = z\}$	$(3, 8, 0)$

where $\mathbb{O}$ is the 8−dimensional Cayley algebra.

Let

$$\mathfrak{a} = \mathbb{R}\xi_1 + \cdots + \mathbb{R}\xi_r, \quad \xi_j = \xi_{c_j}, \quad j = 1, \ldots, r.$$

Then, $\mathfrak{a}$ is a maximal Abelian subspace of $\mathfrak{p}$. Let $\{\beta_j\}_{j=1}^r \subset \mathfrak{a}^*$ be the basis of $\mathfrak{a}^*$ determined by

$$\beta_j(\xi_k) = 2\delta_{j,k}, \ \ 1 \leq j,k \leq r,$$

and define an ordering on $\mathfrak{a}^*$ such that

$$\beta_r > \beta_{r-1} > \cdots > \beta_1 > 0. \tag{7}$$

The restricted root system $\Sigma(\mathfrak{g},\mathfrak{a})$ of $\mathfrak{g}$ relative to $\mathfrak{a}$ is of type C_r or BC_r and it consists of the roots $\pm\beta_j$ $(1 \le j \le r)$ with multiplicity 1, the roots $\pm\frac{1}{2}\beta_j \pm \frac{1}{2}\beta_k$ $(1 \le j \ne k \le r)$ with multiplicity a, and possibly the roots $\pm\frac{1}{2}\beta_j$ $(1 \le j \le r)$ with multiplicity $2b$. The set positive roots $\Sigma^+(\mathfrak{g},\mathfrak{a})$ consists of $\frac{1}{2}(\beta_k \pm \beta_j)$ $(1 \le j < k \le r)$; β_j and $\frac{1}{2}\beta_j$ $(1 \le j \le r)$, while the set of negative roots is $\Sigma^-(\mathfrak{g},\mathfrak{a}) = -\Sigma^+(\mathfrak{g},\mathfrak{a})$.

It follows that ρ, the half sum of the positive roots, is given by

$$\rho = \sum_{j=1}^{r} \rho_j \beta_j,$$

where

$$\rho_j = \frac{b+1+a(j-1)}{2},\ j = 1,\dots,r.$$

Let $\mathfrak{n}^\pm$ be the sum of positive respectively negative roots spaces,

$$\mathfrak{n}^\pm \;=\; \sum_{\beta\in\Sigma^\pm} \mathfrak{g}^\beta = \sum_{1\le j<k\le r} \mathfrak{g}^{\pm\frac{\beta_k\pm\beta_j}{2}} + \sum_{1\le j\le r} \mathfrak{g}^{\pm\beta_j} + \sum_{1\le j\le r} \mathfrak{g}^{\pm\frac{\beta_j}{2}}.$$

The Iwasawa decomposition is then given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^- .$$

Let, as usual, $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, then we have

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+.$$

We let $P = P_{\min} = MAN$ be the minimal parabolic subgroup of G, with M, A and N the corresponding Lie groups with Lie algebras $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}^-$.

Let $\mathfrak{t}_{\mathbb{C}}^-$ be the subspace

$$\mathfrak{t}_{\mathbb{C}}^- = \mathbb{C}D(c_1,\bar{c}_1) + \cdots + \mathbb{C}D(c_r,\bar{c}_r)$$

of $\mathfrak{k}_{\mathbb{C}}$. Then $\mathfrak{t}_{\mathbb{C}}^-$ is Abelian and we extend it to a Cartan subalgebra $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}}^- + \mathfrak{t}_{\mathbb{C}}^+$ of $\mathfrak{k}_{\mathbb{C}}$. The root system $\Psi := \Sigma(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$, when restricted to $\mathfrak{t}_{\mathbb{C}}^-$ is of the form

$$\Psi|_{\mathfrak{t}_{\mathbb{C}}^-} = \Sigma(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}}^-) = \{\pm\frac{1}{2}(\gamma_k\pm\gamma_j),\ 1\le j\ne k\le r\ ;\ \pm\gamma_j,\ \pm\frac{1}{2}\gamma_j,\ 1\le j\le r\}$$

where γ_j are the Harish-Chandra strongly orthogonal roots defined by

$$\gamma_j(D(c_k,\bar{c}_k)) = 2\delta_{jk},\ \ \gamma_j|_{\mathfrak{t}_{\mathbb{C}}^+} = 0,\ \ 1\le j,k\le r.$$

The set of compact roots $\Psi_c := \Sigma(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is such that

$$\Psi_c|_{\mathfrak{t}^-_{\mathbb{C}}} = \{\frac{1}{2}(\gamma_k - \gamma_j),\ 1 \le j \ne k \le r\ ;\ \pm\frac{1}{2}\gamma_j,\ 1 \le j \le r\},$$

and the set of noncompact roots Ψ_n satisfies

$$\Psi_n|_{\mathfrak{t}^-_{\mathbb{C}}} = \{\pm\frac{1}{2}(\gamma_k + \gamma_j),\ 1 \le j \ne k \le r\ ;\ \pm\gamma_j\ ;\ \pm\frac{1}{2}\gamma_j,\ 1 \le j \le r\}.$$

We choose a consistent ordering with (3) and (7)

$$\gamma_r > \gamma_{r-1} > \cdots > \gamma_1.$$

We will also need the set of positive noncompact roots $\Psi_n|^+_{\mathfrak{t}^-_{\mathbb{C}}}$,

$$\Psi_n|^+_{\mathfrak{t}^-_{\mathbb{C}}} = \{\frac{1}{2}(\gamma_k + \gamma_j),\ 1 \le j \ne k \le r\ ;\ \frac{1}{2}\gamma_j,\ \gamma_j,\ 1 \le j \le r\}.$$

2.2. **Bounded symmetric domain of type $\mathbf{I}_{r,r+b}$.** Let $V = M_{r,r+b}(\mathbb{C})$ be the vector space of complex $r \times (r+b)-$matrices. V is a Jordan triple system for the following triple product

$$\{x\bar{y}z\} = xy^*z + zy^*x.$$

Then the endomorphisms $D(z, \bar{v})$ are given by

$$D(z, \bar{v})w = \{z\bar{v}w\} = zv^*w + wv^*z.$$

There is a canonical and natural choice of frames. One considers the standard matrix units $\{e_{i,j},\ 1 \le i \le r,\ 1 \le j \le r+b\}$ and defines $c_j = e_{j,j}$, $1 \le j \le r$. Then the Pierce decomposition $V = \sum_{0 \le j \le k \le r} \oplus V_{j,k}$ of V is given by

$$\begin{aligned} V_{j,j} &= \mathbb{C}c_j,\ 1 \le j \le r, \\ V_{j,k} &= \mathbb{C}e_{j,k} + \mathbb{C}e_{k,j},\ 1 \le j < k \le r, \\ V_{0,j} &= \operatorname{span}\{e_{j,k},\ r < k \le r+b\},\ 1 \le j \le r. \end{aligned}$$

Let

$$\mathbf{I}_{r,r+b} = \{z \in M_{r,r+b}(\mathbb{C})\ :\ I_r - z^*z \gg 0\}$$

where I_r denote the unit matrix of rank r. Then $\mathbf{I}_{r,r+b}$ is a bounded symmetric domain of dimension $r(r+b)$, rank r and genus $2r+b$. The multiplicities are $2b$ and $a = 2$ if $2 \le r$, $a = 0$ if $r = 1$. The domain $\mathbf{I}_{r,r+b}$ is of tube type if and only if $b = 0$. Its Shilov boundary is

$$S = \{z \in M_{r,r+b}(\mathbb{C})\ :\ z^*z = I_r\}.$$

Let $G = SU(r, r+b)$ denote the special unitary group of the hermitian form

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_r\bar{w}_r - z_{r+1}\bar{w}_{r+1} - \cdots - z_{2r+b}\bar{w}_{2r+b}$$

on $\mathbb{C}^{2r+b}$ and write its elements in block form

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}; \quad \mathbf{a} \in M_{r,r}(\mathbb{C}), \quad \mathbf{b} \in M_{r,r+b}(\mathbb{C}) \quad \text{etc.}$$

Then G acts transitively on $\mathbf{I}_{r,r+b}$ via

$$g \cdot z = (\mathbf{a}z + \mathbf{b})(\mathbf{c}z + \mathbf{d})^{-1}, \quad \text{with } g = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}.$$

The subgroup $K = S(U(r) \times U(r+b))$ consisting of elements of the form

$$\begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{d} \end{pmatrix}, \quad \mathbf{a} \in U(r), \ \mathbf{d} \in U(r+b), \quad \mathbf{det}(\mathbf{a})\,\mathbf{det}(\mathbf{d}) = 1$$

is easily seen to be a maximal compact subgroup of G. The Lie algebra $\mathfrak{g}$ of G decomposes into $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{k}$, the Lie algebra of K, consists of all matrices

$$\begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{d} \end{pmatrix}, \ \mathbf{a} \in M_{r,r}(\mathbb{C}), \ \mathbf{d} \in M_{r+b,r+b}(\mathbb{C}), \ \mathbf{a}^* = -\mathbf{a}, \ \mathbf{d}^* = -\mathbf{d}$$

and $\mathfrak{p}$ consists of all matrices

$$\begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix}, \ v \in M_{r,r+b}(\mathbb{C}).$$

The induced vector fields are given respectively by

$$z \mapsto \mathbf{a}z - z\mathbf{d},$$

and

$$z \mapsto \xi_v(z) = v - zv^*z.$$

The complex Lie algebra $\mathfrak{k}_{\mathbb{C}}$ is given by the set of all matrices

$$\begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{d} \end{pmatrix}, \ \mathbf{a} \in M_{r,r}(\mathbb{C}), \ \mathbf{d} \in M_{r+b,r+b}(\mathbb{C}), \ \mathbf{tr}(\mathbf{a}) + \mathbf{tr}(\mathbf{d}) = 0.$$

Hence, $\mathfrak{k}_{\mathbb{C}}$ can be written as the sum

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}^{(1)} \oplus \mathfrak{k}_{\mathbb{C}}^{(2)},$$

where $\mathfrak{k}_{\mathbb{C}}^{(1)}$ and $\mathfrak{k}_{\mathbb{C}}^{(2)}$ are the ideals consisting respectively of the matrices

$$\begin{pmatrix} \mathbf{a} & 0 \\ 0 & -\frac{\mathbf{tr}(\mathbf{a})}{r+b} I_{r+b} \end{pmatrix}, \ \mathbf{a} \in M_{r,r}(\mathbb{C}),$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{d} \end{pmatrix}, \ \mathbf{d} \in M_{r+b,r+b}(\mathbb{C}), \ \mathbf{tr}(\mathbf{d}) = 0.$$

Then, identifying $\mathfrak{k}_{\mathbb{C}}$ as linear transformations of V, we have

$$\mathfrak{k}_{\mathbb{C}} = \text{span}\{D(u, \bar{v}), \quad u, v \in V\},$$

and

$$\mathfrak{k}_{\mathbb{C}}^{(1)} = \text{span}\{D(u, \bar{v})^{(1)}, \;\; u, v \in V\},$$

where the endomorphism $D(u, \bar{v})^{(1)}$ is given by

$$D(u, \bar{v})^{(1)} z = uv^* z.$$

3. The Poisson transform

Let $\mathcal{D}(\Omega)^G$ be the algebra of all invariant differential operators on Ω. Recall the definition of the Harish-Chandra $e_\lambda-$function : e_λ, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is the unique $N-$invariant function on Ω such that

$$e_\lambda(\exp(t_1\xi_1 + \cdots + t_r\xi_r) \cdot 0) = e^{2t_1(\lambda_1+\rho_1)+\cdots+2t_r(\lambda_r+\rho_r)}.$$

Then e_λ are the eigenfunctions of $T \in \mathcal{D}(\Omega)^G$ and we denote $\chi_\lambda(T)$ the corresponding eigenvalues. Denote further

$$\mathcal{M}(\lambda) = \{f \in C^\infty(\Omega);\, Tf = \chi_\lambda(T)f,\, T \in \mathcal{D}(\Omega)^G\}. \tag{8}$$

Recall the parabolic subgroup $P = P_{min}$ introduced in the subsection 2.1. Corresponding to P there is the Poisson transform on the maximal boundary $G/P = K/M$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the *Poisson transform* $\mathcal{P}_{\lambda,K/M}$ is defined by

$$\mathcal{P}_{\lambda,K/M} f(gK) = \int_K e_\lambda(k^{-1}g) f(k) dk$$

on the space $\mathcal{B}(K/M)$ of hyperfunctions on K/M.

It is proved by Kashiwara *et al.* in [9] that for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, if

$$-2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \notin \{1, 2, 3, \ldots\} \tag{9}$$

for all $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$, then the Poisson transform is a G-isomorphism from $\mathcal{B}(K/M)$ onto $\mathcal{M}(\lambda)$.

