FINE HOCHSCHILD INVARIANTS OF DERIVED CATEGORIES FOR SYMMETRIC ALGEBRAS

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Abstract. Let A be a symmetric k-algebra over a perfect field k. Külshammer defined for any integer n a mapping ζ_n on the degree 0 Hochschild cohomology and a mapping κ_n on the degree 0 Hochschild homology of A as adjoint mappings of the respective p-power mappings with respect to the symmetrizing bilinear form. In an earlier paper it is shown that ζ_n is invariant under derived equivalences. In the present paper we generalize the definition of κ_n to higher Hochschild homology and show the invariance of κ and its generalization under derived equivalences. This provides fine invariants of derived categories.

Introduction

Let k be a commutative ring and let A be a k-algebra which is projective as a k-module. If B is a second k-algebra and if the derived categories $D^b(A)$ of bounded complexes of A-modules and $D^b(B)$ of B-modules are equivalent as triangulated categories, then the Hochschild cohomology of A is isomorphic to the Hochschild cohomology of B (Rickard [15]). Analogous statements hold for the cyclic homology (Keller [6]), the K-theory (Thomason-Trobaugh [19]), the fact of being symmetric algebra (cf Rickard [15] for fields and [20] in a more general situation) and others. This is one of the reasons why the derived category $D^b(A)$ is now one of the main tools in representation theory. Nevertheless, most of the invariants are quite difficult to compute, except some small cases like the centre, or the Grothendieck group. Therefore, it is usually quite hard to distinguish two derived categories.

A symmetric k-algebra A is equipped with a symmetric non degenerate bilinear form $(\ , \ ) : A \times A \longrightarrow k$. Denote by KA the commutator subspace, that is the k-linear space generated by the set of $ab - ba$ for $a, b \in A$. If k is a perfect field of characteristic $p > 0$, then Külshammer defined in [10] $T_n(A)^\perp$ to be the orthogonal space to the set of $x \in A$ so that x^{p^n} falls into KA. It turned out that $T_n(A)^\perp$ is a decreasing sequence of ideals in the centre of A. If A and B are symmetric k-algebras with equivalent derived categories, then the centres of A and B are isomorphic, and in [21] we showed that this isomorphism maps $T_n(A)^\perp$ to $T_n(B)^\perp$. This fine ideal structure of the centre of the algebra A gives valuable and computable derived invariants of A. In joint work with Thorsten Holm [5] we are able to apply the invariance of the ideals $T_n(A)^\perp$ to tame blocks of group rings solving delicate questions whether certain parameters in the defining relations of particular algebras lead to different derived categories. Thorsten Holm and Andrzej Skowroński use this new fine invariant to classify all tame domestic symmetric algebras up to derived equivalence [4].

Alejandro Adem asked during the 2005 Oberwolfach conference "Cohomology of finite groups: Interactions and applications" if it is possible to generalize Külshammer's ideals of the centre of A to a derived invariant of higher degree Hochschild cohomology. The purpose of this paper is to answer to this question.

Külshammer shows in [10] that $T_n(A)^\perp$ is the image of a certain mapping $\zeta_n : Z(A) \longrightarrow Z(A)$ which is defined by $(z, a^{p^n}) = (\zeta_n(z), a)^{p^n}$ for all $z \in Z(A)$ and all $a \in A$. This is the way we view $T_n(A)^\perp$ in [21]. Since $KA = Z(A)^\perp$ the dual operation defines a mapping

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$\kappa_n : A/KA \longrightarrow A/KA$ by the equation $(z^{p^n}, a) = (z, \kappa_n(a))^{p^n}$ for all $z \in Z(A)$ and all $a \in A/KA$. As a first result we show that κ_n as well is a derived invariant, observing that $HH_0(A) = A/KA$ which is known to be a derived invariant.

As a first step we are going to show that for any symmetric algebra A one gets a non degenerate pairing

$$(\ , \ )_m : HH^m(A, A) \times HH_m(A, A) \longrightarrow k \ .$$

Since $HH_*(A, A)$ does not have a multiplicative structure, it seems to be impossible to write down the defining relation for an analogue for ζ_n^A on higher Hochschild cohomology. Nevertheless, Hochschild cohomology is a graded commutative ring. Suppose either m is even and p odd or p even and m arbitrary. We are able to show for these parameters that the p^n-power mapping $HH^m(A, A) \longrightarrow HH^{p^n m}(A, A)$ has a right adjoint $\kappa_n^{(m),A} : HH_{p^n m}(A, A) \longrightarrow HH_m(A, A)$ with respect to $(\ , \ )_m$ and $(\ , \ )_{p^n m}$ and moreover $\kappa_n^{(0)} = \kappa_n$. As a main result we show that any derived equivalence F of standard type between A and B induces an isomorphism $HH_m(F)$ on the Hochschild homology and this in turn conjugates $\kappa_n^{(m),A}$ to $\kappa_n^{(m),B}$.

Using a suggestion of Bernhard Keller we also study the p-power map by the Gerstenhaber Lie structure of the Hochschild cohomology. We show that this again is invariant under derived equivalences. Nevertheless, there is no obvious reason why this p-power map should be semilinear. Moreover, it is only defined from degree 1 Hochschild cohomology onwards. Furthermore, we expect that this will be somewhat harder to compute in examples. On the other hand this gives for $p = 2$ a richer structure to the set of derivations on A, and this structure is then a computable derived invariant.

Our paper is organised as follows. In Section 1 we recall the basic constructions concerning Hochschild (co-)homology and their invariance under derived equivalences. In Section 2 we show the derived invariance of Külshammer's mapping κ_n as well as some consequences for derived equivalences between blocks of group rings. Section 3 is devoted to the definition of the generalization of κ_n to higher Hochschild homology, and to show its invariance under derived equivalence. In Section 4 we first recall the Gerstenhaber construction of a Lie algebra structure on the Hochschild cohomology, define for all primes p a structure of a restricted Lie algebra on odd degree Hochschild homology and on the entire Hochschild cohomology for $p = 2$, and show its invariance under derived equivalence.

1. Derived equivalences and Hochschild constructions revisited

1.1. Basic constructions and definitions for Hochschild homology and cohomology. In order to fix notation and also for convenience of the reader we recall in this section the basic notions for Hochschild cohomology and homology. Most of the material in this section can be found in Loday [12], Keller [7] and Stasheff [17].

