INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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**N° 2126**

Décembre 1993

PROGRAMME 4

Robotique,
image
et vision

Rapport de recherche

1993

ISSN 0249-6399

# Time-varying Exponential Stabilization of Nonholonomic Systems in Power Form

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Projet ICARE

Rapport de recherche n° 2126 — Décembre 1993 — 21 pages

**Abstract:** Systems in canonical power form have recently been used to model the kinematic equations of nonholonomic mechanical systems. In [14, 15], McCloskey and Murray have had the idea of using the properties of homogeneous systems to derive exponentially stabilizing continuous time-periodic feedbacks for this class of systems. Motivated by this work, the present study extends a control design method previously proposed by Samson to the design of such homogeneous feedbacks. The approach here followed has the advantage of yielding simple and direct stability proofs. Homogeneity-related results needed for time-varying exponential stabilization are also provided.

**Key-words:** nonholonomic systems, chained and power forms, time-varying stabilization, homogeneous systems

*(Résumé : tsvp)*

This work is partly supported by Esprit Project "PROMotion" 6546

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# Stabilisation Exponentielle Instationnaire des Systèmes Non-holonomes Sous Forme Canonique dite "Power Form"

**Résumé :** Les systèmes sous forme canonique dite *power form* ont été récemment utilisés pour modéliser les équations de systèmes mécaniques non-holonomes. Dans [14, 15], McCloskey et Murray ont eu l'idée d'utiliser les propriétés des systèmes homogènes pour synthétiser des commandes par retour d'état continues, périodiques par rapport au temps, capables de stabiliser asymptotiquement et exponentiellement l'origine pour cette classe de systèmes. Motivée par ce travail, la présente étude décrit une extension, permettant d'obtenir des retours d'état homogènes, d'une méthode de synthèse précédemment proposée par Samson. L'approche ici considérée a l'avantage de conduire à des démonstrations de stabilité simples et directes. Les résultats associés à la propriété d'homogénéité qui permettent d'établir la stabilisation exponentielle dans le cas de systèmes instationnaires sont également fournis.

**Mots-clé :** systèmes non-holonomes, forme "chainée" et "power form", retours d'état instationnaires, systèmes homogènes

# 1 Introduction

Recent studies on mobile robot control have been using so-called *chained form* and equivalent *power form* systems to model the kinematic equations of various nonholonomic wheeled vehicles, see [17, 33, 30, 14, 15, 26] for example. It is now well known that nonholonomic systems, although controllable, are not point-stabilizable by means of continuous feedback $u(x)$ where $x$ is the state [3, 20, 22, 1]. This has motivated the study of alternative solutions, such as piecewise continuous feedback, as first proposed by Bloch and McClamroch [2] in 1989 for a knife edge, and smooth time-varying feedback, as first proposed by Samson [21] in 1990 in the case of a three-dimensional unicycle-type vehicle. The idea of smooth time-varying feedback $u(x,t)$, although little exploited before [21], may in fact be traced back, for one-dimensional systems, to the work of Sontag and Sussmann [28] in 1980, and a complementary contribution by Sontag [29] in 1989. Various developments have followed, for example [5, 4, 31], in the case of piecewise continuous feedback, and [18, 24, 32, 8, 25, 14], in the case of time-varying feedback. The potentialities of time-varying feedback, as a general stabilization technique, has also been explored by Coron [6, 7] who established that almost all controllable systems are stabilizable by continuous time-periodic feedback.

As noted in [21, 23], and subsequently in several other references, the convergence provided by smooth time-periodic feedback is only polynomial and not exponential, by contrast with some discontinuous feedbacks [31]. In [14, 15] McCloskey and Murray proposed to use, for nonholonomic systems in power form, continuous but non-lipschitz time-periodic control laws which make the closed-loop vector field homogeneous. In this case, local stabilisation is equivalent to global stabilization and exponential convergence. However, the stability analysis proposed in [14, 15] is not entirely conclusive, despite demonstrative simulation results. Also, the construction of adequate control laws for $n$-dimensional systems does not seem to be systematic. In [26], the stability proof for a class of exponentially stabilizing continuous time-varying feedbacks is simpler, but is only given for three-dimensional systems. The contribution of the present paper is, on one hand, to propose a systematic way of deriving, for $n$-dimensional systems in power form, a family of homogeneous continuous time-varying feedback laws, the stabilizing properties of which are easily proved, and, on the other hand, to establish properly the homogeity-related results needed for time-varying exponential stabilization.

## 2 Chained forms, power forms and mobile robots

It is now well known [32, 30, 26] that, by an adequate (local) change of coordinates, the kinematic equations of various nonholonomic wheeled mobile robots, such as unicycle-type and car-like vehicles with or without trailers, can be written in the normal (or canonical) form of a *chained system*. A two-input system with a single chain has the form:

$$\left.\begin{array}{rcl} \dot{x}_1 &=& u_1 \\ \dot{x}_2 &=& x_3\, u_1 \\ \dot{x}_3 &=& x_4\, u_1 \\ &\vdots& \\ \dot{x}_j &=& x_3\, u_1 \\ &\vdots& \\ \dot{x}_{n-1} &=& x_n\, u_1 \\ \dot{x}_n &=& u_2 \end{array}\right\} \tag{1}$$

This equation may also be written:

$$\dot{x} \;=\; u_1 f_1(x) \;+\; u_2 f_2(x) \tag{2}$$

with the vector fields $f_1$ and $f_2$:

$$f_1 \;=\; \frac{\partial}{\partial x_1} \;+\; x_3\frac{\partial}{\partial x_2} \;+\; x_4\frac{\partial}{\partial x_3} \;+\; \ldots \;+\; x_n\frac{\partial}{\partial x_{n-1}} \quad ; \quad f_2 \;=\; \frac{\partial}{\partial x_n} \tag{3}$$