We now introduce the Poisson transform on the Shilov boundary. Let $h(z)$ be the unique $K-$invariant polynomial on V whose restriction to $\mathbb{R}c_1 + \cdots + \mathbb{R}c_r$ is given by

$$h(\sum_{j=1}^r t_j c_j) = \prod_{j=1}^r (1 - t_j^2).$$

As h is real-valued, we may polarize it to get a polynomial on $V \times V$, denoted by $h(z, w)$, holomorphic in z and anti-holomorphic in w such that $h(z, z) = h(z)$. Recall that the function h is related to the Bergman operator (see (13) below) by that

$$\mathbf{det}\, b(z, \bar{z}) = h(z, \bar{z})^p.$$

The *Poisson kernel* $P(z,u)$ on $\Omega \times S$ is

$$P(z,u) = \Big(\frac{h(z,z)}{|h(z,u)|^2}\Big)^{\frac{n}{r}}.$$

For a complex number s we define the *Poisson transform* $\mathcal{P}_s\varphi$ on the space $\mathcal{B}(S)$ of hyperfunctions φ on S by

$$(\mathcal{P}_s\varphi)(z) = \int_S P(z,u)^s\varphi(u)d\sigma(u).$$

The kernel $P(z,u)^s$ has the following transformation property

$$P(gz,gu)^s = |J_g(u)|^{-\frac{2ns}{rp}}P(z,u)^s, \quad \forall g \in G \tag{10}$$

where $J_g(u)$ is the Jacobian of g at u.

The kernel $P(z,u)^s$, for $u = e$ is a special case of the $e_\lambda-$function. The Poisson transform $\mathcal{P}_s$ on S can be viewed as a restriction of the Poisson transform $\mathcal{P}_{\lambda,K/M}$. However for fixed s there are various choices of λ and we will find a specific λ so that the above condition (9) is valued when s satisfies (1). Let

$$\xi_c = \xi_1 + \cdots + \xi_r$$

and consider the decomposition

$$\mathfrak{a} = \mathbb{R}\xi_c \oplus \xi_c^\perp = \mathbb{R}\xi_c \oplus \sum_{j=1}^{r-1}\mathbb{R}(\xi_j - \xi_{j+1})$$

under the (negative) Killing form on $\mathfrak{g}$. We denote ξ_c^* the dual vector, $\xi_c^*(\xi_c) = 1$. We extend ξ_c^* to $\mathfrak{a}$ via the orthogonal projection defined above. Observe first that

$$\rho(\xi_c) = n = n\xi_c^*(\xi_c).$$

We have then

$$\mathcal{P}_s f(z) = \mathcal{P}_{\lambda_s,K/M} f(z)$$

where f on S is viewed as a function on K and thus on K/M, $\lambda_s \in \mathfrak{a}_{\mathbb{C}}^*$ is given by

$$\lambda_s = \rho + 2n(s-1)\xi_c^*. \tag{11}$$

Thus

$$\mathcal{P}_s\mathcal{B}(S) \subset \mathcal{P}_{\lambda_s,K/M}\mathcal{B}(K/M) \subset \mathcal{M}(\lambda_s).$$

When s satisfies (1) we have then $\mathcal{P}_{\lambda_s,K/M}\mathcal{B}(K/M) = \mathcal{M}(\lambda_s)$.

4. The second-order Hua operator

We shall define the Hua operator both in terms of the enveloping algebra and using the covariant Cauchy-Riemann operator, the later having the advantage of being geometric and more explicit. To avoid some extra constants we fix and normalize the Killing form B on $\mathfrak{g}_{\mathbb{C}}$ by requiring that on $\mathfrak{p}^+ \times \mathfrak{p}^-$ it is given by,

$$B(u, \bar{v}) = \langle u, v\rangle = \frac{1}{p}\,\mathbf{tr}\, D(u, \bar{v}), \quad u \in V = \mathfrak{p}^+, \bar{v} \in \bar{V} = \mathfrak{p}^-$$

where the trace is computed on the space V. (So the standard Killing form, $(X, Y) \mapsto \mathbf{tr}\,\mathrm{Ad}(X)\,\mathrm{Ad}(Y)$, is $-pB(X, Y)$.)

Let $\{v_j\}$ and $\{v_j^*\}$ be dual bases of $\mathfrak{p}^+$ and $\mathfrak{p}^-$ with respect to the normalized Killing form B. Let $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ be the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. Since $[\mathfrak{p}^+, \mathfrak{p}^-] \subset \mathfrak{k}_{\mathbb{C}}$, the operator

$$\mathcal{H} = \mathcal{H}_{\mathfrak{k}_{\mathbb{C}}} = -\sum_{i,j} v_i v_j^* \otimes [v_j, v_i^*]$$

is an element of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes \mathfrak{k}_{\mathbb{C}}$, and is independent of choice of the basis; it is called the *second-order Hua operator*. If we identify $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ with left invariant differential operators on G, $\mathcal{H}$ defines a homogeneous operator from $C^\infty(G/K)$ to the C^∞−sections of $G \times_K \mathfrak{k}_{\mathbb{C}}$. $\mathcal{H}$ can also be viewed as a differential operator from $C^\infty(G)$ to $C^\infty(G, \mathfrak{k}_{\mathbb{C}})$.

For $X \in \mathfrak{k}_{\mathbb{C}}$, define

$$\mathcal{H}^X = -\sum_j [X, v_j] v_j^* \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}).$$

Let $\mathcal{S}$ be a linear subspace of $\mathfrak{k}$, $\mathcal{S}_{\mathbb{C}}$ its complexification. Let $\{X_j\}$ be a basis of $\mathcal{S}_{\mathbb{C}}$ and $\{X_j^*\}$ be the dual basis with respect to the Killing form B. Then the projection of $\mathcal{H}$ onto $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes \mathcal{S}_{\mathbb{C}}$ is

$$\mathcal{H}_{\mathcal{S}_{\mathbb{C}}} = \sum_j \mathcal{H}^{X_j} \otimes X_j^*.$$

It can also be defined independently of basis, see e.g. [10, Proposition 1].

We need another more explicit and geometric definition of the Hua operator using the covariant Cauchy-Riemann operator studied in [3], [18] and [23]. Recall briefly that the covariant Cauchy-Riemann operator can be defined on any holomorphic Hermitian vector bundle over a Kähler manifold. Trivializing the sections of a homogeneous vector bundle E on the bounded symmetric space Ω as the space $C^\infty(\Omega, E)$ of E-valued functions on Ω, where E is a holomorphic representation of $K_{\mathbb{C}}$, the *covariant Cauchy-Riemann operator* $\bar{\mathbf{D}}$ is defined by

$$\bar{\mathbf{D}} f = b(z, \bar{z}) \bar{\partial} f, \tag{12}$$

where $b(z, \bar{z})$ is the inverse of the Bergman metric, also called *Bergman operator* of Ω, given by

$$b(z, \bar{w}) = 1 - D(z, \bar{w}) + Q(z)Q(\bar{w}). \tag{13}$$

The operator $\bar{\mathbf{D}}$ maps $C^\infty(\Omega, E)$ to $C^\infty(\Omega, V \otimes E)$, with V viewed as the holomorphic tangent space.

Consider the space $C^\infty(\Omega)$ of C^∞-functions on Ω as the sections of the trivial line bundle. The operator ∂ is then well-defined on $C^\infty(\Omega)$ and it maps $C^\infty(\Omega)$ to $C^\infty(\Omega, V') = C^\infty(\Omega, \bar{V})$ with the later identified as the space of sections of the holomorphic cotangent bundle. We can then define the differential operator

$$\mathrm{Ad}_{V\otimes\bar{V}}(\bar{D} \otimes \partial) : C^\infty(\Omega) \to C^\infty(\Omega, \mathfrak{k}_{\mathbb{C}}), \qquad f \mapsto \mathrm{Ad}_{V\otimes\bar{V}}(\bar{D} \otimes \partial f)$$

with $\mathrm{Ad}_{V\otimes\bar{V}} : V \otimes \bar{V} = \mathfrak{p}^+ \otimes \mathfrak{p}^- \to \mathfrak{k}_{\mathbb{C}}$ being the Lie bracket, $u \otimes v \to D(u, v)$. So by the covariant property of ∂ and $\bar{\mathbf{D}}$ (see [23]) we have

$$\mathrm{Ad}_{V\otimes\bar{V}}(\bar{D} \otimes \partial)(f(gz)) = dg(z)^{-1} \mathrm{Ad}_{V\otimes\bar{V}}(\bar{D} \otimes \partial f)(gz),$$

where $dg(z) : V = T_z^{(1,0)} \to T_{gz}^{(1,0)}$ is the differential of the mapping g, which further is $dg(z) = \mathrm{Ad}(dg(z))$ the adjoint action of $dg(z) \in K_{\mathbb{C}}$ on $\mathfrak{k}_{\mathbb{C}}$. It follows easily that this operator and the operator $\mathcal{H}_{\mathfrak{k}_{\mathbb{C}}}$ agree,

$$\mathrm{Ad}_{V\otimes\bar{V}}(\bar{D} \otimes \partial) = \mathcal{H} = \mathcal{H}_{\mathfrak{k}_{\mathbb{C}}}.$$

Symbolically we may write

$$\mathcal{H} = D(b(z, \bar{z})\bar{\partial}, \partial).$$

Using an orthonormal basis $\{e_j\}$ of V with respect to the hermitian scalar product (4), the operator $\mathcal{H}$ can also be written

$$\mathcal{H} f(z) = \sum_{i,j} D(b(z, \bar{z})\bar{e}_i, e_j) \bar{\partial}_i \partial_j f(z).$$

5. The Poisson kernel and the Second-order Hua operator

We will compute the action of the Hua operator on the Poisson kernel. Let us first recall the notion of *quasi-inverse* in the Jordan triple V; see [11]. Let $z \in V$ and $\bar{w} \in \bar{V}$. The element z is called quasi-invertible with respect to $\bar{w}$, if $b(z, \bar{w})$ is invertible. The quasi-inverse of z with respect to $\bar{w}$ is then given by

$$z^{\bar{w}} = b(z, \bar{w})^{-1}(z - Q(z)\bar{w}).$$

For example, in the type $\mathbf{I}_{r,r+b}$ case (see 2.2), let $x, y \in V = M_{r,r+b}(\mathbb{C})$, then

$$b(x, \bar{y})z = (I - xy^*)z(I - yx^*).$$

If $I - xy^*$ is invertible, then the quasi-inverse of x is

$$x^{\bar{y}} = b(x, \bar{y})^{-1}(x - Q(x)\bar{y}) = (I - xy^*)^{-1}(x - xy^*x)(I - y^*x)^{-1} = (I - xy^*)^{-1}x.$$

Fix a Jordan frame $\{c_j\}_{1\leq j\leq r}$ and choose and orthonormal basis $\{e_\alpha\}$ of V consisting of the frame $\{c_j\}_{1\leq j\leq r}$, orthonormal basis of each of the subspaces V_{jk} and an orthonormal basis of each of the subspaces V_{j0}. The following lemma can easily be proved by direct computations.

Lemma 5.1. (1) *For any irreducible bounded symmetric domain* Ω *it holds*
(a) $\sum_{\alpha=1}^{r} D(e_\alpha, \bar{e}_\alpha) = pZ_0$.
(b) $\sum_{e_\alpha\in V_{jk}} D(e_\alpha, \bar{e}_\alpha) = \frac{a}{2}\big[D(c_j, \bar{c}_j) + D(c_k, \bar{c}_k)\big]$.
(2) *If* Ω *is of type* $\mathbf{I}_{r,r+b}$, *then* $\sum_{e_\alpha\in V_{j0}} D(e_\alpha, \bar{e}_\alpha)^{(1)} = bD(c_j, \bar{c}_j)^{(1)}$.

We need also the following lemma.

