

**CHEAPEST IDENTIFICATION EXPERIMENT
WITH GUARANTEED ACCURACY IN THE
PRESENCE OF UNDERMODELING**

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Abstract: This paper considers a recently introduced paradigm for optimal identification experiment design and extends the results to the case of an identification in a model structure which does not contain the true system.

Keywords: experiment design, prediction error identification for control

1. INTRODUCTION

A controller for a real-life system G_0 is usually designed on the basis of a model of G_0 identified using data collected from the true system. When designing the identification experiment, the control engineer often has to make a trade-off between her/his desire of obtaining an accurate model and the economical constraint of keeping the experimental costs low. Obtaining an accurate model requires a long identification experiment and a powerful input signal, while keeping the experimental costs low corresponds to a short experiment time and the excitation of G_0 with a low power signal.

The typical approach to this problem has been to maximize the accuracy of the identified model (possibly with a given, say, control-oriented objective in mind) for a given experiment time and under prespecified constraints on input power (see e.g. (Zarrop, 1979; Ljung, 1999) and references therein). In recent contributions (see e.g. (Bombois *et al.*, 2004b; Bombois *et al.*, 2004a;

Jansson and Hjalmarsson, 2004; Barenthin *et al.*, 2005)), this tradeoff has been addressed from the dual perspective. Indeed, assuming that the experiment time N is fixed, the optimal (open-loop) identification experiment is defined as the experiment on G_0 whose input signal power \mathcal{P}_u is minimized under the constraint that the modeling error $|G_0(e^{j\omega}) - G(e^{j\omega}, \hat{\theta}_N)|$ between the identified model $G(e^{j\omega}, \hat{\theta}_N)$ and the true system $G_0(e^{j\omega})$ is guaranteed to remain below some pre-specified threshold $r_{adm}(\omega)$. In (Bombois *et al.*, 2004b; Bombois *et al.*, 2004a), it is shown how this threshold $r_{adm}(\omega)$ can e.g. be chosen in order to guarantee that the controller designed from the identified model $G(z, \hat{\theta}_N)$ is guaranteed to achieve sufficient performance with the unknown true G_0 . A more detailed comparison between the two approaches for optimal experiment design can be e.g. found in (Gevers and Bombois, 2006).

In the sequel of the paper, we focus on the novel approach for optimal experiment design. Until now, this novel approach has been treated assuming that the modeling error is only due to variance effects i.e. assuming that the model

$G(z, \hat{\theta}_N)$ is identified in a full-order model structure ($\mathcal{S} \in \mathcal{M}$). In this particular case, a probabilistic bound $\alpha(\omega, u)$ on the achievable modeling error $|G_0(e^{j\omega}) - G(e^{j\omega}, \hat{\theta}_N)|$ given an input signal $u(t)$ is obtained a-priori as a function of $u(t)$ via the expression of the variance of the identified model. Thus, the optimal input signal $u(t)$ ($t = 1..N$) is given by:

$$\begin{aligned} & \arg \min_{u(t)} \mathcal{P}_u \\ & \text{subject to } \alpha(\omega, u) < r_{adm}(\omega) \quad \forall \omega \end{aligned} \quad (1)$$

with \mathcal{P}_u the total power of $u(t)$.

In this paper, we extend these results to the case where the identification takes place in a reduced order model structure i.e. a model structure that does not contain the true system ($\mathcal{S} \notin \mathcal{M}$). In this case, the modeling error is made up of two contributions: a variance contribution due to the noise (such as in (1)) and a bias contribution due to the undermodeling. If the model structure is chosen linear in the parameter vector, both contributions can be a-priori estimated by upper-bounds $\alpha(\omega, u)$ and $\beta(\omega, u)$ depending on the to-be-determined input signal (Hakvoort and den Hof, 1997). This leads to the following experiment design problem:

$$\begin{aligned} & \arg \min_{u(t)} \mathcal{P}_u \\ & \text{subject to } \alpha(\omega, u) + \beta(\omega, u) < r_{adm}(\omega) \quad \forall \omega \end{aligned} \quad (2)$$

Due to the relative complexity of the relation between the bias error term $\beta(\omega, u)$ and the to-be-determined input signal $u(t)$, the techniques developed to solve (1) can unfortunately not be transposed to (2). We nevertheless propose a methodology to determine the optimal input signal when the class of input signals, in which the optimization (2) takes place, is restricted to the class of PRBS input signals. Consequently, the optimization (2) is performed on the amplitude and the clock period of the PRBS signal.