Let k be a field and let A be a k-algebra. Consider a complex $\mathbf{B}A$ whose degree n homogeneous component is $(\mathbf{B}A)_n := A^{\otimes(n+2)}$ and whose differential $d_n : (\mathbf{B}A)_n \longrightarrow (\mathbf{B}A)_{n-1}$ is

$$d_n(x_0 \otimes \cdots \otimes x_{n+1}) := \sum_{j=0}^{n} (-1)^j (x_0 \otimes \cdots \otimes x_{j-1} \otimes x_j x_{j+1} \otimes x_{j+2} \otimes \cdots \otimes x_{n+1}).$$

Then the complex $(\mathbf{B}A, d)$ is a projective resolution of A as $A \otimes_k A^{op}$-module.

For a later application in Section 4 we need to extend this construction to

$$\mathbf{B}(A) := k \oplus A[1] \oplus (A \otimes A)[2] \oplus (A \otimes A \otimes A)[3] \oplus \dots$$

where the brackets indicate the degrees of the components. The mappings

$$\begin{array}{rcl} \Delta_n : \mathbf{B}(A)_n & \longrightarrow & \oplus_{1 \le j \le n} \mathbf{B}(A)_j \otimes \mathbf{B}(A)_{n-j} \\ x_1 \otimes \cdots \otimes x_n & \mapsto & \sum_{j=0}^{n} (x_1 \otimes \cdots \otimes x_j) \otimes (x_{j+1} \otimes \cdots \otimes x_n) \end{array}$$

(where it is understood that the boundary cases $j = 0$ and $j = n$ correspond to the element 1_k in the first or the last bracket) compose to a graded co-algebra map $\Delta : \mathbf{B}(A) \longrightarrow \mathbf{B}(A) \otimes \mathbf{B}(A)$ and $\mathbf{B}(A)$ becomes a differential graded co-associative co-algebra. Observe the shift by degree 2 between $\mathbf{B}A$ and $\mathbf{B}(A)$.

We abbreviate in the sequel $A^e := A \otimes A^{op}$ and $A^* = Hom_k(A, k)$.

By definition, for all A^e-modules M one puts

$$HH_n(A, M) = H_n(\mathbf{B}A \otimes_{A\otimes A^{op}} M)$$

and

$$HH^n(A, M) = H^n(Hom_{A\otimes A^{op}}(\mathbf{B}A, M)).$$

1.2. **Some facts on derived equivalences.** Suppose two finite dimensional k-algebras A and B have equivalent derived categories $D^b(A) \simeq D^b(B)$ as triangulated categories. Then, there is a complex Y in $D^b(B \otimes A^{op})$ and a complex X in $D^b(A \otimes B^{op})$ so that

$$F_Y := Y \otimes^{\mathbb{L}}_A - : D^b(A) \longrightarrow D^b(B)$$

is an equivalence and so that

$$X \otimes^{\mathbb{L}}_B - : D^b(B) \longrightarrow D^b(A)$$

is an equivalence quasi-inverse to F_Y. It is known that one may choose X and Y so that both complexes are formed by projective modules if restricted to either side, and that then the left derived tensor product can be replaced by the ordinary tensor product.

Moreover, doing so,

$$F^e_Y := Y \otimes^{\mathbb{L}}_A - \otimes^{\mathbb{L}}_A X : D^b(A \otimes A^{op}) \longrightarrow D^b(B \otimes B^{op})$$

is an equivalence of triangulated categories satisfying

$$F^e_Y(A) \simeq B \ .$$

In [20] we have shown that for quite general algebras, in particular for finite dimensional algebras over fields k, we have

$$F^e_Y(Hom_k(A, k)) \simeq Hom_k(B, k) \ .$$

Moreover, in [21] we have shown that for k a field of characteristic $p > 0$ denoting by $k^{(n)}$ the k-vector space k twisted by the n-th power of the Frobenius automorphism Fr,

$$F^e_Y(Hom_k(A, k^{(n)})) \simeq Hom_k(B, k^{(n)}).$$

In particular, if $A \simeq Hom_k(A, k)$ in $A^e - mod$, then $B \simeq Hom_k(B, k)$ in $B^e - mod$. Such algebras are called symmetric.

Another consequence (cf Rickard[15] or [9]) is that F^e_Y (and therefore F_Y) induces an isomorphism

$$\begin{array}{rcl} HH^m(A, A) = Ext^m_{A^e}(A, A) & = & Hom_{D^b(A^e)}(A, A[m]) \\ & \stackrel{F^e}{\longrightarrow} & Hom_{D^b(B^e)}(B, B[m]) = HH^m(B, B) \ . \end{array}$$

Hochschild homology as well is an invariant of the derived category. This seems to be well known, but I could not find a reference in the literature, and therefore I include a proof. In particular, it is important to have an explicit isomorphism which we will need for our proof. In fact, $\mathbf{B}A$ is a free resolution of A in $A^e - mod$. Since F^e_Y is an equivalence,

$$F^e_Y(\mathbf{B}A) = Y \otimes_A \mathbf{B}A \otimes_A X$$

is a projective resolution of B in $B^e - mod$. The mapping

$$\begin{array}{rcl} \mathbf{B}A \otimes_{A^e} (X \otimes_B Y) & \longrightarrow & (Y \otimes_A \mathbf{B}A \otimes_A Y) \otimes_{B^e} B \\ u \otimes (x \otimes y) & \mapsto & (y \otimes u \otimes x) \otimes 1 \end{array}$$

is well defined as is easily seen. Moreover, an inverse is

$$\begin{array}{rcl}(Y \otimes_A \mathbf{B}A \otimes_A Y) \otimes_{B^e} B & \longrightarrow & \mathbf{B}A \otimes_{A^e} (X \otimes_B Y) \\ (y \otimes u \otimes x) \otimes b & \mapsto & u \otimes (xb \otimes y) = u \otimes (x \otimes by)\end{array}$$

We take the degree m homology of the various complexes and observe since $X \otimes_B Y \simeq A$,

$$H_m(\mathbf{B}A \otimes_{A^e} (X \otimes_B Y)) = Tor_m^{A^e}(A, A) = HH_m(A, A) \ .$$

Moreover,

$$H_m((Y \otimes_A \mathbf{B}A \otimes_A Y) \otimes_{B^e} B) = Tor_m^{B^e}(B, B) = HH_m(B, B) \ .$$

Therefore, also for two k-algebras A and B the *Hochschild homology* is an invariant of the derived category.

