CONJUGACY OF PIECEWISE C^1-HOMEOMORPHISMS OF CLASS P OF THE CIRCLE

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ABSTRACT. We give a characterization of piecewise C^1-homeomorphism of class P of the circle with irrational rotation number and finitely many break points which are piecewise differentiably conjugate to C^1-diffeomorphims. The following properties are equivalent:

i) f is conjugate to a C^1-diffeomorphism of the circle by a piecewise C^1-homeomorphism of class P.

ii) The number of break points of f^n is bounded by some constant that doesn't depend on n.

iii) The product of jumps of f in the break points contained in a same orbit is equal 1.

iv) f is conjugate to a C^1-diffeomorphism of the circle by a piecewise quadratic homeomorphism of class P.

This characterization extend Liousse's Theorem for PL homeomorphisms of the circle ([3]).

1. INTRODUCTION

1.1. **Preliminaries.** Denote by $S^1 = \mathbb{R}/\mathbb{Z}$ the circle and $\pi : \mathbb{R} \longrightarrow S^1$ the canonical projection. Let f be an orientation preserving homeomorphism of S^1. The homeomorphism f admits a lift $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ that is an increasing homeomorphism of $\mathbb{R}$ such that $\pi o \tilde{f} = f o \pi$. Conversely, the projection of such a homeomorphism of $\mathbb{R}$ is an orientation preserving homeomorphism of S^1.

Let $x \in S^1$. We call:

- *orbit* of x by f the subset $O_f(x) = \{f^n(x) : \ n \in \mathbb{Z}\}$
- *positive orbit* from x by f the subset $O^+_f(x) = \{f^n(x) : \ n \in \mathbb{N}\}$
- *negative orbit* from x by f the subset $O^-_f(x) = \{f^n(x) : \ n \in -\mathbb{N}\}$

A segment of the orbit $O_f(d)$ containing d is a subset of the form $\{f^s(d) : -k \leq s \leq n-k\}$, noted $[f^{-k}(d), ..., f^{n-k}(d)]$, $k, \ n \in \mathbb{N}$.

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Historically, the dynamic study of circle homeomorphisms was initiated by H. Poincaré ([4], 1886), he introduced the rotation number of a homeomorphism f of S^1 as

$$\rho(f) = \lim_{n \longrightarrow +\infty} \frac{\tilde{f}^n(x) - x}{n} (mod 1)$$

Poincaré shows that this limit exist and does not depends of x and the lift $\tilde{f}$ of f.

Assuming f is a C^r-diffeomorphism ($r \geq 2$) and $\rho(f)$ is irrational, A. Denjoy ([2]) proved:

Theorem 1.1. *(Denjoy)([2]). Every C^r-diffeomorphism f ($r \geq 2$) with irrational rotation number $\rho(f)$ is topologically conjugate to rotation $R_{\rho(f)}$.*

This means that there exists an orientation preserving homeomorphism h of S^1 such that $f = h^{-1} \circ R_{\rho(f)} \circ h$.

Denjoy noted that this result can be extended (with the same proof) to a large class of circle homeomorphisms: *the class* P.

1.2. **Class** P

Definition 1.2. Let f be a piecewise C^1 orientation preserving homeomorphism of S^1. A *break point* of f is a point of discontinuity of the derivative Df. The homeomorphism f is called of *class* P if it is C^1 except in a finitely or countably many break points admitting left and right derivatives; the derivative $Df : S^1 \longrightarrow \mathbb{R}_+^\star$ has the following properties :

- there exist two constants $0 < a < b < +\infty$ such that:

$$a < Df(x) < b,$$

for all x where Df exists,

$$a < Df_+(c) < b, \textit{ and } a < Df_-(c) < b \textit{ at the break point } c,$$

- $LogDf$ is a bounded variation on S^1

The ratio $\sigma_f(c) := \frac{Df_+(c)}{Df_-(c)}$ is called *jump of* f *in* c.

Notice that homeomorphisms of class P form a sub-group of the group of orientation preserving homeomorphisms of S^1.

As examples of homeomorphisms of class P, we mention:

- the C^2-diffeomorphisms,
- The PL homeomorphisms (i.e. piecewise linear), these are not C^2-diffeomorphisms. An orientation preserving homeomorphism f of S^1 is called a *PL homeomorphism* if f is differentiable except at finite number of break points $(c_i)_{1 \leq i \leq p}$ of S^1 and such that Df is constant on each $]c_i, c_{i+1}[$.

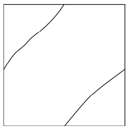

FIGURE 1. General Boshernitzan of last break point 0

The set of PL homeomorphisms is a sub-group of the group of homeomorphisms of class P and contains rotations.