Under the following global change of coordinates:

$$\left.\begin{array}{rcl} y_1 &=& x_1 \\ y_2 &=& x_n \\ y_3 &=& -x_{n-1} + x_1 x_n \\ &\vdots& \\ y_j &=& \sum_{i=0}^{i=j-2} (-1)^i x_1^i x_{n-j+2+i} \\ &\vdots& \\ y_n &=& \sum_{i=0}^{i=n-2} (-1)^i x_1^i x_{2+i} \end{array}\right\} \quad (4)$$

it is simple to verify that the previous chained system is itself equivalent to the following form, called *power form*, introduced in [32]:

$$\left.\begin{array}{rcl} \dot{y}_1 &=& u_1 \\ \dot{y}_2 &=& u_2 \\ \dot{y}_3 &=& y_1 u_2 \\ &\vdots& \\ \dot{y}_j &=& \frac{1}{(j-2)!} y_1^{j-2} u_2 \\ &\vdots& \\ \dot{y}_n &=& \frac{1}{(n-2)!} y_1^{n-2} u_2 \end{array}\right\} \quad (5)$$

i.e.

$$\dot{y} \;=\; u_1 f_1(y) \;+\; u_2 f_2(y) \quad (6)$$

with:

$$f_1 \;=\; \frac{\partial}{\partial y_1} \quad ; \quad f_2 \;=\; \sum_{j=2}^{n} \frac{y_1^{j-2}}{(j-2)!} \frac{\partial}{\partial y_j} \quad (7)$$

In [18], a systematic method, based on Lyapunov and Lasalle techniques, is given to design smooth time-periodic stabilizing control laws for controllable systems without drift meeting a certain Lie Bracket condition [18, Assumption 1], which is here satisfied because the rank of $\{f_1, f_2, [f_1, f_2], \mathrm{ad}_{f_1}^2 f_2, \ldots, \mathrm{ad}_{f_1}^{n-2} f_2\}$ is equal to $n$. In the case where one can find system coordinates for which $f_1$ is a coordinate vector field, the method yields analytic expressions of the control laws. Hence, it is natural to look for coordinates $(y_1, \ldots, y_n)$ for which $f_1 = \frac{\partial}{\partial y_1}$. This gives the change of coordinates (4) which transforms the chain form into the power form.

In [32], another set of stabilizing time-periodic control laws for systems in power form, with a stability analysis relying upon *Center Manifold* techniques, is considered. This already illustrates the fact that there are multiple ways of designing and analysing stabilizing time-varying feedbacks.

In the present paper, we propose a systematic way of designing stabilizing continuous time-varying feedbacks, with the properties of homogeneity and exponential convergence being obtained as a particular case. Based on Lyapunov analysis complemented by Lasalle's invariance principle, the method takes up the techniques used by Samson in several of his articles and yields simple control expressions.

# 3 Homogeneity and exponential stabilization

In this section, after recalling briefly the main definitions concerning homogeneity, we extend a result from [11] to time-varying vector fields, this extension being simplified by arguments taken from [19].

Define, for $\lambda > 0$, the dilation operator $\delta_\lambda$ by:

$$\delta_\lambda(y_1, \ldots, y_n) \quad = \quad (\lambda^{r_1} y_1, \ldots, \lambda^{r_n} y_n) \; . \tag{8}$$

with $r_j > 0$, the "weight" associated with the $j$th coordinate.

Being bound to deal with functions depending not only on the state variables $(y_1, \ldots, y_n)$ but also on the time index $t$, we may affect to time a homogeneity weight $r_0 = 0$ and define the "extended" dilation:

$$\delta_\lambda(t, y_1, \ldots, y_n) \quad = \quad (t, \lambda^{r_1} y_1, \lambda^{r_2} y_2, \ldots, \lambda^{r^n} y_n) \; . \tag{9}$$

Recall that a function $h$ from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}$ is said to be *homogeneous of degree* $\tau$ with respect to the family of dilations $\delta_\lambda$ if

$$h(\, \delta_\lambda(t, y_1, \ldots, y_n)\, ) \quad = \quad \lambda^\tau \, h(t, y_1, \ldots, y_n) \; . \tag{10}$$

and a time-varying vector field is said to be homogeneous of degree $\sigma$ with respect to the family of dilations $\delta_\lambda$ if and only if its $i$th coordinate is a homogeneous function of degree $r_i + \sigma$, i.e.

$$f^i(\, \delta_\lambda(t, y_1, \ldots, y_n)\, ) \quad = \quad \lambda^{r_i + \sigma} \, f^i(t, y_1, \ldots, y_n) \; , \tag{11}$$

where $f^i$ stands for the $i$th component of the vector field $f$. Note that since the dilation does not act on $t$ the homogeneity of a time-varying function (resp. vector field) is equivalent to the homogeneity of the corresponding time-invariant functions (resp. vector fields) obtained by fixing time, for any value of time.

Suppose that $f$ is a homogeneous vector field with degree $\sigma$ and $h$ is a homogeneous function of degree $\tau$, with respect to the same dilation, then:

$$\left.\begin{array}{l} hf \text{ is a homogeneous vector field of degree } \tau+\sigma, \\ \frac{\partial h}{\partial t} \text{ is a homogeneous function of degree } \tau, \\ L_f h \text{ is a homogeneous function of degree } \tau+\sigma. \end{array}\right\} \quad (12)$$

with respect to the considered dilation. (Recall that $L_f h(t,y) = \frac{\partial h}{\partial y}(t,y)f(t,y)$ is the Lie derivative of $h$ along $f$).

Consider now the vector fields $f_1$ and $f_2$ associated with the system in power form (5) and given by (7). It is easy to verify that $f_1$ is homogeneous of degree $-r_1$ with respect to any considered dilation, and that the vector field $f_2$ is only homogeneous with respect to dilations such that:

$$r_j \;=\; r_2 + (j-2)r_1 \text{ for } j \geq 2 \quad (13)$$

The degree of homogeneity of $f_2$ is then $-r_2$. Therefore:

$$\left.\begin{array}{l} f_1 \text{ is a homogeneous vector field of degree } -r_1, \\ f_2 \text{ is a homogeneous vector field of degree } -r_2. \end{array}\right\} \quad (14)$$

with respect to the family of dilations satisfying (13). In the sequel, only such dilations will be considered, with $r_1$ and $r_2$ kept as degrees of freedom which do not need to be specified at this stage. Note however that only the ratio $\frac{r_2}{r_1}$ really matters since, by a change of variable on $\lambda$, it is clear that $r_1$ can be normalized and set equal to one without any loss of generality.

