Density of periodic points, invariant measures and almost equicontinuous points of Cellular Automata

Pierre Tisseur [a]

[a] *Department of Mathematics, Trent University, Peterborough, Ontario, Canada*

Abstract

Revisiting the notion of μ-almost equicontinuous cellular automata introduced by R. Gilman, we show that the sequence of image measures of a shift ergodic measure μ by iterations of a μ-almost equicontinuous cellular automata F, converges in Cesaro mean to an invariant measure μ_c. If the initial measure μ is a Bernouilli measure, we prove that the Cesaro mean limit measure μ_c is shift mixing. Therefore we also show that for any shift ergodic and F-invariant measure μ, the existence of μ-almost equicontinuous points implies that the set of periodic points is dense in the topological support $S(\mu)$ of the invariant measure μ. Finally we give a non trivial example of a couple (μ-equicontinuous cellular automata F, shift ergodic and F-invariant measure μ) which has no equicontinuous point in $S(\mu)$.

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1 Introduction

A one-dimensional cellular automaton (CA) is a discrete mathematical idealization of a space-time physical system. The space, called configuration space, is the set of doubly infinite sequences of elements of a finite set A. The discrete time is represented by the action of a cellular automaton F on this space. Using extensive computer simulations, Wolfram [6] has proposed a first empirical (visual) classification of one dimensional cellular automata. In [3] Gilman propose a formal and measurable classification by roughly dividing the set of cellular automata in two parts, those with *almost equicontinuous*

Email address: `pierre@xaravve.trentu.ca` (Pierre Tisseur).

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points or equicontinuous points and those with *almost expansive points* (partition in order and disorder). The Gilman's classes are defined thanks to a Bernoulli measure μ which correspond to the Wolfram's simulations that use random entry. The measure does not need to be invariant, so the classification can be apply to any cellular automata. In [5], Kurka, introduce a topological classification based on the equicontinuity, sensitivity and expansiveness properties. If a cellular automaton has equicontinuous points, then there exist finite configurations that stop the propagation of the perturbations on the one dimensional lattice. If a cellular automaton has μ-almost equicontinuous points then the probability that a perturbation move to infinity is equal to zero (see [3]). Remark that the class of cellular automata with almost equicontinuous points contains the topological class of CA with equicontinuous points. In this paper we consider the definitions of Gilman (μ expansiveness and μ equicontinuity) in the more general case of shift ergodic measures. We show that under this condition, if a cellular automaton F has μ-equicontinuous points then the sequence $(\mu \circ F^{-n})$ converges in Cesaro mean to an invariant measure μ_c. We prove that F has still μ_c-equicontinuous points and if the initial shift ergodic measure is a Bernouilli measure, then μ_c is a shift mixing measure. Remark that the convergence in Cesaro mean of CA with equicontinuous points in $S(\mu)$ (the topological support of a measure μ), had been done by Blanchard an Tisseur in [1]. In [3] Gilman gives an example of a μ-equicontinuous CA that has no equicontinuous points. The invariant measure μ_c, (limit by Cesaro mean of $(\mu \circ F^n)$) that we can construct (using ou results) for this particular automaton still has μ_c-equicontinuous points, but the restriction of this CA to the topological support of μ_c has equicontinuous points. Using Cesaro mean of images measures, we describe a cellular automaton with a non trivial dynamic which keep the sensitiveness property (no equicontinuous points) if we restrict its action to the topological support $S(\mu_c)$ of the invariant measure μ_c. This example use a "counter" dynamic and is defined thanks to a composition of 5 cellular automata acting on different shifts. In [2] Boyle and Kitchen have shown that closing cellular automata have always a dense set of periodic points. The expansive CA and some cellular automata with equicontinuous points belong to this large class. Here, we prove that if μ is a shift ergodic measure and F is a μ-invariant cellular automaton with μ-equicontinuous points then the set of F-periodic points is dense in the the topological support $S(\mu)$. This result extends a previous result on the density of periodic points of surjective with equicontinuous points, cellular automata acting on a mixing subshift of finite type (see [1]). Finally, even if μ-equicontinuity appears to have a more complex dynamic that equicontinuity for almost all the points, we show that the measurable entropy $h_\mu(F)$ of any cellular automaton F with μ-almost equicontinuous points is equal to zero if the measure μ is shift ergodic and F invariant.

2 Definitions and preliminary results

2.1 Symbolics systems and cellular automata

Let A be a finite set or alphabet. Denote by A^* the set of all concatenations of letters in A. These concatenations are called words. The length of a word $u \in A^*$ is denoted by $|u|$. The set of bi-infinite sequences $x = (x_i)_{i\in\mathbb{Z}}$ is denoted by $A^{\mathbb{Z}}$. A point $x \in A^{\mathbb{Z}}$ is called a configuration. For $i \leq j$ in $\mathbb{Z}$ we denote by $x(i, j)$ the word $x_i \dots x_j$ and by $x(p, \infty)$ the infinite sequence $(v_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$ one has $v_i = x_{p+i-1}$. We endow $A^{\mathbb{Z}}$ with the product topology. The shift $\sigma\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by : $\sigma(x) = (x_{i+1})_{i\in\mathbb{Z}}$. For each integer t and each word u, we call cylinder the set $[u]_t = \{x \in A^{\mathbb{Z}} : x_t = u_1 \dots ; x_{t+|u|} = u_{|u|}\}$. For this topology $A^{\mathbb{Z}}$ is a compact metric space. A metric compatible with this topology can be defined by the distance $d(x, y) = 2^{-i}$ where $i = \min\{|j| \text{ such that } x(j) \neq y(j)\}$. The dynamical system $(A^{\mathbb{Z}}, \sigma)$ is called the full shift. A subshift X is a closed shift-invariant subset X of $A^{\mathbb{Z}}$ endowed with the shift σ. If $\alpha = \{A_1, \dots, A_n\}$ and $\beta = \{B_1, \dots, B_m\}$ are two partitions denote by $\alpha \vee \beta$ the partition $\{A_i \cap A_j\; i = 1 \dots n,\; j = 1, \dots, m\}$. Consider a probability measure μ on the Borel sigma-algebra $\mathcal{B}$ of $A^{\mathbb{Z}}$. If μ is σ-invariant then the topological support of μ (which is the smallest closed subset of mesure 1) is a subshift denoted by $S(\mu)$. The metric entropy $h_\mu(T)$ of a transformation T is an isomorphism invariant between two μ-preserving transformations. Put $H_\mu(\alpha) = \sum_{A\in\alpha} \mu(A) \log \mu(A)$. The entropy of the partition α is defined as $h_\mu(\alpha) = \lim_{n\to\infty} 1/n H_\mu(\vee_{i=0}^{n-1} T^{-i}\alpha)$ and the entropy of (X, T, μ) as $\sup_\alpha h_\mu(\alpha)$. A cellular automaton (CA) is a continuous self-map F on $A^{\mathbb{Z}}$ commuting with the shift. The Curtis-Hedlund-Lyndon theorem states that for every cellular automaton F there exist an integer r and a block map f from A^{2r+1} to A such that: $F(x)_i = f(x_{i-r}, \dots, x_i, \dots, x_{i+r})$. The integer r is called the radius of the cellular automaton. If the block map of a cellular automaton is such that $F(x)_i = f(x_i, \dots, \dots, x_{i+r})$, the cellular automaton is called one-sided and can be extended a map on a two-sided shift $A^{\mathbb{Z}}$ or a map on a one-sided shift $A^{\mathbb{N}}$. If X is a subshift of $A^{\mathbb{Z}}$ and one has $F(X) \subset X$, the restriction of F to X determines a dynamical system (X, F); it is called a cellular automaton on X. For example, given any shift invariant measure we can consider the restriction of the cellular automaton $(F, A^{\mathbb{Z}})$ to $(F, S(\mu))$. A closed subset of $Y \subset A^{\mathbb{Z}}$ (not necessarily shift-invariant) such that $F(Y) \subset Y$ is said F-invariant.

2.2 Almost equicontinuous points of cellular automata

In [3] Gilman shows that for a Bernoulli measure μ it is possible to divide the cellular automata set in three following classes: The class of CA where there exists equicontinuous points, the class of CA with μ-almost equicontinuous points but without equicontinuous point and the class of almost expansive CA. In this section we recall the topological and measurable definitions for the expansiveness and equicontinuous classes of cellular automata F of radius r acting on set $A^{\mathbb{Z}}$ where A is a finite set. Since all the Gilman proof use the shift ergodicity of the Bernoulli measure, we extend the initial definitions to any shift ergodic measure.

For any real $\epsilon > 0$, define $D(x, \epsilon)$ the set of the point y such that for all $i \in \mathbb{N}$ one has $d(F^i(x), F^i(y)) \leq \epsilon$ and for all positive integer n write $B_n(x) = D(x, 2^{-n})$. Let μ be any probability measure on $A^{\mathbb{Z}}$ and for any $x \in A^{\mathbb{Z}}$ define the cylinder set $C_n(x)$ as the set of y such $y_i = x_i$ with $-n \leq i \leq n$. We use these two types of set to rewrite the following definitions.