Analytic homeomorphisms are not of class P. By J.C. Yoccoz [5], they satisfy the conclusion of Denjoy theorem but not Denjoy theory.

Definition 1.3. A piecewise C^1-homeomorphism B of class P of S^1 with two break points x_0 and $B(x_0)$ is called a *general Boshernitzan* of S^1 of *last break point* $B(x_0)$. If $B(x_0) = 0$, B is called a *standard Boshernitzan of last break point* 0 (see Figure 1).

If we conjugate a standard Boshernitzan B_0 of last break point 0 by a rotation R_{x_0}, the homeomorphism $R_{x_0} \circ B_0 \circ R_{x_0}^{-1}$ is a general Boshernitzan of last break point $B(x_0)$ and conversely.
The lift $\tilde{B}_0$ (mod 1) of B_0 on $[0,1[$ is a general exchange of 2 intervals on $[0,1[$. The affine Boshernitzan correspond to affine exchange of 2 intervals on $[0,1[$.

Notice that Boshernitzan was the first who studied these examples in order to built examples of "rational" PL homeomorphisms with irrational rotation numbers (cf. [1]).

Denote by
- $C(f)$ the set of break points of f.
- $\pi_s(f)$ the product of jumps of f in the break points of f :

$$\pi_s(f) = \prod_{c \in C(f)} \sigma_f(c).$$

Definition 1.4. ([3]) A piecewise C^1-homeomorphism f of class P of S^1 has the *property* D if the product of jumps of f in the break points of f contained in a same orbit is equal 1.

In particular, if f has the property D then $\pi_s(f) = 1$. Conversely, if $\pi_s(f) = 1$ and if all break points of f are contained in a same orbit then f has the property D. If f is a PL homeomorphism we have always $\pi_s(f) = 1$ and therefore if all the break points of f are contained in a same orbit then f satisfies property D.

Proposition 1.5. *(Invariance of π_s by piecewise C^1 conjugacy).*

Let f, g be two orientation preserving piecewise C^1- homeomorphisms of class P of S^1. If f and g are piecewise C^1 conjugate then $\pi_s(f) = \pi_s(g)$.

Proof. Let H be the piecewise C^1-homeomorphism which conjugate f to g: $Hof = goH$. For $x \in S^1$, we define $\pi_s(f)(x) = \prod\limits_{y \in O_f(x)} \sigma_f(y)$ the product of jumps of f in the break points contained in the orbit of x. Since $O_g(H(x)) = H(O_f(x))$ then

$$\pi_s(g)(H(x)) = \prod_{z \in O_g(H(x))} \sigma_g(z) = \prod_{y \in O_f(x)} \sigma_g(H(y)).$$

Since $\sigma_g(H(y)) = \frac{\sigma_H(f(y))\sigma_f(y)}{\sigma_H(y)}$ then

$$\pi_s(g)(H(x)) = \prod_{y \in O_f(x)} \frac{\sigma_H(f(y))\sigma_f(y)}{\sigma_H(y)} = \pi_s(f)(x) \prod_{y \in O_f(x)} \frac{\sigma_H(f(y))}{\sigma_H(y)}.$$

Or $\prod\limits_{y \in O_f(x)} \frac{\sigma_H(f(y))}{\sigma_H(y)} = \prod\limits_{k \in \mathbb{Z}} \frac{\sigma_H(f(f^k(x)))}{\sigma_H(f^k(x))} = \frac{\prod\limits_{k \in \mathbb{Z}} \sigma_H(f^k(x))}{\prod\limits_{k \in \mathbb{Z}} \sigma_H(f^k(x))} = 1.$

Hence $\pi_s(g)(H(x)) = \pi_s(f)(x)$ and therefore $\pi_s(g) = \pi_s(f)$. ☐

In the sequel, we suppose that homeomorphisms f of class P considered have only a finite number of break points i.e. $C(f)$ is finite.

Our main result is the following :

Theorem 1.6. *Let f be a piecewise C^1-homeomorphism of class P of the circle with irrational rotation number. The following properties are equivalent:*

i) f is conjugate to a C^1-diffeomorphism of the circle through a piecewise C^1-homeomorphism.

ii) The number of break points of f^n is bounded by some constant that doesn't depend on n.

iii) f satisfies property D.

iv) f is conjugate to a general Boshernitzan (of last break point $B(x_0)$ such that $\sigma_B(x_0) \times \sigma_B[B(x_0)] = 1$) or a C^1-diffeomorphism of the circle through a PL homeomorphism.

v) f is conjugate to a C^1-diffeomorphism of the circle through a piecewise quadratic homeomorphism.