Suppose that the system inputs $u_1$ and $u_2$ are assigned to be some functions of $y$ and $t$, periodic with respect to time, then the vector field associated with the closed-loop system is the time-varying vector field $u_1 f_1 + u_2 f_2$, and the closed-loop equation is the ordinary differential equation

$$\dot{y} \;=\; u_1(t,y)f_1(y) \;+\; u_2(t,y)f_2(t,y) \;\; . \quad (15)$$

The following proposition (the second part of it actually) is an extension to time-varying systems of a result stated in [11] (lemma 2.1).

**Proposition 1** *Suppose that $u_1(t,y)$ and $u_2(t,y)$ are continuous with respect to $y$ and infinitely differentiable with respect to $t$, periodic with period $T > 0$, and that*

$$\left.\begin{array}{ll}(t,x) \mapsto u_1(t,y) & \text{is homogeneous of degree } r_1 \\ (t,x) \mapsto u_2(t,y) & \text{is homogeneous of degree } r_2,\end{array}\right\} \tag{16}$$

*then the closed-loop vector field is homogeneous of degree zero. Furthermore, if the origin is locally asymptotically stable for (15), then it is globally exponentially stable with respect to the dilation in the sense that there exists $K > 0$ and $a > 0$ such that, along any solution $y(t)$ of the closed-loop system:*

$$\rho(y(t)) \quad \leq \quad K\, e^{-at}\, \rho(y(0)) \tag{17}$$

*where $\rho$ is the homogeneous norm given by:*

$$\rho(y) \quad = \quad \left(\sum_{j=1}^{n} |y_j|^{\frac{p}{r_j}}\right)^{\frac{1}{p}} \tag{18}$$

*with $p > 0$.*

Note that the exponential rate of convergence of the solutions to zero, given by $a$, does not depend on the value attributed to $p$, while the constant $K$ may depend on $p$.

**Proof :** The fact that the closed-loop vector field is homogeneous of degree zero is an obvious consequence of (16), (14) and (12). Then, according to the Proposition 4 of the Appendix, which is a slight generalization of [19, Theorem 2], local stabilization of the origin implies the existence of a positive Lyapunov function $\overline{V}(t,y)$, homogeneous of degree one, of class $\mathcal{C}^\infty$ except at $y=0$, continuous everywhere (from Proposition 4, the function is only of class $\mathcal{C}^0$ everywhere due to its degree of homogeneity chosen equal to one), and such that:

$$\dot{\overline{V}}(t,y) \quad = \quad \frac{\partial \overline{V}}{\partial t}(t,y) \;+\; \sum_{i=1}^{n} \frac{\partial \overline{V}}{\partial x_i}(u_1(t,y) f_1^i(y) + u_2(t,y) f_2^i(t,y) \;<\; 0 \quad \forall x \neq 0 \tag{19}$$

Now, since $\dot{\overline{V}}$ may be viewed as $\frac{\partial \overline{V}}{\partial t} + L_{u_1 f_1 + u_2 f_2} \overline{V}$ and since $L_{u_1 f_1 + u_2 f_2}$ is a homogeneous vector field of degree zero, one deduces from (12) that $\dot{\overline{V}}$ is a homogeneous time-varying periodic function with the same degree as $\overline{V}$. It is thus of degree one. From there, it is not difficult to establish that there are two positive numbers $K_1$ and $K_2$ such that:

$$K_1\,\overline{V}(t,y) \;\leq\; -\dot{\overline{V}}(t,y) \;\leq\; K_2\,\overline{V}(t,y) \quad \forall y \neq 0 \tag{20}$$

Indeed, the function $-\frac{\dot{\overline{V}}(t,y)}{\overline{V}(t,y)}$ is well defined and strictly positive away from $y = 0$, and it is homogeneous of degree zero. Therefore, it takes the same value at $(t,y)$ and at $\delta_\lambda(t,y)$ for all $\lambda > 0$ so that one may restrict his attention to $y \in S_\rho^{n-1}$, where $S_\rho^{n-1}$ is the unit sphere associated with the "homogeneous norm" $\rho$ defined by (18). Moreover, time-periodicity allows one to consider that time lives on the compact set $S^1 = \mathbb{R}/T\mathbb{Z}$, instead of $\mathbb{R}$. On the compact set $S^1 \times S_\rho^{n-1}$, the smooth and strictly positive function $-\frac{\dot{\overline{V}}}{\overline{V}}$ reaches its minimum, denoted as $K_1$, and its maximum, denoted as $K_2$; both numbers being strictly positive.

By using the same arguments, one shows that there are positive numbers $K_3$ and $K_4$ such that

$$K_3\,\rho(y) \;\leq\; \overline{V}(t,y) \;\leq\; K_4\,\rho(y) \quad \forall y \neq 0 \tag{21}$$

Now, from (20): $\dot{\overline{V}} \leq -K_1\,\overline{V}$, and thus: $\overline{V}(t,y(t)) \;\leq\; \overline{V}(0,y(0))\; e^{-K_1 t}$, along any system's solution. From (21), one then obtains:
$\rho(y(t)) \;\leq\; \frac{K_4}{K_3}\, e^{-K_1 t}\, \rho(y(0))$, which is the relation (17) with $K = K_4/K_3$ and $a = -K_1$. ∎

# 4 A class of time-varying stabilizing control laws for systems in power form

We describe in this section a way of designing time-varying stabilizing feedbacks. Depending on the choice of some design parameters, one obtains control laws which are either smooth, or homogeneous and only continuous, with exponential convergence of the closed-loop system's solutions to zero in the latter case.

This is a two-step approach. The first step consists in looking at the $n-1$ dimensional time-varying system obtained by forgetting the first equation in (5) and considering $y_1$ as a (given) function of time. Via a Lyapunov-like analysis, a control $u_2(y_1(t), y_2, \ldots, y_n)$ capable of stabilizing the reduced system when $y_1(t)$ is endowed with some properties, is then derived. The second step consists in determining $u_1$ in order to meet these properties and also ensure the convergence of $y_1$ to zero. This approach is basically the one used in [21], [24], [26].

Let us thus proceed and consider any function $W_1(y_2, \ldots, y_n)$ such that:

$$\left.\begin{array}{l} W_1 \text{ is continuously differentiable on } I\!\!R^{n-1} \text{ and of class } \mathcal{C}^2 \text{ on } I\!\!R^{n-1} - \{0\}, \\ W_1(0) = 0 \text{ and } W_1(x) > 0 \text{ for } x \neq 0, \\ \frac{\partial W_1}{\partial x}(x) \neq 0 \text{ for } x \neq 0 \\ W_1 \text{ is proper.} \end{array}\right\} \tag{22}$$

(by *proper* it is meant that the preimage of a compact set is compact, i.e. the set $\{x \,/\, W_1(x) \geq K\}$ is bounded for all $K > 0$).