Definitions 1 *Equicontinuity*

-A point $x \in A^{\mathbb{Z}}$ is called an equicontinuous point if for all positive integer n there exist another positive integer m such that $B_n(x) \supset C_n(x)$.

-A cellular automata is an almost equicontinuous CA if there exists equicontinuous points.

-A cellular automaton is equicontinuous is all the points $x \in A^{\mathbb{Z}}$ are equicontinuous points.

-A point x is called a μ-almost equicontinuous if $\mu(B_n(x)) > 0$ for all $n \geq r$.

-If μ is a shift ergodic measure, a cellular automaton F is μ-almost equicontinuous if there exists some μ-almost equicontinuous point x.

Definitions 2 *Expansiveness*

-A Cellular automaton is positively expansive if there exists a positive integer n such that for all $x \in A^{\mathbb{Z}}$ one has $B_n(x) = \{x\}$.

-A cellular automaton F is almost expansive if there exists a positive integer n such that for all $x \in A^{\mathbb{Z}}$, $\mu(B_n(x)) = 0$ where μ is a shift ergodic measure.

-A cellular automaton is sensitive if for all points $x \in A^{\mathbb{Z}}$ and each integer $m > 0$ one has $C_m(x) \subsetneq B_r(x)$.

Remark 1 *If x is an equicontinuous point for a CA F which belong to the topological support $S(\mu)$ of some probability measure μ, then x is also a μ-equicontinuous point for F. If the measure μ is shift ergodic, then F is a μ-equicontinuous CA. If there exists a point x and integer $m > 0$ such that $C_m(x) \subset B_r(x)$ then $x(-m, m)$ is called a blocking word. In [3] Gilman prove the partition of the space of CA into two classes (μ-equicontinuous and μ-*

equicontinuous classes) with respect to a Bernoullli measure μ . The partition in μ-equicontinuous and μ-expansive CA (see [3]) is still true for any shift ergodic measure μ.

The following result appears in [3] (in a slightly different form and for Bernoulli measures).

Proposition 1 *[3] Let μ be a shift-ergodic measure and F a cellular automaton with radius r. The following are properties are equivalent:*

(i) There exist a point $x \in A^{\mathbb{Z}}$ such that $\mu(B_r(x)) > 0$.

(ii) The set of μ-equicontinuous points has measure 1 for F.

(iii) Almost all points x verify that for any integer $m \geq 0$ one has

$$\lim_{n\to\infty} \left(\mu\left(C_n(x) \cap B_m(x)\right)/(\mu(C_n(x))\right) = 1.$$

Questions 1 *Is it possible that there exist some non μ-equicontinuous point x such that there exist an integer $m \geq r$ with $\mu(B_m(x)) > 0$?*

The next topological result is due to Gilman (see [4]).

Proposition 2 *[4] If there exist a point x and an integer $m \neq 0$ such that $B_n(x) \cap \sigma^m B_n(x) \neq \emptyset$ with $n \geq r$ then the common sequence $(F^i(y)(-n,n))_{i\in\mathbb{N}}$ of all points $y \in B_n(x)$ is ultimatly periodic.*

Proof:

Since all the elements of $B_n(x)$ share the same sequence $(F^i(x)(-n,n))_{i\in\mathbb{N}}$ and the orbit under F of each shift periodic point is ultimatly periodic, it is sufficient to show that $B_n(x)$ contains a shift periodic element. Take an integer n greater than the radius r of F and pick two points $z_1 = z_1^- x(-n,n) z_1^+$ and $z_2 = z_2^- x(-n,n) z_2^+$ in $B_n(x)$. As the automaton depends on a local rule of radius r, each point $z_{(i,j)} = z_i^- x(-n,n) z_j^+$ belongs to $B_n(x)$ for all $(i,j) \in \{1,2\} \times \{1,2\}$. Pick a point $y_1 = y_1^- x(-n,n) y_1^+ = y_1^{-*} x(-n+m, n+m) y_1^{+*}$ in $B_n(x) \cap \sigma^m B_n(x)$ and remark that the point $y_2 = y_1^{-*} x(-n,n) y_1^+$ belongs to $B_n(x)$. Suppose without loosing generalities that $m > 0$ and define the word $w = x(-n, -n+m)$. By construction we get $y_2(-n, -n+2m) = ww$ and since $y_2 \in \cup_{i=0}^{2} \sigma^{-i} B_n(x)$ we obtain that $\sigma^{-m} y_2 \in B_n(x)$. Repeating the same process, we obtain for all positive integer k a point $y_k \in \cup_{i=0}^{k-1} \sigma^{-i} B_n(x)$ such that $y_k(-n, -n+km) = w^k$. As $B_n(x)$ is a closed and compact set, we can conclude by saying that the σ periodic point $y = \lim_{i\to\infty} \sigma^{-im} y_{2i}$ belong to $B_n(x)$.

$\square$

In [4] Gilman state the following result using the ergodic properties of any Bernoulli measure μ.

Proposition 3 *[4] Let μ be a shift ergodic measure. If a cellular automaton F has a μ-equicontinuous point, then for all $\epsilon > 0$ there exists a F-invariant closed set Y such that $\mu(Y) > 1 - \epsilon$ and the restriction of F to Y is equicontinuous.*

Squetch of the Proof:

Let x be a μ-equicontinuous point and m a positive integer greater than the radius r of the cellular automaton F. Since $\mu(B_m(x)) > 0$, and μ is a shift ergodic measure, there exists a set S of measure 1 such that all points y in S intersect infinitively often the set $B_n(x)$. From Proposition 2 for each positive integer k the sequences $(F^n(y)(-k,k))_{n\in\mathbb{N}}$ are ultimatly periodic. Let $Y_{(p(k))}$ be the set of point y such that each sequence $(F^n(y)(-k,k))_{n\in\mathbb{N}}$ are periodic of period p and preperiod $p(k)$. As the measure μ is shift ergodic, for each real $\epsilon > 0$ there exists a map $p_\epsilon \colon \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have $\mu(Y_{(p_\epsilon(n))}) > 1 - \epsilon \times 2^{-n}$. The set $Y = \lim_{n\to\infty} \cap_{i=1}^{n} Y_{(p(i))}$ is closed since it is an intersection of the closed sets $Y_{(p(i)}$ and $\mu(Y) > 1 - \epsilon$. Each point $y \in Y$ is clearly an equicontinuous point.

$\square$

3 Main results

3.1 Measure entropy and density of the set of periodic points

Proposition 4 *If a cellular automaton F have some μ-equicontinuous points with μ a F-invariant and shift-ergodic measure then its measure entropy $h_\mu(F)$ is equal to zero.*

Proof:

Let α_p be the partition of $A^{\mathbb{Z}}$ by the $2p+1$ central coordinates. Two points x and y belong to the same element of α_p if and only if $x(-p,p) = y(-p,p)$. Let $\alpha_n^p(x)$ be the element of the partition $\alpha_p \cap F^{-1}\alpha_p \dots F^{-n+1}\alpha_p$ which contains x. Clearly for all $n \in \mathbb{N}$ we have $\alpha_p^n(x) \supset B_p(x)$. From Proposition 1, there exist a set of points Z with measure 1 such that if $y \in Z$ then $\mu(B_m(y)) > 0$ for all integer $m \geq 0$. This implies that for almost all y and positive integer p, we have $\lim_{n\to\infty} \frac{-\log \mu(\alpha_p^n(y))}{n} \leq \lim_{n\to\infty} \frac{-\log \mu(B_p(x))}{n} = 0$. Using the Shannon theorem which tell that

$$h_\mu(F,\alpha_p) = \int_{A^{\mathbb{Z}}} \lim_{n\to\infty} \frac{-\log\mu(\alpha_p^n(y))}{n} d\mu(y),$$

we can conclude that $h_\mu(F) = \sup_{\alpha_p} h_\mu(F,\alpha_p) = 0$.

□

Remark 2 *From remark 1, all cellular automata F which have equicontinuous points in the topological support of a shift ergodic measure μ, verify : $h_\mu(F) = 0$.*

Proposition 5 *Let μ be a shift ergodic and cellular automaton F, invariant measure . If F has μ-almost equicontinuous points then the set of F-periodic points is dense in the topological support $S(\mu)$.*

Proof:

We are going to show that for any point $z \in S(\mu)$ and positive integer p we can construct a σ and F periodic point $\bar{w} = ^\infty w^\infty$ such that there exists an integer $r \leq k \leq |w|$ with $w(k, 2p+1+k) = z(-p,p)$. We recall that r is the radius of F. Since there is a μ-almost equicontinuous point $x \in S(\mu)$ then $\mu(B_r(x)) > 0$. Since μ is shift ergodic, there exist positive integers i and j such that $\mu\Big(C_p(z) \cap \sigma^{-(i+p)} B_r(x) \cap \sigma^{j+p} B_r(x)\Big) > 0$. To simplify, write $S = C_p(z) \cap \sigma^{-(i+p)} B_r(x) \cap \sigma^{j+p} B_r(x)$ and pick a point $y \in S$. From Proposition 2, there exist a shift periodic point $\overline{w} \in S$ such that $\overline{w}(-r-i-p, j+p) = w = y(-r-i-p, j+p)$. From the Poincaré recurrence theorem, there exists an integer m such that $\mu\left(S \cap F^{-m} S\right) > 0$. This implies that there exist a point $y \in S$ such that $F^m(y)(-r-i-p, r+j+p) = y(-r-i-p, r+j+p)$. But from the definition of $B_r(x)$, all the points $y \in S$ share the same sequence $(F^i(y)(-r-i-p, j+p)) = (F^i(\overline{w})(-r-i-p, j+p))_{i\in\mathbb{N}}$. Since the common σ period of $\overline{w}$ is $r+i+j+2p+1$ we obtain that $F^m(\overline{w}) = \overline{w}$. Arguing that the sequence of images by a cellular automaton of any shift periodic point is ultimatly periodic we can assert that $(F^i(\overline{w}))$ is a periodic sequence and conclude.