Lemma 5.2. *Let* $\bar{w} \in \bar{V}$. *For any complex number* s, *the holomorphic and the anti-holomorphic differential of the function* $z \mapsto h(z, \bar{w})$ *are given by*

$$\partial h(z, \bar{w})^s = -s\, h(z, \bar{w})^s \bar{w}^z \;\; ; \;\; \bar{\partial} h(z, \bar{w})^s = -s\, h(z, \bar{w})^s w^{\bar{z}}.$$

Proof. This is a consequence of the formula

$$\bar{w}^z = -\partial \log \mathbf{det}\, b(z, \bar{w})^{\frac{1}{p}} = -\partial \log h(z, \bar{w}),$$

see [25, Proposition 3.1]. □

Theorem 5.3. *For* u *fixed in* S, *the function*

$$z \mapsto P_{s,u}(z) := P(z, u)^s$$

satisfies the following differential equation

$$\mathcal{H}P_{s,u}(z) = \big[(\frac{n}{r}s)^2 D(b(z, \bar{z})(z^{\bar{z}} - u^{\bar{z}}), \bar{z}^z - \bar{u}^z) - (\frac{n}{r}sp)Z_0\big]P_{s,u}(z).$$

Proof. Choose a basis $\{e_\alpha\}$ as in Lemma 5.1. Then

$$\mathcal{H}P_{s,u}(z) = \sum_{\alpha,\beta} D(b(z,\bar{z})e_\alpha, \bar{e}_\beta)\bar{\partial}_\alpha\partial_\beta P_{s,u}(z).$$

According to Lemma 5.2,

$$\partial P_{s,u}(z) = \partial\Big[\frac{h(z,z)}{|h(z,u)|^2}\Big]^{\frac{n}{r}s} = -\frac{n}{r}s\Big[\frac{h(z,z)}{|h(z,u)|^2}\Big]^{\frac{n}{r}s}\big[\bar{z}^z - \bar{u}^z\big]$$

where we have identified $(\mathfrak{p}^-)'$ with $\mathfrak{p}^+$ by the Hermitian form (4). Performing one more time differentiation, we get

$$\bar{\partial}\partial P_{s,u}(z) = (\frac{n}{r}s)^2 P_{s,u}(z)\big[z^{\bar{z}} - u^{\bar{z}}\big] \otimes \big[\bar{z}^z - \bar{u}^z\big] - \frac{n}{r}sP_{s,u}(z)\bar{\partial}\big[\bar{z}^z - \bar{u}^z\big].$$

Moreover,

$$\bar{\partial}\big[\bar{z}^z - \bar{u}^z\big] = \bar{\partial}[\bar{z}^z] = \bar{\partial}\partial \log h(z,\bar{z})^{-1} = b(z,\bar{z})^{-1}\mathrm{Id},$$

where Id is the identity form in $(\mathfrak{p}^+)' \otimes (\mathfrak{p}^-)'$. Hence,

$$\bar{\partial}\partial P_{s,u}(z) = (\frac{n}{r}s)^2 P_{s,u}(z)\big[z^{\bar{z}} - u^{\bar{z}}\big] \otimes \big[\bar{z}^z - \bar{u}^z\big] - (\frac{n}{r}s)b(z,\bar{z})^{-1}\mathrm{Id}.$$

Consequently

$$\begin{aligned}\mathcal{H}P_{s,u}(z) &= \Big[(\frac{n}{r}s)^2 \sum_{\alpha,\beta}\langle z^{\bar{z}} - u^{\bar{z}}, e_\alpha\rangle\langle \bar{z}^z - \bar{u}^z, \bar{e}_\beta\rangle D(b(z,\bar{z})e_\alpha, \bar{e}_\beta) \\ &\quad -(\frac{n}{r}s)\sum_\alpha D(e_\alpha, \bar{e}_\alpha)\Big]P_{s,u}(z) \\ &= \Big[(\frac{n}{r}s)^2 D(b(z,\bar{z})(z^{\bar{z}} - u^{\bar{z}}), \bar{z}^z - \bar{u}^z) - (\frac{n}{r}s)pZ_0\Big]P_{s,u}(z),\end{aligned}$$

since $\sum_\alpha D(e_\alpha, \bar{e}_\alpha) = pZ_0$.

□

If Ω is of tube type, then the genus p is given by $p = 2\frac{n}{r}$ and Theorem 5.3 becomes :

Corollary 5.4. *Let Ω be a tube type domain. For any $u \in S$, the function $z \mapsto P_{s,u}(z)$ satisfies the Hua equation*

$$\mathcal{H}P_{s,u}(z) = 2(\frac{n}{r})^2 s(s-1)P_{s,u}(z)I, \tag{14}$$

where I is the identity operator.

This corollary has been proved also by Faraut and Korányi in [4, Theorem XIII.4.4]. Notice that the first factor 2 in (14) is because in this case our Hua operator is twice the Hua operator of Faraut and Korányi. In fact we are using the definition $[u, \bar{v}] = D(u, \bar{v})$ so that for tube domain it is twice the "square" operator $\Box$ of Faraut and Korányi.

In [15, Theorem 4.1] Shimeno gives the following characterization of the image of the Poisson transform for tube type domains :

Theorem 5.5. *Let Ω be a tube type domain. Suppose $s \in \mathbb{C}$ satisfies the following condition*

$$-4[1 + j\frac{a}{2} + \frac{n}{r}(s-1)] \notin \{1, 2, 3, \cdots\} \text{ for } j = 0 \text{ and } 1.$$

A smooth function f on Ω is the Poisson transform $\mathcal{P}_s$ of a hyperfunction on S if and only if f satisfies the following Hua equation

$$\mathcal{H}f = 2(\frac{n}{r})^2 s(s-1) f Z_0.$$

This is a slight different formulation of Shimeno's result. In fact, if s' denotes the Shimeno's parameter, then our parameter s is

$$s = \frac{r}{2n}\big(s' + \frac{n}{r}\big).$$

6. The main result for type $\mathbf{I}_{r,r+b}$ domains

In this section we restrict ourself to the case $\Omega = \mathbf{I}_{r,r+b}$. Recall that in subsection 2.2 we have fixed a decomposition $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}^{(1)} \oplus \mathfrak{k}_{\mathbb{C}}^{(2)}$. We let $\mathcal{H}^{(1)}$ be the first component of the Hua operator $\mathcal{H}$.

Symbolically $\mathcal{H}^{(1)}$ is given by

$$\mathcal{H}^{(1)} = D(b(z, \bar{z})\bar{\partial}, \partial)^{(1)},$$

and can be identified with the operator

$$(I_r - zz^*)\bar{\partial}_z \cdot (I_{r+b} - z^*z) \cdot {}^t\partial_z$$

introduced by Hua [7], since in this case $b(z, \bar{z})v = (I - zz^*)v(I - z^*z)$.

We state now the main theorem of this section.

Theorem 6.1. *Suppose $s \in \mathbb{C}$ satisfies the following condition*

$$-4[b + 1 + j + (r+b)(s-1)] \notin \{1, 2, 3, \cdots\} \text{ for } j = 0 \text{ and } 1. \tag{15}$$

A smooth function f on $\mathbf{I}_{r,r+b}$ is the Poisson transform $\mathcal{P}_s(\varphi)$ of a hyperfunction φ on S if and only if f satisfies the following Hua equation

$$\mathcal{H}^{(1)} f = (r+b)^2 s(s-1) f I_r, \tag{16}$$

where I_r is the identity matrix of rank r.

Note here that the constant $r + b = \frac{n}{r}$ for the domain $\mathbf{I}_{r,r+b}$.

6.1. **The necessity of the Hua equation (16).** To show the necessity of the Hua equation it is sufficient to show that the function $P_{s,u}$ satisfies (16) for every $u \in S$.

Proposition 6.2. *If Ω is of type $\mathbf{I}_{r,r+b}$, then*

$$\mathcal{H}^{(1)} P_{s,u}(z) = (r+b)^2 s(s-1) P_{s,u}(z) I_r.$$

Proof. It is sufficient to prove the formula at $z = 0$. Specifying the result of Theorem 5.3 to the type $\mathbf{I}_{r,r+b}$ domain we get for any $u \in S$,

$$\mathcal{H}^{(1)} P_{s,u}(0) = [(\frac{n}{r}s)^2 D(u,u)^{(1)} - (\frac{n}{r}sp) Z_0^{(1)}] P_{s,u}(0).$$

Now, obviously $D(u,u)^{(1)} = I_r$ and $Z_0^{(1)} = \frac{r+b}{2r+b} I_r$. Therefore,

$$\mathcal{H}^{(1)} P_{s,u}(0) = (r+b)^2 s(s-1) P_{s,u}(0) I_r.$$

□

6.2. **The Hua operator and the eigenfunctions of invariant differential operators.** We give first the expression for the *radial part* of the Hua operator $\mathcal{H}^{(1)}$, i.e. its restriction to $K-$invariant functions. We fix a Jordan frame $\{c_j\}_{j=1}^r$, then every element of V can be written as

$$z = k \sum_{j=1}^{r} t_j c_j,$$

with $k \in K$, and $t_j \geq 0$. If f is a function on Ω invariant under K, we write

$$f(z) = F(t_1, \ldots, t_r).$$

The function F is a symmetric function of the variables $t_1, \ldots, t_r$, defined on the unit cube $0 \leq t_j < 1$.

Proposition 6.3. *Let Ω be the type $\mathbf{I}_{r,r+b}$ domain. Let f be $\mathcal{C}^2$ and $K-$invariant function, then for $a = \sum_{j=1}^r t_j c_j$,*

$$\mathcal{H}^{(1)} f(a) = \sum_{j=1}^{r} \mathcal{H}_j F(t_1, \ldots, t_r) D(c_j, \bar{c}_j)^{(1)}, \tag{17}$$

where the scalar-valued operators $\mathcal{H}_j$ are given by

$$\mathcal{H}_j = (1-t_j^2)^2 (\frac{\partial^2}{\partial t_j^2} + \frac{1}{t_j}\frac{\partial}{\partial t_j}) +$$

$$+ \sum_{k \neq j} (1-t_j^2)(1-t_k^2) \Big[\frac{1}{t_j - t_k} (\frac{\partial}{\partial t_j} - \frac{\partial}{\partial t_k}) + \frac{1}{t_j + t_k} (\frac{\partial}{\partial t_j} + \frac{\partial}{\partial t_k}) \Big] +$$

$$+2b(1-t_j^2)\frac{1}{t_j}\frac{\partial}{\partial t_j}.$$

Proof. The proof is similar to the proof of [4, Theorem XIII.4.7] and we will only show how one can compute the last term of the radial part $\mathcal{H}_j^{(1)}$, namely $2b(1-t_j^2)\frac{1}{t_j}\frac{\partial}{\partial t_j}$. Let $\{e_\alpha\}$ be an orthonormal basis of V consisting of the frame $\{c_j\}_{j=1}^r$, an orthonormal basis of each of the subspaces V_{jk} and an orthonormal basis of each of the subspaces V_{j0}; and let $z_\alpha = x_\alpha + iy_\alpha$ be the complex coordinates. Let f be a function on Ω and fix $a = \sum_{k=1}^r t_k c_k$. Then,

$$\mathcal{H}^{(1)}f(a) = \sum_{\alpha,\beta} D(b(a,\bar{a})e_\alpha, \bar{e}_\beta)^{(1)}\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta}f(a).$$

For any $X \in \mathfrak{g}$ and any $v \in V$, it is known that

$$\partial_{Xv}\partial_{Xv}f + \partial_{X^2v}f = 0.$$

We will apply this formula for different elements in $\mathfrak{g}$. Suppose $e_\alpha = e_\beta \in V_{j,0}$. For the element $X = i(D(e_\alpha, \bar{c}_j) + D(c_j, \bar{e}_\alpha)) \in \mathfrak{k}$, we have

$$Xa = i(D(e_\alpha,\bar{c}_j) + D(c_j,\bar{e}_\alpha))a = iD(e_\alpha,\bar{c}_j)a = iD(a,\bar{e}_\alpha)e_\alpha = it_je_\alpha,$$

and

$$X^2a = X(it_je_\alpha) = -t_j(D(e_\alpha,\bar{c}_j) + D(c_j,\bar{e}_\alpha))a = -t_jc_j.$$

Therefore

$$\partial_{it_je_\alpha}\partial_{it_je_\alpha}f(a) + \partial_{-t_jc_j}f(a) = 0,$$

which implies

$$\frac{\partial^2}{\partial y_\alpha^2}f = \frac{1}{t_j}\frac{\partial}{\partial t_j}F.$$

Similarly, For $X = D(e_\alpha,\bar{c}_j) - D(c_j,\bar{e}_\alpha) \in \mathfrak{k}$, we have

$$Xa = (D(e_\alpha,\bar{c}_j) - D(c_j,\bar{e}_\alpha))a = t_je_\alpha,$$

and

$$X^2a = X(t_je_\alpha) = t_j(D(e_\alpha,\bar{c}_j) - D(c_j,\bar{e}_\alpha))e_\alpha = -t_jc_j.$$

Hence,

$$\partial_{t_je_\alpha}\partial_{t_je_\alpha}f(a) + \partial_{-t_jc_j}f(a) = 0.$$

From this we obtain

$$\frac{\partial^2}{\partial x_\alpha^2}f = \frac{1}{t_j}\frac{\partial}{\partial t_j}F.$$

Summarizing, we find on $V_{j,0}$,

$$4\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta}f = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ (\frac{\partial^2}{\partial x_\alpha^2} + \frac{\partial^2}{\partial y_\alpha^2})f = 2\frac{1}{t_j}\frac{\partial}{\partial t_j}F & \text{if } \alpha = \beta \end{cases}$$

Furthermore,

$$D(b(a,\bar{a})e_\alpha,\bar{e}_\alpha)^{(1)} = D((1-t_j^2)e_\alpha,\bar{e}_\alpha)^{(1)} = (1-t_j^2)D(e_\alpha,\bar{e}_\alpha)^{(1)}.$$

Hence,

$$\begin{aligned}
&\sum_{j=1}^{r}\sum_{e_\alpha,e_\beta\in V_{j,0}} D(b(a,\bar{a})e_\alpha,\bar{e}_\beta)^{(1)}\frac{\partial^2}{\partial z_\alpha\partial\bar{z}_\beta}f(a)\\
&=\sum_{j=1}^{r}\sum_{e_\alpha\in V_{j,0}}(1-t_j^2)D(e_\alpha,\bar{e}_\alpha)^{(1)}2\frac{1}{t_j}\frac{\partial F}{\partial t_j}\\
&=2\sum_{j=1}^{r}(1-t_j^2)\frac{1}{t_j}\frac{\partial F}{\partial t_j}\sum_{e_\alpha\in V_{j,0}}D(e_\alpha,\bar{e}_\alpha)^{(1)}\\
&=2b\sum_{j=1}^{r}(1-t_j^2)\frac{1}{t_j}\frac{\partial F}{\partial t_j}D(c_j,\bar{c}_j)^{(1)},
\end{aligned}$$

since we already proved in Lemma 5.1, that $\sum_{e_\alpha\in V_{j,0}} D(e_\alpha,\bar{e}_\alpha)^{(1)} = bD(c_j,c_j)^{(1)}$. This finishes the proof. □

The next proposition claims that the Hua equation (16) for $s\in\mathbb{C}$ is sufficient for f being an eigenfunction of $\mathcal{D}(\Omega)^G$. A similar result for general tube domains is proved in [15].