It is clear that $W_1$ is non-increasing along the solutions if $u_2$ is chosen such that $u_2 L_{f_2} W_1 \leq 0$, no matter how $u_1$ is chosen. This may be achieved, for example, by taking:

$$u_2 \;=\; -g_2 \,\text{sign}(L_{f_2} W_1)\; |L_{f_2} W_1|^{\alpha} \tag{23}$$

with $\alpha > 0$ and $g_2 > 0$.

Irrespective of the behavior of $y_1$ (considered as a function of time, governed by $u_1$), this control guarantees boundedness of $(y_2, \ldots, y_n)$ and eventually convergence to zero when $y_1(t)$ is suitably chosen. To this purpose, one may try to determine a continuous feedback $u_1$ (which has to be time dependent since continuous pure-state feedbacks cannot stabilize the complete system) which keeps $y_1$ bounded, makes it converge to zero, and produces a $y_1(t)$ which causes $(y_2, \ldots, y_n)$ to go to zero. A simple choice, among other possibilities, is:

$$u_1 \;=\; -g_1 y_1 \;+\; h(\,W_1(y_2, \ldots, y_n)\,)\,\sin\frac{2\pi t}{T}\;. \tag{24}$$

with $g_1 > 0$.

**Proposition 2** *For any function $W_1$ satisfying (22) and any continuous real function $h$ such that*

$$\left.\begin{array}{l} h \textit{ is of class } \mathcal{C}^1 \textit{ on } ]0, +\infty), \\ h(s) = 0 \;\Leftrightarrow\; s = 0 \;, \end{array}\right\} \tag{25}$$

*the control law (24)-(23) globally asymptotically stabilizes the origin for the system (5).*

**Proof :** First, it is proved that all the system's solutions are bounded uniformly with respect to initial conditions. This in turn implies that the solutions exist over $[0, +\infty)$. They are also unique in view of the regularity conditions imposed upon the functions $W_1$ and $h$.

The boundedness of the system's solutions comes from that $W_1(y_2(t), \ldots, y_n(t))$ is nonnegative and nonincreasing. It is thus bounded. This implies, from (22), that $(y_2(t), \ldots, y_n(t))$ is uniformly bounded, and since

$$\dot{y}_1 \quad = \quad -g_1 y_1 \; + \; h(\, W_1(y_2(t), \ldots, y_n(t)) \; \sin \frac{2\pi t}{T} \tag{26}$$

is the equation of a Hurwitz (stable) linear system perturbed by the additive uniformly bounded term $h(W_1(y_2(t), \ldots, y_n(t))) \sin \frac{2\pi t}{T}$, $y_1(t)$ is also uniformly bounded. Moreover $W_1(y_2(t), \ldots, y_n(t))$ converges to some limit value, and $\dot{W}_1(y_1(t), \ldots, y_n(t))$, being uniformly continuous with respect to $t$, tends to zero.

We then apply LaSalle's invariance principle for time-invariant systems [13, Theorem 2]. This can be done by incorporating time in the system state vector ($\dot{t} = 1$). In doing so, at least one of the state components (the time coordinate) is unbounded so that it may seem that Lasalle's principle does not apply in this case. This difficulty is presently overcomed in the following way: since the time-dependence is periodic, it is possible to replace all occurences of $t$ in the control laws, and thus in the closed-loop vector field, by the "angle" $\theta$ which lives in $\mathbb{R}/(2\pi/T)\mathbb{Z}$ and is thus naturally bounded. This is in fact equivalent to incorporating $sin \frac{2\pi t}{T}$ and $cos \frac{2\pi t}{T}$ in the state vector, instead of $t$. Now, by Lasalle's principle, since all the trajectories $(\theta(t), y_1(t), \ldots, y_n(t))$ are bounded, and since the function $W_1(y_2, \ldots, y_n)$ decreases along these trajectories, all solutions converge to the set consisting of the reunion of the trajectories for which $\dot{W}_1(y_1, \ldots, y_n)$ is identically zero.

Consider such a trajectory $(\theta(t), y_1(t), \ldots, y_n(t))$. Since $\dot{W}_1$ is $u_2 L_{f_2} W_1$, and $u_2$ is given by (23), $u_2$ is identically zero on this trajectory. Therefore, in view of the system's equations (5), $y_2(t), \ldots, y_n(t)$ are constant. $W_1$ is also constant on the considered trajectory, since $\dot{W}_1 = 0$. Let $c$ denote its value.

By integration of (26), $y_1(t)$ is given by

$$y_1(t) \;=\; A\cos(\frac{2\pi t}{T}+\varphi) \;+\; B\,e^{-g_1 t} \tag{27}$$

where the constant $A$, $B$, and $\varphi$ depend on $h(c)$ and $y_1(0)$, and $A$ is equal to zero if and only if $h(c)=0$, i.e. if and only if $W_1$ is itself equal to zero. Now, $\dot{W}_1=0$ is equivalent to $L_{f_2}W_1=0$, i.e.

$$\sum_{j=2}^{n} \frac{y_1^{j-2}}{(j-2)!}\frac{\partial W_1}{\partial y_j} \;=\; 0 \;\;, \tag{28}$$

which, from (27), implies

$$\sum_{j=2}^{n} \frac{\left(A\cos(\frac{2\pi t}{T}+\varphi) \;+\; B\,e^{-g_1 t}\right)^{j-2}}{(j-2)!}\frac{\partial W_1}{\partial y_j}(y_2,\ldots,y_n) \;=\; 0 \tag{29}$$

where all the coefficients $\frac{\partial W_1}{\partial y_j}$ are constant, since they depend only on $y_2,\ldots,y_n$.

If $A=B=0$, the relations (26)-(27) imply that $y_1$ and $h(W_1)$ are equal to zero and, from (25), that $W_1=0$. Hence, from (22), $y_j=0$, $(j=2,\ldots,n)$.