□

3.2 Invariant measures as limit of Cesaro means

Proposition 2 also allows us to prove a Cesaro mean convergence result.

Theorem 3 *Let μ be a shift-ergodic measure. If a cellular automaton F has some μ-almost equicontinuous points then $(\mu \circ F^{-n})$ converges vaguely in*

Césaro mean under F to an invariant measure μ_c.

Proof

To show that the sequence of measure $(\frac{1}{n}\sum_{i=0}^{n-1}\mu\circ F^{-i})_{n\in\mathbb{N}} = (\mu_n)_{n\in\mathbb{N}}$ converge vaguely in measure we need to show that for all $x\in S(\mu)$ and $m\in\mathbb{N}$ the sequence $(\mu_n(C_m(x)))_{n\in\mathbb{N}}$ converges. Since there exist a point z and an integer $m>0$ with $\mu(B_m(z))>0$ where μ a shift ergodic measure, we get $\lim_{n\to\infty}\mu(\cup_{i=-n}^{n}\sigma^{-i}B_m(z)) = 1$. Using the same arguments than in Proposition 3 we can assert that there exists a set $Y_{(\mathbf{P}_\epsilon(k))}$ of measure greater than $1-\epsilon$ such that all the sequences $(F^n(y)(-k,k))_{n\in\mathbb{N}}$ are eventually periodic with preperiod $pp_\epsilon(k)$ and period $p_\epsilon(k)$ if $y\in Y_{(\mathbf{P}_\epsilon(k),k)}$ and $\mathbf{P}_\epsilon(k) = (pp_\epsilon(k), p_\epsilon(k))$. Hence for all $x\in A^{\mathbb{Z}}$

$$\begin{aligned}\mu_n(C_m(x)\cap Y_{(\mathbf{P}_\epsilon(k))}) = {} & \tfrac{1}{n}\textstyle\sum_{i=0}^{pp_\epsilon(k)-1}\mu\left(F^{-i}\left(C_m(x)\right)\cap Y_{(\mathbf{P}_\epsilon(k))}\right)\\ & +\tfrac{1}{n}\textstyle\sum_{i=pp_\epsilon(k)}^{n-1}\mu\left(F^{-i}\left(C_m(x)\right)\cap Y_{(\mathbf{P}_\epsilon(k))}\right).\end{aligned}$$

The first term tends to 0; using periodicity one gets

$$\lim_{n\to\infty}\mu_n(C_m(x)\cap Y_{(\mathbf{P}_\epsilon(k))}) = \frac{1}{p_\epsilon(k)}\sum_{i=0}^{p_\epsilon(k)-1}\mu\left(F^{-(i+pp_\epsilon(k))}(C_m(x)\cap Y_{(\mathbf{P}_\epsilon(k))}\right).$$

Clearly if $k\geq m$ we have $\lim_{\epsilon\to 0}\mu_n(C_m(x)\cap Y_{(\mathbf{P}_\epsilon(k))}) = \mu_n(C_m(x))$. The convergence is uniform with respect to ϵ since for all x and $m\in\mathbb{N}$

$$\left|\mu_n(C_m(x)\cap Y_{(\mathbf{P}_\epsilon(k))}) - \mu_n(C_m(x))\right| \leq \frac{n\epsilon}{n} = \epsilon.$$

Consequently, letting ϵ going to 0 and assuming that $k\geq m$, we get the result by inversing the limits

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu\circ F^{-i}(C_m(x)) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\lim_{\epsilon\to 0}\mu\circ F^{-i}(C_m(x)\cap Y_{(\mathbf{P}_\epsilon(k))})$$

$$= \lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu\circ F^{-i}(C_m(x)\cap Y_{(\mathbf{P}_\epsilon(k))})$$

$$= \lim_{\epsilon\to 0}\frac{1}{p_\epsilon(k)}\sum_{i=0}^{p_\epsilon(k)-1}\mu\left(F^{-(i+pp_\epsilon(k))}(C_m(x))\cap Y_{(\mathbf{P}_\epsilon(k))}\right) = \mu_c(C_m(x)).$$

We denote by μ_c the Cesaro mean limit of $(\mu\circ F^n)_{n\in\mathbb{N}}$. □

In the following Proposition and Corrolary, we suppose that μ_c is a Probability measure on $A^{\mathbb{Z}}$ which came from the Cesaro mean of $(\mu\circ F^{-n})_{n\in\mathbb{N}}$. Remark that the three results of this subsection, remain true for cellular automata with equicontinuous points in $S(\mu)$.

Proposition 6 *If μ is a Bernouilli measure and F a μ-equicontinuous CA, then the Cesaro mean measure μ_c is a shift mixing measure.*

Proof

It is enough to show that for any points x and y and positive integer m and n we have $\lim_{t\to\infty} \mu_c(C_m(x) \cap \sigma^{-t} C_n(y)) = \mu_c(C_m(x)) \times \mu_c(C_n(y))$. From the proof of Theorem 3, we get for any $k \geq \max\{m, n\}$

$$\mu_c(C_m(x)) = \lim_{\epsilon\to 0} \frac{1}{p_\epsilon(k)} \sum_{i=0}^{p_\epsilon(k)-1} \mu\left(F^{-(i+pp_\epsilon(k))}(C_m(x)) \cap Y_{(\mathbf{P}_\epsilon(k))}\right)$$

and

$$\mu_c(C_n(y)) = \lim_{\epsilon\to 0} \frac{1}{p_\epsilon(k)} \sum_{i=0}^{p_\epsilon(k)-1} \mu\left(F^{-(i+pp_\epsilon(k))}(C_n(y)) \cap Y_{(\mathbf{P}_\epsilon(k))}\right).$$

Remark that for all $y \in Y_{(\mathbf{P}_\epsilon(k),k)}$, the sequence $\left(F^{pp_\epsilon(k)+n}(y)(-k,k)\right)_{n\in\mathbb{N}}$ is periodic. Since μ is a Bernouilli measure it follows that when $t \geq 2k+1+2pp_\epsilon(k) \times r$ we obtain for all $i \in \mathbb{N}$

$$\mu\left(F^{-i}(C_m(x)) \cap Y_{(\mathbf{P}_\epsilon(k))} \cap F^{-i}(\sigma^{-t}C_n(y)) \cap \sigma^{-t}Y_{(\mathbf{P}_\epsilon(k))}\right)$$

$$= \mu\left(F^{-i}(C_m(x)) \cap Y_{(\mathbf{P}_\epsilon(k))}\right) \times \mu\left(F^{-i}(\sigma^{-t}C_n(y)) \cap \sigma^{-t}Y_{(\mathbf{P}_\epsilon(k))}\right).$$

We write $t_\epsilon = 2k+1+2pp_\epsilon(k) \times r$. Remark that for all $t \in \mathbb{N}$ and $\epsilon > 0$, we have $\mu(Y_{(\mathbf{P}_\epsilon(k))} \cap \sigma^{-t} Y_{(\mathbf{P}_\epsilon(k)))}) > 1 - 2\epsilon$.