Proposition 6.4. *Let Ω be the type $\mathbf{I}_{r,r+b}$ domain. Let $s\in\mathbb{C}$ and let λ_s be given by (11). Suppose f on Ω satisfies the Hua equation (16). Then f is an eigenfunction of all $T\in\mathcal{D}(\Omega)^G$ with eigenvalues $\chi_{\lambda_s}(T)$.*

Proof. Let f be a function on Ω solution of the Hua equation. Let $g\in G$, then the function

$$\Phi(z) = \int_K f(gk\cdot x)dk,\ z\in\Omega$$

is a $K-$biinvariant solution of differential equations (17). Thus by a result of Yan [22][1], f is proportional to the unique spherical function

$$\varphi_{\lambda_s}(z) = \int_K e_{\lambda_s}(k\cdot z)dk$$

in $\mathcal{M}(\lambda_s)$, i.e. $f(z) = c\varphi_{\lambda_s}(z)$. It is easy to see that $c = f(g\cdot 0)$, then

$$\int_K f(gk\cdot z)dk = \varphi_{\lambda_s}(z)f(g\cdot 0);$$

[1]Roughly speaking, the the differential equations used in [22] is obtained from (17) by the change of coordinates $x_j = -t_j^2/1-t_j^2$.

and consequently, by [6, Proposition 2.4, Chapter IV], f is a joint eigenfunction of all $T \in \mathcal{D}(\Omega)^G$ with eigenvalues $\chi_{\lambda_s}(T)$. □

6.3. **The sufficiency of the Hua equation (16).** We suppose in the rest of section 6 that $s \in \mathbb{C}$ satisfies the condition (15) and that f satisfies the sufficient condition (16) in Theorem 6.1. It follows immediately from Proposition 6.4 that $f \in \mathcal{M}(\lambda_s)$, and thus by Kashiwara *et al.* [9], f is the Poisson transform of a function φ on the Furstenberg boundary G/P_{min}, $f = \mathcal{P}_s(\varphi)$. To prove that φ is a function on the Shilov Boundary S, we follow a method by Berline and Vergne [1] (see also [10]), the reader is refereed that paper for some general arguments.

We need first two elementary lemmas for general bounded symmetric domain Ω; the first one gives explicit formulas for the root spaces $\mathfrak{g}^\alpha$, $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ and can easily be deduced from the Peirce decomposition (see [11], [19] and [21]). The second is essentially stated in [1] in terms of the Cayley transform, it has however an easier form in terms of the Jordan triple and can easily be proved using the first. To state them we need some notational preparation. Recall the quadratic map $z \to Q(z)$ given in section 2. For the fixed Jordan frame $\{c_j\}$ and the corresponding Peirce decomposition (5), the map

$$\tau : z \to \tau(z) = Q(e)\bar{z}$$

where $e = c_1 + \ldots + c_r$, defines a real involution of V_2 and thus a real form

$$A(e) = \{z \in V \; ; \; \tau(z) = z\}$$

of V_2; let $V_2 = A(e) \oplus iA(e)$ be corresponding decomposition with $A(e)$ being a real Jordan algebra. Let

$$B(e) = \{z \in V \; ; \; \tau(z) = -z\},$$

then $A(e) = iB(e)$. For $1 \leq j \leq k \leq r$, let

$$V_{jk} = A_{jk} \oplus B_{jk}$$

be the decomposition of the space V_{jk} into real and imaginary part relative to the real form $A(e)$.

Lemma 6.5. *The root spaces $\mathfrak{g}^\alpha$, $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ are explicitly given as follows :*

$$\mathfrak{g}^{\pm\beta_j} = \mathbb{R}(\xi_{ic_j} \mp 2iD(c_j, \bar{c}_j)),$$

$$\mathfrak{g}^{\frac{\beta_k - \beta_j}{2}} = \{\xi_a + D(c_k - c_j, \bar{a}) \; ; \; a \in A_{jk}\},$$

$$\mathfrak{g}^{\pm\frac{\beta_k + \beta_j}{2}} = \{\xi_b \mp D(c_k + c_j, \bar{b}) \; ; \; b \in B_{jk}\},$$

$$\mathfrak{g}^{\pm\frac{\beta_j}{2}} = \{\xi_v \pm (D(c_j, \bar{v}) - D(v, \bar{c}_j); v \in V_{0j}\}.$$

for $1 \leq j, k \leq r$.

Lemma 6.6. *The corresponding root spaces for the positive compact roots* $\frac{\gamma_k-\gamma_j}{2}$, $1 \le j < k \le r$ *and* $\frac{1}{2}\gamma_j$, $1 \le j \le r$ *are given by*

$$\mathfrak{k}_{\mathbb{C}}^{\frac{\gamma_k-\gamma_j}{2}} = \{D(c_k, \bar{v});\ v \in V_{jk}\} = \{D(v, \bar{c}_j);\ v \in V_{jk}\},$$

and

$$\mathfrak{k}_{\mathbb{C}}^{\frac{\gamma_j}{2}} = \{D(c_j, \bar{v});\ v \in V_{j0}\}.$$

For the matrix domain Ω of type $\mathbf{I}_{r,r+b}$ in $M_{r,r+b}(\mathbb{C})$ we choose as in subsection 2.2 an explicit frame $\{c_j\} = \{e_{j,j}\}$ consisting of diagonal matrices, viewed as a $r \times (r+b)-$matrices. Let now $\{v_\alpha\}$ be an orthonormal basis of $V = M_{r,r+b}(\mathbb{C})$ consisting of root vectors. The dual basis vectors are $v_\alpha^* = \bar{v}_\alpha$.

Recall that

$$\mathcal{H}^{(1)} = \mathcal{H}^{(1)}_{\mathfrak{t}_{\mathbb{C}}} + \mathcal{H}^{(1)}_{\mathfrak{k}_{\mathbb{C}}^+} + \mathcal{H}^{(1)}_{\mathfrak{k}_{\mathbb{C}}^-},$$

therefore, the system (16) implies in particular

$$\mathcal{H}^{(1)}_{\mathfrak{k}_{\mathbb{C}}^+} f = 0. \tag{18}$$

However,

$$\mathcal{H}^{(1)}_{\mathfrak{k}_{\mathbb{C}}^+} = \sum_{k>j} \mathcal{H}^{(1)}_{\mathfrak{k}_{\mathbb{C}}^{(\gamma_k-\gamma_j)/2}} + \sum_{j=1}^{r} \mathcal{H}^{(1)}_{\mathfrak{k}_{\mathbb{C}}^{\gamma_j/2}}.$$

Lemma 6.7. *We have,*

$$\big(\mathfrak{k}_{\mathbb{C}}^{\frac{\gamma_j}{2}}\big)^{(1)} = 0.$$

Proof. Indeed, using Lemma 6.6, let $D(c_j, \bar{v}) \in \mathfrak{k}_{\mathbb{C}}^{\frac{\gamma_j}{2}}$, with $v = e_{j,j+m} \in V_{j,0}$ $(m > 0)$. Then

$$D(c_j, \bar{v})^{(1)} = e_{j,j} e^*_{j,j+m} = 0.$$

□

Hence, from (18) it follows

$$\sum_{k>j} \mathcal{H}^{(1)}_{\mathfrak{k}_{\mathbb{C}}^{(\gamma_k-\gamma_j)/2}} f = 0. \tag{19}$$

Lemma 6.8. *We have*

$$\big(\mathfrak{k}_{\mathbb{C}}^+\big)^{(1)} = \sum_{k>j} \big(\mathfrak{k}_{\mathbb{C}}^{\frac{\gamma_k-\gamma_j}{2}}\big)^{(1)},$$

and the right hand side is a linear direct sum, namely the spaces $\big(\mathfrak{k}_{\mathbb{C}}^{\frac{\gamma_k-\gamma_j}{2}}\big)^{(1)}$ *are linearly independent.*

Proof. This follows easily from Lemma 6.6 by observing that $D(c_k, \bar{v}) \mapsto D(c_k, \bar{v})^{(1)}$ is a linear homomorphism, and that $D(c_k, \bar{v})^{(1)} = \bar{\alpha} e_{k,j}$ for $v = \alpha e_{j,k} + b e_{k,j} \in V_{j,k}$.

□

We conclude from (19) and the above lemma that

$$\mathcal{H}^{(1)}_{\mathfrak{k}_{\mathbb{C}}^{(\gamma_k-\gamma_j)/2}} f = 0 \tag{20}$$

for any positive compact root $\frac{\gamma_k - \gamma_j}{2}$.

Let $\Psi_c^{+,(i)}$ be the set positive compact roots in $\mathfrak{k}_{\mathbb{C}}^{(i)}$, for $i = 1, 2$. Then (20) implies

$$\mathcal{H}^{\beta} f = 0, \quad \text{for } \beta \in \Psi_c^{+,(1)} \text{ with } \beta \equiv \frac{\gamma_k - \gamma_j}{2} \ (k > j)$$

where $\mathcal{H}^{\beta}$ is the component of $\mathcal{H}$ given by

$$\mathcal{H}^{\beta} = \sum_{\alpha \in \Psi_n^+} [E_{\beta}, v_{\alpha}] \bar{v}_{\alpha}$$

and E_{β} is the root vector of β.

Now we fix for the rest of this section $\beta \in \Psi_c^{+,(1)}$ such that

$$\beta|_{\mathfrak{t}_{\mathbb{C}}^-} = \frac{\gamma_j - \gamma_{j-1}}{2}.$$

The root vector E_{β} has the form $E_{\beta} = D(c_j, \bar{w})$ with $w = e_{j,j-1}$ or $w = e_{j-1,j}$ being one of the basis vectors $\{v_{\alpha}\}$. Observe that $[E_{\beta}, v_{\alpha}] = 0$ unless α is in the set $\Psi_1 \cup \Psi_2 \cup \Psi_3$ where

$$\begin{aligned}
\Psi_1 &= \{\alpha \in \Psi_n^+;\ \alpha|_{\mathfrak{t}_{\mathbb{C}}^-} = \tfrac{\gamma_k + \gamma_{j-1}}{2},\ k \le j-1\}, \\
\Psi_2 &= \{\alpha \in \Psi_n^+;\ \alpha|_{\mathfrak{t}_{\mathbb{C}}^-} = \tfrac{\gamma_k + \gamma_{j-1}}{2},\ k \ge j\}, \\
\Psi_3 &= \{\alpha \in \Psi_n^+;\ \alpha|_{\mathfrak{t}_{\mathbb{C}}^-} = \tfrac{\gamma_{j-1}}{2}\}.
\end{aligned}$$

Consider the Poincaré-Birkhoff-Witt decomposition

$$\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}} + \mathcal{U}(\mathfrak{a}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}^-)$$

and let π be the projection

$$\pi : \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}} + \mathcal{U}(\mathfrak{a}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}^-) \to \mathcal{U}(\mathfrak{a}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}^-)$$

The function f is now viewed as a function on $G = NAK$, and the group A will be identified as $(\mathbb{R}^+)^r$. Under this identification, f satisfies furthermore the equation

$$\mathcal{R}(\pi(\mathcal{H}^{\beta}))f = 0,$$

where $\mathcal{R}$ is the mapping from $\mathcal{U}(\mathfrak{a}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}})$ to differential operators on NA defined by

$$\mathcal{R}(\xi_k) = t_k \frac{\partial}{\partial t_k}, \quad \mathcal{R}(X_{-\alpha}) = t^{\alpha} X_{-\alpha},$$

for $\xi_k \in \mathfrak{a}$, $1 \leq k \leq r$, and $X_{-\alpha} \in \mathfrak{n}$ identified with the corresponding left-invariant differential operator.

