

Products of ratios of consecutive integers

Régis de la Bretèche, Carl Pomerance & Gérald Tenenbaum

For Jean-Louis Nicolas, on his sixtieth birthday

1. Introduction

Let $\{\varepsilon_n\}_{1 \leq n < N}$ be a finite sequence with each $\varepsilon_n \in \{0, \pm 1\}$, and write

$$\frac{a}{b} = \prod_{1 \leq n < N} \left(\frac{n}{n+1} \right)^{\varepsilon_n},$$

where the fraction is in its smallest terms. Now, define $A(N)$ as the maximal value of a as $\{\varepsilon_n\}_{1 \leq n < N}$ runs through all possible 3^{N-1} sequences of $0, \pm 1$, and let $B(N)$ denote the corresponding value of b . (Note that maximizing b instead of a would lead to $B(N)/A(N)$.) We obviously have $A(N) \leq N!$, hence $\log A(N) \leq N \log N$ for all N . In [7], it is shown by an elegant “near-tiling” of the integers in $[1, N]$ with triples $n, 2n, 2n+1$ that

$$\log A(N) \leq \left\{ \frac{2}{3} + o(1) \right\} N \log N.$$

Further, a brief argument of M. Langevin is presented that

$$\log A(N) \geq \{ \log 4 + o(1) \} N.$$

Our aim in this article is to establish the true order of magnitude for $\log A(N)$. Put

$$k(c) := 1 + 2 \log(1 - 2c) - \frac{2}{c} \log \left(1 + \frac{2c^2}{1 - 3c} \right),$$

$$K(c) := 2 \int_0^c k(u) du, \quad K := \max_{0 < c < 1/5} K(c) \approx 0.107005.$$

Theorem 1.1. *For large N , we have*

$$(1.1) \quad \log A(N) \geq \{K + o(1)\} N \log N.$$

Let $P(n)$ denote the largest prime factor of a positive integer n with the convention that $P(1) = 1$. The lower bound (1.1) is an easy consequence of the estimate stated in the following result.

Theorem 1.2. *For $c \in [0, 1]$, $x \geq 1$, let $S(x, c)$ denote the number of those integers n not exceeding x such that $\min\{P(n), P(n+1)\} > x^{1-c}$. Then, for any fixed $c_0 \in]0, \frac{1}{5}[$ and uniformly for $c \in [0, c_0]$, $x \rightarrow \infty$, we have*

$$(1.2) \quad S(x, c) \leq 2x \int_0^c \log \left(\frac{1-v}{1-v-2c} \right) \frac{dv}{1-v} + o(x).$$

Remark. Under a suitably strong form of the Elliott–Halberstam hypothesis, we get the better bound

$$(1.3) \quad S(x, c) \leq x \{ \log(1-c) \}^2 + o(x)$$

for $0 \leq c \leq \frac{1}{2}$. This would yield the value $K = 4/\sqrt{e} - 2 \approx 0.426123$ in Theorem 1.1. See Section 3 for further methodological remarks.

The bound (1.3) is probably optimal, in fact it is likely to be the case that $S(x, c) = x \{ 1 - \varrho(1/\{1-c\}) \}^2 + o(x)$ for $0 \leq c < 1$, where ϱ is the Dickman–de Bruijn function. An even stronger statement is suggested in [3].

Note that (1.1) follows from (1.2) by selecting $\varepsilon_n = 1$ if $P(n) > N^{1-c}$ and $P(n) > P(n+1)$, $\varepsilon_n = -1$ if $P(n+1) > N^{1-c}$ and $P(n+1) > P(n)$ and $\varepsilon_n = 0$ in all other cases. Indeed, with these choices for ε_n , we obtain that for each prime $p > N^{1-c} \geq N^{1/2}$, the exponent on p in the prime factorization of the rational number $A(N)/B(N)$ is

$$\sum_{\substack{n < N \\ P(n)=p}} 2 - \sum_{\substack{n < N \\ P(n+1) > P(n)=p}} 2 - \sum_{\substack{n < N \\ P(n) > P(n+1)=p}} 2.$$