Let $A = \liminf_{t\to\infty} \mu_c(C_m(x) \cap \sigma^{-t} C_n(y)) =$

$$\liminf_{t\to\infty} \lim_{n\to\infty} \lim_{\epsilon\to 0} \frac{1}{n} \sum_{i=0}^{n-1} \mu\left(F^{-i}(C_m(x)) \cap \sigma^{-t}C_n(y)) \cap Y_{(\mathbf{P}_\epsilon(k))} \cap \sigma^{-t}Y_{(\mathbf{P}_\epsilon(k))}\right).$$

Using similar arguments of those arising in the proof of Theorem 3, for the convergence with respect to n and the uniform convergence with respect to ϵ, we can write that A is equal to

$$\liminf_{t\to\infty} \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu\left(F^{-i}(C_m(x)) \cap \sigma^{-t}C_n(y)) \cap Y_{(\mathbf{P}_\epsilon(k))} \cap \sigma^{-t}Y_{(\mathbf{P}_\epsilon(k))}\right).$$

Using the uniform convergence with respect to ϵ, we can invert the limitinf with respect to t and the limit with respect to ϵ and obtain that A is equal to

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu\left(F^{-i}(C_m(x))\cap\sigma^{-t_\epsilon}C_n(y))\cap Y_{(\mathbf{P}_\epsilon(k))}\cap\sigma^{-t_\epsilon}Y_{(\mathbf{P}_\epsilon(k))}\right)$$

$$=\lim_{\epsilon\to 0}\frac{1}{p_\epsilon(k)}\sum_{i=0}^{p_\epsilon(k)-1}\mu\left(F^{-i+pp_\epsilon(k)}\left(C_m(x))\cap\sigma^{-t_\epsilon}C_n(y)\right)\cap Y_{(\mathbf{P}_\epsilon(k))}\cap\sigma^{-t_\epsilon}Y_{(\mathbf{P}_\epsilon(k))}\right).$$

Using the independance of the finite sets $\{F^{-i+pp_\epsilon(k)}\left(C_m(x)\right))\cap Y_{(\mathbf{P}_\epsilon(k))}\, i\in\mathbb{N}\}$ and the image by σ^{-t_ϵ} of $\{F^{-i+pp_\epsilon(k)}\left(C_n(y)\right))\cap Y_{(\mathbf{P}_\epsilon(k))}\, i\in\mathbb{N}\}$ with respect to the measure μ it follows that $A=\mu_c(C_m(x))\times\mu_c(C_n(y))$. If we substitute lim sup instead of lim inf in A we obtain the same result, so we can conclude that $\lim_{t\to\infty}\mu_c(C_m(x)\cap\sigma^{-t}C_n(y))=\mu_c(C_m(x))\times\mu_c(C_n(y))$.

$\square$

Remark 4 *If the initial measure μ is the direct product of a Bernouilli measure and some atomic measure on a point of the type $\overline{a}=\ldots aaaa\ldots$ which consists in the repetition of the same letter a in the alphabet A, we obtain that the limit measure μ_c is shift mixing using the same arguments than in Theorem 3.*

Proposition 7 *If $(S(\mu),F)$ is a μ-almost equicontinuous cellular automaton and the Cesaro mean limit measure $\mu_c=\frac{1}{n}\sum\mu\circ F^{-i}$ is shift mixing, then μ_c is still μ_c-equicontinuous for $(S(\mu_c),F)$.*

Proof

Since μ_c is a shift ergodic measure, it remains to show that there exist a point z and an integer $m>0$ such that $\mu_c(B_m(z))>0$. If F is μ equicontinuous then there exist a point x and a positive integer m such that $\mu(B_m(x))>0$. As μ is a shift ergodic measure, there exists an integer $t>0$ such that $B_m(x)\cap\sigma^{-t}B_m(x)\neq\emptyset$. From Proposition 3, there exist positive integers $pp>0$ and $p>0$ such that $(F^{n+pp}(x)(-m,m))_{n\in\mathbb{N}}$ is periodic of period p. Let $z=F^{pp}(x)$, we can conclude arguing that

$$\mu_c(B_m(z))\geq\lim_{p\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu(F^{-i}(B_m(z)\cap B_m(x)),$$

$$\mu_c(B_m(z))\geq\frac{1}{p}\sum_{i=0}^{p-1}\mu(F^{-i-pp}(B_m(z)\cap B_m(x))$$

and

$$\mu_c(B_m(z))\geq\frac{1}{p}\mu(F^{-pp}(B_m(z)\cap B_m(x))=\frac{1}{p}\mu(B_m(x))>0.$$

$\square$

3.3 Example of μ-equicontinuous CA without equicontinuous points

When we study and compare some measurable (given an invariant measure μ) and topological properties of the dynamic of cellular automata, it is natural to consider the dynamical system $(F, S(\mu))$ which is the restriction to the topological support $S(\mu)$ of the action of a cellular automaton F. The automaton F is a surjective map $S(\mu) \to S(\mu)$ and from Proposition 5 the set of periodic points in $S(\mu)$ is dense when μ is also shift ergodic and F has μ-equicontinuous points. In the case of basic examples of μ-equicontinuous CA (see the example given in [3]), the systems $(F, S(\mu))$ always contains equicontinuous points even if $(F, A^{\mathbb{Z}})$ may not have these kind of points.

Roughly, if μ is a shift ergodic measure, to have μ-equicontinuous points without equicontinuous points requires that there exist some ‘perturbations’ that can move to infinity but the probability that these perturbations move to infinity is equal to zero. One way to get theses properties for an automaton $(F, S(\mu))$, is that F generate permanently ‘propagating structures” of different sizes. The “length of life” of the “propagating structures” depends on their size. This is roughly the dynamic of the following cellular automaton F_e.

3.3.1 Definition of the cellular automaton F_e

The automaton F_e we consider act on $X = X_0 \times X_1 \times X_2$ where $X_0 = \{0,1\}^{\mathbb{Z}}$, $X_1 = \{E_0, E_1, E_2, E_3, 0, R, L\}^{\mathbb{Z}}$ and $X_2 = \{0,1\}^{\mathbb{Z}}$. We define F_e as the composition of 5 other cellular automata $F_e = F_3 \circ F_2 \circ F_1 \circ F_p^{X_0} \circ F_p^{X_1}$. To simplify we write $\hat{E} = \{E_0, E_1, E_2, E_3\}$ and $\overline{E} = \{0, L, R\}$.

In the following we denote by $x = (x^0, x^1, x^2)$ any point $x \in X$ where $x^i \in X_i$ with $0 \leq j \leq 2$. The letter in position i in the restriction of $x \in X$ to X_j is denoted by x_i^j. We denote by $\mathbf{1}_S(x)$ the map which is equal to one if $x \in S$ and zero otherwise.

The automaton F_1 is the identity on $X_0 \times X_1$ and its restriction to X_2 came from the following block map f_1 of radius 3

$$f_1(x_{i-3}^2, \dots, x_i^2, \dots, x_{i+3}^2) = \mathbf{1}_{\{1\}}(x_{i-3}^2) \times \mathbf{1}_{\{1\}}(x_{i-2}^2) \times \mathbf{1}_{\{1\}}(x_{i-1}^2).$$

The automaton F_2 is still the identity on $X_0 \times X_1$ but its action on X_2 depends on X_1. The block map f_2 is defined by:

$$f_2 \begin{pmatrix} x_{i-2}^1, \dots, x_i^1, \dots, x_{i+2}^1 \\ x_{i-2}^2, \dots, x_i^2, \dots, x_{i+2}^2 \end{pmatrix} = \begin{pmatrix} x_i^1 \\ x_i^2 \vee_{j=0}^2 \mathbf{1}_{\{E_0\}}(x_{i-j}^1) \end{pmatrix}$$

where $\vee_{i=0}^2 \mathbf{1}_{\{E_0\}} x_{i-j}^1$ is equal to 1 when at least one x_{i-j}^1 is equal to 1.

The automaton F_3 is the identity map on $X_0 \times X_2$ and is defined thanks to a local rule f_3 on X_1.

The block map f_3 is defined by the following rules :

$$
\begin{aligned}
f_3(x^1_{i-11}, \dots x^1_i, \dots x^1_{i+11}) &= R \text{ if } x^1_{i-10}, \dots x^1_i, \dots x^1_{i+k} = R0^{11+k} \\
&\quad \text{where } m = \min\{10, \min\{k \mid x_{i+k} \in \hat{E}\}\} \\
&= L \text{ if } x^1_{i-k}, \dots x^1_i, \dots x^1_{i+10} = 0^{11+k}L \\
&\quad \text{where } m = \min\{10, \min\{k \mid x_{i-k} \in \hat{E}\}\} \\
&= R \text{ if } \exists\, 0 \leq k, j \leq 9 \text{ such that} \\
&x^1_{i-j-k}, \dots, x^1_i, \dots, x^1_{i+10} = E^* 0^k L 0^{j+10} \\
&\quad \text{with } j + 2k + 1 = 10 \text{ and } E^* \in \hat{E}. \\
&= L \text{ if } \exists\, 0 \leq k, j \leq 9 \text{ such that} \\
&x^1_{i-10}, \dots, x^1_i, \dots, x^1_{i+j+k} = 0^{10+j} R 0^k E^* \\
&\quad \text{with } j + 2k + 1 = 10 \text{ and } E^* \in \hat{E}.
\end{aligned}
$$

Consider now the case where the central coordinate x_i is an element of $\hat{E}$.

$$
\begin{aligned}
&\text{For each } i \in \{0, 1, 2, 3\} \\
&f_3(x^1_{i-10}, \dots, E_i, \dots x^1_{i+10}) = E_{i+1} \text{ if } x^1_{i-k}, \dots, x^1_i = R0^{k-1}E_i \text{ with } 0 \leq k \leq 9 \\
&\qquad\qquad\qquad\qquad\qquad \text{where the addition `} i+1 \text{' is made modulo 4.}
\end{aligned}
$$

For all the other cases where the central coordinate x_i is an element E^* in $\hat{E}$ we have $f_3(x^1_{i-10}, \dots, E^*, \dots x^1_{i+10}) = E^*$. In all the other cases that have not been discribed above we have $f_3(x^1_{i-10}, \dots, x^1_i, \dots x^1_{i+10}) = 0$.