We will prove that the operator $t^{-\frac{1}{2}(\beta_j - \beta_{j-1})}\mathcal{R}(\pi(\mathcal{H}^{\beta}))$ has analytic coefficient near $t = 0$ and study the induced equation of $t^{-\frac{1}{2}(\beta_j - \beta_{j-1})}\mathcal{R}(\pi(\mathcal{H}^{\beta}))f = 0$.

6.3.1. *The projection of the Hua operator in the PBW-decomposition.* We will compute the Poincaré-Birkhoff-Witt components of the Hua operators as an element in the universal algebra of $\mathfrak{g}_{\mathbb{C}}$. Let now Ω be a general bounded symmetric domain.

Lemma 6.9. *The Iwasawa decomposition of $v \in \mathfrak{p}^+$ and $\bar{v} \in \mathfrak{p}^-$ in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} + \mathfrak{n}^-_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}$ is given as follows :*

(1) *For $v \in V_{kj}$, $r \geq k > j \geq 1$,*

$$\begin{aligned} v &= \zeta_v + \zeta'_v - D(v, \bar{c}_k), \\ \bar{v} &= \eta_{\bar{v}} + \eta'_{\bar{v}} - D(c_k, \bar{v}) \end{aligned}$$

where $\zeta_v, \eta_{\bar{v}} \in \mathfrak{g}_{\mathbb{C}}^{-\frac{\beta_k - \beta_j}{2}}$, $\zeta'_v, \eta'_{\bar{v}} \in \mathfrak{g}_{\mathbb{C}}^{-\frac{\beta_k + \beta_j}{2}}$ are given by

$$\begin{aligned} \zeta_v &= \frac{1}{2}[v - \overline{\tau(v)} + D(v, \bar{c}_k - \bar{c}_j)], \\ \zeta'_v &= \frac{1}{2}[v + \overline{\tau(v)} + D(v, \bar{c}_j + \bar{c}_k)], \\ \eta_{\bar{v}} &= \frac{1}{2}[\bar{v} - \tau(v) + D(c_j - c_k, \bar{v})], \\ \eta'_{\bar{v}} &= \frac{1}{2}[\tau(v) + \bar{v} + D(c_j + c_k, \bar{v})]. \end{aligned}$$

(2) *For $v = c_j \in V_{jj}$, $1 \leq j \leq r$,*

$$\begin{aligned} c_j &= \frac{1}{2}\xi_j - \frac{1}{2}\zeta_j - D(c_j, \bar{c}_j), \\ \bar{c}_j &= -\frac{1}{2}\xi_j - \frac{i}{2}\zeta_j - D(c_j, \bar{c}_j), \end{aligned}$$

with

$$\zeta_j = i[\xi_{ic_j} + 2D(c_j, \bar{c}_j)].$$

(3) *For* $v \in V_{j0}$, $1 \le j \le r$,

$$\begin{aligned} v &= \zeta_v - D(v, \bar{c}_j), \\ \bar{v} &= \eta_{\bar{v}} - D(c_j, \bar{v}), \end{aligned}$$

with

$$\begin{aligned} \zeta_v &= v + D(v, \bar{c}_j) \in \mathfrak{n}_{\mathbb{C}}^{-\frac{\beta_j}{2}} \\ \eta_{\bar{v}} &= \bar{v} + D(c_j, \bar{v}) \in \mathfrak{n}_{\mathbb{C}}^{-\frac{\beta_j}{2}}. \end{aligned}$$

We denote by $\pi_{\mathfrak{n}_{\mathbb{C}}^0}$ the projection onto the nilpotent subalgebra

$$\mathfrak{n}_{\mathbb{C}}^0 = \sum_{k>j\ge 1} \mathfrak{g}_{\mathbb{C}}^{-\frac{\beta_k - \beta_j}{2}}$$

in the Iwasawa decomposition of $\mathfrak{g}_{\mathbb{C}}$,

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} + \sum_{k\ge j\ge 0} \mathfrak{g}_{\mathbb{C}}^{-\frac{\beta_k + \beta_j}{2}} + \mathfrak{n}_{\mathbb{C}}^0.$$

Then, it follows from Lemma 6.9

$$\pi_{\mathfrak{n}_{\mathbb{C}}^0}(\bar{v}) = -\pi_{\mathfrak{n}_{\mathbb{C}}^0}(\tau(v)), \tag{21}$$

which we will need in the next proposition.

Return back to type $\mathbf{I}_{r,r+b}$ domains. We compute now the projection $\pi(\mathcal{H}^\beta)$ of $\mathcal{H}^\beta$. Recall that the $\beta-$root vector is $E_\beta = D(c_j, \bar{w})$, with $w = e_{j,j-1}$ or $w = e_{j-1,j}$.

Proposition 6.10. *The projection* $\pi(\mathcal{H}^\beta)$ *is given by*

$$\begin{aligned} \pi(\mathcal{H}^\beta) = &\sum_{k=1}^{j-2} \sum_{v_\alpha \in V_{k,j-1}} (\zeta_{\{c_j \bar{w} v_\alpha\}} + \zeta'_{\{c_j \bar{w} v_\alpha\}})(\eta_{\bar{v}_\alpha} + \eta'_{\bar{v}_\alpha}) + j\eta_{\bar{w}} + j\eta'_{\bar{w}} \\ &+ (\zeta_{\tau(w)} + \zeta'_{\tau(w)})(-\frac{1}{2}\xi_{j-1} - \frac{i}{2}\zeta_{j-1}) \\ &+ \sum_{k=j+1}^{r} \sum_{v_\alpha \in V_{k,j-1}} (\zeta_{\{c_j \bar{w} v_\alpha\}} + \zeta'_{\{c_j \bar{w} v_\alpha\}})(\eta_{\bar{v}_\alpha} + \eta'_{\bar{v}_\alpha}) \\ &+ (\frac{1}{2}\xi_j - \frac{1}{2}\zeta_j)(\eta_{\bar{w}} + \eta'_{\bar{w}}) \\ &+ \sum_{v_\alpha \in V_{j-1,0}} \zeta_{\{c_j \bar{w} v_\alpha\}} \eta_{\bar{v}_\alpha} + J_\beta \end{aligned}$$

where the last term

$$J_\beta = \begin{cases} b(\eta_{\bar{w}} + \eta'_{\bar{w}}) & \text{if} \quad w = e_{j-1,j} \\ 0 & \text{if} \quad w = e_{j,j-1} \end{cases}$$

and where the sum $\sum_{v_\alpha\in V_{kj}}$ *is taken over the orthonormal basis* $\{v_\alpha\}$ *of* $V_{k,j}$.

Proof. We compute the projection $\sum_{\alpha\in\Psi_n^+}\pi([E_\beta, v_\alpha]v_\alpha^*)$. For $\alpha\in\Psi_n^+$ we know that $[E_\beta, v_\alpha]=0$ unless $\alpha\in\Psi_1\cup\Psi_2\cup\Psi_3$.

- *Case I* : $\alpha\in\Psi_1$, with $\alpha|_{\mathfrak{t}_{\mathbb{C}}^-}=\frac{\gamma_k+\gamma_{j-1}}{2}$. Then $v_\alpha\in V_{k,j-1}$ and

$$v_{\beta+\alpha}:=[E_\beta, v_\alpha]=[D(c_i,\bar w), v_\alpha]=D(c_i,\bar w)v_\alpha\in V_{j,k}.$$

By the previous Lemma, for $k<j-1$,

$$v_{\beta+\alpha}=\zeta_{v_{\beta+\alpha}}+\zeta'_{v_{\beta+\alpha}}-D(v_{\beta+\alpha},c_j)$$

and modulo the ideal $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}$,

$$\begin{aligned}[E_\beta, v_\alpha]v_\alpha^*=v_{\beta+\alpha}\bar v_\alpha &\equiv (\zeta_{v_{\beta+\alpha}}+\zeta'_{v_{\beta+\alpha}})(\eta_{\bar v_\alpha}+\eta'_{\bar v_\alpha})-[D(v_{\beta+\alpha},c_j),\bar v_\alpha]\\ &= (\zeta_{v_{\beta+\alpha}}+\zeta'_{v_{\beta+\alpha}})(\eta_{\bar v_\alpha}+\eta'_{\bar v_\alpha})+\overline{D(c_j,\bar v_{\beta+\alpha})v_\alpha}.\end{aligned}$$

To find the last term we note first that for any Jordan triple system [11],

$$[D(v_\alpha,\bar v_\alpha), D(w,\bar c_j)]=D(v_\alpha,\overline{D(c_j,\bar w)v_\alpha})-D(D(w,\bar c_j)v_\alpha,\bar v_\alpha);$$

we let it act on c_j and then sum over v_α

$$\sum_{v_\alpha\in V_{k,j-1}}D(c_j,D(c_j,\bar w)v_\alpha)v_\alpha=\frac{a}{2}(D(c_{j-1},c_{j-1})+D(c_k,c_k))w$$

by using Proposition 5.1. It is further w, since $a=2$ for type **I** domains. Thus

$$\sum_{\substack{v_\alpha\in V_{k,j-1}\\ k<j-1}}[E_\beta,v_\alpha]v_\alpha^*\equiv\sum_{\substack{v_\alpha\in V_{k,j-1}\\ k<j-1}}(\zeta_{v_{\beta+\alpha}}+\zeta'_{v_{\beta+\alpha}})(\eta_{\bar v_\alpha}+\eta'_{\bar v_\alpha})+(j-2)\eta_{\bar w}+(j-2)\eta'_{\bar w}$$

Now consider $v_\alpha\in V_{k,j-1}$ with $k=j-1$, namely $v_\alpha=c_{j-1}$. Correspondingly

$$[E_\beta, v_\alpha]=[D(c_j,\bar w), c_{j-1}]=Q(c_j+c_{j-1})\bar w=Q(e)\bar w=\tau(w)$$

with $w \to \tau(w) = Q(e)\bar{w}$ the involution on V_2. Modulo $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}$

$$\begin{aligned}
[E_\beta, v_\alpha]\bar{v}_\alpha = \tau(w)\bar{c}_{j-1} &= (\zeta_{\tau(w)} + \zeta'_{\tau(w)} - D(\tau(w), c_j))\bar{c}_{j-1} \\
&\equiv (\zeta_{\tau(w)} + \zeta'_{\tau(w)})(-\frac{1}{2}\xi_{j-1} - \frac{i}{2}\zeta_{j-1}) - D(\tau(w), c_j)\bar{c}_{j-1} \\
&\equiv (\zeta_{\tau(w)} + \zeta'_{\tau(w)})(-\frac{1}{2}\xi_{j-1} - \frac{i}{2}\zeta_{j-1}) + \overline{D(c_j, \tau(w))c_{j-1}} \\
&\equiv (\zeta_{\tau(w)} + \zeta'_{\tau(w)})(-\frac{1}{2}\xi_{j-1} - \frac{i}{2}\zeta_{j-1}) + \overline{w} \\
&\equiv (\zeta_{\tau(w)} + \zeta'_{\tau(w)})(-\frac{1}{2}\xi_{j-1} - \frac{i}{2}\zeta_{j-1}) + \eta_{\bar{w}} + \eta'_{\bar{w}}.
\end{aligned}$$

• *Case II* : $\alpha \in \Psi_2$, with $\alpha|_{\mathfrak{t}^-_{\mathbb{C}}} = \frac{1}{2}(\gamma_{j-1} + \gamma_k)$, $k \geq j$. Consider $k > j$ first. Similar to the previous case we have, modulo $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}$,

$$\begin{aligned}
[E_\beta, v_\alpha]\bar{v}_\alpha &\equiv (\zeta_{\{c_j\bar{w}v_\alpha\}} + \zeta'_{\{c_j\bar{w}v_\alpha\}})(\eta_{\bar{v}_\alpha} + \eta'_{\bar{v}_\alpha}) + \overline{D(c_k, \overline{D(c_j, \bar{w})v_\alpha})v_\alpha} \\
&\equiv (\zeta_{\{c_j\bar{w}v_\alpha\}} + \zeta'_{\{c_j\bar{w}v_\alpha\}})(\eta_{\bar{v}_\alpha} + \eta'_{\bar{v}_\alpha})
\end{aligned}$$

since $D(c_k, \overline{D(c_j, \bar{w})v_\alpha})v_\alpha = 0$ by the Peirce rule $\{V_{kk}\bar{V}_{kj}V_{k,j-1}\} = \{0\}$ and $D(c_j, \bar{w})v_\alpha \in V_{jk}$.