Corollary 1.7. *Under the hypothesis of Theorem 1.6, if the number of break points of f^n is bounded by some constant that doesn't depend on n then f is conjugate to a C^1-diffeomorphism of the circle through a piecewise quadratic homeomorphism.*

It follows from Theorem 1.6:

Corollary 1.8. *([3]) Let f be a PL homeomorphism of the circle with irrational rotation number α. The following properties are equivalent:*

i) f is conjugate to the rotation R_α through a piecewise C^1-homeomorphism,

ii) The number of break points of f^n is bounded by some constant that doesn't depend on n.

iii) f is conjugate to an affine Boshernitzan of last break point 0 or a rotation with rotation number α through a PL homeomorphism,

iv) f is conjugate to R_α through a piecewise C^∞ (analytic) homeomorphism.

2. Homeomorphisms of class P with property D

Let $n \in \mathbb{N}$. Denote by $C(f^n)$ the set of break points of f^n. We have $C(f^n) \subset \{f^{-k}(c),\ k = 0, 1, ..., n-1;\ c \in C(f)\}$.

Let $x \in S^1$, we have :

$$Df^n(x) = Df(x) \times Df(f(x)) \times ... \times Df(f^{n-1}(x)).$$

The jump of f^n in x is then :

$$\sigma_{f^n}(x) = \sigma_f(x) \times \sigma_f(f(x)) \times ... \times \sigma_f(f^{n-1}(x)).$$

Definition 2.1. Let $c \in C(f)$. A *maximal connection* of c is a segment $M = [f^{-p}(c), ..., f^q(c)]$ of the orbit $O_f(c)$ which contains all the break points of f contained on $O_f(c)$ and such that $f^{-p}(c)$ (resp. $f^q(c)$) is the first (resp. last) break point of f on $O_f(c)$.

Hence, the negative orbit $O_f^-(f^{-p}(c))$(resp. positive orbit $O_f^+(f^q(c))$) from $f^{-p}(c)$(resp. from $f^q(c)$) doesn't contain any break point of f.

The following properties are trivial:

- Two break points of f are on the same maximal connection, if and only if, they are on the same orbit.

- Two distinct maximal connections are disjoints.

There is a finite number of maximal connections ; denoted by
$M_1 = [f^{-N_1}(c_1), ..., c_1], ..., M_p = [f^{-N_p}(c_p), ..., c_p]$ where $c_1, ..., c_p \in C(f)$.
So, we have the decomposition :
$C(f) = \coprod_{i=1}^{p} C_i$ where $C_i = C(f) \cap M_i,\ i = 1, ..., p$.

This is equivalent to say that: for every $i = 1, ..., p$, the product of jumps of f in the break points of C_i is equal 1 (i.e. $\prod\limits_{d \in C_i} \sigma_f(d) = 1$). We have also $\prod\limits_{d \in C_i} \sigma_f(d) = \prod\limits_{d \in M_i} \sigma_f(d)$. So, f satisfies the property D means that $\prod\limits_{d \in C_i} \sigma_f(d) = 1$.

Proposition 2.2. *Let f be a homeomorphism of class P of the circle with irrational rotation number α. Then f has property D if and only if the number of break points of f^n is bounded by some constant N_0 that doesn't depend on n.*

Proof. Suppose that the number of break points of f^n is bounded by a constant N_0. Let $d \in C(f)$ and $M = [f^{-p}(d), ..., f^q(d)]$ be the maximal connection of d. Lets show that $\prod\limits_{\delta \in M} \sigma_f(\delta) = 1$:

Let $n \in \mathbb{N}$ fixed. We have :

$$C(f^{n+1}) \subset \{f^{-k}(c) : c \in C(f),\ 0 \leq k \leq n\},$$

and

$$\sigma_{f^{n+1}}[f^{-k}(d)] = \sigma_f[f^{-k}(d)] \times ... \times \sigma_f[f^{n-k}(d)]$$

We let :

$$J_n := \{0 \leq k \leq n : \sigma_{f^{n+1}}[f^{-k}(d)] = 1\}.$$

By hypothesis :

$$card\{0 \leq k \leq n\ ;\ \sigma_{f^{n+1}}[f^{-k}(d)] \neq 1\} \leq N_0.$$

Then

$$card(J_n) \geq n + 1 - N_0.$$

For n such that $n > N_0 + p$, we have $card(J_n) > p + 1 \geq card(J)$ where $J = \{0 \leq k \leq p\ : \sigma_{f^{n+1}}[f^{-k}(d)] = 1\}$. Therefore, J est strictly contained in J_n. Let $K_n := J_n - J$ and let k_n be the minimum of K_n. Since
$n - k_n + 1 \geq card(K_n)$ and $card(K_n) \geq\ n - (N_0 + p)$ then
$p + 1 \leq k_n \leq p + N_0 + 1$ for $n > p + N_0$.