If either $A$ or $B$ is nonzero, the functions $g_j(t):\;t\mapsto\left(A\cos(\frac{2\pi t}{T}+\varphi)\;+\;B\,e^{-g_1 t}\right)^{j-2}$ are linearly independent so that the constant coefficients $\frac{\partial W_1}{\partial y_j}$ in (29) must all be equal to zero. From (22), this again implies $y_j=0$, $(j=2,\ldots,n)$ and $W_1=0$.

Hence, all solutions for which $L_{f_2}W_1$ is identically zero are contained in the set defined by $y_2=0,\ldots,y_n=0$. From LaSalle's invariance principle, this implies that all trajectories converge to this set. Therefore, for any trajectory, $y_j(t)$ $(j=2,\ldots,n)$ and $W_1(t)$ tend to zero, and so does $h(W_1(t))$. In view of (26), this in turn implies that $y_1(t)$ tends to zero. ∎

This stabilization result is true for any $W_1$ and $h$ meeting the assumptions (22) and (25) respectively.

In particular, if $h$ is differentiable everywhere and $\alpha$ is chosen larger than one (or $\alpha=1$ with $W_1$ being twice differentiable everywhere), then the obtained time-varying controls are differentiable everywhere.

Another possibility, motivated by Proposition 1, consists in choosing the functions $W_1$ and $h$ so that:

$$\left.\begin{array}{l} W_1 \text{ is homogeneous of degree } q > r_2 + (n-2)r_1 \ , \\ h(s) \ = \ g_0|s|^{r_1/q}h(s) \text{ with } g_0 \neq 0, \\ \alpha \ = \ \dfrac{r_2}{q - r_2} \ . \end{array}\right\} \tag{30}$$

In this case, one easily verifies that the control laws $u_1$ and $u_2$ considered previously are homogeneous of degree $r_1$ and $r_2$ respectively, so that Proposition 1 applies. Note that $h$, and thus $u_1$, cannot be differentiable everywhere due to the ratio $r_1/q$ which is smaller than one in this case. The result is summarized in the following proposition:

**Proposition 3** *If $h$ and $W_1$ satisfy (22) and (30), then the control law given by (24)-(23) makes the origin globally exponentially stable with respect to the considered family of dilations $\delta_\lambda$, in the sense of the inequality (17).*

To illustrate this result, one may, for example, choose:

$$W_1(y_2, \ldots, y_n) \ = \ \sum_{j=2}^{n} a_j |y_j|^{\frac{q}{r_2 + (j-2)r_1}} \tag{31}$$

with $a_j > 0$ $(j = 2, \ldots, n)$ and $q > r_2 + (n-2)r_1$.

$W_1$ is then homogeneous of degree $q$ and continuously differentiable. One also has in this case:

$$L_{f_2} W_1 \ = \ \sum_{j=2}^{n} \frac{q}{r_2 + (j-2)r_1} \frac{a_j}{(j-2)!} y_1^{j-2} sign(y_j) |y_j|^{\frac{q}{r_2+(j-2)r_1} - 1} \tag{32}$$

Explicit expressions for $u_1$ and $u_2$ are then obtained from (24) and (23).

We have simulated this solution for a four-dimensional $(n = 4)$ system in power form representing the kinematics of a car-like vehicle. This simulation thus illustrates the possibility of exponentially stabilizing such a vehicle about a desired configuration by means of a continuous time-varying feedback. We chose:

$$r_1 = r_2 = 1 \ , \quad q = 6 \ , \tag{33}$$

which yields:

$$\begin{aligned} W_1(y) &= a_2 y_2^6 \; + \; a_3 |y_3|^3 \; + \; a_4 y_4^2 \\ L_{f_2} W_1(y) &= 6 a_2 y_2^5 \; + \; 3 a_3 \mathrm{sign}(y_3) y_3^2 y_1 \; + \; a_4 y_4 y_1^2 \end{aligned}$$

and the controls:

$$\begin{aligned} u_1 &= -g_1 \, y_1 \; + \; g_0 \sqrt[6]{W_1(y)} \\ u_2 &= -g_2 \, sign(L_{f_2} W_1(y)) \, \sqrt[5]{|L_{f_2} W_1(y)|} \end{aligned} \tag{34}$$

In the simulation, the gains are:

$$\left.\begin{aligned} a_2 &= 0.5 & g_0 &= 0.4 \\ a_3 &= 10^4 & g_1 &= 3 \\ a_4 &= 1.5 \times 10^6 & g_2 &= 0.1 \end{aligned}\right\} \tag{35}$$

Recall that the state components $x_1$ and $x_2$ of the equivalent four-dimensional chained system may physically be interpreted as the cartesian coordinates of the vehicle's point located at mid-distance of the rear wheels (see [30], or in a more general framework [26], for example). In view of the change of coordinates (4), these cartesian coordinates are thus given by $y_1$ and $y_4 - y_1 y_3 + \frac{1}{2} y_1^2 y_2$ respectively. The vehicle's orientation angle $\theta$ is given by $arctan(x_3)$ $(= arctan(-y_3 + y_1 y_2))$, $u_2$ is the velocity of the front steering wheel angle, and the vehicle's translational velocity $v$ is equal to $\frac{u_1}{cos(\theta)}$.

**Fig. 1** shows the vehicle's cartesian motion, from the initial configuration $(x = 0,\ y = 1,\ \theta = 0)$ to the goal configuration $(x = y = \theta = 0)$.

**Fig. 2** shows the convergence of the vehicle's coordinates to zero.

**Fig. 3** shows the control inputs $u_1$ and $u_2$ versus time.

Finally, **Fig. 4** shows the linear decreasing of $Log(\rho(y))$ which illustrates the exponential convergence of $\rho(y)$ to zero.

## APPENDIX

(Existence of homogeneous Lyapunov functions for asymptotically stable homogeneous time-varying systems)

The purpose of this appendix is to extend to homogeneous periodic time-varying systems a theorem by Rosier [19, theorem 2], itself a generalization of a theorem by Hahn [9], stating that a continuous homogeneous time-invariant vector field with a unique globally asymptotically stable equilibrium admits a homogeneous Lyapunov function. This result is the core of the exponential convergence property associated with asymptotically stable homogeneous vector fields of degree zero (see the proof of Proposition 1).