Thus,

$$\log A(N) \geq \sum_{\substack{n < N \\ P(n) > N^{1-c}}} 2 \log P(n) - \sum_{\substack{n < N \\ P(n), P(n+1) > N^{1-c}}} 2 \log \min\{P(n), P(n+1)\}.$$

We have

$$\begin{aligned} \sum_{\substack{n \leq N \\ P(n), P(n+1) > N^{1-c}}} 2 \log \min\{P(n), P(n+1)\} &= \int_0^c (1-u) \log N \, dS(N, u) \\ &= (\log N) \left\{ (1-c)S(N, c) + \int_0^c S(N, u) \, du \right\}, \end{aligned}$$

and, since the number of $n < N$ with $P(n) > N^{1-c}$ is $N \log\{1/(1-c)\} + o(N)$ uniformly for $0 \leq c \leq 1/2$,

$$\sum_{\substack{n < N \\ P(n) > N^{1-c}}} \log P(n) = cN \log N + o(N).$$

We thus obtain

$$\begin{aligned} \log A(N) &\geq 2(\log N) \left\{ cN - (1-c)S(N, c) - \int_0^c S(N, u) \, du + o(N) \right\} \\ &\geq 2N(\log N) \{g(c) + o(1)\}, \end{aligned}$$

where we have set

$$g(c) := c - (1-c)f(c) - \int_0^c f(u) \, du, \quad \text{with} \quad f(u) := 2 \int_0^u \log \left(\frac{1-v}{1-v-2u} \right) \frac{dv}{1-v}.$$

We check by computation that $g'(c) = k(c)$. This implies the desired estimate.

2. Proof of Theorem 1.2

We employ the Rosser–Iwaniec sieve. A slightly better numerical bound could be obtained from a more sophisticated sieve method. We do not pursue such an improvement here but provide precise indications in Section 3. We refer to [5], [6] for a complete reference of the Rosser–Iwaniec coefficients and merely recall the property we shall use. We denote by γ the Euler constant, and we let p run over primes.

Lemma 2.1. *Let \mathcal{Q} denote a set of primes, let $z \geq 2$ and write $Q(z) := \prod_{p \leq z, p \in \mathcal{Q}} p$. There exists a sequence $\{\lambda_d\}_{d=1}^{\infty}$ of real numbers, vanishing for $d > z$ or $\mu(d) = 0$, satisfying $\lambda_1 = 1$, $|\lambda_d| \leq 1$, $\mu * \mathbf{1} \leq \lambda * \mathbf{1}$, and such that for any number $\alpha > 0$,*

$$\sum_{d|Q(z)} \frac{\lambda_d w(d)}{d} \leq \prod_{\substack{p \leq z \\ p \in \mathcal{Q}}} \left(1 - \frac{w(p)}{p}\right) \left\{2e^\gamma + O_\alpha\left(\frac{1}{(\log z)^{1/3}}\right)\right\},$$

uniformly for all multiplicative functions w satisfying

$$\begin{aligned} \text{(i)} \quad & 0 < w(p) < p \quad (p \in \mathcal{Q}), \\ \text{(ii)} \quad & \prod_{u < p \leq v, p \in \mathcal{Q}} \left(1 - \frac{w(p)}{p}\right)^{-1} \leq \frac{\log v}{\log u} \left(1 + \frac{\alpha}{\log u}\right) \quad (2 \leq u \leq v \leq z). \end{aligned}$$

If n is counted by $S(x, c)$, then $n = ap_1 = bp_2 - 1$, where p_1 and p_2 are primes greater than x^{1-c} . Then a and b are obviously coprime, and moreover $2|ab$. We need an upper bound for the number $Z(a, b)$ of admissible pairs (p_1, p_2) for given a, b . Let C be a sufficiently large constant and set $z := (x/a)^{1/2}b^{-1}(\log x)^{-C}$. If \mathcal{Q} is the set of all primes not dividing a and with $\{\lambda_d\}_{d=1}^{\infty}$ the sequence from Lemma 2.1, we plainly have

$$\begin{aligned} Z(a, b) &\leq \sum_{\substack{p_1 \leq x/a \\ ap_1 \equiv -1 \pmod{b}}} \mu * \mathbf{1}((ap_1 + 1)/b, Q(z)) \\ \text{(2.1)} \quad &\leq \sum_{d|Q(z)} \lambda_d \sum_{\substack{p_1 \leq x/a \\ ap_1 \equiv -1 \pmod{bd}}} 1. \end{aligned}$$