As

$$\sigma_{f^{n+1}}[f^{-k_n}(d)] = \sigma_f[f^{-p}(d)] \times ... \times \sigma_f[f^{n-k_n}(d)]$$

then

$$\sigma_f[f^{-p}(d)] \times \sigma_f[f^{-p+1}(d)] \times ... \times \sigma_f[f(d)] = \sigma_{f^{n+1}}[f^{-k_n}(d)]$$

for every $n > p + q + N_0$. It follows,

$$\sigma_f[f^{-p}(d)] \times \sigma_f[f^{-p+1}(d)] \times ... \times \sigma_f[f^q(d)] = \prod_{\delta \in M} \sigma_f(\delta) = 1.$$

Conversely, suppose that for every $d \in C(f)$, the product of jumps of f in the break points of the maximal connection of d is equal 1. Lets show that the number of break points of f^n is bounded.

Let $n \in \mathbb{N}$. The point of discontinuity of Df^n are among $f^{-k}(c_i)$, $k = 0, 1, ..., n + N_i - 1$, $i = 1, ..., p$.
The jump of f^n in these points are :

$$\sigma_{f^n}[f^{-k}(c_i)] = \sigma_f[f^{-k}(c_i)] \times ... \times \sigma_f(c_i) \times ... \times \sigma_f[f^{n-k-1}(c_i)].$$

If $k \geq N_i$ and $n-k-1 \geq 0$ then the segment of orbit $[f^{-k}(c_i), ..., f^{n-k-1}(c_i)]$ contains the maximal connection of c_i. Hence, $\sigma_{f^n}[f^{-k}(c_i)] = 1$. So, a necessary condition for $f^{-k}(c_i)$ to be a discontinuity point of Df^n is that $0 \leq k < N_i$ or $n-1 < k \leq n+N_i+1$. Therefore, the number of discontinuity points of Df^n which are contained in the orbit of c_i is less then $2N_i$.
We conclude that the number of discontinuity points of Df^n is bounded by $2(N_1 + ... + N_p)$. □

To a piecewise C^1-homeomorphism f of class P of S^1 that satisfies property D, we associate the real :

$$\pi(f) = \prod_{i=1}^{p} \prod_{k=0}^{N_i - 1} \sigma_{f^{N+1}}(f^{-k}(c_i))$$

where $N = \underset{1 \leq i \leq p}{max}(N_i)$.

Remark. If f is a homeomorphism that satisfies the property D, then $\pi_s(f) = 1$ but one can have $\pi(f) \neq 1$ as can be shown by the following example:

If B is a standard Boshernitzan of last break point 0 and which satisfies the property D, we have : $\pi_s(B) = 1$ and $\pi(B) = \sigma_{B^2}(0) = \sigma_B(0) \times \sigma_B(B(0))$.
If $B(0) \neq B^{-1}(0)$, then $\sigma_B(B(0)) = 1$ and $\pi(B) = \sigma_B(0) \neq 1$.
If $B(0) = B^{-1}(0)$, then $\pi(B) = \sigma_B(0)\sigma_B(B^{-1}(0)) = \sigma_B(0)\sigma_B(B(0)) = 1$.

Proposition 2.3. *Let f be a piecewise C^1-homeomorphism of class P of the circle. If f satisfies property D then :*

i) if $\pi(f) = 1$ then f is conjugate to a C^1-diffeomorphism of the circle through a PL homeomorphism.

ii) if $\pi(f) \neq 1$ then f is conjugate to a general Boshernitzan B (of last break point $B(x_0)$ such that $\pi_s(B) = \sigma_B(x_0) \times \sigma_B[B(x_0)] = 1$ and $\pi(B) = \pi(f)$) through a PL homeomorphism.

Proof. **Case i) $\pi(f) = 1$.**

Let H be the PL homeomorphism with the following properties: :

i) $H(0) = 0$,
ii) $f^{-k}(c_i),\ i = 1, ..., p;\ k = 0, ..., N_i - 1$ are the break points of H,
iii) the jumps of H are given by : $\sigma_H(f^{-k}(c_i)) = \sigma_{f^{N+1}}(f^{-k}(c_i))$ where $N = \max\limits_{1 \leq i \leq p} N_i$.