Returning now to the global picture of optimal experiment design (i.e. the classical and the novel approach), let us note that this paper is, to our knowledge, the first contribution on optimal experiment design which deals with undermodeling using exact formulas for the modeling error. Another contribution for optimal experiment design with $\mathcal{S} \notin \mathcal{M}$ is presented in (Ljung, 1999)[Chapter 4.5], but that result is based on an approximation of the variance error which is only valid for systems of infinite order.

2. PE IDENTIFICATION ASPECTS

We consider the identification of a linear time-invariant single input single output system:

$$\mathcal{S} : y(t) = G_0(z)u(t) + e(t) \quad (3)$$

Here, to simplify the notations, the additive noise $e(t)$ is assumed to be the realization of a white noise of variance σ_e^2 . However, colored noise can be treated as well (see (Hakvoort and den Hof, 1997)). As proposed in (Hakvoort and den Hof, 1997), the reduced-order model structure that we will use for the identification is linear in the parameter i.e.

$$\mathcal{M} = \{G(z, \theta) = \Lambda(z)\theta \mid \theta \in \mathbf{R}^{k \times 1}\} \quad (4)$$

where $\Lambda(z) = (\Lambda_1(z) \Lambda_2(z) \dots \Lambda_k(z))$ is a row vector containing the first k elements of a series of basis functions $\Lambda_i(z)$ ($i = 1..k$). Examples of those basis functions are the FIR basis with $\Lambda_i(z) = z^{-i}$ and the Laguerre basis with $\Lambda_i(z) = z^{-1} \frac{\sqrt{1-\xi^2}}{1-\xi z^{-1}} \left(\frac{-\xi+z^{-1}}{1-\xi z^{-1}} \right)^{i-1}$ for some pole ξ .

Without loss of generality, the true transfer function $G_0(z)$ can be written as an infinite expansion of the chosen series of basis functions $\Lambda_i(z)$ ($i = 1..k$): $G_0(z) = \sum_{i=1}^{\infty} g_0(i) \Lambda_i(z)$. Consequently, $G_0(z)$ can be divided into the sum of a model lying in \mathcal{M} and an undermodeling part $G_0^{tail}(z)$:

$$G_0(z) = \Lambda(z)\theta_0 + \overbrace{\sum_{i=k+1}^{\infty} g_0(i) \Lambda_i(z)}^{G_0^{tail}(z)} \quad (5)$$

with $\Lambda(z)$ as in (4) and $\theta_0 \in \mathbf{R}^{k \times 1} \triangleq (g_0(1) g_0(2) \dots g_0(k))^T$.

To identify a model $G(z, \hat{\theta}_N) = \Lambda(z)\hat{\theta}_N$ of the true system $G_0(z)$, an input signal $u(t)$ ($t = 1..N$) of length N is applied to (3) and the corresponding output signal $y(t)$ ($t = 1..N$) is measured. Based on this IO data set, the parameter vector $\hat{\theta}_N$ of the identified model is computed as follows: $\hat{\theta}_N = R^{-1} \frac{1}{N} \sum_{t=1}^N \phi(t)y(t)$ with $R = \frac{1}{N} \sum_{t=1}^N \phi(t)\phi^T(t)$ and $\phi^T(t) = \Lambda(z)u(t)$.

As said in the introduction, our objective in this paper is, for a fixed data length N , to determine the optimal input signal $u(t)$ for the identification. The optimal input signal is here defined as the one solving the experiment design problem presented in the introduction. This experiment design problem uses an upper bound on the modeling error as a function of the input signal $u(t)$. In the identification framework presented above, this bound can be determined provided we possess the following prior information on the true system (3) (Hakvoort and den Hof, 1997) (see that paper for a discussion on how this prior information can be obtained).

Assumption 1. Consider the true system (3) and the expression (5) for $G_0(z)$. We possess reliable

upper bounds $\bar{\sigma}_e^2$, $\bar{g}(i)$ for the unknown variance σ_e^2 and for the unknown coefficients $g_0(i)$ in $G_0^{tail}(z)$ i.e. $\sigma_e^2 \leq \bar{\sigma}_e^2$ and $|g_0(i)| < \bar{g}(i) \forall i \geq k+1$. Moreover, we assume that the series of coefficients $\bar{g}(i)$ at least die at an exponential rate in i .