2. Symmetric algebras and the Külshammer-κ

Given a perfect field k of characteristic p and a symmetric k-algebra A with symmetrizing bilinear form $(\ ,\ )_A = (\ ,\ )$, Külshammer defined in [10] a mapping $\zeta_n = \zeta_n^A : Z(A) \longrightarrow Z(A)$ by $(z, a^{p^n})_A = (\zeta_n(z), a)_A^{p^n}$, for all $z \in HH^0(A)$ and $a \in HH_0(A)$.

The bilinear form $(\ ,\ )_A$ induces an identification $A \simeq Hom_k(A, k)$ as $A \otimes_k A^{op}$-bimodules.

In [20] we showed that $F^e(Hom_k(A, k)) \simeq Hom_k(B, k)$ and therefore we get an induced non degenerate symmetric bilinear form $(\ ,\ )_B$ on B, making B a symmetric algebra as well. In [21] we showed that ζ_n on A induces the mapping ζ_n on B by the derived equivalence F and F^e, if one chooses the symmetrizing form $(\ ,\ )_B$ on B.

We would like to mention a consequence which is implied by [21].

Corollary 2.1. *Suppose k is a perfect field of characteristic p, suppose G and H are finite groups, B_G is a block of kG with defect group D_G and suppose B_H is a block of kH of defect group D_H. If $D^b(kG) \simeq D^b(kH)$, then the exponent of D_G and the exponent of D_H coincide.*

Proof. This follows from the fact that the ideals $im(\zeta_n^{B_G})$ and $im(\zeta_n^{B_H})$ are mapped to each other by the isomorphism $Z(B_G) \simeq Z(B_H)$, and that by a result due to Külshammer [11, formulae (17), (47) and (78)], the exponent of D_G is the smallest integer n so that $im(\zeta_n^{B_G}) = im(\zeta_{n+1}^{B_G})$, and likewise for H. ∎

Remark 2.2. Suppose k is an algebraically closed field of characteristic p, suppose G and H are finite groups, suppose R is a complete discrete valuation domain of characteristic 0 with $R/rad(R) = k$ and field of fractions K. Suppose B_G is a block of RG with defect group D_G and suppose B_H is a block of RH of defect group D_H. If $D^b(B_G) \simeq D^b(B_H)$, then the orders of the defect groups coincide: $|D_G| = |D_H|$. The argument is the following construction[1] of Cliff, Plesken and Weiss [2]. Let B be a block of RG with defect group D and let Λ be the centre of B. Define inductively $\Lambda_0 := \Lambda$ and

$$\Lambda_{i+1} := \{x \in K\Lambda \mid x \cdot rad(\Lambda_i) \subseteq rad(\Lambda_i)\} \ .$$

Cliff, Plesken and Weiss show for algebraically closed fields k [2, Theorem 3.4] that

$$min\{s \mid \Lambda_s = \Lambda_{s+1}\} = |D|.$$

Külshammer gives a second, in some sense dual mapping κ_n defined by the equation $(z^{p^n}, a) = (z, \kappa_n(a))^{p^n}$, for $z \in HH^0(A)$, and for $a \in HH_0(A)$.

Proposition 2.3. *The mapping $\kappa_n^A : HH_0(A) \longrightarrow HH_0(A)$ is invariant under a derived equivalence $F : D^b(A) \longrightarrow D^b(B)$ in the sense that under the induced isomorphism $HH_0(F) : HH_0(A) \longrightarrow HH_0(B)$ one has*

$$HH_0(F) \circ \kappa_n^A \circ HH_0(F)^{-1} = \kappa_n^B$$

if one chooses the induced bilinear form $(\ ,\ )_B$ on B.

[1]This argument arose during a discussion with Gabriele Nebe in Aachen in October 2005. I am very grateful for her kind hospitality and for giving me this reference

Proof. Examining this relation, κ_n fits in the commutative diagram

$$\begin{array}{ccccc} A\otimes_{A\otimes_k A^{op}} A & \simeq & ((A\otimes_{A\otimes_k A^{op}} A)^*)^* & \simeq & Hom_k(Hom_{D^b(A\otimes_k A^{op})}(A,A),k) \\ & & & & \downarrow ((Fr^k)_*)^n \\ \uparrow \kappa_n & & & & Hom_k(Hom_{D^b(A\otimes_k A^{op})}(A,A),k^{(n)}) \\ & & & & \uparrow ((\mu_p)^*)^n \\ A\otimes_{A\otimes_k A^{op}} A & \simeq & ((A\otimes_{A\otimes_k A^{op}} A)^*)^* & \simeq & Hom_k(Hom_{D^b(A\otimes_k A^{op})}(A,A),k) \end{array}$$

where Fr is the Frobenius automorphism on k, and where

$$\mu_p : Z(A) \ni z \mapsto z^p \in Z(A).$$

Using $Z(A) = Hom_{A^e}(A,A)$, the mapping μ_p corresponds to

$$Hom_{A^e}(A,A) \ni f \mapsto \underbrace{f\circ f\circ\cdots\circ f\circ f}_{p \text{ factors}} \in Hom_{A^e}(A,A)\ .$$

Since F^e is a functor it is clear that $F^e(\mu_p^A) = \mu_p^B$. The rest of the proof is exactly analogous to the one in [21].

Hence F^e maps κ_n^A to κ_n^B with respect to the induced identification $B \simeq Hom_k(B,k)$ and the proof is finished. ■

This result will be generalised to higher Hochschild homology in Theorem 1. Though the proof there covers the present proposition it seems to be useful to have a short independent proof here.

We use Külshammer's description of the image and the kernel of κ_n to get a nice invariant of derived categories. Let $P_n(ZA) := < z^{p^n} \mid z \in Z(A) >_{k-\text{space}}$. Then

$$P_n(ZA)^\perp/KA = \{x \in A/KA \mid (z^{p^n}, x) = 0\ \forall z \in Z(A)\}$$

and

$$T_n(ZA)^\perp/KA = \{x \in A/KA \mid \forall\ z \in Z(A) : z^{p^n} = 0 \Rightarrow (z,x) = 0\}.$$

Now, $T_n(ZA)^\perp/KA$ is a $Z(A)$-submodule of the $Z(A)$-module A/KA. Indeed, for $x \in T_n(ZA)^\perp/KA$ and $y \in Z(A)$, one gets for any $z \in Z(A)$ with $z^{p^n} = 0$ also $(yz)^{p^n} = 0$ and so, $(z, yx) = (yz, x) = 0$ as well.