Such homeomorphism exist since

$$\prod_{i=1}^{p} \prod_{k=0}^{N_i-1} \sigma_H(f^{-k}(c_i)) = \pi(f) = 1$$

Take $G = H \circ f \circ H^{-1}$. A priori, the break points of G are :

- the break points of H^{-1} :

$$H(f^{-k}(c_i)), ;\ k = 0, ..., N_i - 1,\ i = 1, ..., p,$$

- the image by H of break points of f :

$$H(f^{-k}(c_i)),\ k = 0, ..., N_i,\ i = 1, ..., p,$$

- the image by $H \circ f^{-1}$ of break points of H :

$$H \circ f^{-1}(f^{-k}(c_i)) = H(f^{-k}(c_i)) \ with\ k = 1, ..., N_i.$$

Therefore the possible break points of G are among :

$$H(f^{-k}(c_i)), k = 0, ..., N_i,\ i = 1, ..., p.$$

Compute the jumps of G in these points:

$$\sigma_G[H(f^{-k}(c_i))] = \frac{\sigma_H[f(f^{-k}(c_i))] \times \sigma_f(f^{-k}(c_i))}{\sigma_H(f^{-k}(c_i))}$$

- If $k = 1, ...N_i - 1$, we have by definition :

$$\sigma_H(f^{-k}(c_i)) = \sigma_{f^{N+1}}(f^{-k}(c_i))$$

and since $N \geq k$ then :

$$\sigma_{f^{N+1}}(f^{-k}(c_i)) = \prod_{s=0}^{N} \sigma_f(f^{s-k}(c_i)) = \prod_{s=-k}^{N-k} \sigma_f(f^s(c_i)) = \prod_{s=-k}^{0} \sigma_f(f^s(c_i)).$$

Hence :

$$\sigma_H[f(f^{-k}(c_i))] = \sigma_H[f^{-(k-1)}(c_i)] = \prod_{s=-k+1}^{0} \sigma_f(f^s(c_i)).$$

Then : $\sigma_G[H(f^{-k}(c_i))] = 1$.

- If $k = 0$, we obtain :

$$\sigma_G[H(f^{-k}(c_i))] = \sigma_G[H(c_i)] = \frac{\sigma_H[f(c_i)] \times \sigma_f(c_i)}{\sigma_H(c_i)}$$

We have $\sigma_G[H(c_i)] = 1$ since $\sigma_H(c_i) = \sigma_f(c_i)$ and $\sigma_H[f(c_i)] = 1$ (since $f(c_i)$ is not a break point of H.)

- If $k = N_i$, we have :

$$\sigma_G[H(f^{-k}(c_i))] = \sigma_G[H(f^{-N_i}(c_i))] = \frac{\sigma_H[f(f^{-N_i}(c_i))] \times \sigma_f(f^{-N_i}(c_i))}{\sigma_H(f^{-N_i}(c_i))}$$

We have $\sigma_H(f^{-N_i}(c_i)) = 1$ (since $f^{-N_i}(c_i)$ is not a break point of H).
Also,

$$\sigma_H[f(f^{-N_i}(c_i))] \times \sigma_f(f^{-N_i}(c_i)) = \prod_{s=-N_i+1}^{0} \sigma_f(f^s(c_i)) \times \sigma_f(f^{-N_i}(c_i)) = \prod_{s=0}^{N_i} \sigma_f(f^{-s}(c_i)) = 1$$

since

$$\sigma_H[f(f^{-N_i}(c_i))] = \sigma_H[f^{-(N_i-1)}(c_i)] = \prod_{s=-N_i+1}^{0} \sigma_f(f^s(c_i))$$

It follows that : $\sigma_G[H(f^{-N_i}(c_i))] = 1$.

We conclude that G has no break point, G is then a C^1-diffeomorphism.

Case ii) $\pi(f) \neq 1$.

We choose a point c such that :

$c \neq f^{-k}(c_i)$, $c \neq f(c)$, $c \neq f^{N+1}(f^{-k}(c_i))$, $i = 1, ..., p$, $k = 0, ..., N_i$.

To obtain the PL homeomorphism H that satisfies properties i), ii), and iii), we need to add a break point c to H such that: $\sigma_H(c) = \frac{1}{\pi(f)}$.

The break points of H are :

$$f^{-k}(c_i), \; (i = 1, ..., p, \; k = 0, ..., N_i - 1), \; \textit{and } c$$

and the jumps in these points are :

$$\sigma_H(f^{-k}(c_i)) = \sigma_{f^{N+1}}(f^{-k}(c_i)), \; \sigma_H(c) = \frac{1}{\pi(f)}.$$

We let $G = H \circ f \circ H^{-1}$. The possible break points of G are among: $H(c),\ H \circ f^{-1}(c),\ H(f^{-k}(c_i)),\ k = 0, ..., N_i,\ i = 1, ..., p.$

- the jump of G in $H(c)$:

$$\sigma_G[H(c)] = \frac{\sigma_H[f(c)] \times \sigma_f(c)}{\sigma_H(c)}.$$

We have : $\sigma_f(c) = 1$ since $c \neq f^{-k}(c_i)$, $i = 1, ..., p$, $k = 0, ..., N_i$.