Based on this assumption, we now deduce a computable upper bound on the achievable modeling error $G_0(z) - G(z, \hat{\theta}_N)$ and its relation with the chosen input signal $u(t)$. For this purpose, observe that, in the framework presented above,

$$G_0(z) - G(z, \hat{\theta}_N) = G_0^{tail}(z) - \Lambda(z) \left(R^{-1} \frac{1}{N} \sum_{t=1}^N \phi(t) (x(t) + e(t)) \right) \quad (6)$$

with $x(t) = G_0^{tail}(z)u(t)$. The modeling error $G_0(z) - G(z, \hat{\theta}_N)$ can here be clearly divided into two different additive contributions: a contribution $\Delta_1(z)$ due to the noise $e(t)$ (the so-called variance error) and a contribution $\Delta_2(z)$ due to the undermodeling (the so-called bias error):

$$\begin{aligned} \Delta_1(z) &= -\Lambda(z) \left(R^{-1} \frac{1}{N} \sum_{t=1}^N \phi(t) e(t) \right) \\ \Delta_2(z) &= G_0^{tail}(z) - \Lambda(z) \left(R^{-1} \frac{1}{N} \sum_{t=1}^N \phi(t) x(t) \right) \end{aligned} \quad (7)$$

As can be seen in (7), the variance error $\Delta_1(z)$ is equal to $-\Lambda(z)\epsilon$ with $\epsilon \in \mathbf{R}^{k \times 1}$ a zero-mean random variable with covariance matrix $P_\epsilon = \frac{\sigma_e^2}{N} R^{-1}$. In the expression of P_ϵ , the only unknown element is σ_e^2 . However, based on Assumption 1, we see that $P_\epsilon \leq \frac{\bar{\sigma}_e^2}{N} R^{-1}$. Based on this, we can use classical results (see e.g. (Bombois *et al.*, 2005)) to determine an upper bound for the modulus of the variance error at each ω . This upper bound is valid up to a certain user-chosen probability level:

$$|\Delta_1(e^{j\omega})| < \sqrt{\frac{\bar{\sigma}_e^2 \chi}{N}} \lambda_1(T(\omega) R^{-1} T^T(\omega)) \quad (8)$$

with $T(\omega) = (Re(\Lambda^T(e^{j\omega})) Im(\Lambda^T(e^{j\omega})))^T$, χ a constant depending on the chosen probability level and $\lambda_1(A)$ the largest eigenvalue of A . The quantity $\alpha(\omega, u)$ is dependent on the input signal $u(t)$ used during the identification through the matrix R .

A bound on the bias error $\Delta_2(z)$ will now be deduced. Using the expression of $G_0^{tail}(z)$ in (5), the bias error $\Delta_2(z)$ can be rewritten as:

$$\Delta_2(z) = \sum_{i=k+1}^{\infty} g_0(i) \underbrace{\left(\Lambda_i(z) - \Lambda(z) \left(R^{-1} \frac{1}{N} \sum_{t=1}^N \phi(t) x_i(t) \right) \right)}_{\mathcal{B}_i(z, u)} \quad (9)$$

with $x_i(t) = \Lambda_i(z)u(t)$. The bias error is thus an expansion of transfer functions $\mathcal{B}_i(z, u)$ dependent on $u(t)$ through $x_i(t) = \Lambda_i(z)u(t)$, $\phi(t)$ and R . Using (9) and Assumption 1, an upper bound for the real part of $\Delta_2(e^{j\omega})$ is given by $\beta_R(\omega, u) = \sum_{i=k+1}^{\infty} \bar{g}(i) |Re(\mathcal{B}_i(e^{j\omega}, u))|$ and an upper bound for its imaginary part by $\beta_I(\omega, u) = \sum_{i=k+1}^{\infty} \bar{g}(i) |Im(\mathcal{B}_i(e^{j\omega}, u))|$. Both bounds can be calculated to within arbitrary accuracy due to the exponential decay rate of $\bar{g}(i)$ (Hakvoort and den Hof, 1997). Consequently, a computable upper bound for the modulus $|\Delta_2(e^{j\omega})|$ of the bias error is:

$$|\Delta_2(e^{j\omega})| \leq \sqrt{\overbrace{\beta_R^2(\omega, u) + \beta_I^2(\omega, u)}^{\beta(\omega, u)}} \quad (10)$$

Note that this upper bound $\beta(\omega, u)$ is a slightly less conservative upper bound than the one in the original paper (Hakvoort and den Hof, 1997) where the bias error (9) is divided into two contributions which are subsequently bounded separately; introducing an unnecessary conservatism.