Corollary 2.4. *Let $F : D^b(A) \longrightarrow D^b(B)$ be an equivalence of standard type between the derived categories of the symmetric k-algebras A and B over a perfect field k. Then*

- *the isomorphism $HH_0(F) : A/KA \longrightarrow B/KB$ maps $P_n(ZA)^\perp/KA$ to $P_n(ZB)^\perp/KB$.*
- *the isomorphism $HH_0(F) : A/KA \longrightarrow B/KB$ maps $T_n(ZA)^\perp/KA$ to $T_n(ZB)^\perp/KB$ as submodules over the centres of the algebras.*

Proof. This is a consequence of Proposition 2.3, the fact that our isomorphisms are functorial, and hence preserve the natural structure of $A/KA = HH_0(A,A)$ as $HH^0(A,A) = Z(A)$-module, and the fact that the first module $P_n(ZA)^\perp/KA$ is the kernel of κ_n^A, whereas $T_n(ZA)^\perp/KA$ is the image of κ_n^A, as was shown by Külshammer [11, (52),(53)]. ■

Remark 2.5. Since the analogous statement for the, in some sense dual, mapping ζ proved to be extremely useful for distinguishing derived categories, such as the classification of tame domestic symmetric algebras by Holm-Skowroński [4] or to fix some of the open parameters in the derived equivalence classification of tame blocks of group rings [5], there is quite some hope that this corollary is as useful as was the method in [21]. Indeed, this new quite sophisticated invariant is explicitly computable in case the Hochschild homology is known as vector space, but additional structure is lacking.

A first step for the generalisation is the following lemma. Instead of taking degree 0 homology, one takes higher degree homology, and finds again a bilinear form.

Lemma 2.6. *Let A be a symmetric k-algebra with symmetrizing form $(\ ,\ )$. Then, there is a non degenerate bilinear form*

$$(\ ,\ )_m : HH^m(A,A) \otimes HH_m(A,A) \longrightarrow k$$

so that $(\ ,\ )_0 = (\ ,\ )|_{Z(A)\otimes A/KA}$. This form gives an isomorphism

$$Hom_k(HH^m(A,A),k) \simeq HH_m(A,A)$$

and any such k-linear isomorphism induces a non degenerate bilinear form.

Proof. Since $(-\otimes_{A\otimes A^{op}} A, Hom_k(A,-))$ is an adjoint pair of functors between k-vector spaces and $A\otimes A^{op}$-modules,

$$Hom_k(\mathbf{B}A \otimes_{A\otimes A^{op}} A, k) \simeq Hom_{A\otimes A^{op}}(\mathbf{B}A, Hom_k(A,k))$$

Taking homology of these complexes, and using that $Hom_k(-,k)$ is exact and contravariant, gives

$$\begin{array}{rcl} Hom_k(HH_m(A,A),k) & \simeq & Hom_k(H_m(\mathbf{B}A \otimes_{A\otimes A^{op}} A),k) \\ & \simeq & H^m(Hom_k(\mathbf{B}A \otimes_{A\otimes A^{op}} A,k)) \\ & \simeq & H^m(Hom_{A\otimes A^{op}}(\mathbf{B}A, Hom_k(A,k))) \\ & = & HH^m(A, Hom_k(A,k)) \end{array}$$

Since A is symmetric, $A \simeq Hom_k(A,k)$ as $A\otimes A^{op}$-modules and we get an isomorphism

$$HH^m(A,A) \xrightarrow{\varphi_n} Hom_k(HH_m(A,A),k)$$

as k-vector spaces. Now, put for any $f \in HH^m(A,A)$ and $x \in HH_m(A,A)$

$$(f,x)_m := (\varphi_m(f))(x)\ .$$

It is clear that $(\ ,\ )_m$ is bilinear since φ_m is k-linear. The form $(\ ,\ )_m$ is non degenerate since φ is an isomorphism. In case $m=0$ we find back the form $(\ ,\ )$ which was used to identify A with $Hom_k(A,k)$.

The last part of the statement is well known.

This finally proves the lemma. ■

3. The cup product κ

Now, $HH^*(A,A)$ carries a natural graded commutative ring structure given by cup product. We shall describe how this allows to define, just as in the construction of κ_n, a semilinear p-power map $\mu_p^{(m)}$ and by this means a higher degree $\kappa_n^{(m)} : HH_{p^n m}(A,A) \longrightarrow HH_m(A,A)$ with $\kappa_n = \kappa_n^{(0)}$.

Let $m \in 2\mathbb{N}$. Then, for any $n\in\mathbb{N}$ and any $x \in HH_{p^n m}(A,A)$ one gets a k-linear map

$$\begin{array}{rcl} HH^m(A,A) & \longrightarrow & k \\ f & \mapsto & Fr_p^{-n}\left((f^{p^n},x)_{p^n m}\right)\ . \end{array}$$

This is hence an element in $Hom_k(HH^m(A,A),k)$. Now, by Lemma 2.6

$$\begin{array}{rcl} HH_m(A,A) & \simeq & Hom_k(HH^m(A,A),k) \\ x & \mapsto & (f \mapsto (f,x)_m) \end{array}$$

Hence, for all $n\in\mathbb{N}$ and for all $x\in HH_{p^n m}(A,A)$ there is a unique $\kappa_n^{(m)}(x) \in HH_m(A,A)$ so that for all $f \in HH^m(A,A)$ one has

$$\left(f^{p^n},x\right)_{p^n m} = \left(\left(f,\kappa_n^{(m)}(x)\right)_m\right)^{p^n}.$$

We defined for all $n\in\mathbb{N}$ and all $m\in 2\mathbb{N}$ a mapping

$$\kappa_n^{(m),A} = \kappa_n^{(m)} : HH_{p^n m}(A,A) \longrightarrow HH_m(A,A)$$

so that $\kappa_n^{(0),A} = \kappa_n$. In case A is clear from the context, we denote $\kappa_n^{(m),A} = \kappa_n^{(m)}$.

Remark 3.1. Observe that since the Hochschild cohomology ring is *graded commutative*, for p odd the mapping

$$HH^{2m+1}(A,A) \ni f \mapsto f^p \in HH^{p\cdot(2m+1)}(A,A)$$

is 0 for $m \in \mathbb{N}$. Of course, the 0-mapping is semilinear as well. If $p=2$, then being graded commutative is just the same as being commutative, and so the restriction on m is not necessary.

Theorem 1. *Let A be a finite dimensional symmetric k-algebra over the field k of characteristic $p>0$. Let B be a second algebra so that $D^b(A) \simeq D^b(B)$ as triangulated categories. Let p be a prime and let $m \in \mathbb{N}$. Then, there is a standard equivalence $F : D^b(A) \simeq D^b(B)$, and any such standard equivalence induces an isomorphism $HH_m(F) : HH_m(A,A) \longrightarrow HH_m(B,B)$ of all Hochschild homology groups satisfying*

$$HH_m(F) \circ \kappa_n^{(m),A} \circ HH_{p^n m}(F)^{-1} = \kappa_n^{(m),B} .$$

Proof. By Remark 3.1 in case m odd we may assume $p=2$.