We have also $\sigma_H[f(c)] = 1$ since $f(c)$ is not a break point of H ; indeed :

- $f(c) \neq c$
- $f(c) \neq f^{-k}(c_i))$, $k = 0, ..., N_i - 1$ (otherwise, $c = f^{-s}(c_i),\ s = 1, ..., N_i$, which is impossible by hypothesis.).

It follows that: $\sigma_G[H(c)] = \frac{1}{\sigma_H(c)} \neq 1$. $H(c)$ is then a break point of G.

- The jump of G in $H \circ f^{-1}(c)$:

$$\sigma_G[H \circ f^{-1}(c)] = \frac{\sigma_H(c) \times \sigma_f[f^{-1}(c)]}{\sigma_H[f^{-1}(c)]}$$

We have : $\sigma_H[f^{-1}(c)] = 1$ since :

- $f^{-1}(c) \neq c$
- $f^{-1}(c) \neq f^{-k}(c_i))$ since :

Otherwise, $c = f^{1-k}(c_i),\ k = 0, ..., N_i - 1$. For $k = 1, ..., N_i - 1$, this is impossible by hypothesis. For $k = 0$, $c = f(c_i)$ is also impossible since by hypothesis, $c \neq f^{N+1}(f^{-k}(c_i))$, $i = 1, ..., p$, $k = 0, ..., N_i$, in particular for $k = N_i$, $c \neq f(c_i)$. It follows that $\sigma_G[H \circ f^{-1}(c)] = \sigma_H(c) \neq 1$. So, $H \circ f^{-1}(c)$ is a break point of G.

- the jump of G in $H \circ f^{-k}(c_i)$: by the compute above, we have $\sigma_G[H \circ f^{-k}(c_i)] = 1$, $i = 1, ..., p$.

Therefore, G has exactly two break points :
$x_0 := H \circ f^{-1}(c)$ and $G(x_0) = H(c)$ with $\sigma_G(x_0) \times \sigma_G[G(x_0)] = 1$. The homeomorphism G is then a Boshernitzan of S^1 of last break point $H(c)$. Let's show that $\pi(B) = \pi(f)$:

We have
$\pi(B) = \sigma_B(H(c)) = \sigma_{H \circ f \circ H^{-1}}(H(c)) = \frac{\sigma_H(f(c)) \times \sigma_f(c)}{\sigma_H(c)} = \frac{1}{\frac{1}{\pi(f)}} = \pi(f).$ □

Corollary 2.4. *Let f be a piecewise C^1-homeomorphism of class P of the circle. If the break points of f are on the same orbit with $\pi_s(f) = \pi(f) = 1$ then f is conjugate to a C^1-diffeomorphism of the circle through a PL homeomorphism.*

Proof. This follows from the fact that if all the break points of f are on the same orbit and $\pi_s(f) = 1$ then f has the property D and so, we conclude by proposition 2.3, i). □

3. The conjugacy lemmas

Lemma 3.1. *For every $b \in S^1$, for every $\delta \in]0,1[$, there exist a piecewise quadratic orientation preserving homeomorphism $H_{\delta,b}$ of S^1 with only one break point b such that: $\sigma_{H_{\delta,b}}(b) = \delta$.*

Proof. If $\delta \in]0,1[$ we put

$$\tilde{H}_\delta(x) = (\frac{1-\delta}{1+\delta})x(x + \frac{2\delta}{1-\delta}),\ x \in [0,1[.$$

One can check that $\tilde{H}_\delta(0) = 0$, $\tilde{H}_\delta(1) = 1$ and that $\tilde{H}_\delta$ is an increasing homeomorphism of $[0,1[$ with $(D\tilde{H}_\delta)_+(0) = \frac{2\delta}{1+\delta}$ and $(D\tilde{H}_\delta)_-(1) = \frac{2}{1+\delta}$.

Let $t \in [0,1[$ such that $p(t) = b$ and consider the map :

$$\tilde{H}_{\delta,t}(x) = \tilde{H}_\delta(x-t) + t,\ if\ x \in [t,\ t+1[.$$

$\tilde{H}_{\delta,t}$ is extended to an increasing homeomorphism, denoted also by $\tilde{H}_{\delta,t}$ on $\mathbb{R}$ and defined by :

$\tilde{H}_{\delta,t}(x) = \tilde{H}_{\delta,t}(x-k)$, if $x \in [t+k, t+k+1[$, $k \in \mathbb{Z}$.