Now let $k = j$, then $[E_\beta, v_\alpha] = D(c_j, \bar{w})v_\alpha = D(v_\alpha, \bar{w})c_j = \langle v_\alpha, w\rangle c_j$, which vanishes except when $v_\alpha = w$ and in that case,

$$[E_\beta, v_\alpha]\bar{v}_\alpha = c_j\bar{v}_\alpha = (\frac{1}{2}\xi_j - \frac{1}{2}\zeta_j - D(c_j, \bar{c}_j))\bar{w},$$

and

$$[E_\beta, v_\alpha]\bar{v}_\alpha \equiv (\frac{1}{2}\xi_j - \frac{1}{2}\zeta_j)(\eta_{\bar{w}} + \eta'_{\bar{w}}) + \eta_{\bar{w}} + \eta'_{\bar{w}}.$$

• *Case III* : $\alpha \in \Psi_3$, with $\alpha|_{\mathfrak{t}^-_{\mathbb{C}}} = \frac{1}{2}\gamma_{j-1}$, and the root vector $v_\alpha \in V_{j-1,0}$. In this case, we have,

$$\begin{aligned}
[E_\beta, v_\alpha]\bar{v}_\alpha = D(c_j, \bar{w})v_\alpha\bar{v}_\alpha &\equiv (\zeta_{D(c_j,\bar{w})v_\alpha} - D(D(c_j, \bar{w})v_\alpha, c_j))\eta_{\bar{v}_\alpha} \\
&\equiv \zeta_{D(c_j,\bar{w})v_\alpha}\eta_{\bar{v}_\alpha} + \overline{D(c_j, \overline{D(c_j, w)v_\alpha})v_\alpha}.
\end{aligned}$$

However by the commutator relation (JP15) in [11] we have

$$[D(w, \bar{v}_\alpha), D(c_j, \bar{c}_j)] = D(D(w, \bar{v}_\alpha)c_j, \bar{c}_j) - D(c_j, \overline{D(v_\alpha, \bar{w})c_j}) = -D(c_j, \overline{D(v_\alpha, \bar{w})c_j})$$

since $D(w, \bar{v}_\alpha)c_j = 0$ by the Peirce rule that $D(w, \bar{v}_\alpha)c_j \in \{V_{j,j-1}\bar{V}_{j-1,0}V_{jj}\} = \{0\}$, thus

$$D(c_j, \overline{D(c_j, \bar{w})v_\alpha})v_\alpha = [D(c_j, c_j), D(w, \bar{v}_\alpha)]v_\alpha = D(c_j, c_j)D(w, \bar{v}_\alpha)v_\alpha = D(v_\alpha, \bar{v}_\alpha)w$$

since $D(c_j, c_j)v_\alpha = 0$.

It is easy to see, by direct matrix computation, that,

$$\sum_{v_\alpha \in V_{j-1,0}} \overline{D(v_\alpha, \bar{v}_\alpha) w} = \begin{cases} b\bar{w} & \text{if} \quad w = e_{j-1,j} \\ 0 & \text{if} \quad w = e_{j,j-1}. \end{cases}$$

Hence, modulo $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}$

$$\sum_{v_\alpha \in V_{j-1,0}} \overline{D(v_\alpha, \bar{v}_\alpha) w} \equiv \begin{cases} b(\eta_{\bar{w}} + \eta'_{\bar{w}}) & \text{if} \quad w = e_{j-1,j} \\ 0 & \text{if} \quad w = e_{j,j-1} \end{cases}$$

Consequently,

$$\sum_{v_\alpha \in V_{j-1,0}} [E_\beta, v_\alpha] v_\alpha^* \equiv \sum_{v_\alpha} \zeta_{D(c_j, \bar{w}) v_\alpha} \eta_{\bar{v}_\alpha} + \begin{cases} b(\eta_{\bar{w}} + \eta'_{\bar{w}}) & \text{if} \quad w = e_{j-1,j} \\ 0 & \text{if} \quad w = e_{j,j-1} \end{cases}$$

and this finishes the proof. □

6.3.2. *The induced equations.* We apply now the theory of boundary values of eigenfunctions of $\mathcal{D}(\Omega)^G$ on symmetric spaces, see [9], [12], [13], [16].

We identify the space G/K with NA and A with $(\mathbb{R}^+)^r$. It follows from Proposition 6.10 that the operator $t^{-\frac{1}{2}(\beta_j - \beta_{j-1})}\mathcal{R}[\pi(\mathcal{H}^\beta)]$ has analytic coefficients near $t = 0$, then the induced equation for the differential equation $t^{-\frac{1}{2}(\beta_j - \beta_{j-1})}\mathcal{R}[\pi(\mathcal{H}^\beta)]f = 0$ is

$$\lim_{t \to 0} t^{\lambda - \rho} t^{-\frac{1}{2}(\beta_j - \beta_{j-1})} \mathcal{R}[\pi(\mathcal{H}^\beta)] t^{\rho - \lambda} (\mathcal{B}_\lambda f), = 0$$

where $t = (t_1, t_2, \cdots, t_r) \in A = (\mathbb{R}^+)^r$, and

$$t^\mu = t_1^{\mu(\xi_1)} \cdots t_r^{\mu(\xi_r)}$$

for $\mu \in \mathfrak{a}_{\mathbb{C}}^*$. Here $\mathcal{B}_\lambda f$ is the boundary value of f.

Proposition 6.11. *The boundary value $\mathcal{B}_\lambda f$ of f satisfies the following induced equation*

$$\mathcal{R}[\zeta_{\tau(w)}](\mathcal{B}_\lambda f) = 0. \tag{22}$$

Observe, using (21), that the induced equation (22) is equivalent to the following one

$$\mathcal{R}[\eta_{\bar{w}}](\mathcal{B}_\lambda f) = 0.$$

Proof. Let us compute the limit of the differential operator

$$t^{\lambda - \rho} t^{-\frac{1}{2}(\beta_j - \beta_{j-1})} \mathcal{R}[\pi(\mathcal{H}^\beta)] t^{\rho - \lambda}$$

when $t \to 0$. We will consider each term in the projection $\pi(\mathcal{H}^\beta)$.

- The differential operator corresponding to $j(\eta_{\bar{w}}+\eta'_{\bar{w}})$ is
$$t^{\lambda-\rho}t^{-\frac{1}{2}(\beta_j-\beta_{j-1})}j\big[t^{\frac{1}{2}(\beta_j-\beta_{j-1})}\mathcal{R}(\eta_{\bar{w}})+t^{\frac{1}{2}(\beta_j+\beta_{j-1})}\mathcal{R}(\eta'_{\bar{w}})\big]t^{\rho-\lambda},$$
and its limit when $t\mapsto 0$ is
$$j\mathcal{R}(\eta_{\bar{w}}). \tag{23}$$
- Consider the quadratic term $(\zeta_{\tau w}+\zeta'_{\tau w})(-\frac{1}{2}\xi_{j-1}-\frac{i}{2}\zeta_{j-1})$. The corresponding differential operator is
$$t^{\lambda-\rho}t^{-\frac{1}{2}(\beta_j-\beta_{j-1})}\big[\big(t^{\frac{1}{2}(\beta_j-\beta_{j-1})}\mathcal{R}(\zeta_{\tau(w)}+t^{\frac{1}{2}(\beta_j+\beta_{j-1})}\mathcal{R}(\eta'_{\tau(w)})\big)\times$$
$$\big(-\tfrac{1}{2}t_{j-1}\tfrac{\partial}{\partial t_{j-1}}-\tfrac{i}{2}t^{\beta_{j-1}}\mathcal{R}(\zeta_{j-1})\big)\big]t^{\rho-\lambda}$$
$$\big(\mathcal{R}(\zeta_{\tau(w)})+t^{\beta_{j-1}}\mathcal{R}(\zeta'_{\tau(w)})\big)\big(-\tfrac{1}{2}(\rho-\lambda)(\xi_{j-1})-\tfrac{i}{2}t^{\beta_{j-1}}\mathcal{R}(\zeta_{j-1})\big).$$
Its limit when $t\to 0$ is
$$-\frac{1}{2}(\rho-\lambda)(\xi_{j-1})\mathcal{R}(\zeta_{\tau(w)}).$$
- For the quadratic term $(\frac{1}{2}\xi_j-\frac{1}{2}\zeta_j)(\eta_{\bar{w}}-\eta'_{\bar{w}})$, the corresponding differential operator is
$$t^{\lambda-\rho}t^{-\frac{\beta_j-\beta_{j-1}}{2}}\big[\big(\tfrac{1}{2}t_j\tfrac{\partial}{\partial t_j}-\tfrac{1}{2}t^{\beta_j}\mathcal{R}(\zeta_j)\big)\big(t^{\frac{\beta_j-\beta_{j-1}}{2}}\mathcal{R}(\eta_{\bar{w}})+t^{\frac{\beta_j+\beta_{j-1}}{2}}\mathcal{R}(\eta'_{\bar{w}})\big)\big]t^{\rho-\lambda}.$$
Its limit is
$$\lim_{t\to 0}t^{\lambda-\rho}t^{-\frac{\beta_j-\beta_{j-1}}{2}}\big[\tfrac{1}{2}\tfrac{\partial}{\partial t_j}\big(t^{\frac{\beta_j-\beta_{j-1}}{2}}t^{\rho-\lambda}\mathcal{R}(\eta_{\bar{w}})+t^{\frac{\beta_j+\beta_{j-1}}{2}}t^{\rho-\lambda}\mathcal{R}(\eta'_{\bar{w}})\big)\big]$$
$$=\lim_{t\to 0}t^{\lambda-\rho}t^{-\frac{\beta_j-\beta_{j-1}}{2}}\big[\tfrac{1}{2}\big(\tfrac{\beta_j-\beta_{j-1}}{2}(\xi_j)+(\rho-\lambda)(\xi_j)\big)t^{\frac{\beta_j-\beta_{j-1}}{2}}t^{\rho-\lambda}\mathcal{R}(\eta_{\bar{w}})$$
$$+\tfrac{1}{2}\big(\tfrac{\beta_j+\beta_{j-1}}{2}(\xi_j)+(\rho-\lambda)(\xi_j)\big)t^{\frac{\beta_j+\beta_{j-1}}{2}}t^{\rho-\lambda}\mathcal{R}(\eta'_{\bar{w}})\big],$$
which is
$$\frac{1}{2}[1+(\rho-\lambda)(\xi_j)]\mathcal{R}(\eta_{\bar{w}}).$$
- The induced equation corresponding to the last term of the projection $\pi(\mathcal{H}^\beta)$ is
$$\begin{cases} b\mathcal{R}(\eta_{\bar{w}}) \\ 0 \end{cases} \tag{24}$$
- Now, it is easy to see, using the same computations, that the induced equation of the remaining terms of $\pi(\mathcal{H}^\beta)$ is zero.

It follows now from (23) – (24) and (21), that the boundary value $\mathcal{B}_\lambda f$ of f satisfies
$$C_1\mathcal{R}(\zeta_{\tau(w)})(\mathcal{B}_\lambda f)=0$$

where C_1 is given by

$$C_1 = \frac{1}{2}(\rho - \lambda)(\xi_j - \xi_{j-1}) + \begin{cases} \frac{1}{2} + j + b \\ \frac{1}{2} + j \end{cases}$$

Now if $C_1 \neq 0$, then the induced equation is $\mathcal{R}(\zeta_{\tau(w)})(\mathcal{B}_\lambda f) = 0$. On the other hand, if $C_1 = 0$, we may replace f by $t^{\kappa(\frac{\beta_j - \beta_{j-1}}{2})} f$ for sufficiently large $\kappa > 0$, consider the differential operator

$$t^{-\frac{1}{2}(\beta_j - \beta_{j-1})} t^{\kappa(\frac{\beta_j - \beta_{j-1}}{2})} \mathcal{R}[\pi(\mathcal{H}^\beta)] t^{-\kappa(\frac{\beta_j - \beta_{j-1}}{2})},$$

and we still prove that $\mathcal{R}(\zeta_{\tau(w)})(\mathcal{B}_\lambda f) = 0$, see also [15]. □

We continue the proof of the necessity condition of the Hua equation. We now get $\mathcal{R}(\zeta_{\tau(w)})(\mathcal{B}_{\lambda_s} f) = 0$ for any root β such that $\beta \equiv \frac{1}{2}(\gamma_j - \gamma_{j-1})$. Since $\{\frac{1}{2}(\gamma_j - \gamma_{j-1}),\ 2 \leq j \leq r\}$ is the set of simple roots of the system $\{\frac{1}{2}(\gamma_k - \gamma_j),\ 1 \leq j < k \leq r\}$, it follows that $\mathcal{R}(\zeta_{\tau(w)})(\mathcal{B}_{\lambda_s} f) = 0$ for any $w \in V_{j,k}$. However, by Lemma 6.9, $\operatorname{span}\{\zeta_{\tau(w)}\ ;\ w \in V_{j,k}\}$ is $\mathfrak{n}^0_{\mathbb{C}}$. Thus $\mathcal{B}_{\lambda_s} f \in \mathcal{B}(S)$. This finishes the proof of Theorem 6.1.