Again let $\mathbf{B}A$ be the bar resolution. Then the complex computing the Hochschild homology is $\mathbf{B}A \otimes_{A^e} A$. Since A is finite dimensional, we compute for any integer ℓ

$$\begin{aligned} H_\ell(\mathbf{B}A \otimes_{A^e} A) &\simeq Hom_k(H^\ell(Hom_k(\mathbf{B}A \otimes_{A^e} A, k)), k) \\ &\simeq Hom_k(H^\ell(Hom_{A^e}(\mathbf{B}A, Hom_k(A,k))), k) \end{aligned}$$

just using the standard adjointness formulas between hom and tensor functors. Finally, using the bilinear form (,) on A we get

$$\begin{aligned} H_\ell(\mathbf{B}A \otimes_{A^e} A) &\simeq Hom_k(H^\ell(Hom_{A^e}(\mathbf{B}A, Hom_k(A,k))), k) \\ &\simeq Hom_k(H^\ell(Hom_{A^e}(\mathbf{B}A, A)), k) \ . \end{aligned}$$

We discover

$$HH_\ell(A,A) \simeq Hom_k(Ext^\ell_{A^e}(A, Hom_k(A,k)), k) \simeq Hom_k(HH^\ell(A,A), k)$$

and get diagram

$$\begin{array}{ccccc} HH_{p^n m}(A,A) & \xrightarrow{\simeq} & Hom_k(Ext^{p^n m}_{A^e}(A, Hom_k(A,k)), k) & \xrightarrow{\simeq} & Hom_k(HH^{p^n m}(A,A), k) \\ & & & & {\scriptstyle ((\mu_p^{(m)})^n)^*} \downarrow \\ \downarrow \kappa_n^{(m)} & & & & Hom_k(HH^m(A,A), k^{(n)}) \\ & & & & {\scriptstyle (Fr_p^n)_*} \uparrow \\ HH_m(A,A) & \xrightarrow{\simeq} & Hom_k(Ext^m_{A^e}(A, Hom_k(A,k)), k) & \xrightarrow{\simeq} & Hom_k(HH^m(A,A), k) \end{array}$$

which we easily see to be commutative by what we observed previously.

We need to show that applying $HH_*(F)$ (resp. $HH^*(F)$) to the various mapping spaces for the Hochschild (co-)homology of A gives the analogous mapping for B.

Let $X \in D^b(A \otimes B^{op})$ be a two-sided tilting complex with inverse $Y \in D^b(B \otimes A^{op})$. We may and will assume that X and Y are complexes being projective on the left and projective on the right. Then, we may replace the left derived tensor product by the ordinary tensor product. Let

$$F_X := Y \otimes_A - \otimes_A X : D^b(A \otimes A^{op}) \longrightarrow D^b(B \otimes B^{op})$$

Then, F_X is a triangle equivalence, and in particular, $F_X(\mathbf{B}A)$ is a resolution of B in $D^b(B \otimes B^{op})$. So, F_X induces a commutative diagram

$$\begin{array}{ccc} Hom_k(HH_{p^n m}(A,A), k) & \longleftarrow & Ext^{p^n m}_{A^e}(A,A) \\ \downarrow & & \downarrow \\ Hom_k(HH_{p^n m}(B,B), k) & \longleftarrow & Ext^{p^n m}_{B^e}(B,B) \end{array}$$

where the bottom row is again given by the adjointness formula between Hom and tensor functors.

Moreover, $F_X(Hom_k(A,k)) = Hom_k(B,k)$ as was shown in [20]. This implies that an isomorphism $A \simeq Hom_k(A,k)$ induces an isomorphism $B \simeq Hom_k(B,k)$ as bimodules. Therefore, the induced diagram

$$\begin{array}{ccc} Hom_k(Ext^{p^n m}_{A^e}(A, Hom_k(A,k)),k) & \longrightarrow & Hom_k(Ext^{p^n m}_{A^e}(A,A),k) \\ \downarrow F_X & & \downarrow F_X \\ Hom_k(Ext^{p^n m}_{B^e}(B, Hom_k(B,k)),k) & \longrightarrow & Hom_k(Ext^{p^n m}_{B^e}(B,B),k) \end{array}$$

is commutative.

Now, we know that the cup product on $HH^*(A,A)$ is the composition of mappings in $Ext^*_{A^e}(A,A) = Hom_{D^b(A\otimes A^{op})}(A, A[*])$. Therefore, applying F_X to any of this composition of mappings is going to give again the composition of mappings on $Ext^*_{B^e}(B,B) = Hom_{D^b(B\otimes B^{op})}(B, B[*])$. Since the mapping $\mu_p^{(m)}$ is the k-linear dual of this, again F_X induces a commutative diagram

$$\begin{array}{ccc} Hom_k(HH^{p^n m}(A,A),k) & \stackrel{\mu_p^{(m),A}}{\longrightarrow} & Hom_k(HH^m(A,A),k^{(n)}) \\ \downarrow F_X & & \downarrow F_X \\ Hom_k(HH^{p^n m}(B,B),k) & \stackrel{\mu_p^{(m),B}}{\longrightarrow} & Hom_k(HH^m(B,B),k^{(n)}) \end{array}$$

Since F_X acts on the contravariant variable of $Hom_k(HH^m(A,A),k)$ and in the space of semilinear mappings $Hom_k(HH^m(A,A),k^{(n)})$, we get

$$\begin{array}{rl} Hom_k(F_X,k)\circ Hom_k(HH^m(A,A),(\mathrm{Fr}_k)) = & \\ = & Hom_k(HH^m(A,A),(\mathrm{Fr}_k))\circ Hom_k(F_X,k)\ . \end{array}$$

This shows the theorem. ■

We observe first properties analogous to those in Külshammer [11]. For this put

$$T_n^{(m)}(HH^m(A,A)) := \{x \in HH^m(A,A) \mid x^{p^n} = 0\}.$$

Proposition 3.2. *Suppose k is a perfect field of characteristic $p > 0$ and that A is a symmetric finite dimensional k-algebra. Then, denoting by $\perp_m$ the orthogonality with respect to the pairing $(\ ,\ )_m$,*

(1) *$\kappa_n^{(m)}$ is k-semilinear,*
(2) *$\kappa_{n+\ell}^{(m)} = \kappa_\ell^{(m)} \circ \kappa_n^{(p^\ell m)}$*
(3) *$im(\kappa_n^{(m)}) = \left(T_n^{(m)}\right)^{\perp_m}$.*
(4) *$ker\ \kappa_n^{(m)} = \{x^{p^n} \mid x \in HH^m(A,A)\}^{\perp_{p^n m}}$*

Proof. The first statement comes from the construction in the proof of Theorem 1 of κ as composition of semilinear mappings.