Then $\tilde{H}_{\delta,t}$ is an increasing homeomorphism of $\mathbb{R}$ which satisfies :

- $\tilde{H}_{\delta,t}(x+1) = \tilde{H}_{\delta,t}(x) + 1$,
- $t+k,\ k \in \mathbb{Z}$ are the break points of $\tilde{H}_{\delta,t}$,
- $\sigma_{\tilde{H}_{\delta,t}}(t+k) = \delta$.

As a consequence, $\tilde{H}_{\delta,t}$ induces by projection the piecewise quadratic homeomorphism $H_{\delta,t}$ of S^1 with only one break point b such that
$\sigma_H(b) = \delta$. □

Lemma 3.2. *Let f be a piecewise C^1-homeomorphism of class P of S^1 with irrational rotation number and having $p+1$ break points $x_0, x_1, ..., x_p$ ($p \geq 1$) contained in a same orbit. Then there exists a piecewise quadratic homeomorphism H with one break point x_1 such that for $F = H \circ f \circ H^{-1}$ we have :*

a) if there exist $i_0 \in \{0,1,...,p\}$ such that $f(x_{i_0}) = x_1$ then :

a-1) $C(F) = \{y_i = H(x_i),\ i = 0,2,...,p\}$ if $\sigma_f(x_1) \times \sigma_f(x_{i_0}) \neq 1$.
a-2) $C(F) = \{y_i = H(x_i),\ i = 0,2,...,p, i \neq i_0\}$ if $\sigma_f(x_1) \times \sigma_f(x_{i_0}) = 1$.

b) if for every $i = 0,1,...,p,\ f(x_i) \neq x_1$ then
$C(F) = \{y_i = H(x_i),\ i = 0,2,...,p\} \cup \{Hof^{-1}(x_1)\}$. Moreover, if $x_1 = f^k(x_0)$, $k \geq 2$ then $y_1 = F^{k-1}(y_0)$.

Proof. By Lemma 3.1, let H be the piecewise quadratic homeomorphism with one break point x_1 such that $\sigma_H(x_1) = \sigma_f(x_1)$. Take $F = H \circ f \circ H^{-1}$. The eventually break points of F are:

$$H(x_i),\ i = 0, 1, ..., p,\ H \circ f^{-1}(x_1).$$

- The jumps of F in these points are:

$$\sigma_F[H(x_i)] = \frac{\sigma_H[f(x_i)] \times \sigma_f(x_i)}{\sigma_H(x_i)}$$

Case a-1): there exist $i_0 \in \{0, 1, ..., p\}$ such that $f(x_{i_0}) = x_1$ with $\sigma_f(x_1) \times \sigma_f(x_{i_0}) \neq 1$.

In this case, $f(x_i) \neq x_1$ for $i \neq i_0$ and $H \circ f^{-1}(x_1) = H(x_{i_0})$. Hence

- $\sigma_F[H(x_i)] = \sigma_f(x_i) \neq 1$, for $i \notin \{1,\ i_0\}$:

indeed, $\sigma_H(x_i) = 1$ since x_i is not a break point of H, and $\sigma_H[f(x_i)] = 1$ since $f(x_i) \neq x_1$.

- $\sigma_F[H(x_1)] = 1$:

indeed, $\sigma_F[H(x_1)] = \frac{\sigma_H[f(x_1)] \times \sigma_f(x_1)}{\sigma_H(x_1)}$, $\sigma_H[f(x_1)] = 1$ since $f(x_1) \neq x_1$ (x_1 is not a periodic point) and $\frac{\sigma_f(x_1)}{\sigma_H(x_1)} = 1$.

- $\sigma_F[H(x_{i_0})] = \sigma_f(x_1) \times \sigma_f(x_{i_0}) \neq 1$:

indeed, $\sigma_F[H(x_{i_0})] = \frac{\sigma_H(x_1) \times \sigma_f(x_{i_0})}{\sigma_H(x_{i_0})}$, $\sigma_H(x_{i_0}) = 1$ since $x_{i_0} \neq x_1$ and $\sigma_H(x_1) = \sigma_f(x_1)$.

One conclude that F has p break points:

$$y_i = H(x_i),\ i = 0, 2, ..., p.$$

contained in the orbit of y_0.

Case a-2): there exist $i_0 \in \{0, 1, ..., p\}$ such that $f(x_{i_0}) = x_1$ with $\sigma_f(x_1) \times \sigma_f(x_{i_0}) = 1$.