The automaton $F_p^{X_0}$ is the identity on X_2 and its action on $X_0 \times X_1$ came from the following local rule:

$$
f_p^{X_0}\begin{pmatrix} x^0_{i-10}, \dots, x^0_i, \dots, x^0_{i+10} \\ x^1_{i-10}, \dots, x^1_i, \dots, x^1_{i+10} \end{pmatrix} = \begin{pmatrix} f_p^{(X_0,0)} \\ \mathbf{1}_0(x^0_{i-1})x^1_i + \mathbf{1}_1(x^0_{i-1})\left[\mathbf{1}_{\overline{E}}(x^1_i)x^1_i \vee \mathbf{1}_{\hat{E}}(x^1_i)E_1\right] \end{pmatrix}
$$

where

$$
\begin{aligned}
f_p^{(X_0,0)} &= 1 \text{ if } x^1_{i-1} \in \hat{E} \\
&= 1 \text{ if } x^1_i \notin \hat{E} \text{ ; } x^1_{i-n}, \ldots x^1_{i+m} = 0^{m+n+1} \text{ where } m = \min \\
&\{10, \min\{k \mid x_{i-k} \in \hat{E}\}\} \text{ and } n = \min\{10, \min\{k \mid x_{i+k} \in \hat{E}\}\} \\
&\text{ and } \exists\, 0 \leq l \leq \lceil \tfrac{m-1}{2} \rceil \text{ such that } x^0_{i-l} = 1 \\
&= 0 \text{ otherwise.}
\end{aligned}
$$

The automaton $F_p^{X_1}$ is the identity on X_0 and X_2 and its action on X_1 is given by the local rule $f_p^{X_1}$:

$$f_p^{X_1}(x^1_{i-152}, ..., x^1_i, ..., x^1_{i+152}) = \mathbf{1}_{\overline{E}}(x^1_i)x^1_i + \mathbf{1}_{\hat{E}}(x^1_i)x^1_i \times \prod_{j=-152}^{152} \mathbf{1}_{\overline{E}}(x^1_{i+j}).$$

3.3.2 The dynamic of F_e

In this subsection, we describe the global dynamic of F_e by showing the contribution of each of the 5 cellular automata.

THE DYNAMIC OF F_1

The action of F_1 on X_2 is only the shift of consecutive sequences (or "trains") of letters "1" of one coordinate to the right and the destruction of the two last letters "1" at the left side of this *train*.

Action of F_1 on a finite configuration of X_1:

$$01111110000 \overset{F_1}{\mapsto} 00001111000 \overset{F_1}{\mapsto} 00000001100 \overset{F_1}{\mapsto} 0000000000$$

Remark that a *train* of 1 with a length $2k+1$ will move of k coordinates to the right before collapsing.

THE DYNAMIC OF F_2

The action of the cellular automaton F_2 is to 'create' a sequence of three letters '1' in X_2 when there is a letter E_0 in x^1_i.

$$\text{Example: } \binom{E_0}{0}\binom{*}{0}\binom{*}{0} \overset{f_2}{\mapsto} \binom{E_0}{1}\binom{*}{1}\binom{*}{1} \overset{f_2}{\mapsto} \binom{E_0}{1}\binom{*}{1}\binom{*}{1}.$$

The symbol * replace any letter in $\{0, L, R\}$.

Action of $F_2 \circ F_1$

If $F_e^i(x^1) = E_0$ for $0 \leq i \leq n$ then there is at least a *train of 1* of length $n+3$ moving to the left of at least $\lceil \frac{n+3}{2} \rceil$ coordinates.

Action of $F_2 \circ F_1$ (* is any letter in $\hat{E} \cup \overline{E}$)

$$\begin{array}{rll} & {}^{\infty}0E_0 * * * * * * * * * * \dots & x^0 \\ & {}^{\infty}0000000000 \cdots & x^1 \\ (F_2 \circ F_1)^n \downarrow & & \\ & {}^{\infty}0E_0 * * * * * * * * * * \dots & F^n(x)^0 \\ & \underbrace{{}^{\infty}01111 \cdots 1}_{n+3 \text{ times}} 000 \cdots & F^n(x)^1 \end{array}$$

The dynamic of $F_p^{X_1}$

The cellular automaton $F_p^{X_1}$ is a projection of X to $X' = X_0 \times X_1' \times X_2$ where X_1' is the subshift of finite type where there is at least $l_m = 152$ letters in $\overline{E}$ between 2 letters in $\hat{E}$. Since in X_1' there no 2 consecutive letters in $\hat{E}$, it induce a special kind of dynamic (*counter dynamic*) due to the action of F_3 on X_1'. In the following (except in a part of the proof of Proposition 8) we will consider the action of F_e on X' rather than X.

The 'counter' dynamic of F_3 on X_1'

Under the action of F_3, a letter R surrounded by 10 letters 0 in the right and left coordinates moves of 10 coordinates to the right. A letter L also surrounded by 10 letters 0 moves of 10 coordinates to the left. When a letter R surrounded by only letters 0 and one letter $E_i \in \hat{E}$, reaches a position situated at less than 10 coordinates to the right side of the letter E_i, the letter E_i changes in E_{i+1}, (the addition is made modulo 4) and the letter L becomes a letter R and start to move to the left side. When a letter L surrounded by only letters 0 and one letter $E_i \in \hat{E}$ at a coordinate situated at less than 10 coordinates to the left, the letter R becomes a letter L without modify the letter E_i.

Now consider a pattern of the form $E^* *^l E^*$ where $* \in \overline{E}$ and $E^* \in \hat{E}$. First remark that for any points x which contains this pattern, the evolution under F_3 of the subpattern $*^l E^*$ depends only on the initial $E^* *^l E^*$. Since the block map f_3 applied on a word $u \in \{\hat{E} \cup \overline{E}\}^{23}$ gives a letter 0 when appears more than one letter R or L in u, it is clear that after $\lceil \frac{l}{10} \rceil$ iterations it remains at most only one letter L or R in any pattern $E^* *^l E^*$ in $F^{\lceil \frac{l}{10} \rceil}(x)$.

We call *counter* of size $i+j+1$, any finite configuration of the form $E^*0^i * 0^j E^*$ and *void counter* any pattern of the form $E^*0^kE^*$ where $k \in \mathbb{N}$, $E^* \in \hat{E}$ and $* \in \{R, L\}$. The action of F_3 on the $l+1$ last coordinates of *counters* of size l is a rotation. The letter R move to the right and change in L in the neighborhood of the last letter E_i which change in E_{i+1}. Then the L return to the left side and change in R in the neighborhood of the first E^*. The period of a *counter* of size l is approximatly equal to $\frac{l}{5}$ (between $\lfloor \frac{l}{5} \rfloor$ and $\lceil \frac{l}{5} \rceil$). This period corresponds to the number of iterations needed for the commuting letters L, R to go and return in the neighborhood (less than 10 coordinates) of the first $E^* \in \hat{E}$.

Let's see a typical evolution of a *counter*.

$$\begin{array}{l}
{}^{\infty}\!* E^*0R0^{140}000000000RLE_3 \overset{F_3^{14}}{\longmapsto} {}^{\infty}\!*E^*000^{140}R0000000000E_3 \overset{F_3}{\hookleftarrow} \\
\hookrightarrow {}^{\infty}\!*E^*000^{140}0000000000RE_3 \overset{F_3}{\longmapsto} {}^{\infty}\!*E^*000^{140}R0000000000E_0 \overset{F_3^{14}}{\hookleftarrow} \\
\hookrightarrow {}^{\infty}\!*E^*0R0^{140}0000000000E_0 \overset{F_3}{\longmapsto} {}^{\infty}\!*E^*00000000L0^{140}0E_0 \overset{F_3^{14}}{\hookleftarrow} \\
\hookrightarrow {}^{\infty}\!*E^*000^{140}000000000RE_0 \overset{F_3}{\longmapsto} {}^{\infty}\!*E^*000^{140}L000000000E_1
\end{array}$$

Considering the additional action of $F_3 \circ F_2 \circ F_1$ and the set X', the set of *counters* split in two subsets : The *void counters* (which represent the patterns of the form $E^*0^kE_i$ with $i \in \{1, 2, 3\}$) and the *counters* whose patterns is of the form $E^*0^kE_0$. In the second case, under the action of $F_3 \circ F_2$, the last letter E_0 generate a continuous flow of letters 1 moving to the right side. In the following section we will see that under the action of $F_p^{X_0}$, the *void counters* of the type $E^*0^kE_0$ will disapear after less than $\lceil \frac{k}{5} \rceil$ iterations. Remark that since a finite configuration ($E^*0^kE_0$) generate a continuous flow (or an infinite length *train*) of letters 1, there would exist equicontinuous points for (F_e, X) (and for $(F_e, S(\mu_c))$ see section 3.3.2) without the action of $F_p^{X_0}$.