Let us put, for real $y \geq 2$ and integers q, r with $q \geq 1$,

$$\pi(y; q, r) := \sum_{\substack{p \leq y \\ p \equiv r \pmod{q}}} 1, \quad E(y; q) := \max_{(r, q)=1} |\pi(y; q, r) - \text{li}(y)/\varphi(q)|.$$

We apply Lemma 2.1 to the multiplicative function $d \mapsto d\varphi(b)/\varphi(bd)$. Using the fact that $(a, bd) = 1$ for each $d | Q(z)$, and noticing that c bounded below $1/5$ ensures that $z \geq b$ when x is large enough, we deduce that

$$\text{(2.2)} \quad Z(a, b) \leq M(a, b) + R(a, b)$$

with

$$R(a, b) := \sum_{d \leq z} E(x/a; bd)$$

and

$$\begin{aligned} M(a, b) &:= \sum_{d|Q(z)} \frac{\lambda_d \operatorname{li}(x/a)}{\varphi(bd)} \\ &\leq \{2e^\gamma + o(1)\} \frac{\operatorname{li}(x/a)}{\varphi(b)} \prod_{\substack{p \leq z \\ p \nmid ab}} \left(1 - \frac{1}{p-1}\right) \prod_{\substack{p \leq z \\ p|b}} \left(1 - \frac{1}{p}\right) \\ &= \{2e^\gamma + o(1)\} \frac{\operatorname{li}(x/a)}{b} \prod_{\substack{p \leq z \\ p \nmid ab}} \left(\frac{p-2}{p-1}\right). \end{aligned}$$

Now we observe that, uniformly as x tends to ∞ and a, b vary in the specified ranges,

$$\prod_{\substack{p \leq z \\ p > 2}} \left(\frac{p-2}{p-1}\right) = 2 \prod_{\substack{p \leq z \\ p > 2}} \frac{p(p-2)}{(p-1)^2} \prod_{p \leq z} \left(1 - \frac{1}{p}\right) \sim \frac{2e^{-\gamma}}{A \log z}$$

where

$$A := \prod_{p > 2} \left(1 + \frac{1}{p(p-2)}\right).$$

Therefore, writing

$$h(n) := \prod_{\substack{p|n \\ p > 2}} \left(\frac{p-1}{p-2}\right),$$

we obtain that the estimate

$$(2.3) \quad M(a, b) \leq \frac{\{8 + o(1)\} h(ab) x}{Aab \log(x/a) \log(x/ab^2)}$$

holds uniformly for $a \leq x^c$, $b \leq x^c$, $(a, b) = 1$, as $x \rightarrow \infty$.

Let $\tau(m)$ denote the number of divisors of m . By the Bombieri–Vinogradov theorem, we have, with $X_a := (x/a)^{1/2} (\log x)^{-C}$,

$$\begin{aligned} \sum_{b \leq x^c} R(a, b) &\leq \sum_{m \leq X_a} \tau(m) E(x/a; m) \\ &\ll \left\{ \sum_{m \leq X_a} E(x/a; m) \sum_{m \leq X_a} \tau(m)^2 E(x/a; m) \right\}^{1/2} \ll \frac{x}{a(\log x)^2}, \end{aligned}$$

where we have used the trivial estimate $E(x/a; m) \ll x/am$ and the well-known fact that $\sum_{m \leq x} \tau(m)^2/m \ll (\log x)^4$. Therefore, we obtain from (2.2) and (2.3)

$$(2.4) \quad \begin{aligned} S(x, c) &\leq \sum_{\substack{a \leq x^c, b \leq x^c \\ (a,b)=1, 2|ab}} Z(a, b) \\ &\leq \frac{8 + o(1)}{A} x \sum_{a \leq x^c} \frac{h(a)}{a \log(x/a)} \sum_{\substack{b \leq x^c \\ 2|ab \\ (b,a)=1}} \frac{h(b)}{b \log(x/ab^2)} + O\left(\frac{x}{\log x}\right). \end{aligned}$$