Combining (8) and (10) and using the triangle inequality, we obtain that, modulo an user-chosen probability level,

$$|G_0(e^{j\omega}) - G(e^{j\omega}, \hat{\theta}_N)| \leq \alpha(\omega, u) + \beta(\omega, u) \quad \forall \omega. \quad (11)$$

This bound deduced from the data is made up of the sum of two terms. As already mentioned earlier, both terms are dependent on the input signal $u(t)$ used during the identification; but in different ways. The dependence of the bias term $\beta(\omega, u)$ on $u(t)$ is only relative. Indeed, the bias term remains unchanged if the signal $u(t)$ is multiplied by any constant scaling factor σ i.e. $\beta(\omega, \sigma u) = \beta(\omega, u) \forall \sigma \in \mathbf{R}^+$. Unlike $\beta(\omega, u)$, the dependence of the variance term $\alpha(\omega, u)$ is not only relative. Indeed, when the signal $u(t)$ is multiplied by a scaling factor σ , the variance error term is multiplied by $1/\sigma$: $\alpha(\omega, \sigma u) = \frac{1}{\sigma} \alpha(\omega, u)$. These two important properties for the sequel of this paper follow from (8), (10) and (9) combined with the definitions of R , $\phi(t)$, and $x_i(t)$.

Note finally that the bound in (11) can be computed before the identification experiment based on the prior information given in Assumption 1 and the signal $u(t)$ ($t = 1 \dots N$) that will be used for the identification.

3. EXPERIMENT DESIGN PROBLEM

As said in the introduction, our objective in this paper is, for a fixed data length N , to determine the least powerful input signal for the identification of a model of the true system while guaranteeing that the modeling error remains at each frequency below a pre-specified threshold $r_{adm}(\omega)$.

In the framework defined in Section 2, this constraint on the modeling error can be expressed as follows:

$$\beta(\omega, u) + \alpha(\omega, u) < r_{adm}(\omega) \quad \forall \omega. \quad (12)$$

with $\alpha(\omega, u)$ and $\beta(\omega, u)$ as given in (8) and in (10), respectively.

The experiment design problem becomes therefore:

$$\begin{aligned} & \arg \min_{u(t) \in \mathcal{U}} \mathcal{P}_u \\ & \text{subject to (12)} \end{aligned} \quad (13)$$

with \mathcal{U} a parametrized class of input signals which is preferably the broadest possible.

Due to complexity of the relation between $u(t)$ (and/or its spectrum $\Phi_u(\omega)$) and the bias error contribution $\beta(\omega, u)$, the techniques developed to solve this experiment design problem in the case of a full order model structure (i.e. $\beta(\omega, u) = 0$) and the classes \mathcal{U} used in these techniques can unfortunately not be transposed to (13).

We propose nevertheless a methodology to design our input signal in an optimal way. For this purpose, it is required to restrict the class \mathcal{U} of $u(t)$, in which the optimization (13) takes place, to a class with a very limited amount of optimization parameters. Among these, we here focus on a very widely-used and rather broad type of input signals: the class of pseudo-random binary sequences PRBS (Ljung, 1999). A PRBS signal is a finite-length deterministic signal characterized by only two parameters: its amplitude σ and its clock period ν . A PRBS signal $u(t)$ of amplitude σ is a binary signal taking its value uniquely in $\{+\sigma, \sigma\}$. The clock period ν is an integer ≥ 1 indicating the periodicity of the switches between σ or $-\sigma$: a PRBS signal with clock period $\nu = 1$ is allowed to change its value at each time sample while a PRBS signal with a clock period $\nu > 1$ is constant for ν samples.