**Proposition 4** *Suppose that $x = 0$ is an asymptotically stable equilibrium of*

$$\dot{x} = F(t,x) , \tag{36}$$

*with $F \in \mathcal{C}^0(I\!R \times I\!R^n, I\!R^n)$ being $T$-periodic ($F(t+T,x) = F(t,x)$) and homogeneous of degree $\tau$ with respect to the dilation $\delta_\lambda$ defined in (8).*

*Then, for any $\alpha > 0$, and any integer $p$ such that $p < \frac{\alpha}{\max\{r_j\}}$, there exists a function $\overline{V}(t,x)$ which satisfies:*

1. *$\overline{V} \in \mathcal{C}^p(I\!R \times I\!R^n, I\!R) \cap \mathcal{C}^\infty(I\!R \times I\!R^n - \{0\}, I\!R)$*
2. *$\overline{V}$ is $T$-periodic : $\overline{V}(t+T,x) = \overline{V}(t,x)$*
3. *$\overline{V}$ is homogeneous of degree $\alpha$ with respect to the dilation $\delta_\lambda$ :*

$$\overline{V}(\delta_\lambda(t,x)) = \lambda^\alpha \overline{V}(t,x) \tag{37}$$

4. *$\overline{V}(t,0) = 0$ for all $t$, $\overline{V}(t,x) > 0$ for $x \neq 0$ and $\overline{V}$ is proper with respect to $x$ (for all $K$, the set of $x$'s such that $\overline{V}(t,x) \leq K$ for at least one $t$ is bounded).*
5. *$\dot{\overline{V}}(t,x) = \frac{\partial \overline{V}}{\partial t}(t,x) + \sum_{i=1}^n \frac{\partial \overline{V}}{\partial x_n} f_n(t,x) < 0 \quad \forall x \neq 0.$*

**Sketch of proof :** The only difference with [19, theorem 2] is the periodic time-dependence of both $F$ and $\overline{V}$. We may go along the same lines as in [19]:

- Using homogeneity and periodicity, local stability of $\dot{x} = F(t,x)$ implies "strong stability" on $I\!R^n$ (similar to [19, proposition 1]).

- The next step consists in using a theorem from [10] in order to exhibit a smooth non homogeneous Lyapunov function. Actually, the case of periodic time-dependence is considered in [10, theorem 7]. The function $V(t,x)$ so

obtained has all the properties of the function $\overline{V}(t,x)$ described in Proposition 4, except for homogeneity and that it is $\mathcal{C}^\infty$ everywhere.
- One may then proceed with the construction of a homogeneous function $\overline{V}$ from the non-homogeneous function $V$, as done in [19, proposition 2]. This construction is the following. Let $a \in \mathcal{C}^\infty(I\!R, I\!R)$ be a nondecreasing function on $I\!R$, constant and equal to 0 on $(-\infty, 1]$, and constant and equal to 1 on $[2, +\infty)$. $\overline{V}$ is then defined by:

$$\overline{V}(t,x) \quad = \quad \begin{cases} \displaystyle\int_0^{+\infty} \frac{1}{\mu^{\alpha+1}} a\left(V\left(\delta_\mu(t,x)\right)\right) \mathrm{d}\mu & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{38}$$

One easily verifies that $\overline{V}$ is time-periodic with the same period as $V$ and that it is homogeneous with degree $\alpha$ (by a change of integration variable). The subsequent arguments given by Rosier to prove his theorem remain valid when $\overline{V}$ is time-periodic. The fact that $\overline{V}(t,x)$ is usually not differentiable more than $p$ times at $x = 0$, with $p < \frac{\alpha}{\max\{r_j\}}$, basically comes from that the function $s \mapsto s^{\alpha/r}$ ($\alpha > 0$, $r > 0$, $\alpha/r$ not in $I\!N$), which is homogeneous of degree $\alpha$ with respect to the dilation $\delta_\lambda(s) = \lambda^r s$, is only of class $\mathcal{C}^q$ with $q < \alpha/r$. ∎

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Fig.1: Motion of the vehicle