We have for $\nu = 0$ or 1

$$(2.5) \quad \sum_{\substack{b \geq 1 \\ (b,a)=1}} \frac{h(2^\nu b)}{b^s} = H(s) G_a(s) \zeta(s) \quad (\Re s > 1)$$

where

$$H(s) := \prod_{p>2} \left(1 + \frac{1}{p^s(p-2)}\right), \quad G_a(s) := \left(1 - \frac{\varepsilon(a)}{2^s}\right) \prod_{\substack{p|a \\ p>2}} \left(\frac{1 - p^{-s}}{1 + p^{-s}/(p-2)}\right),$$

with $\varepsilon(a) = 1$ if a is even, $\varepsilon(a) = 0$ if a is odd. The functions H and G_a can be analytically continued to the half-plane $\Re s > 0$. Note that $H(1) = A$, $G_a(1) = 2^{-\varepsilon(a)} h(a)^{-1}$. By estimates of Selberg–Delange type (see [8], chap. II.5), (2.5) yields in turn

$$\sum_{\substack{b \leq y \\ (b,a)=1}} h(2^\nu b) \sim \frac{Ay}{2^{\varepsilon(a)} h(a)} \quad (y \rightarrow \infty),$$

and

$$\sum_{\substack{b \leq x^c \\ (a,b)=1, 2|ab}} \frac{h(b)}{b \log(x/ab^2)} = \frac{A}{4h(a)} \log\left(\frac{1 - v_a}{1 - 2c - v_a}\right) + o(1) \quad (x \rightarrow \infty)$$

and $v_a := (\log a)/\log x$. Substituting this back into (2.4), we arrive at

$$\begin{aligned} S(x, c) &\leq \{2 + o(1)\} x \sum_{a \leq x^c} \frac{1}{a \log(x/a)} \log\left(\frac{1 - v_a}{1 - 2c - v_a}\right) \\ &= \{2 + o(1)\} x \int_0^c \log\left(\frac{1 - v}{1 - 2c - v}\right) \frac{dv}{1 - v}. \end{aligned}$$

□

We remark that with a little more care, the bound $1/5$ in the theorem may be replaced with $1/3$.

3. Further remarks

In [3] it is shown that if N is large, then for at least $0.0099N$ values of $n \leq N$ we have $P(n) > P(n+1)$, and for at least $0.0099N$ values of $n \leq N$ we have $P(n) < P(n+1)$. It follows from Theorem 1.2 that each inequality occurs on a set of integers n of lower asymptotic density

$$\log \left(\frac{1}{1-c} \right) - 2 \int_0^c \log \left(\frac{1-v}{1-v-2c} \right) \frac{dv}{1-v}$$

for each value of c with $0 < c < 1/5$. The maximum of this expression is greater than 0.05544 so we have majorized the result from [3]. Presumably, the set E of integers n with $P(n) > P(n+1)$ has asymptotic density $1/2$. A general theorem of Hildebrand [4] also implies that E has positive lower asymptotic density, but we did not check the numerical value that can be derived from this result.

In [3] it is shown that $P(n) < P(n+1) < P(n+2)$ holds infinitely often, and it was conjectured that so too $P(n) > P(n+1) > P(n+2)$ holds infinitely often. This conjecture was recently proved by Balog in [1].

We observe that the maximal value $A(N)$ corresponds to a sequence $\varepsilon = \{\varepsilon_n\}_{1 \leq n < N}$ where $\varepsilon_n \in \{-1, 1\}$.