ACTION $F_p^{X_0}$ AND $F_p^{X_0} \circ F_3$ on ${}^{\infty}0^{\infty} \times X'_1$

Consider a point $x \in^{\infty} 0^{\infty} \times X'_1 \times X_2$. If $x^1_i = E^*$ ($E^* \in \hat{E}$ and $x \in X'_1$) then $F_p^{X_0}(x^0_{i+1}) = 1$. If there is no letter R or L in the word $x^1_i, \ldots x^1_{i+k}$, letters 1 will appear in position $i+1$ to $i+k$ in x^0 at the next iteration. Then consider some cylinder in X'_1 of the form $[E^{(*,1)}0^j * 0^kE^{(*,2)}]_i \cap X'_1 = [E^{(*,1)}0^j * 0^kE^{(*,2)}]_i^{X'_1}$ ($i, j, k \in \mathbb{N}$; $* \in \{R, L\}$; $E^{(*,\binom{1}{2})} \in \hat{E}$). Since the letter

R and L move two times faster than the letters 1 in X_0 we obtain for each $x \in^\infty 0^\infty \times [E^{(*,1)}0^j * 0^k E^{(*,2)}]_i^{X_1'} \times X_2$ that $y^0_{i+t_1} \cdots y^0_{i+t_2} = 0^{\lceil \frac{t_1}{2} \rceil}$ when $y = F_p^{X_0} \circ F_3(x)$, $t_1 = \lceil \frac{2(j+k+1)}{3} \rceil$ and $t_2 = j+k+1$. It follows that in this case, $F_p^{X_0}$ does not modify the last letter $E^{(*,2)}$. So $F_p^{X_0}$ does not affect the *counters* of the type $E^{(*,1)}0^i * 0^j E^{(*,2)}$ in ${}^\infty 0^\infty \times X_1' \times X_2$ where $* \in \{R, L\}$. Nevertheless, if $x \in^\infty 0^\infty \times [E^{(*,1)}0^l E^{(*,2)}]_i^{X_1} \times X_2$, after $\lceil \frac{l}{5} \rceil$ iterations, a letter 1 will appear in position $i+l+1$ in x^0 and at the next iteration the final letter $E^{*,2}$ will be fixed to E_1. Hence after a while ($\lceil \frac{l}{5} \rceil$) all the patterns $E^* 0^l E^{(*,2)}$ with $E^{(*,2)} \neq E_1$ will disapear. In the more general case ($x \in X'$), the action of $F_p^{X_0}$ on X_1' will be the identity after a while (which means after the iterations of F_e). Notice that the dynamic of the restriction of $F_p^{X_0}$ to ${}^\infty 0^\infty \times X_1'$ is enough for the results of subsection 3.3.3 .

Typical actions of $F_p^{X_0}$ on ${}^\infty 0^\infty \times X_1$ with a *void counter* of size $10n$:

$$\left(F_p^{X_0}\right)^n \left({}^\infty \binom{0}{0}\binom{0}{E_2}\binom{0}{0}\binom{\cdots}{\cdots}\binom{0}{0}\binom{0}{E_0}\right) = \left(F_p^{X_0}\right)^{n-1} \left({}^\infty \binom{0}{0}\binom{0}{E_2}\binom{1}{0}\binom{0}{0}\binom{\cdots}{\cdots}\binom{0}{E_0}\right)$$

$$= F_p^{X_0} \left({}^\infty \binom{0}{0}\binom{0}{E_2}\binom{1}{0}\binom{1}{0}\binom{\cdots}{\cdots}\binom{1}{0}\binom{0}{E_0}\right) =^\infty \binom{0}{0}\binom{0}{E_2}\binom{1}{0}\binom{1}{0}\binom{\cdots}{\cdots}\binom{1}{0}\binom{0}{E_1}.$$

Action of $F_3 \circ F_2 \circ F_1$ on $X_1' \circ X_2$

Consider the *counter* of size l : $\mathcal{C}_l$ of the form $E^{(*,1)}0^j * 0^k E^{(*,2)}$ ($j+k+1 = l$, $* \in \{R, L\}$ and $E^{(*,1)}, E^{(*,1)} \in \hat{E}$). Under the action of $F_3 \circ F_2 \circ F_1$, the last letter $E^{(*,2)}$ stay in the state E_0 during a period of $\frac{l}{5} \leq P_l \leq \frac{l}{5}+1$. Taking into consideration the action of $F_2 \circ F_1$, the rotation of the *counter* $\mathcal{C}_l$ will generate periodically a *train of 1* of length $P_l + 3$ (see action of $F_2 \circ F_3$). During $3P_l$ iterations, the *counter* $\mathcal{C}_l$ is in the non emmitting phase (the last letter is in $\{E_i | \, 1 \leq i \leq 3\}$).

Action of F_e on counters

Since a *counter* $\mathcal{C}_l$ generate a *train of 1* with an approximatly length $\frac{l}{5}$, it can influence some patterns situated at $(\frac{l}{5}+3+(\frac{l}{5}+3)/2 = \frac{3l}{10}+9/2)$ coordinates to the right of the right extremity of the *counter*. Recall that a *train of 1* looses 2 elements and moves of 1 coordinate in one iteration. Remark that using some concatenation process (with other *counters*), the *train of 1* generated by a *counter* $\mathcal{C}_l$ may produce perturbations further than $\frac{3l}{10} + \frac{9}{2}$ coordinates. Roughly, a *train of 1* of length t looses $2l$ elements when it cross a second *counter* of size l to the right side and can gain $\frac{l}{5}+3$ elements (with a good synchronization) thanks to this second counter.

The Figure 1 represents a typical action of F_e on 5 *counters* $\mathcal{C}_{2100}$, $\mathcal{C}_{304}$ and three $\mathcal{C}_{152}$. The first *counter* of size 152 is a *void counter* which never generate any sequence of 1 in X_1. To the left side there is one large *counter* of size 3000 which is in a non emmitting state (last letter not an E_0). For simplification,

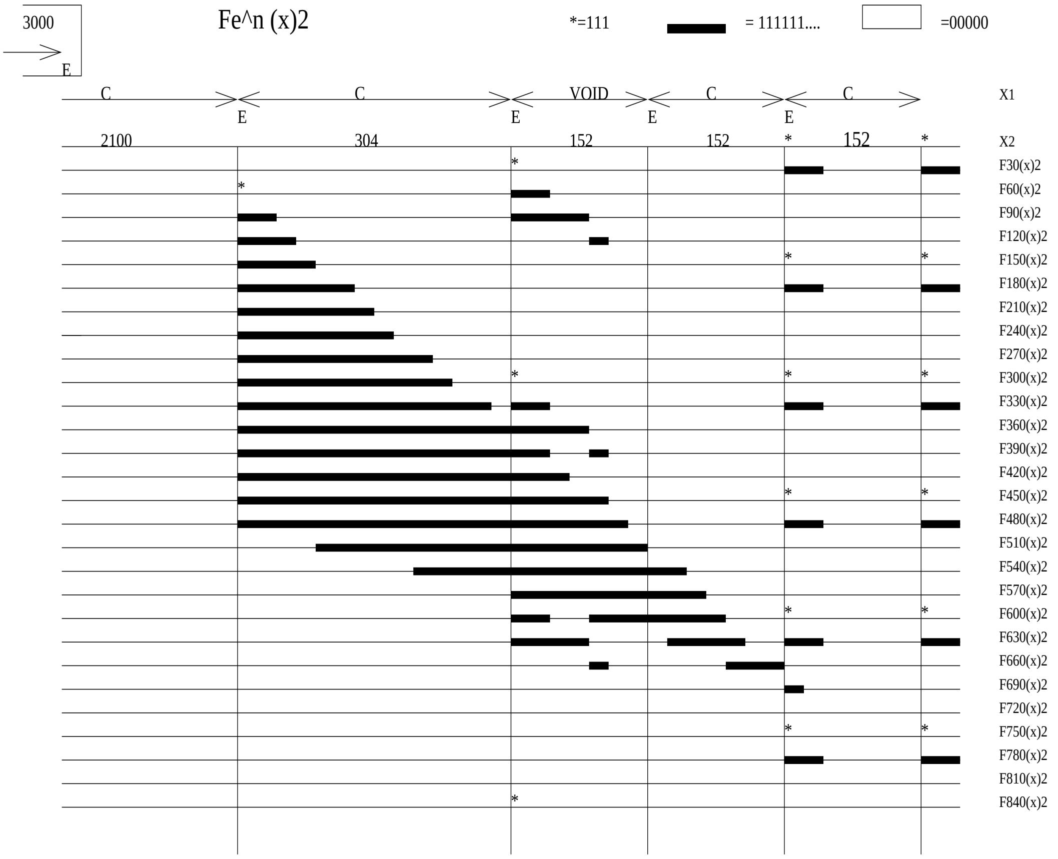

Fig. 1. An illustration of the dynamic of F_e on 5 *counters* and the resulting dynamic of *train of 1* in X_2. Remark that each line represents 30 iterations of F_e. The black horizontal lines represent the *trains of 1* and blank ones, the sequences of 0. The extremity of the counters are delimited by arrows.

we do not specify the states of the *counters* and their evolution because the interesting part of their dynamic can be deduce from the evolution of the *train of 1* in X_1. Remark that the non 'emmitting period' of the *counters* last 3 times more than the 'emmitting' one.