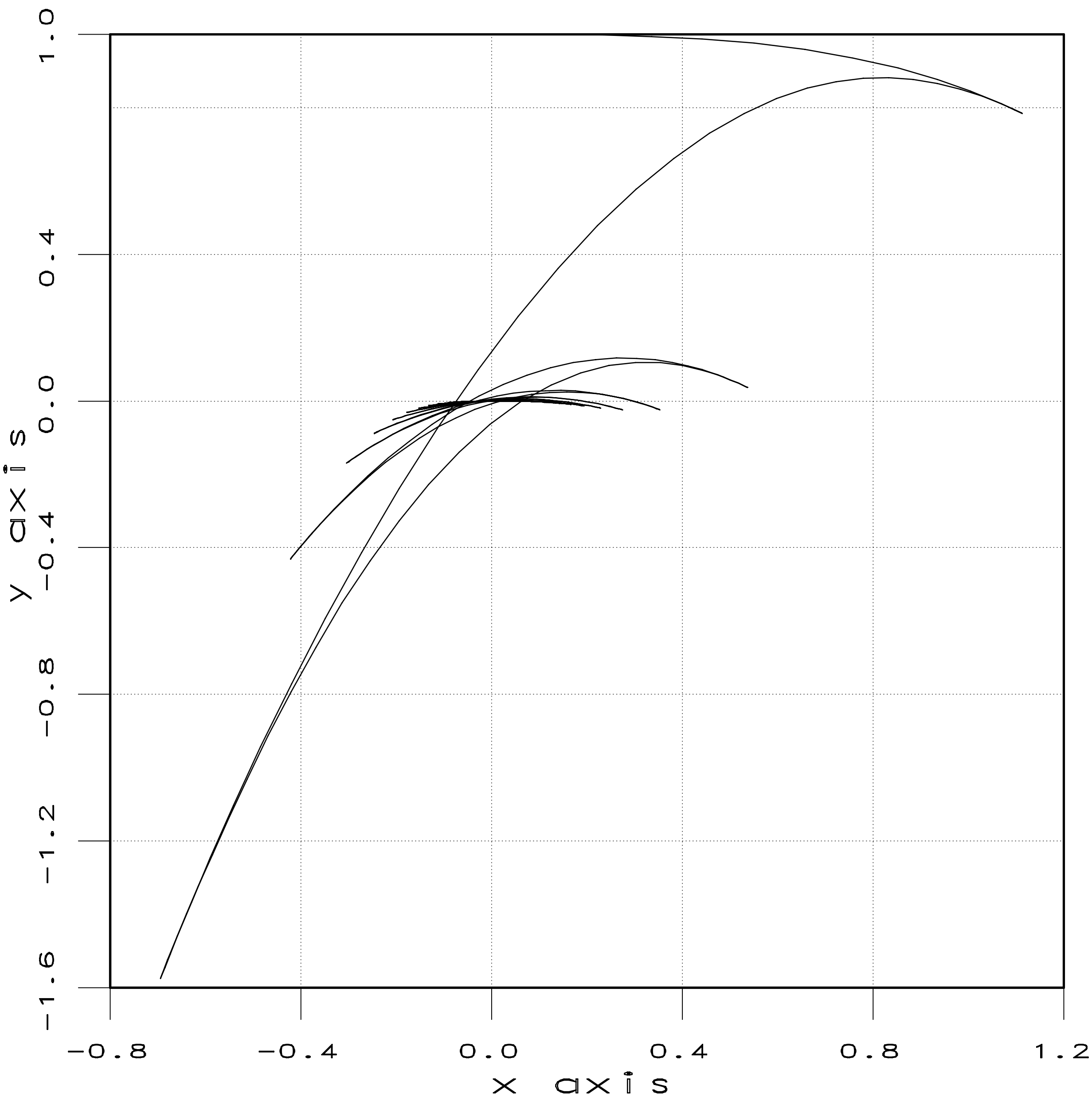
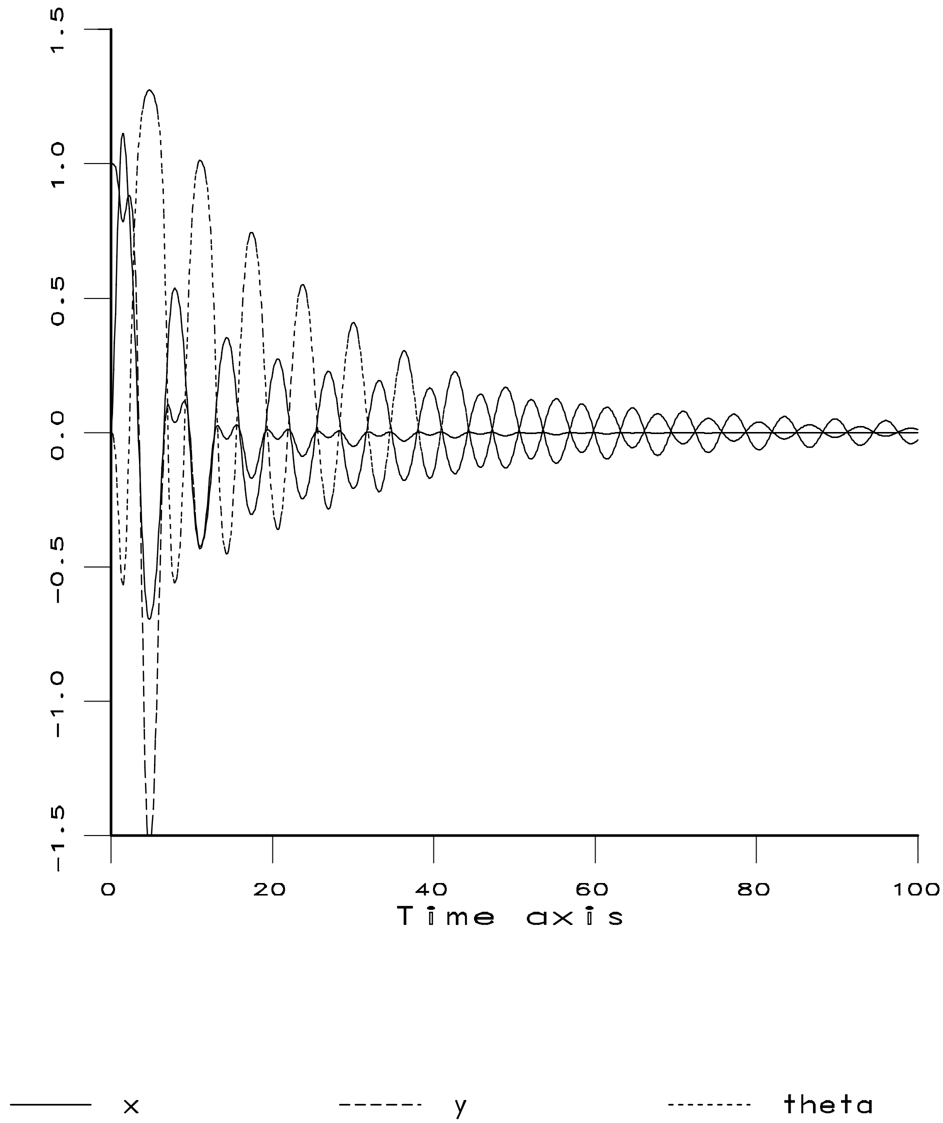