7. General non-tube domains

7.1. **Third-order Hua operators.** We first construct two third-order Hua operators using the covariant C-R operator $\bar{\mathbf{D}}$ and the covariant connection ∇. Let again E be as in section 4 a homogeneous holomorphic vector bundle on Ω. On E there is a Hermitian structure defined by using the Bergman operator $b(z, z)$ as an element in $K_{\mathbb{C}}$ and thus the corresponding Hermitian connection $\nabla : C^\infty(\Omega, E) \to C^\infty(\Omega, T' \otimes E)$, where T' is the cotangent bundle. Under the decomposition $T'_z = (T')_z^{(1,0)} + (T')_z^{(0,1)}$ we have $\nabla = \nabla^{(1,0)} + \bar{\partial}$ with

$$\nabla^{(1,0)} : C^\infty(\Omega, E) \to C^\infty(\Omega, (T')^{(1,0)} \otimes E) = C^\infty(\Omega, \mathfrak{p}^- \otimes E),$$

using our identification that $(T')_z^{(1,0)} = \mathfrak{p}^-$. (The operator $\nabla^{(1,0)}$ is denoted by $\mathcal{D}$ in [23].) Note that on the space $C^\infty(\Omega)$ of sections of the trivial bundle we have $\nabla^{(1,0)} = \partial$.

We now define the third-order Hua operators $\mathcal{W}$ and $\mathcal{U}$ on $C^\infty(\Omega, E)$ by

$$\mathcal{W} f = \mathrm{Ad}_{\mathfrak{p}^+ \otimes \mathfrak{k}_{\mathbb{C}} \to \mathfrak{p}^+} \left(\bar{\mathbf{D}} (\mathrm{Ad}_{\mathfrak{p}^+ \otimes \mathfrak{p}^- \to \mathfrak{k}_{\mathbb{C}}} (\bar{\mathbf{D}} \nabla^{(1,0)} f)) \right),$$

and

$$\mathcal{U} f = \mathrm{Ad}_{\mathfrak{k}_{\mathbb{C}} \otimes \mathfrak{p}^+ \to \mathfrak{p}^+} \left(\mathrm{Ad}_{\mathfrak{p}^- \otimes \mathfrak{p}^+ \to \mathfrak{k}_{\mathbb{C}}} (\nabla^{(1,0)} \bar{\mathbf{D}}) \bar{\mathbf{D}} f \right).$$

It follows from the covariant properties of the C-R operator $\bar{\mathbf{D}}$ and $\nabla^{(1,0)}$ that we have

$$\mathcal{W}(f(gz)) = \mathrm{Ad}(dg(z)^{-1}) \mathcal{W} f(gz).$$

These operators can also be defined by using the enveloping algebra. For our purpose we consider E to be the trivial representation and the space $C^\infty(\Omega)$ of smooth functions identified as right K-invariant functions on G. Let $\{v_\alpha\}$ be a basis of $\mathfrak{p}^+$ consisting of root vectors and $\{v_\alpha^*\}$ be the dual basis of $\mathfrak{p}^-$. Then we have

$$\mathcal{W}f = \sum_{\alpha,\beta,\gamma} v_\alpha^* v_\beta^* v_\gamma \tilde{f} \otimes [v_\alpha, [v_\beta, v_\gamma^*],$$

$$\mathcal{U}f = \sum_{\alpha,\beta,\gamma} v_\gamma v_\alpha^* v_\beta^* \tilde{f} \otimes [[v_\gamma^*, v_\alpha], v_\beta],$$

where $\tilde{f}$ is the lift of f to G.

Remark 7.1. *The third-order Hua operator defined by Berline and Vergne in* [1] *is, in terms of the above notation*

$$\mathcal{V} = \sum_{\alpha,\beta,\gamma} v_\alpha v_\beta^* v_\gamma \otimes [v_\alpha^*, [v_\beta, v_\gamma^*]] = Ad_{\mathfrak{p}^-\otimes\mathfrak{k}_\mathbb{C}\to\mathfrak{p}^-} \left(\nabla^{(1,0)} Ad_{\mathfrak{p}^+\otimes\mathfrak{p}^-\to\mathfrak{k}_\mathbb{C}}(\bar{\mathbf{D}}\nabla^{(1,0)})\right),$$

up to some non-zero constant. So it is different from our $\mathcal{W}$ *and* $\mathcal{U}$*. For explicit computations the operators* $\mathcal{W}$ *and* $\mathcal{U}$ *are somewhat easier to handle as the operator* $\bar{\mathbf{D}}$ *has a rather explicit formula (12) on different holomophic bundles* [3], *whereas the formula for* $\nabla^{(1,0)}$ *depends on the metric on the bundles* [23]. *Note also that the first* $\nabla^{(1,0)}$ *and the second* $\nabla^{(1,0)}$ *in* $\mathcal{V}$ *are different as they are acting on different bundles.*

Denote

$$c = 2(n+1) + \frac{1}{n}(a^2-4)\dim(\mathcal{P}^{(1,1)}) \tag{25}$$

where $\dim(\mathcal{P}^{(1,1)})$ is the dimension of the irreducible subspace of holomorphic polynomials on V with lowest weight $-\gamma_1-\gamma_2$. For any $s\in\mathbb{C}$, put $\sigma = \frac{n}{r}s$. Our main result for non-tube domains is

Theorem 7.2. *Let* Ω *be a general non-tube domain. If* $f = \mathcal{P}_s(\varphi)$ *is the Poisson transform of a hyperfunction* φ *on* S*. Then*

$$\Big(\mathcal{U} - \frac{-2\sigma^2+2p\sigma+c}{\sigma(2\sigma-p-b)}\mathcal{W}\Big)f = 0. \tag{26}$$

Conversely, suppose s *satisfies the following condition*

$$-4[b+1+j\frac{a}{2}+\frac{n}{r}(s-1)] \notin \{1,2,\cdots\}, \text{ for } j=0 \text{ and } 1.$$

Let f *be in* $\mathcal{M}(\lambda_s)$ *and suppose* f *satisfies (26). Then* $f=\mathcal{P}_s(\varphi)$ *is the Poisson transform of a hyperfunction* φ *on* S.

7.2. **The necessity of the Hua equation (26).** Consider the operator

$$\mathcal{Y} = \mathcal{U} - \frac{-2\sigma^2 + 2p\sigma + c}{\sigma(2\sigma - p - b)}\mathcal{W}.$$

Proposition 7.3. *Let $s \in \mathbb{C}$. We have*

$$\mathcal{W}P_{s,u}(z) = P_{s,u}(z)\sigma^2(2\sigma - p - b)u$$

and

$$\mathcal{U}P_{s,u}(z) = P_{s,u}(z)\sigma(-2\sigma^2 + 2p\sigma + c)u.$$

In particular, for $\sigma \neq 0, \frac{p+b}{2}$

$$\mathcal{Y}P_{s,u}(z) = 0,$$

and the image $f = \mathcal{P}_s(\varphi)$ of the Poisson transform of a hyperfunction φ on S satisfies

$$\mathcal{Y}f = 0.$$

Proof. We compute first $\mathcal{W}$ on $P_s(z,u)$. By the covariant property of $\mathcal{W}$ and transformation property (10) of the kernel $P_s(z,u)$ we need only to prove that the formula is valid at $z = 0$. Proceeding as in the proof of Theorem 5.3 using Lemma 5.2, we have

$$\bar{\mathbf{D}}\nabla^{(1,0)}P_s(z,u) = \sigma^2 P_{s,u}(z)\big[b(z,z)(z^{\bar{z}} - u^{\bar{z}})\big] \otimes \big[\bar{z}^z - \bar{u}^z\big] - \sigma P_{s,u}(z)\mathrm{Id}.$$

Its image under $-\mathrm{Ad}_{V\otimes\bar{V}\to\mathfrak{k}_{\mathbb{C}}}$ is,

$$-\mathrm{Ad}_{V\otimes\bar{V}\to\mathfrak{k}_{\mathbb{C}}}\bar{\mathbf{D}}\nabla^{(1,0)}P_s(z,u) = \sigma^2 P_{s,u}(z)D(b(z,z)(z^{\bar{z}}-u^{\bar{z}}), \bar{z}^z-\bar{u}^z) - \sigma P_{s,u}(z)pZ_0,$$

since

$$-\mathrm{Ad}_{V\otimes\bar{V}\to\mathfrak{k}_{\mathbb{C}}}(u\otimes\bar{v}) = -[u,\bar{v}] = D(u,\bar{v})$$

and

$$-\mathrm{Ad}_{V\otimes\bar{V}\to\mathfrak{k}_{\mathbb{C}}}\mathrm{Id} = pZ_0.$$

To compute $\bar{\mathbf{D}}\mathrm{Ad}_{V\otimes\bar{V}\to\mathfrak{k}_{\mathbb{C}}}\bar{\mathbf{D}}\nabla^{(1,0)}P_s(z,u)$ at $z = 0$ we observe that for any function f, $\bar{\mathbf{D}}f(0) = \bar{\partial}f(0)$ and the later is the coefficient of $\bar{z}$ in the expansion of f near $z = 0$. By direct computation we find

$$\begin{aligned}-\bar{\mathbf{D}}\mathrm{Ad}_{V\otimes\bar{V}\to\mathfrak{k}_{\mathbb{C}}}\bar{\mathbf{D}}\nabla^{(1,0)}P_s(z,u)_{|z=0} &= \sigma^3 u\otimes D(u,\bar{u}) - p\sigma^2 u\otimes Z_0\\ &+\sigma^2\textstyle\sum_j v_j\otimes[D(Q(u)\bar{v}_j,\bar{u}) - D(u,\bar{v}_j)].\end{aligned}$$

where $\{v_j\}$ is an orthonormal basis of V. Now for $v \in V, X \in \mathfrak{k}_{\mathbb{C}}$, $\mathrm{Ad}_{V\otimes\mathfrak{k}_{\mathbb{C}}\to}(v\otimes X) = [v,X] = -Xv$, where Xv is the defining action of $\mathfrak{k}_{\mathbb{C}}$ on V. We get

$$\begin{aligned}&\mathrm{Ad}_{V\otimes\mathfrak{k}_{\mathbb{C}}\to V}\bar{\mathbf{D}}\mathrm{Ad}_{V\otimes\bar{V}\to\mathfrak{k}_{\mathbb{C}}}\bar{\mathbf{D}}\nabla^{(1,0)}P_s(z,u)_{|z=0}\\ &\qquad= 2\sigma^3 u - p\sigma^2 u + \sigma^2\textstyle\sum_j[D(Q(u)\bar{v}_j,\bar{u})v_j - D(u,\bar{v}_j)v_j].\end{aligned}$$

The sum can further be evaluated by using the Peirce decomposition with respect to u, and we obtain eventually

$$\mathrm{Ad}_{V\otimes\mathfrak{k}_{\mathbb{C}}\to V}\bar{\mathbf{D}}\mathrm{Ad}_{V\otimes\bar{V}\to\mathfrak{k}_{\mathbb{C}}}\bar{\mathbf{D}}\nabla^{(1,0)}P_s(z,u)_{|z=0}=\sigma^2(2\sigma-p-b)u.$$

This proves the first formula.

For the second formula we have first

$$\bar{\mathbf{D}}P_s(z,u)=\sigma P_s(z,u)b(z,z)(z^{\bar{z}}-u^{\bar{z}})$$

and,

$$\nabla^{(1,0)}\bar{\mathbf{D}}\bar{\mathbf{D}}P_s(z,u)(0)_{|z=0}=\sigma\sum_{j,k}\partial_{v_k}\bar{\partial}_{v_j}\left(P_s(z,u)b(z,z)(z^{\bar{z}}-u^{\bar{z}})\right)_{|z=0}.$$

Performing the differentiation we find then

$$\nabla^{(1,0)}\bar{\mathbf{D}}\bar{\mathbf{D}}P_s(z,u)_{|z=0}=\sigma^3\bar{u}\otimes u\otimes u$$
$$-\sigma^2\textstyle\sum_k(\bar{v}_k\otimes(v_k\otimes u+u\otimes v_k))-\sigma\sum_{j,k}\bar{v}_k\otimes v_j\otimes D(v_k,\bar{v}_j)u.$$

So that

$$\mathcal{U}P_s(z,u)_{|z=0}=-\sigma^3D(u,\bar{u})u$$
$$+\sigma^2\textstyle\sum_k\left[D(v_k,\bar{v}_k)u+D(u,\bar{v}_k)v_k\right]+\sigma\sum_{j,k}D(v_j,\bar{v}_k)D(v_k,\bar{v}_j)u.$$

Again $\sum_k D(v_k,\bar{v}_k)v=pv$ for $v\in V$ and

$$\sum_{j,k}D(v_j,\bar{v}_k)D(v_k,\bar{v}_j)u=cp$$

by [24, Lemma 2.5] with c given as in (25). □

In particular, if $s=1$, $\mathcal{W}P_{s,u}(z)=0$; the similar result, $\mathcal{V}P_{s,u}(z)=0$ for the Hua operator $\mathcal{V}$ was proved by Berline and Vergne [1, Proposition 3.3].