A PRBS signal $u(t)$ ($t = 1 \dots N$) of amplitude σ and clock period ν has a total power $\mathcal{P}_u = \frac{1}{N} \sum_{t=1}^N u^2(t) = \sigma^2$ and a power spectrum $\Phi_u(\omega)$ approximately given by $\frac{\sigma^2}{\nu} \frac{1 - \cos(\nu\omega)}{1 - \cos(\omega)}$. Based on this expression, we see that a PRBS signal with $\nu = 1$ has (approximately) the flat spectrum of a white noise and that, for increasing values of ν , there is a shift of signal power to the low frequent part as can be seen in Figure 1.

To sum up, the clock period ν of the PRBS signal allows to shape the frequency content of the power spectrum $\Phi_u(\omega)$ of $u(t)$, without changing its total power \mathcal{P}_u and the amplitude σ of a PRBS signal allows to adapt the total power level \mathcal{P}_u of $u(t)$ without changing its frequency content.

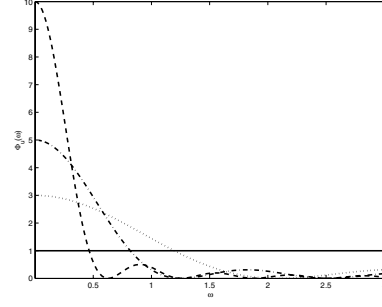


Fig. 1. $\frac{1}{\nu} \frac{1 - \cos(\nu\omega)}{1 - \cos(\omega)}$ for $\nu = 1$ (solid), for $\nu = 3$ (dotted), for $\nu = 5$ (dashdot) and for $\nu = 10$ (dashed)

Based on the considerations above and denoting $p_\nu(t)$ ($t = 1 \dots N$) the PRBS signal¹ of clock period ν and amplitude $\sigma = 1$, the class of all PRBS signals $u(t)$ of length N is given by:

$$\mathcal{U} = \{u(t) = \sigma p_\nu(t) \mid \sigma \in \mathbf{R}^+ \text{ and } \nu \in \mathbf{N} \geq 1\} \quad (14)$$

Since we restrict attention to the class \mathcal{U} of PRBS input signals and since $\mathcal{P}_u = \sigma^2$ for that type of signals, solving experiment design problem (13) is equivalent to determining the clock period and the amplitude of the PRBS signal $u(t) = \sigma p_\nu(t)$ ($t = 1 \dots N$) with the smallest amplitude σ which still guarantees the constraint (12). In the sequel, we present a methodology to solve this experiment design problem.

If we denote, for each value of ν , by $\sigma_{opt}(\nu)$ the minimal value for the amplitude of a PRBS signal of clock period ν which guarantees (12) i.e.

$$\begin{aligned} & \sigma_{opt}(\nu) = \arg \min_{\sigma} \sigma \\ & \text{subject to } \beta(\omega, \sigma p_\nu) + \alpha(\omega, \sigma p_\nu) < r_{adm}(\omega) \quad \forall \omega \end{aligned} \quad (15)$$

Then, the solution of the experiment design problem (13) within the class of PRBS signals is a PRBS signal $u_{opt}(t) = \sigma_{opt} p_{\nu_{opt}}(t)$ with a clock period ν_{opt} and an optimal amplitude σ_{opt} given respectively by:

$$\begin{aligned} \nu_{opt} &= \arg \min_{\nu} \sigma_{opt}(\nu) \\ \sigma_{opt} &= \sigma_{opt}(\nu_{opt}) \end{aligned} \quad (16)$$

The next proposition gives a methodology to compute the quantity $\sigma_{opt}(\nu)$ which is necessary to determine ν_{opt} and σ_{opt} as shown in (16).

Proposition 1. Consider a fixed clock period ν and generate the unitary-amplitude PRBS signal $p_\nu(t)$ ($t = 1 \dots N$) corresponding to this clock period ν . Then, the solution $\sigma_{opt}(\nu)$ of (15) is given by:

¹ More precisely, $p_\nu(t)$ is defined as the (possibly truncated) PRBS generated in Matlab by `idinput(N,'prbs',[0,(1/ν)])`. This precision ensures the uniqueness of $p_\nu(t)$ (given ν).

$$\sigma_{opt}(\nu) = \sup_{\omega} \frac{\alpha(\omega, p_{\nu})}{r_{adm}(\omega) - \beta(\omega, p_{\nu})} \quad (17)$$

where $\alpha(\omega, p_{\nu})$ and $\beta(\omega, p_{\nu})$ are the variance and bias terms (8) and (10) computed with $u(t) = p_{\nu}(t)$ ($t = 1 \dots N$).