In this case, we have as in the case a-1), $\sigma_F[H(x_i)] = \sigma_f(x_i) \neq 1$, for $i \neq 1,\ i_0$, $\sigma_F[H(x_1)] = 1$ and $\sigma_F[H(x_{i_0})] = \sigma_f(x_1) \times \sigma_f(x_{i_0}) = 1$.

One conclude that F has $p - 1$ break points:

$$y_i = H(x_i),\ 0 \leq i \leq p, i \neq 1, i_0$$

contained in the orbit of y_0.

Case b): for every $i = 0, 1, ..., p$, $f(x_i) \neq x_1$.

- we have $\sigma_F[H(x_i)] = \sigma_f(x_i) \neq 1$:

indeed, $\sigma_H(x_i) = 1$ since x_i is not a break point of H, and $\sigma_H[f(x_i)] = 1$ since $f(x_i) \neq x_1$.

- $\sigma_F[H(x_1)] = 1$:

indeed, $\sigma_F[H(x_1)] = \frac{\sigma_H[f(x_1)]\times\sigma_f(x_1)}{\sigma_H(x_1)}$, $\sigma_H[f(x_1)] = 1$ since $f(x_1) \neq x_1$ and $\frac{\sigma_f(x_1)}{\sigma_H(x_1)} = 1$.

- $\sigma_F[H \circ f^{-1}(x_1)] = \sigma_H(x_1) \neq 1$:

indeed, $\sigma_F[H \circ f^{-1}(x_1)] = \frac{\sigma_H(x_1)\times\sigma_f(f^{-1}(x_1))}{\sigma_H(f^{-1}(x_1))}$, $\sigma_H(f^{-1}(x_1)) = 1$ since $f(x_1) \neq x_1$ and $\sigma_f(f^{-1}(x_1)) = 1$ since $f^{-1}(x_1) \neq x_i$, $i = 0, 1, ..., p$.

Therefore F has $p+1$ break points:

$$y_1 = H \circ f^{-1}(x_1),\ y_i = H(x_i),\ i = 0, 2, ..., p,$$

Moreover, if $x_1 = f^k(x_0)(k \geq 2)$ then $y_1 = F^{k-1}(y_0)$ since

$$y_1 = H \circ f^{-1}(x_1) = H \circ f^{k-1}(x_0) = F^{k-1}(H(x_0)) = F^{k-1}(y_0).$$

□

Corollary 3.3. *Let B be a general Boshernitzan of last break point $x_1 = B(x_0)$ and with irrational rotation number. Then:*

i) if $\sigma_B(x_0) \times \sigma_B(x_1) = 1$ then B is conjugate to a C^1-diffeomorphism through a piecewise quadratic homeomorphism with one break point x_1.

ii) if $\sigma_B(x_0) \times \sigma_B(x_1) \neq 1$ then B is conjugate to a homeomorphism of class P with a unique break point through a piecewise quadratic homeomorphism with one break point x_1.

Proof. This follows from Lemma 3.2, case a) for $p = 1$. □

4. Proof of Theorem 1.6 and Corollary 1.8

Proof of Theorem 1.6.

i) $\Longrightarrow$ *ii)*. If H is the piecewise C^1-homeomorphism which conjugate f to a diffeomorphism F: $f = H^{-1} \circ F \circ H$ then $f^n = H^{-1} \circ F^n \circ H$. As H and H^{-1} have the same number l of break points then f^n has at most $2l$ break points for every $n \in \mathbb{Z}$.

ii) $\Longrightarrow$ *iii)* is the proposition 2.2.

iii) $\Longrightarrow$ *iv)* is the proposition 2.3.

iv) $\Longrightarrow$ *v)* follows from proposition 2.3 and corollary 3.3, i).

v) $\Longrightarrow$ *i)* is obvious.

Proof of Corollary 1.8. The implications *i)* $\Rightarrow$ *ii)* and *ii)* $\Rightarrow$ *iii)* are consequence of Theorem 1.6. The implication *iv)* $\Rightarrow$ *i)* is clear. The implication *iii)* $\Rightarrow$ *iv)* follows from the fact that if B is the affine Boshernitzan of last break point 0 with slopes (λ, λ') then the piecewise analytic homeomorphism h with one break point defined by its restriction to $[0, 1[$ of its lift (denoted also by h) $h(x) := \frac{(\frac{\lambda}{\lambda'})^x - 1}{\frac{\lambda}{\lambda'} - 1}$ conjugate B to the rotation R_α where $\alpha = \frac{Log\lambda}{Log\lambda - Log\lambda'}$.

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