Fig.2: Convergence of x, y, and theta



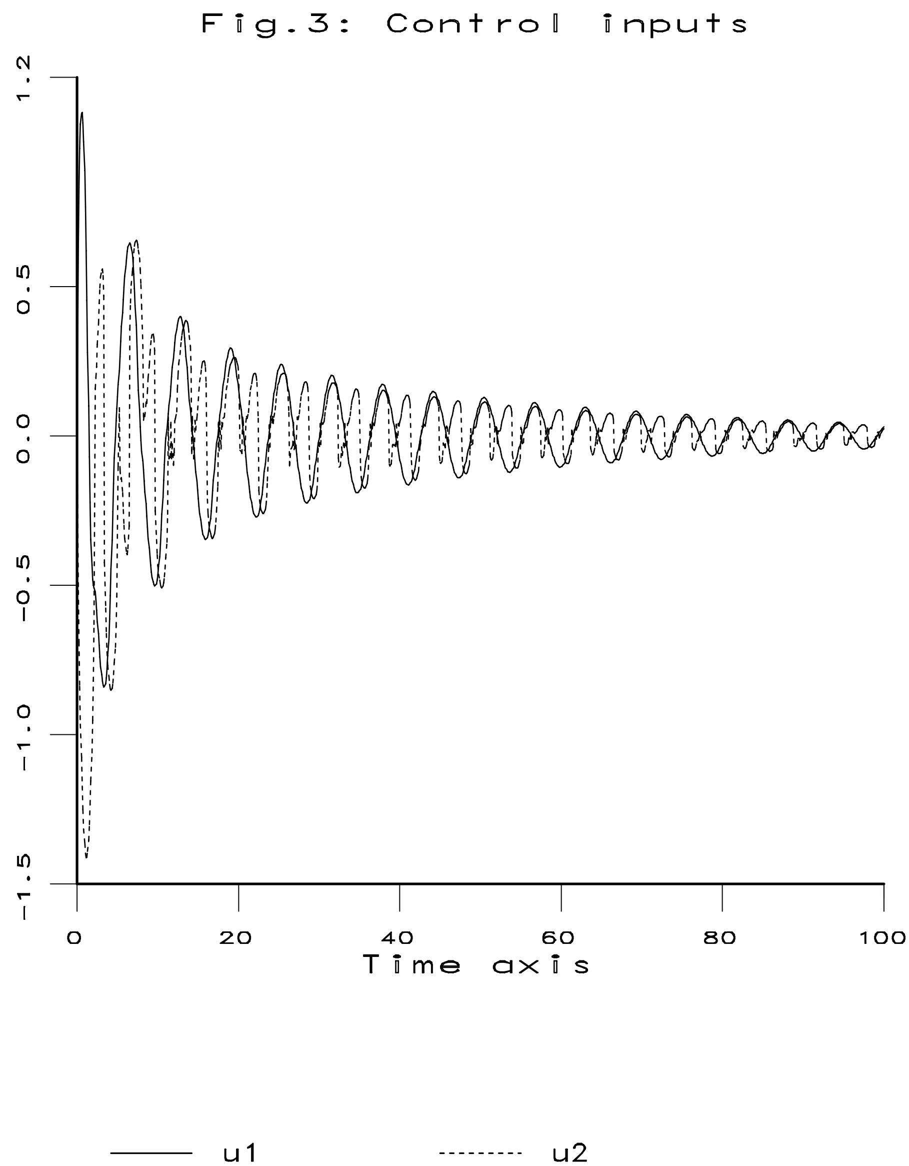
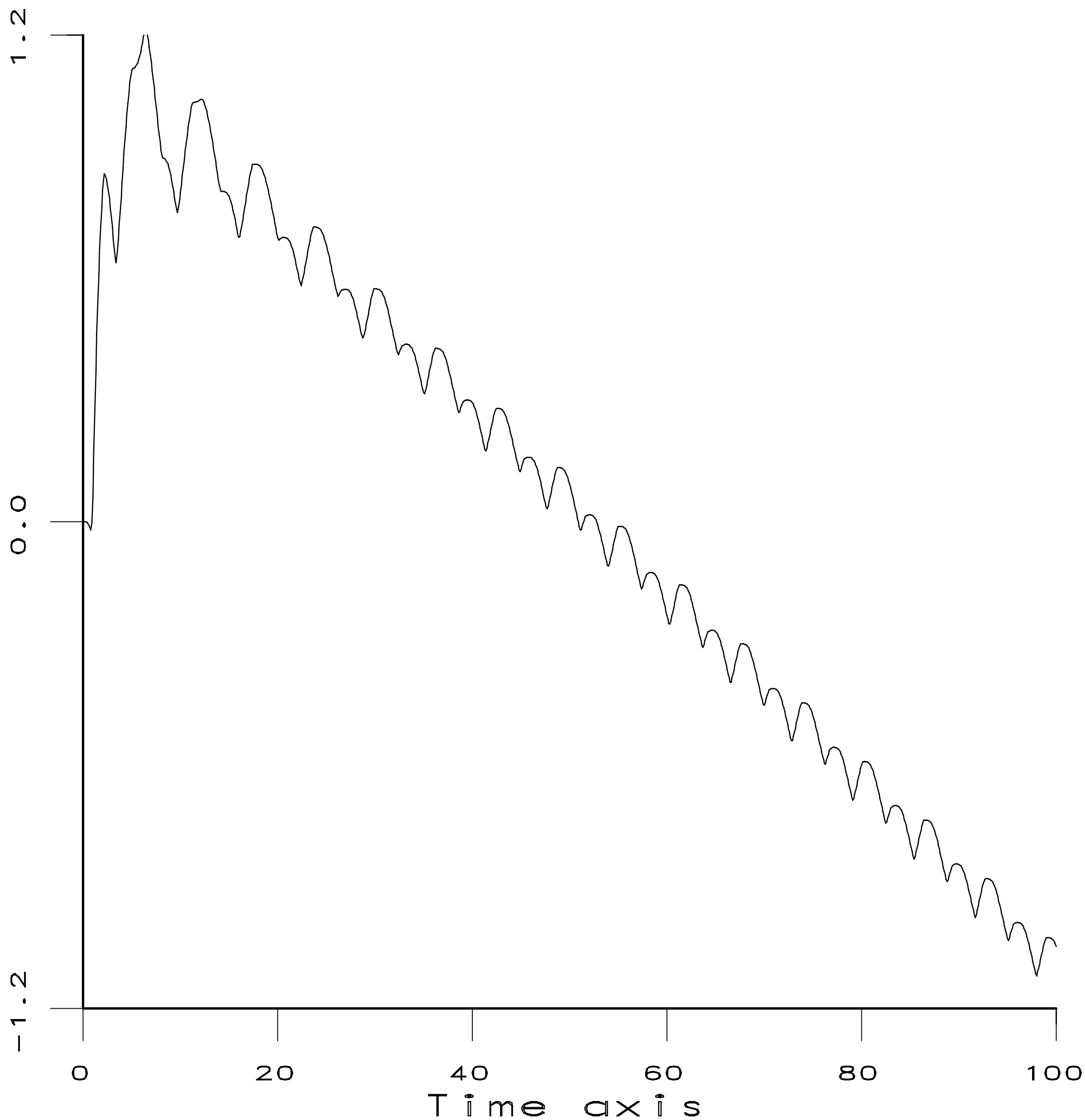


Fig.3: Control inputs
1.2
0.5
0.0
-0.5
-1.0
-1.5
0
20
40
60
80
100
Time axis
u1
u2

Fig.4: Log(homogeneous norm) versus time



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Éditeur

INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)

ISSN 0249-6399