Proof. Based on the remark at the end of Section 2, the constraint $\beta(\omega, u) + \alpha(\omega, u) < r_{adm}(\omega)$ for $u(t) = \sigma p_{\nu}(t)$ (see (15)) can be rewritten as follows:

$$\beta(\omega, p_{\nu}) + \frac{1}{\sigma} \alpha(\omega, p_{\nu}) < r_{adm}(\omega) \quad (18)$$

It is then obvious that the minimal amplitude σ for which (18) holds at each ω is given by (17). ■

Proposition 1 gives a methodology to compute the optimal amplitude $\sigma_{opt}(\nu)$ for a given value of the clock period ν . This optimal amplitude $\sigma_{opt}(\nu)$ is typically different for different values of ν . Using Proposition 1, this quantity can be computed for all possible values of the integer ν . The clock period and the amplitude of the signal $u(t) \in \mathcal{U}$ solving the experiment design problem (13) can thereafter be determined using (16). Note that, due to the typical absence of local minima in the function $\sigma_{opt}(\nu)$, the optimization problem $\nu_{opt} = \arg \min_{\nu} \sigma_{opt}(\nu)$ can be generally tackled using a classical dichotomy technique; therefore preventing us from the rather cumbersome need to compute $\sigma_{opt}(\nu)$ for all possible values of the integer ν .

Remark 1. The class \mathcal{U} of inputs considered above (see (14)) covers the range of low-frequency dominant spectra including the flat spectra. Consequently, high-frequency dominant spectra are not considered in this class. This drawback can be circumvented by performing the optimization above not only for the class \mathcal{U} of PRBS signals but also for a class of input signals including high-frequency dominant spectra. A possible choice for this class could be the set of input signals generated by filtering a white noise through a high-pass filter (e.g. a Butterworth filter of order 1) with cut-off frequency ν . An input signal within this class could be described by $u(t) = \sigma h_{\nu}(t)$ where $h_{\nu}(t)$ a normalized signal with total power = 1 generated by a filter with cut-off frequency ν and σ is a positive scaling factor. Based on this description, the optimal σ and ν can be determined using a similar procedure as for the PRBS signals since \mathcal{P}_u is here also given by σ^2 . Note that $u(t) = \sigma h_{\nu}(t)$ is no longer a binary signal.

Remark 2. The optimal input signal within the class of PRBS signals (and/or in the class of high-frequency dominant signal $u(t) = \sigma h_{\nu}(t)$) can be obtained since, the optimal σ for a given ν being easily computable, the optimization on the unique other scalar variable ν can be done via a

line search. This approach can of course only be applied if the total amount of decision variable is limited. Consequently, the approach can not be applied for the broader parametrizations (e.g. multisines) that are generally used in the absence of undermodeling and which generally contains a large amount of decision variables.

4. NUMERICAL ILLUSTRATION

In this section, we will illustrate the results presented in this paper. We consider that the true system is given by: $G_0(z) = 0.1766\Lambda_1(z) + 0.1187\Lambda_2(z) + 0.02\Lambda_3(z)$ where $\Lambda_i(z)$ are here Laguerre basis functions with a pole at $\xi = 0.8147$ (see below (4)). The considered model structure \mathcal{M} neglects the third term of the expansion i.e. \mathcal{M} is given by (4) with $\Lambda(z) = (\Lambda_1(z) \Lambda_2(z))$. The bound on $|g_0(3)|$ required in Assumption 1 is here chosen equal to 0.025 and the bound $\bar{\sigma}_e^2 = \sigma_e^2$.

We wish to determine the optimal signal of length $N = 500$ solving experiment design problem (13) when \mathcal{U} is the class of PRBS signals and with a largest admissible modeling error $r_{adm}(\omega)$ computed in order to guarantee that the controller designed with the identified model achieves an acceptable sensitivity function when applied to the true system (see (Bombois *et al.*, 2004a)). The obtained frequency function $r_{adm}(\omega)$ is represented in Figure 2.