Remark 5 *Recall that 152 is the minimum size of a counter thanks to the action of* $F_p^{X_1}$. *This minimum size is required to simplify the proof of Proposition 9 by using quantitative arguments on flows rather than study the complex dynamic of concatenations of train of 1.*

3.3.3 The topological and measurable properties of F_e

Proposition 8 *There exist a shift mixing and* F_e*-invariant measure* μ_c *such that the cellular automaton* F_e *has* μ_c*-equicontinuous points.*

Proof

Let μ_1 be the uniform Bernouilli measure on $\hat{E} \cup \overline{E}$ and $\mu_I = \delta_{^\infty 0^\infty} \times \mu_1 \times \delta_{^\infty 0^\infty}$ where $\delta_{^\infty 0^\infty}$ is the atomic measure on the fix points $^\infty 0^\infty$ in X_0 and X_2. The measure μ_c we consider is simply the Cesaro limit of some converging subsequence $(\mu_I \circ F^{-n})_{n \in \mathbb{N}}$. Since μ_I is the direct product of a Bernouilli measure and atomic measures on shift invariant points, Theorem 3, Proposition 6 and Remark 4 tell us that $(\mu_I \circ F^{-n})_{n \in \mathbb{N}}$ converges to a shift mixing measure if there exists some μ_I-equicontinuous points. Furthemore from Proposition 7, if there exists μ_I-equicontinuous points, there also exists μ_c-equicontinuous points. So to prove that F_e is a μ_c-equicontinuous CA we only need to show that F_e contains a μ_I-equicontinous point.

To complete the proof we are going to show that there exists a point x and an integer $m \geq r$ such that $\mu_I(B_m(x)) > 0$. For each $l \in \mathbb{N}$, denote by $\mathcal{C}_l[i]$ the union of all the sets $^\infty 0^\infty \times [U]_i^{X_1'} \times ^\infty 0^\infty$ where $U = [E^{(*,1)} 0^j * 0^k E^{(*,2)}]_i^{X'1_1}$ is a cylinder in X_1, * replace one letter in $\{L, R\}$, $E^{(*,\binom{1}{2})}$ are letters in $\hat{E}$ and the positive integers verify $j+k+1 = l$. Let $\overline{\mathcal{C}_k[i]}$ be the union of sets $({}^\infty 0^\infty \times [E^* 0^k E^*]_i^{X_1'} \times ^\infty 0^\infty)$ where E^* replace any letter in $\hat{E}$. We call respectively *counters* in position i and *void counters* in positions i the sets $\mathcal{C}_l[i]$ and $\overline{\mathcal{C}_l[i]}$. Next we denote by $\mathcal{C}_l^*[i]$ the union of all the sets $^\infty 0^\infty \times [E^{(*,1)} *^l E^{(*,2)}]_i \times ^\infty 0^\infty$ where $* \in \overline{E}$ and $E^{(*,\binom{1}{2})} \in \hat{E}$. From section 3.3.2, each element in $\mathcal{C}_l^*[i]$ will enter in a *counter* $\mathcal{C}_l[i]$ or $\overline{\mathcal{C}_l[i]}$ after less than $\frac{l}{5}$ iterations. Remark that the iterations of each element y in a *precounters* $\mathcal{C}_l^*[i]$ will never generate a *train of 1* from its coordinate $i+l+1$ longer than $\frac{l}{5}+3$ because the letter E^* in position $i+l+1$ never stay in the emmitting state E_0 for more than $\frac{l}{5}$ iterations.

Consider $x_0 = ({}^\infty 0^\infty, ^\infty 0^\infty, ^\infty 0^\infty)$ and for each $k \in \mathbb{N}$, pick a point $x_k \in [0^k]_{(-k-1-r)} \cap B_r(x_0)$. In the following, we will prove that there exist integers $k > 0$ such that $\mu_I(B_r(x_0)) = \mu_I(B_r(x_k)) > 0$ by showing that

$$\mu_I([0^k]_{(-k-1-r)} \cap B_r(x_k)^{\complement}) < \mu_I([0^k]_{(-k-1-r)}).$$

The set $[0^k]_{(-k-1-r)} \cap B_r(x_k)^{\complement}$ is the set of points that contains *counters* in the left side of $[0^k]_{(-k-1-r)}$ that are able to generate *trains of 1* which move to the right and cross completly the k coordinates betwen $-k-r-1$ to r (the *trains* of "1" enter in the central coordinates $([-r, r])$). Now, consider the map $S(p)$ which gives the minimum size of the *counter* $\mathcal{C}_p^*[-p-l-k-r]$ in order that it produce *trains of 1* that move further than the coordinate $-r$. Recall that $l_m = 152$ is the minimum size of the *counters* due to the projection of $F_p^{X_1}$. Clearly the set$[0^k]_{(-k-1-r)} \cap B_r(x_k)^{\complement}$ is a subset of $\mathbb{S}_k = \left\{\cup_{i=S(0)}^{\infty} \mathcal{C}_i^*[-i-k-1-r] \cup_{p=l_m}^{\infty} \{\cup_{j=S(p)}^{\infty} \mathcal{C}_j^*[-j-p-k-1-r]\}\right\} \cap [0^k]_{(-k-1-r)}$. Remark that we do not try to identified the cases where appears between the coordinates $-p-r-1$ to $-r-1$ *precounters* $\mathcal{C}_j^*[-j-p-k-1-r]\}$ big enought to influence the coordinates $[-r, r]$. In order to find a good upper bound for $\mu_I(\mathbb{S}_k)$ we are going to show that $(S(p))_{p\in\mathbb{N}}$ is a strictly increasing sequence. Let $T_l = \frac{l}{5} + 3$ be the original length of the *trains of 1* generated by a *precounter* $\mathfrak{C}_j^*[-j-p-k-1-r]$. When the left extremity of the *train of 1* arrive in position $-k-1-r$ it has lost $2p$ elements and gain some other ones due to possible concatenations with other *precounters* situated between the coordinates $-p-k-1-r$ and $-k-1-r$. Clearly, the maximum gains in terms of concatenation of letters 1 arrives when there is only one *precounter* of size $p-1$ between the coordinates $-p-k-1-r$ and $-k-1-r$. In this case the remaining *train of 1* called $\mathfrak{T}$ which came from $\mathfrak{C}_j^*[-j-p-k-1-r]$ would have a length of $T_l - 2p + \frac{p-1}{5} + 3$. Since the right front of this *train* is at coordinate $k-1-r+\frac{p}{5}+3$, it needs to cross $k - (\frac{p}{5}+3)$ coordinates to influence the central coordinates $(-r, r)$. As $\mathfrak{T}$ has a size of $T_l - 2p + \frac{p-1}{5} + 3$, it can cross at most $(T_l - 2p + \frac{p}{5} + 3)/2 = \frac{l}{10} + 3 - \frac{11p}{10}$ coordinates and since it has a right extremity in position $k-1-r+\frac{p}{5}+3$, it follows that $S(p) \geq 10(k+p)$.

Under the action of $F_p^{X_1}$, a letter $E^* \in \hat{E}$ will be change in 0 is there is another element of $\hat{E}$ situated in a neighborhood of l_m coordinates. This implies that for all integer $l \geq l_m$ and $i \in \mathbb{N}$ we have $\mu_I(\mathfrak{C}_l^*[i]) = (\hat{q})^2 \times (\overline{q})^{2l_m+l}$ where $\overline{q} = \mu_I(\cup_{E^*\in\overline{E}}[E^*]_i)$ and $\hat{q} = \cup_{*\in\hat{E}}[*]_i$ for some integer i. We set $q_l = \mu_I(\mathfrak{C}_l^*[i])$ and $q^* = [0^k]_{(-k-1-r)}$ and we remark that if $l \geq l_m$, for all $n \in \mathbb{N}$ we have $\mu_I(\mathfrak{C}_{l+n}[i]) = q_l \times \overline{q}^n$. Since $\mathbb{S}_k =$

$$\left\{\cup_{i=S(0)}^{\infty}\mathfrak{C}_i^*[-i-k-1-r] \cup_{p=l_m}^{\infty} \{\cup_{j=S(p)}^{\infty}\mathfrak{C}_j^*[-j-p-k-1-r]\}\right\} \cap [0^k]_{(-k-1-r)}$$

we obtain that

$$\mu_I(\mathbb{S}_k) \leq q^* \left(q_{S(0)} \sum_{i=0}^{\infty} \overline{q}^i + q_{S(l_m)} \sum_{j=0}^{\infty} \overline{q}^j (\sum_{i=0}^{\infty} \overline{q}^i) \right)$$

$$= q^* q_{S(0)} \left(\frac{1}{1-q} + \overline{q}^{10l_m} (\frac{1}{1-\overline{q}})^2 \right).$$

Since $S(0) = 10k$, there will be an integer $k \geq 0$ such that

$$q_{S(0)} \left(\frac{1}{1-\overline{q}} + \overline{q}^{10l_m} (\frac{1}{1-\overline{q}})^2 \right) < 0$$

which prove that $\mu_I([0^k]_{(-k-1-r)} \cap B_r(x_k)^{\complement}) < \mu_I([0^k]_{(-k-1-r)}) = q^*$. Then we can conclude arguing that there exists an integer k such that

$$\mu_I(B_r(x_k)) = \mu_I(x_0) > 0.$$

□

Remark 6 *It is possible to give a simpler proof of Proposition 8 using only the limit measure μ_c but in this case we can not show that $(\mu_I \circ F^{-n})_{n\in\mathbb{N}}$ is* *a converging sequence. We conjecture that the limit in Cesaro mean of the sequence* $(\mu_u \circ F_e^{-n})_{n\in\mathbb{N}}$ *where* μ_u *is measure of the uniform measure on* X *will be identical to the measure* μ_c *defined in the Proof of Proposition 8. In the following* μ_c *will always denote the limit in Cesaro mean of the sequence* $(\mu_I \circ F_e^{-n})_{n\in\mathbb{N}}$.