The second statement again is implied by the following argument.

$$\begin{array}{rcl} (f, \kappa_{n+\ell}^{(m)}(x))_m & = & (f^{p^{n+\ell}}, x)_{p^{n+\ell}m} \\ & = & (f^{p^\ell}, \kappa_n^{(p^\ell m)}(x))_{p^\ell m} \\ & = & (f, \kappa_\ell^{(m)}(\kappa_n^{(p^\ell m)}(x)))_m \end{array}$$

The third statement is shown as follows: The defining equation

$$(x^{p^n}, y)_{p^n m} = \left((x, \kappa_n^{(m)}(y))_m\right)^{p^n}$$

and the fact that $(\ ,\ )_{p^n m}$ is non degenerate show that $im(\kappa_n^{(m)})^{\perp_m} = T_n^{(m)}$. Since $(\ ,\ )_m$ is non degenerate, we may take the orthogonal spaces of these and get the result.

The fourth statement comes directly from the defining equation

$$(x^{p^n}, y)_{p^n m} = \left((x, \kappa_n^{(m)}(y))_m\right)^{p^n}$$

as well. ■

Remark 3.3. We see that the kernel and the image of $\kappa_n^{(m)}$ are very much linked to the set of nilpotent elements of the Hochschild cohomology. Snashall and Solberg conjectured [16] that the Hochschild cohomology ring of any finite dimensional algebra is finitely generated modulo the ideal generated by nilpotent elements.

Corollary 3.4. *We get* $dim(im(\kappa_n^{(m)})) = dim(HH^m(A, A)) - dim(T_n^{(m)})$.

Proof. This is an immediate consequence of the third statement of Proposition 3.2. ■

In general even degree Hochschild cohomology rings of symmetric algebras contain nilpotent elements, but are not necessarily entirely nilpotent. As an example I refer to the article Erdmann and Holm [3, Section 4] where Hochschild cohomology rings of self-injective Nakayama algebras, which includes the Hochschild cohomology of Brauer tree algebras, are computed. There nilpotent elements arise in even Hochschild degrees, though the even degree Hochschild cohomology modulo the nilpotent radical is not zero in general. In particular, $\kappa_n^{(m)}$ is neither zero nor surjective in general.

Remark 3.5. In the joint paper [1] with Bessenrodt and Holm we showed that for the degree zero Hochschild homology one may pass from a possibly non-symmetric algebra A to its trivial extension $\mathbb{T}A$. Rickard showed in [14] that whenever the algebras A and B are derived equivalent then also the trivial extension algebras $\mathbb{T}A$ and $\mathbb{T}B$ are derived equivalent. In degree 0 it is then possible to interpret the mappings κ and ζ on the degree 0 (co-)homology of $\mathbb{T}A$ in terms of A only. One might ask if an analogous construction is possible for $\kappa_n^{(m)}$ as well. The obvious fact that the Hochschild homology of A is a direct factor of the Hochschild homology of $\mathbb{T}A$ might give a natural definition. Nevertheless, there are quite a number of technical problems, such as the fact that Rickard gives a one-sided tilting complex only whereas a two-sided complex is needed for our method. Moreover, on a more practical level, in order to be able to compute $\kappa_n^{(m)}$ via the trivial extension method, one needs at the present stage at least parts of the multiplicative structure of the Hochschild cohomology of $\mathbb{T}A$. Even for rather small algebras A its trivial extension $\mathbb{T}A$ usually will have quite complicate cohomology. A significant simplification is needed and at the moment I do not see clearly how one can cope with these difficulties.

4. Stasheff-Quillen's construction of the Gerstenhaber structure and the Gerstenhaber κ

We recall first a most helpful construction of the Gerstenhaber bracket appearing in a slightly implicit fashion in Quillen [13] and very explicitly in Stasheff [17]. I learned the construction in discussions from Bernhard Keller [7, Section 4.7]. This construction shows that the Gerstenhaber bracket can be defined using a homological construction on the bar complex. For the reader's convenience we give the the construction in some detail.

4.1. **Stasheff-Quillen's construction.** Let

$$Coder(\mathbf{B}(A), \mathbf{B}(A)) := \{D \in End_{A\otimes A^{op}}(\mathbf{B}(A)) | \Delta \circ D = (id_{\mathbf{B}(A)} \otimes D + D \otimes id_{\mathbf{B}(A)}) \circ \Delta\}$$

be the coderivations. Since $\mathbf{B}(A)$ is graded, $Coder(\mathbf{B}(A), \mathbf{B}(A))$ is graded as well. Denote by $Coder^n(\mathbf{B}(A), \mathbf{B}(A))$ the degree n coderivations. The vector space $Coder(\mathbf{B}(A), \mathbf{B}(A))$ is a graded Lie algebra with Lie bracket being the commutator. Now (cf e.g. Stasheff [17, Proposition]),

$$Coder(\mathbf{B}(A), \mathbf{B}(A)) \simeq Hom_{A\otimes A^{op}}(\mathbf{B}A, A)[1] \ .$$

The isomorphism is induced by composing an $f \in Coder(\mathbf{B}(A), \mathbf{B}(A))$ with the projection τ on the degree 1 component A of $\mathbf{B}(A)$ so that $\gamma_A(f) := \tau \circ f \in Hom_{A\otimes A^{op}}(\mathbf{B}A, A)[1]$. So, there is a unique $d_A \in Coder^1(\mathbf{B}(A), \mathbf{B}(A))$ with $\tau \circ d_A = m_A$ for m_A being the multiplication map $m_A : A \otimes_k A \longrightarrow A$. m_A being associative is equivalent to $d_A^2 = 0$. Hence, $Coder(\mathbf{B}(A), \mathbf{B}(A))$ is a differential graded Lie algebra.