In order to solve (13), we use the procedure presented in Section 3. The optimal amplitude $\sigma_{opt}(\nu)$ for a PRBS of clock period ν has been computed via (17) for a large amount of different values of ν . Table 1 gives the result for some significant values of ν . Based on these results, we conclude that the optimal input signal is a PRBS of amplitude $\sigma_{opt} = 0.47$ and clock period $\nu_{opt} = 24$. Note that no improvement is booked by considering in addition high-frequency dominant spectra (such as proposed in Remark 1 at the end of Section 3)

ν	$\sigma_{opt}(\nu)$
1	2.31
10	0.77
20	0.50
24	0.47
30	0.56
50	0.67
100	1.40

Table 1. $\sigma_{opt}(\nu)$ for some significant values of ν

An important observation we can make from Table 1 is that the amplitude necessary to meet the modeling error constraint (12) is five times larger for the classically-used PRBS (the one with $\nu = 1$) than for the PRBS with the optimal clock period $\nu = 24$. This phenomenon which will be

further discussed in the sequel was also observed in the absence of undermodeling (see (Barenthin *et al.*, 2005)).

Verification. We have generated the optimal PRBS signal $u(t) = 0.47p_{\nu=24}(t)$ ($t = 1 \dots 500$) and computed the bound $\alpha(\omega, u) + \beta(\omega, u)$ on the modeling error using (10) and (8). This bound is compared with the threshold $r_{adm}(\omega)$ in Figure 2. As was expected, we observe that (12) is fulfilled.

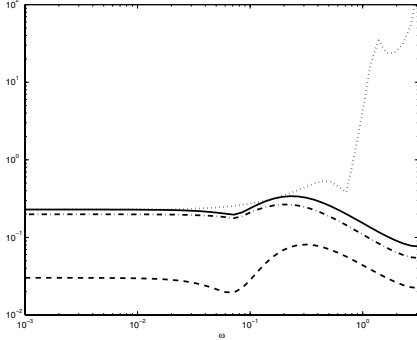


Fig. 2. $u(t) = 0.47p_{\nu=24}(t)$: $\alpha(\omega, u)$ (dashdot), $\beta(\omega, u)$ (dashed), $\alpha(\omega, u) + \beta(\omega, u)$ (solid) and $r_{adm}(\omega)$ (dotted)

For the sake of comparison, we have computed the bound $\alpha(\omega, u) + \beta(\omega, u)$ for the signal $u(t) = 0.47p_{\nu=1}(t)$ i.e. an input signal with the same amplitude 0.47 as the optimal input signal but with a clock period $\nu = 1$. Based on Table 1, we know that the modeling error constraint (12) will not be fulfilled since the minimal amplitude for this purpose is 2.31. This is confirmed in Figure 3. It is interesting to compare Figures 2 and 3. The total power of the input signals leading to these two figures is exactly the same. However, for Figure 2, this power is mainly distributed in the frequencies below π/ν_{opt} while, for Figure 3, this power is distributed over the whole frequency range. This concentration of $\Phi_u(\omega)$ in the low frequencies leads in Figure 2 to bias and variance terms much smaller in this frequency range (i.e. the one where $r_{adm}(\omega)$ is small) than in Figure 3.

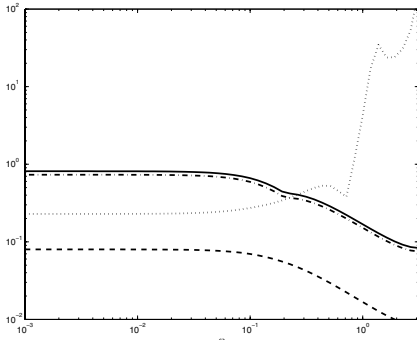


Fig. 3. $u(t) = 0.47p_{\nu=1}(t)$: $\alpha(\omega, u)$ (dashdot), $\beta(\omega, u)$ (dashed), $\alpha(\omega, u) + \beta(\omega, u)$ (solid) and $r_{adm}(\omega)$ (dotted)

5. CONCLUSIONS

In this paper, the least powerful PRBS input signals for an identification guaranteeing that the modeling error remains below some threshold has been determined for identification in a reduced order model structure. Note that the results developed here for open-loop identification can be easily extended to closed-loop identification by adapting Section 2 accordingly (see (Hakvoort and den Hof, 1997)). Note also that, as usual in optimal experiment design, the optimal input signal is designed based on a fair amount of prior information on the true system. Consequently, the results should always be handled with caution, but they nevertheless give useful guidelines on how to choose the input signal to identify a model which is suitable for a robust control procedure.

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