Proposition 3.1. *Let $N \geq 1$. There exists $\{\varepsilon_n\}_{1 \leq n < N} \in \{-1, 1\}^{N-1}$ such that*

$$\frac{A(N)}{B(N)} = \prod_{1 \leq n < N} \left(\frac{n}{n+1} \right)^{\varepsilon_n}.$$

Remark. Let $A_{0,1}(N)$ (respectively $A_{-1,1}(N)$, $A_{-1,0}(N)$) the maximum of numerators where the exponents ε_n are restricted to $\{0, 1\}$ (respectively $\{-1, 1\}$, $\{-1, 0\}$). By the proposition, we have $A_{-1,1}(N) = A(N)$ and

$$\log A_{0,1}(N) = \frac{1}{2} \log A(N) + O(\log N) = \log A_{-1,0}(N) + O(\log N).$$

For example, if $\{\varepsilon_n\}_{1 \leq n < N} \in \{0, 1\}^{N-1}$, we have $\{2\varepsilon_n - 1\}_{1 \leq n < N} \in \{-1, 1\}^{N-1}$. Since the constant sequence -1 gives the numerator N , we deduce the result.

Proof. Take a sequence $\{\varepsilon_n\}_{1 \leq n < N} \in \{-1, 0, 1\}^{N-1}$ where some $\varepsilon_n = 0$. Write the associated product as A/B with $(A, B) = 1$. If we let $\varepsilon_n = 1$, the new numerator is

$$\frac{A}{(A, n+1)} \times \frac{n}{(B, n)},$$

while if we let $\varepsilon_n = -1$, the new numerator is

$$\frac{A}{(A, n)} \times \frac{n+1}{(B, n+1)}.$$

Assuming both of these expressions are smaller than A , we obtain

$$n < (A, n+1)(B, n) \quad \text{and} \quad n+1 < (A, n)(B, n+1).$$

Multiplying these inequalities and using $(A, B) = (n, n+1) = 1$ we obtain

$$n(n+1) < (AB, n(n+1)),$$

a contradiction. So we may choose $\varepsilon_n \in \{\pm 1\}$ without decreasing the associated numerator. With this method we can replace each 0 value with ± 1 and the value of the associated numerator will not decrease. \square

As pointed out in Section 2, the result of Theorem 1.2 can be improved by using more sophisticated sieve methods. Indeed, in the present treatment, we sift integers of the form $(ap_1 + 1)/b$ for fixed a and b , and then sum over a after having disposed of the b -sum by the Bombieri–Vinogradov theorem, thus applying this result to a sequence of length x/a and hence of distribution level roughly $\sqrt{x/a}$. We could, alternatively, sift, for fixed b , the sequence comprising all $(ap_1 + 1)/b$ of a piece. We could then restrict the study to, say, $x/\log x < ap_1 \leq x$. This corresponds to the Dirichlet convolution of two characteristic functions that are of Siegel–Walfisz type, and thus has distribution level $x^{1/2+o(1)}$ — see, e.g., [2] theorem 0(b). Such an approach yields, for any given c_0 , $0 < c_0 < \frac{1}{4}$, the upper bound

$$(3.1) \quad S(x, c) \leq 2x \log(1 - 2c) \log(1 - c) + o(x)$$

uniformly for $0 \leq c \leq c_0$. In turn, this improvement allows the better value $K \approx 0.112945$ in Theorem 1.1, and ups the density in the first paragraph of this section to 0.05866. We warmly thank Étienne Fouvry for these observations.

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Régis de la Bretèche
 École Normale Supérieure
 Mathématiques et Applications
 45, rue d’Ulm
 75230 Paris Cedex 05, France
 breteche@ens.fr

Carl Pomerance
 Department of Mathematics
 Dartmouth College
 Hanover, NH 03755
 USA
 carlp@math.dartmouth.edu

Gérald Tenenbaum
 Institut Élie Cartan
 Université Henri Poincaré Nancy 1
 BP 239
 54506 Vandœuvre Cedex, France
 gerald.tenenbaum@ciril.fr