Lemma 4.1. *(Keller, personal communication)*

- *Suppose k is a field. Then,*
$$D \in Coder^{2n+1}(\mathbf{B}(A), \mathbf{B}(A)) \Rightarrow D^2 \in Coder^{2\cdot(2n+1)}(\mathbf{B}(A), \mathbf{B}(A)).$$
- *Suppose k is a field of characteristic $p > 0$. Then,*
$$D \in Coder^{2n}(\mathbf{B}(A), \mathbf{B}(A)) \Rightarrow D^p \in Coder^{2pn}(\mathbf{B}(A), \mathbf{B}(A)).$$

Proof of the first statement:

$$\begin{aligned}
\Delta \circ D^2 &= (id_{\mathbf{B}(A)} \otimes D + D \otimes id_{\mathbf{B}(A)})^2 \circ \Delta \\
&= id_{\mathbf{B}(A)} \otimes D^2 + (D \otimes id_{\mathbf{B}(A)})(id_{\mathbf{B}(A)} \otimes D) + (id_{\mathbf{B}(A)} \otimes D)(D \otimes id_{\mathbf{B}(A)}) + D^2 \otimes id_{\mathbf{B}(A)} \\
&= id_{\mathbf{B}(A)} \otimes D^2 - D \otimes D + D \otimes D + D^2 \otimes id_{\mathbf{B}(A)}
\end{aligned}$$

Proof of the second statement:

$$\begin{aligned}
\Delta \circ D^p &= (id_{\mathbf{B}(A)} \otimes D + D \otimes id_{\mathbf{B}(A)})^p \circ \Delta \\
&= \left(id_{\mathbf{B}(A)} \otimes D^p + \left(\sum_{j=1}^{p-1} \binom{p}{j} \cdot (D^j \otimes D^{p-j}) \right) + D^p \otimes id_{\mathbf{B}(A)} \right) \circ \Delta \\
&= \left(id_{\mathbf{B}(A)} \otimes D^p + D^p \otimes id_{\mathbf{B}(A)} \right) \circ \Delta
\end{aligned}$$

■

Lemma 4.2. *Let k be a field of characteristic p. Let $D \in Coder^n(\mathbf{B}(A), \mathbf{B}(A))$.*

(1) *If $p = 2$ and $n \in \mathbb{N}$, then the mapping $D \mapsto D^2$ induces a mapping*
$$HH^{n+1}(A, A) \longrightarrow HH^{2n+1}(A, A)$$
(2) *If $p > 2$ and $n = 2m \in 2\mathbb{N}$, then the mapping $D \mapsto D^p$ induces a mapping*
$$HH^{2m+1}(A, A) \longrightarrow HH^{2pm+1}(A, A)$$

Proof. Let $D \in Coder^n(\mathbf{B}(A), \mathbf{B}(A))$. Then, $D^p \in Coder^{pn}(\mathbf{B}(A), \mathbf{B}(A))$. The differential in the Hochschild cohomology complex $Hom_{A^e}(\mathbf{B}A, A)$ corresponds to the commutator $[d_A, -]$ where $Coder^1(\mathbf{B}(A), \mathbf{B}(A)) \ni d_A$ comes from $\tau \circ d_A = m_A : A \otimes A \longrightarrow A$ being the multiplication in the algebra A. We would like to show that the p-power operation induces a genuine operation on Hochschild cohomology.

For this, it is immediate that
$$[d_A, D] = 0 \Rightarrow [d_A, D^p] = 0.$$
Hence, the p-power operation induces an operation on the cycles of the Hochschild cohomology complex.

We need to show moreover that for any $E \in Coder^{n-1}(A, A)$ one has
$$(D + [d_A, E])^p \in D^p + im([d_A, -]).$$

(1) Let $p = 2$. Then, since $d_A^2 = 0$, we get $[d_A, E]^2 = [d_A, E[d_A, E]]$ and therefore
$$(D + [d_A, E])^2 = D^2 + D[d_A, E] + [d_A, E]D + [d_A, E[d_A, E]].$$
We need to show that $D[d_A, E] + [d_A, E]D \in im([d_A, -])$. But,
$$D[d_A, E] + [d_A, E]D = [d_A, [D, E]] + [[d_A, D], E]$$

and whenever D is a Hochschild cocycle, then $[d_A, D] = 0$ and therefore

$$(D + [d_A, E])^2 = D^2 + [d_A, [D, E] + E[d_A, E]] .$$

This shows the statement for $p = 2$.

(2) Let $p > 2$. Again using that $d_A^2 = 0$ one sees that for any positive integer n one has

$$[d_A, E]^n = (d_A E)^n - \left(\sum_{j=1}^{n-1} (d_A E)^j (E d_A)^{n-j} \right) + (-1)^n (E d_A)^n$$

But, using as well that $d_A^2 = 0$, for the n-fold Lie-bracket one gets the same result

$$[d_A, E[d_A, \dots, E[d_A, E] \dots]] = (d_A E)^n - \left(\sum_{j=1}^{n-1} (d_A E)^j (E d_A)^{n-j} \right) + (-1)^n (E d_A)^n$$

So, there is an element $X_n(E)$ with $[d_A, E]^n = [d_A, X_n(E)]$ and hence $[d_A, E]^p \in im([d_A, -])$. Moreover, just as in the case $p = 2$, for any Hochschild cocycle D one has $[d_A, D] = 0$ and one gets

$$(D + [d_A, E])^p \in D^p + im([d_A, -]) .$$

This finishes the proof. ∎

4.2. **The p-restricted Lie structure and its derived invariance.** We recall the definition of a restricted Lie algebra (cf e.g.[18, Chapter 2 Section 1]).

Definition 1. Let L be a Lie algebra over a field k of characteristic $p > 0$. Denote

$$(ad\ (a \otimes X + b \otimes 1))^{p-1} (a \otimes 1) = \sum_{i=1}^{p-1} i \cdot s_i(a, b) \otimes X^i \in L \otimes k[X].$$

A mapping $[p] : L \longrightarrow L$ is called a *p-mapping* if

(1) $ad\ a^{[p]} = (ad\ a)^p$
(2) $(\alpha a)^{[p]} = \alpha^p a^{[p]}$
(3) $(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b)$

A Lie algebra L together with a p-mapping $[p]$ is then called a *p-restricted Lie algebra.*

The following proposition should be well known to the experts, but I could not find a reference. So, I include the short proof.