7.3. **The sufficiency of the Hua equation (26).** The idea of the proof is similar to that in section 6, and many technical computations on the various decomposition involving the third-order Hua operators $\mathcal{W}$ and $\mathcal{U}$ are parallel to those in [1] for the Berline-Vergne's Hua operator $\mathcal{V}$, so we will not present all details.

Suppose hereafter that $f\in\mathcal{M}(\lambda_s)$ satisfies (26). We first observe that the operator $\mathcal{U}$ can also be written as

$$\mathcal{U}=\sum_{\alpha,\beta,\gamma}v_\gamma v_\alpha^* v_\beta^*\otimes[v_\alpha,[v_\beta,v_\gamma^*]]$$

since $[[v_\gamma^*, v_\alpha], v_\beta] = [v_\alpha, [v_\beta, v_\gamma^*]]$ by the Jacobi identity and by $[v_\alpha, v_\beta] = 0$. Writing

$$\mathcal{U} = \sum \mathcal{U}^\delta \otimes v_\delta, \qquad \mathcal{W} = \sum \mathcal{W}^\delta \otimes v_\delta,$$

with

$$\mathcal{U}^\delta \otimes v_\delta = \sum_{\alpha+\beta-\gamma=\delta} v_\gamma v_\alpha^* v_\beta^* \otimes v_\delta, \qquad \mathcal{W}^\delta \otimes v_\delta = \sum_{\alpha+\beta-\gamma=\delta} v_\alpha^* v_\beta^* v_\gamma \otimes v_\delta, \tag{27}$$

we have, modulo $\mathcal{U}(\mathfrak{g}_\mathbb{C})\mathfrak{k}_\mathbb{C}$,

$$\mathcal{U}^\delta \otimes v_\delta - \mathcal{W}^\delta \otimes v_\delta = \Big(\sum_{\alpha+\beta-\gamma=\delta} |C_{\alpha,\beta,\gamma}|^2\Big) v_\delta^* \otimes v_\delta$$

where $C_{\alpha,\beta,\gamma}$ are given by $[v_\alpha, [v_\beta, v_\gamma^*]] = C_{\alpha,\beta,\gamma} v_\delta$.

Writing $\mathcal{Y} = \sum_\delta \mathcal{Y}^\delta \otimes v_\delta$ as above with $\mathcal{Y}^\delta \in \mathcal{U}(\mathfrak{g})$. We have then, modulo $\mathcal{U}(\mathfrak{g}_\mathbb{C})\mathfrak{k}_\mathbb{C}$

$$\mathcal{Y}^\delta = C_1 v_\delta^* + C_2 \mathcal{W}^\delta$$

with

$$C_1 := \sum_{\alpha+\beta-\gamma=\delta} |C_{\alpha,\beta,\gamma}|^2, \quad C_2 := 1 - \frac{-2\sigma^2 + 2p\sigma + c}{\sigma(2\sigma - p - b)}.$$

Thus

$$\mathcal{Y}^\delta f = 0. \tag{28}$$

for any non-compact root $\delta \equiv \frac{\gamma_j + \gamma_{j-1}}{2}$ modulo $\mathfrak{t}_\mathbb{C}^-$, by our assumption on f. We will henceforth fix one such δ, and study the induced equation of (28).

Recall projection π from $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ onto $U(\mathfrak{a}_\mathbb{C} + \mathfrak{n}_\mathbb{C}^-)$. Here we will not be able to find explicit formula for $\pi(\mathcal{Y}^\delta)$ as in Proposition 6.10. Nevertheless we can compute the induced equation.

Consider the decomposition of $\mathcal{U}(\mathfrak{a}_\mathbb{C} + \mathfrak{n}_\mathbb{C}^-)$ under $\mathfrak{a}_\mathbb{C}$,

$$\mathcal{U}(\mathfrak{a}_\mathbb{C} + \mathfrak{n}_\mathbb{C}^-) = \sum_{p \in \Pi^-} \mathcal{U}(\mathfrak{a}_\mathbb{C} + \mathfrak{n}_\mathbb{C}^-)_p \tag{29}$$

where

$$\Pi^- = \{p = \sum_{\beta \in \Sigma^-(\mathfrak{g},\mathfrak{a})} c_\beta \beta, \;\; 0 \le c_\beta, \;\; c_\beta \in \mathbb{Z}\}$$

is root lattice of $\Sigma^-(\mathfrak{g}, \mathfrak{a})$.

Lemma 7.4. *Let* $\alpha+\beta-\gamma=\delta\equiv\frac{\gamma_j+\gamma_{j-1}}{2}$ *be as in (27). Decomposing* $\pi(\bar v_\alpha \bar v_\beta v_\gamma)\in\mathcal{U}(\mathfrak{a}_{\mathbb{C}}+\mathfrak{n}_{\mathbb{C}}^-)$ *according to (29),*

$$\pi(\bar v_\alpha \bar v_\beta v_\gamma)=\sum_{p\in\Pi^-}\pi(\bar v_\alpha \bar v_\beta v_\gamma)_p, \tag{30}$$

we have that $p\le -\frac{\beta_j-\beta_{j-1}}{2}$, *for any* p *appearing in (30) so that* $\pi(\bar v_\alpha \bar v_\beta v_\gamma)_p\neq 0$.

Proof. The lemma can be proved by a case by case computation of the projection by using Lemma 6.9, and is essentially contained in [1]. We sketch another somewhat more systematic method. We denote the Iwasawa decomposition as $\bar v_\alpha=\pi(\bar v_\alpha)+y$ with $y\in\mathfrak{k}_{\mathbb{C}}$. Thus

$$\bar v_\alpha \bar v_\beta v_\gamma=\pi(\bar v_\alpha)\bar v_\beta v_\gamma+y\bar v_\beta v_\gamma.$$

The Iwasawa projection of the first term is

$$\pi(\bar v_\alpha)\pi(\bar v_\beta v_\gamma).$$

The projection of the second term is

$$\pi([y,\bar v_\beta]v_\gamma)+\pi(\bar v_\beta[y,v_\gamma]).$$

Observe by Lemma 6.9 that the element y is a positive compact root vector, so that all these projections involved are of the form $\pi(\bar v_\delta v_\epsilon)$ with v_δ and v_ϵ being non-compact positive root vectors. Our lemma reduces to the following claim, which can be proved easily by using Lemma 6.9:

The weights p of $\pi(\bar v_\delta v_\epsilon)$ satisfy the inequality

$$p\le -\frac{\beta_j-\beta_{j-1}}{2}+\frac{\beta_k-\beta_{k'}}{2}$$

if $\delta-\epsilon=\frac{\gamma_j+\gamma_{j-1}}{2}-\frac{\gamma_k+\gamma_{k'}}{2}$ with $k>k'$,

$$p\le -\frac{\beta_j-\beta_{j-1}}{2}$$

if $\delta-\epsilon=\frac{\gamma_j+\gamma_{j-1}}{2}-\gamma_k$, and

$$p\le -\frac{\beta_j-\beta_{j-1}}{2}+\frac{\beta_k}{2}$$

if $\delta-\epsilon=\frac{\gamma_j+\gamma_{j-1}}{2}-\frac{\gamma_k}{2}$. □

From this it follows that the operator $t^{-\frac{1}{2}(\beta_j-\beta_{j-1})}\mathcal{R}[\pi(\mathcal{Y}^\delta)]$ has analytic coefficients in t near $t=0$ and thus the induced equation

$$\lim_{t\to 0}t^{\lambda-\rho}t^{\frac{1}{2}(\beta_j-\beta_{j-1})}\mathcal{R}[\pi(\mathcal{Y}^\delta)]t^{\rho-\lambda}(\mathcal{B}_\lambda f)=0 \tag{31}$$

of the equation $\mathcal{Y}^\delta f=0$ is well-defined.

Consider next the eigenspace decomposition of the space $\mathfrak{n}_{\mathbb{C}}^{-}$ under the element $\frac{1}{2}\xi_c = \frac{1}{2}\sum_{j=1}^{r}\xi_j$:

$$\mathfrak{n}_{\mathbb{C}}^{-} = \mathfrak{n}_{\mathbb{C}}^{-1} + \mathfrak{n}_{\mathbb{C}}^{-\frac{1}{2}} + \mathfrak{n}_{\mathbb{C}}^{0}$$

with

$$\mathfrak{n}_{\mathbb{C}}^{-1} = \sum_{k\geq j\geq 1} \mathfrak{g}_{\mathbb{C}}^{-\frac{\beta_k+\beta_j}{2}},\ \mathfrak{n}_{\mathbb{C}}^{-\frac{1}{2}} = \sum_{k\geq 1} \mathfrak{g}_{\mathbb{C}}^{-\frac{\beta_k}{2}},\ \mathfrak{n}_{\mathbb{C}}^{0} = \sum_{k\geq j} \mathfrak{g}_{\mathbb{C}}^{-\frac{\beta_k-\beta_j}{2}};$$

and correspondingly

$$\mathcal{U}(\mathfrak{a}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}^{-}) = \mathcal{U}(\mathfrak{a}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}^{-})(\mathfrak{n}_{\mathbb{C}}^{-1} + \mathfrak{n}_{\mathbb{C}}^{-\frac{1}{2}}) + \mathcal{U}(\mathfrak{a}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}^{0}). \tag{32}$$

Let

$$\pi(\mathcal{Y}^{\delta}) = \mathcal{Y}_1 + \mathcal{Y}_0$$

be the decomposition of $\pi(\mathcal{Y}^{\delta})$ according to (32). As the decompositions (29) and (32) are consistent, we see that the weights p that appear in the decomposition of $\mathcal{Y}_1$ according to (29) satisfies $p \leq -\delta$, and $p \leq \mu$ for μ such that $\mathfrak{n}_{\mathbb{C}}^{\mu} \subset \mathfrak{n}_{\mathbb{C}}^{-1} + \mathfrak{n}_{\mathbb{C}}^{-\frac{1}{2}}$. The first implies that the induced equation is well-defined, and the second that the induced equation (31) now reduces to

$$\lim_{t\to 0} t^{\lambda-\rho} t^{-\frac{1}{2}(\beta_j-\beta_{j-1})} \mathcal{R}[\mathcal{Y}_1] t^{\rho-\lambda}(\mathcal{B}_\lambda f) = 0. \tag{33}$$

The element $\mathcal{Y}_0$ can be found along the same lines as in [1], where the constant term $C_1\zeta_{v_\delta}$ was found.

Lemma 7.5. *The element* $\mathcal{Y}_0$ *is given by*

$$\left(C_1 + C_2\left[\frac{1}{2}(-\xi_j^2 - \xi_{j-1}^2 + \xi_j\xi_{j-1}) + C_1'\xi\right]\right)\zeta_{v_\delta}$$

where $\xi \in \mathfrak{a}_{\mathbb{C}}$, C_1' *is some constants independent of* λ.

In particular the induced equation (33) is of the form

$$(C_1 + C_2 D(\lambda))\mathcal{R}(\zeta_{v_\delta})(\mathcal{B}_\lambda f) = 0.$$

where

$$D(\lambda) = \tfrac{1}{2}(-(\rho-\lambda)(\xi_j)^2 - (\rho-\lambda)(\xi_{j-1})^2 + (\rho-\lambda)(\xi_j)(\rho-\lambda)(\xi_{j-1})) \\ + C_1'(\rho-\lambda)(\xi).$$

Observe first that $C_1 > 0$. If $C_2 = 0$ it follows immediately that $\mathcal{R}(\zeta_{v_\delta})(\mathcal{B}_\lambda f) = 0$, so we need only to consider the case $C_2 \neq 0$. If $C_1 + C_2 D(\lambda) \neq 0$ we get again $\mathcal{R}(\zeta_{v_\delta})(\mathcal{B}_\lambda F) = 0$. Finally if $C_1 + C_2 D(\lambda) = 0$ we may replace f by $t^{\kappa\gamma_j}$ for sufficiently large κ and still prove that $\mathcal{R}(\zeta_{v_\delta})(\mathcal{B}_\lambda f) = 0$; see [15]. This completes the proof.

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