Proposition 9 *The cellular automaton* $(F_e, S(\mu_c))$ *is sensitive (has no equicontinuous point in the topological support* $S(\mu_c)$*).*

Proof

Suppose that there exist an equicontinuous point. We need to show that there exists a point $x \in S(\mu_c)$ and an integer m such that $C_m(x) \subset B_r(x)$. First remark that if for some x there exist integers $i > 0$, such that $F^i(x)_0^2 = 0$, then for all $\epsilon = 2^{-m}$ there exists $y \in C_m(x)$ such that $F^i(y)^2 = 1$. This implies that if there exists $x \in A^{\mathbb{Z}}$ and $m > 0$ such that $C_m(x) \subset B_r(x)$, then for all $y \in C_m(x)$ and $i \in \mathbb{N}$, one has $F^i(y)^2 = 1$ (condition (*)). Hence, if there exists an equicontinuous point, there exists a finite configuration that produce a continuous and permanent 'flow of 1'. Let's try to construct such a configuration. Recall that thanks to the action of $F_p^{X_0}$ (see section 3.3.2), after a while (k iterations), there is no *void counter* (patterns of the type $E^*0^kE_0$ in X_1' with $E^* \in \hat{E}$) able to generate a continuous flow of letters '1'. It follows that $S(\mu_c) \cap X$ does not contains any cylinder of the type $[E^*0^kE_0]_i$ ($i \in \mathbb{Z}, k \in \mathbb{N}$). From section 3.3.2, after a while (a period less than $\frac{l}{10}$) there is at most one letter in $\{R, L\}$ between two occurences of letters in $\hat{E}$. Since we search for a finite configuration that produce a permanent 'flow of 1', we can only consider the *counters* $\mathcal{C}$ and *void counter* $\overline{\mathcal{C}}$ and as we are going to use a quantitative arguments on flow of 1, we will only consider the real *counters* $\mathcal{C}$.

If we suppose that there exists an equicontinuous point, then there is a finite sequence of k consecutive counters $\mathcal{C}_{l_k}, \mathcal{C}_{l_{k-1}}, \ldots \mathcal{C}_{l_0}$ that generate a continuous flow of letters 1 (there exists a point $x \in \mathcal{C}_{l_k}, \mathcal{C}_{l_{k-1}}, \ldots \mathcal{C}_{l_0}$ such that $\forall n \in \mathbb{N}$ $F_e^n(x)(-r, r) = 1^{2r+1}$). First remark (see the analysis of the F_3 dynamic in subsection 3.3.2) that a *train* of 1 generated by a *counter* $\mathcal{C}_{l_k}$ will reach the second emittor of the neighbour *counter* $\mathcal{C}_{l_0}$ if for all $1 \leq i \leq k$, one has $l_{i-1} \leq \frac{3l_i}{10} + \frac{9}{2}$. Recall that each *counter* $\mathcal{C}_{l_i}$ generate a *train* of 1, which last at most P_i iterations at the coordinate 0 and do not influence the central coordinates during at least $4P_i$ iterations. Remark that in the case of *void counter* of type $[E^*0^kE_1]$ (E^* is an element of $\hat{E}$), no *train of '1'* are generated. Were are going to show that there exist integers $k > 0$ such that $F^{i+k}(x)^2 \neq 1$ when $0 \leq i \leq 4P_k$ and $\mathcal{C}_{l_k}, \mathcal{C}_{l_{k-1}} \ldots \mathcal{C}_{l_0}$ is the finite sequence of *counters* which appears in the left coordinates of x. In order to do that, we will show that, considering all the *trains* generated by the k counters, it always remains some holes, during a period of $3P_k$ where P_k is the period of the first and largest

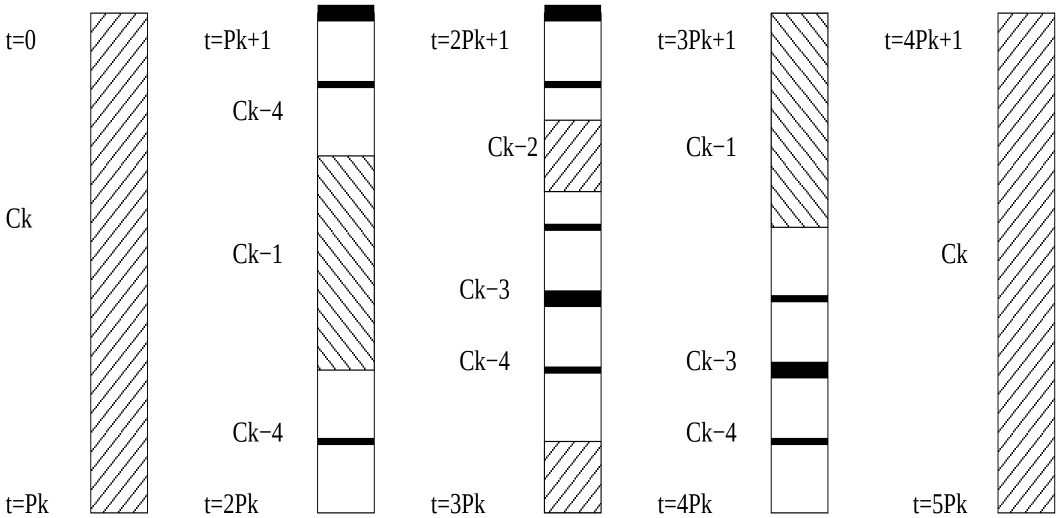

Fig. 2. The flow of 1 generated by a sequence of k *counters*

counter $\mathcal{C}_{l_k}$. Without loosing generalities, we can suppose that the train of 1 generated by the first counter of size l_k arrive in coordinate 0 at $t = 0$ and last at most P_k iterations (we do not know a priori the global dynamic). For time $t = P_k + 1$ to $4 \times P_k$, there is no train of 1 due to this first *counter* that pass through the central coordinate. The *train* of 1 generated by the second counter from the left : $\mathcal{C}_{l_{k-1}}$ last at most P_{k-1} iterations and its effect stops for a period of $3P_{k-1}$ in the interval time $t = P_k + 1$ to $t = 4P_k$. Clearly, between $t = P_k + 1$ and $t = 4P_k$, if P_{k-1} is small enought, there is at least one interval of length at least $3P_{k-1}$ that will be not affected by the two first *counters* if $3P_k - (2 \times 3 + 1)(P_{k-1}) \geq 0$. This interval is minimum when the *train* of 1 generated by the second *counter* pass exactly in the middle of the interval $[P_k + 1, 4P_k]$ between 2 *trains* of the first *counter*. The condition $3P_k - (2\times 3+1)(P_{k-1}) \geq 0$ is equivalent to $P_{k-1} \leq \frac{3P_k}{7}$ and since $l_{k-1} \leq \frac{3l_k}{10} + \frac{9}{2}$, $P_k = \frac{l_k}{5} + 3$, then l_k must be greater than $\frac{39\times 35}{9} \approx 152$. Since, the propagation of the *trains* of 1 from one counter to the other, requires that $l_{i-1} \leq \frac{3l_i}{10} + \frac{9}{2}$ and thanks to the automaton $F_p^{X_1}$, all the *counters* have a size $l \geq 152$, the condition $P_{i-1} \leq \frac{3P_i}{7}$ is true for all $1 \leq i \leq k$ and repeating k times the first process there will remain intervals of length $3P_0$ between $P_k + 1$ and $4P_k$ that will be not affected by any of the *trains* of 1 generated by the k *counters*. Since for any sequence of *counters* $\mathcal{C}_{l_k}, \mathcal{C}_{l_{k-1}}, \ldots \mathcal{C}_{l_0}$ and points y that contains such counters, there exists an integer $n \in \mathbb{N}$ such that $F_e^n(y) \neq 1$, we can conclude. □

Remark 7 *Clearly, The cellular automaton (F_e, X) has no equicontinuous point too.*

Questions

- Is it possible to construct an infinite sequence that generate a continuous flow for the particular automaton F_e?

-In general, what are the conditions on the counters of automata similar to F_e in order that they can not produce equicontinuous points?

-Is it possible to find a μ-equicontinuous CA without equicontinuous point for a F-invariant measure μ whose topological support $S(\mu)$ is a subshift of finite type ?

-Is there an example similar to the one we have presented in the case of an only two states cellular automaton?

-The dynamic of the example F_e given in this paper (propagation of trains) seems to appear in different simulations of one dimensional cellular automata (even in the two states case like the class studied by Wolfram, see [6]). How common are the CA with the same properties?

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