Proposition 4.3.
- *For any field k of characteristic 2 the differential graded Lie algebra $Coder^*(\mathbf{B}(A), \mathbf{B}(A))$ and its homology $HH^{*+1}(A, A)$ are 2-restricted Lie algebras under the Gerstenhaber construction.*
- *For any field k of characteristic $p > 2$ the sum of the odd degree Hochschild cohomology groups $\bigoplus_{n \in \mathbb{N}} HH^{2n+1}(A, A)$ is a p-restricted Lie algebra under the Gerstenhaber construction.*

Proof. As we have seen in Lemma 4.1, for $p = 2$ the square of any coderivation is a coderivation. Moreover by Lemma 4.1, for any prime p the p-power of an even degree coderivation is a coderivation. So, the Gerstenhaber p-power induces a mapping

$$Hom_{A^e}((\mathbf{B}A)^{2m+1}, A) \longrightarrow Hom_{A^e}((\mathbf{B}A)^{2pm+1}, A).$$

Let us show the first property $(ad\ a)^p = ad\ a^{[p]}$. The Lie structure on the Hochschild cohomology is given by the commutator bracket on the coderivations $Coder(\mathbf{B}(A), \mathbf{B}(A))$.

The mapping $a \mapsto a^{[p]}$ is given by taking the ordinary composition of mappings. Hence,

$$\begin{aligned}(ad\ a)^p(y) &= \underbrace{[a,[a,\ldots,[a,}_{p \text{ factors}}\ y\underbrace{]\ldots]]}_{p} \\ &= [a^p, y] \\ &= \left(ad\ a^{[p]}\right)(y)\end{aligned}$$

The second property is trivial, since $(\alpha a)^p = \alpha^p a^p$ in $Coder(\mathbf{B}(A), \mathbf{B}(A))$.

The third property is done exactly analogously to the first example following Lemma 1.2 in [18, Chapter 2]. ∎

Proposition 4.4. *Let A and B be k-algebras over a field k. Suppose $D^b(A) \simeq D^b(B)$ as triangulated categories.*

- *If the characteristic of k is 2 then $HH^*(A,A)$ and $HH^*(B,B)$ are isomorphic as restricted Lie algebras.*
- *If the characteristic of k is $p > 2$, then the Lie algebras consisting of odd degree Hochschild cohomologies $\bigoplus_{n\in\mathbb{N}} HH^{2n+1}(A,A)$ and $\bigoplus_{n\in\mathbb{N}} HH^{2n+1}(B,B)$ are isomorphic as restricted Lie algebras.*

Proof. The fact that the Gerstenhaber structure is preserved is shown by Keller in [7].

We need to show that the isomorphism maps the p-power maps to each other. Let $X \in D^b(A \otimes_k B^{op})$ be a twosided tilting complex which we will assume to be formed by modules projective on either side. Let $Y = Hom_k(X,k) \in D^b(B \otimes_k A^{op})$ be the inverse complex.

We first suppose $p = 2$. Then, as above, the functor

$$Y \otimes_A - \otimes_A X : D^b(A \otimes_k A^{op}) \longrightarrow D^b(B \otimes_k B^{op})$$

is an equivalence. So, this functor induces an isomorphism

$$\begin{array}{rcl}\varphi_X : Hom_{A^e}(\mathbf{B}A, A) & \longrightarrow & Hom_{B^e}(Y \otimes_A \mathbf{B}A \otimes_A X, Y \otimes_A A \otimes_A X) \\ & & \| \\ & & Hom_{B^e}(\mathbf{B}B, B)\end{array}$$

since $Y \otimes_A \mathbf{B}A \otimes_A X$ is a projective resolution of B in the category of $B \otimes B^{op}$-modules, and therefore homotopy equivalent to $\mathbf{B}B$, and since $Y \otimes_A A \otimes_A X \simeq B$ in $D^b(B \otimes_k B^{op})$. We know by the discussion in Section 4.1 that there is an isomorphism γ_A

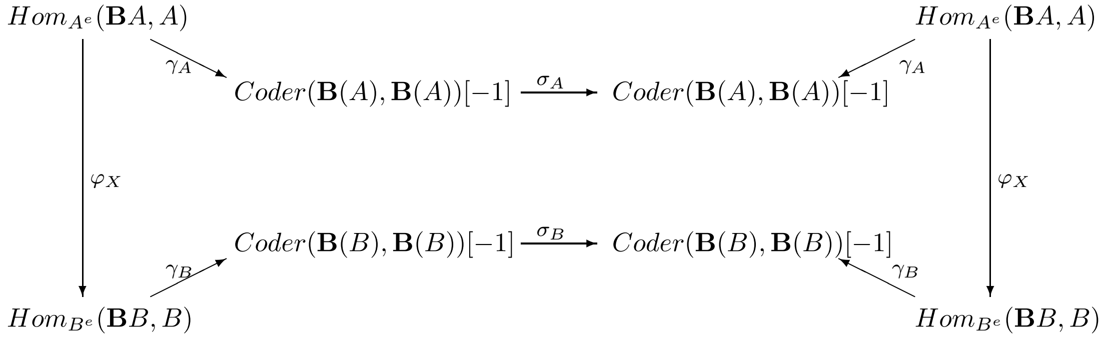
$$\begin{array}{rcl}Coder(\mathbf{B}(A), \mathbf{B}(A))[-1] & \stackrel{\gamma_A}{\longrightarrow} & Hom_{A\otimes_k A^{op}}(\mathbf{B}A, A) \\ D & \mapsto & \tau \circ D\end{array}$$

where τ is the projection mapping $\mathbf{B}(A) \longrightarrow A$ on the degree 1 component A of $\mathbf{B}(A)$. Denote by σ_A the square map on $Coder(\mathbf{B}(A), \mathbf{B}(A))$ and likewise by σ_B the squaring on $Coder(\mathbf{B}(B), \mathbf{B}(B))$.

We need to show that

$$\gamma_B^{-1} \circ \sigma_B \circ \gamma_B = \varphi_X \circ \gamma_A^{-1} \circ \sigma_A \circ \gamma_A \circ \varphi_X^{-1}$$

or in other words that the diagram below is commutative.



But this is obvious.

Suppose now $p > 2$. Then, the p-power operation σ_p is only defined on the space $Coder^{2\mathbb{N}}(\mathbf{B}(A), \mathbf{B}(A))$. Restricting therefore to only odd degree Hochschild cocycles, the same proof then holds.

This finishes the proof of the Proposition. ■

Remark 4.5. (1) In [8] Bernhard Keller shows that a derived equivalence of standard type even preserves the structure of $Hom_{A^e}(\mathbf{B}A, A)$ as B_∞-algebra.
(2) Since there is no obvious reason why the p-power mapping should be additive in general, neither semilinear, it seems to be difficult to get an analogue for Gerstenhaber structures to Külshammer's mappings κ_n.

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