

## FINITE DIFFERENCE SCHEMES AS A MATRIX EQUATION

CLAIRE DAVID \*

**Abstract.**

Finite difference schemes are here solved by means of a linear matrix equation. The theoretical study of the related algebraic system is exposed, and enables us to minimize the error due to a finite difference approximation.

**Key words.** Finite difference schemes, Sylvester equation

**AMS subject classifications.** 15A15, 15A09, 15A23

**1. Introduction: Scheme classes.** Finite difference schemes used to approximate linear differential equations are usually solved by means of a recursive calculus. We here propose a completely different approach, which uses an equivalent matrix equation.

Consider the transport equation:

$$(1.1) \quad \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0, \quad x \in [0, L], \quad t \in [0, T]$$

with the initial condition  $u(x, t = 0) = u_0(x)$ .

**PROPOSITION 1.1.**

*A finite difference scheme for this equation can be written under the form:*

$$(1.2) \quad \alpha u_i^{n+1} + \beta u_i^n + \gamma u_i^{n-1} + \delta u_{i+1}^n + \varepsilon u_{i-1}^n + \zeta u_{i+1}^{n+1} + \eta u_{i-1}^{n-1} + \theta u_{i-1}^{n+1} + \vartheta u_{i+1}^{n-1} = 0$$

where:

$$(1.3) \quad u_l^m = u(lh, m\tau)$$

$l \in \{i-1, i, i+1\}$ ,  $m \in \{n-1, n, n+1\}$ ,  $j = 0, \dots, n_x$ ,  $n = 0, \dots, n_t$ ,  $h, \tau$  denoting respectively the mesh size and time step ( $L = n_x h$ ,  $T = n_t \tau$ ).

The Courant-Friedrichs-Lewy number (*cfl*) is defined as  $\sigma = c\tau/h$ .

*A numerical scheme is specified by selecting appropriate values of the coefficients  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta$  and  $\vartheta$  in equation (1.2). Values corresponding to numerical schemes retained for the present works are given in Table 1.1.*

The number of time steps will be denoted  $n_t$ , the number of space steps,  $n_x$ . In general,  $n_x \gg n_t$ .

---

\*Université Pierre et Marie Curie-Paris 6, Laboratoire de Modélisation en Mécanique, UMR CNRS 7607, Boîte courrier n°162, 4 place Jussieu, 75252 Paris cedex 05, France ([david@lmm.jussieu.fr](mailto:david@lmm.jussieu.fr)).

TABLE 1.1  
Numerical scheme coefficient.

Name	$\alpha$	$\beta$	$\gamma$	$\delta$	$\epsilon$	$\zeta$	$\eta$	$\theta$	$\vartheta$
Leapfrog	$\frac{1}{2\tau}$	0	$\frac{-1}{2\tau}$	$\frac{c}{2h}$	$\frac{-c}{2h}$	0	0	0	0
Lax	$\frac{1}{\tau}$	0	0	$\frac{-1}{2\tau} + \frac{c}{2h}$	$\frac{-1}{2\tau} - \frac{c}{2h}$	0	0	0	0
Lax-Wendroff	$\frac{1}{\tau}$	$\frac{-1}{\tau} + \frac{c^2\tau}{h^2}$	0	$\frac{(1-\sigma)c}{2h}$	$\frac{-(1+\sigma)c}{2h}$	0	0	0	0
Crank-Nicolson	$\frac{1}{\tau} + \frac{c}{h^2}$	$\frac{-1}{\tau} + \frac{c}{h^2}$	0	$\frac{-c}{h^2}$	$\frac{c}{h^2}$	0	$\frac{-c}{h^2}$	$\frac{-c}{h^2}$	0

The paper is organized as follows. The equivalent matrix equation is exposed in section 2. Specific properties of the involved matrices are exposed in section 3. A way of minimizing the error due to the finite difference approximation is presented in section 4.

## 2. The Sylvester equation.

**2.1. Matricial form of the finite differences problem.** Let us introduce the rectangular matrix defined by:

$$(2.1) \quad U = [u_i^n]_{1 \leq i \leq n_x - 1, 1 \leq n \leq n_t}$$

**THEOREM 2.1.** *The problem (1.2) can be written under the following matricial form:*

$$(2.2) \quad M_1 U + U M_2 + \mathcal{L}(U) = M_0$$

where  $M_1$  and  $M_2$  are square matrices respectively  $n_x - 1$  by  $n_x - 1$ ,  $n_t$  by  $n_t$ , given by:

$$(2.3) \quad M_1 = \begin{pmatrix} \beta & \delta & 0 & \dots & 0 \\ \varepsilon & \beta & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \beta & \delta \\ 0 & \dots & 0 & \varepsilon & \beta \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & \gamma & 0 & \dots & 0 \\ \alpha & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma \\ 0 & \dots & 0 & \alpha & 0 \end{pmatrix}$$

the matrix  $M_0$  being given by:

$$(2.4) \quad M_0 = \begin{pmatrix} -\gamma u_1^0 - \varepsilon u_0^1 - \eta u_0^0 - \theta u_0^2 - \vartheta u_2^0 & -\varepsilon u_0^2 - \eta u_0^1 - \theta u_0^3 & \dots & \dots & -\varepsilon u_0^{n_t} - \eta u_0^{n_t-1} \\ -\gamma u_2^0 - \eta u_1^0 - \vartheta u_3^0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\gamma u_{n_x-2}^0 - \eta u_{n_x-2}^0 - \vartheta u_{n_x-1}^0 & 0 & \dots & \dots & 0 \\ -\gamma u_{n_x-1}^0 - \delta u_{n_x}^1 - \eta u_{n_x-2}^0 - \zeta u_{n_x}^2 - \vartheta u_{n_x}^0 & -\delta u_{n_x}^2 - \zeta u_{n_x}^3 - \vartheta u_{n_x}^1 & \dots & \dots & -\delta u_{n_x}^{n_t} - \vartheta u_{n_x}^{n_t-1} \end{pmatrix}$$

and where  $\mathcal{L}$  is a linear matricial operator which can be written as:

$$(2.5) \quad \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$$

where  $\mathcal{L}_1$ ,  $\mathcal{L}_2$ ,  $\mathcal{L}_3$  and  $\mathcal{L}_4$  are given by:

$$(2.6) \quad \mathcal{L}_1(U) = \zeta \begin{pmatrix} u_2^2 & u_2^3 & \dots & u_2^{n_t} & 0 \\ u_3^2 & u_3^3 & \dots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{n_x-1}^2 & u_{n_x-1}^3 & \dots & u_{n_x-1}^{n_t} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \mathcal{L}_2(U) = \eta \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & u_1^1 & u_1^2 & \dots & u_1^{n_t-1} \\ 0 & u_1^0 & u_1^1 & \dots & u_2^{n_t-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & u_{n_x-2}^1 & u_{n_x-2}^2 & \dots & u_{n_x-2}^{n_t-1} \end{pmatrix}$$

$$(2.7) \quad \mathcal{L}_3(U) = \theta \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ u_1^2 & u_1^3 & \dots & u_1^{n_t} & 0 \\ u_2^2 & u_2^3 & \dots & u_2^{n_t} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{n_x-2}^2 & u_{n_x-2}^3 & \dots & u_{n_x-2}^{n_t} & 0 \end{pmatrix} \quad \mathcal{L}_4(U) = \vartheta \begin{pmatrix} 0 & u_2^1 & u_2^2 & \dots & u_2^{n_t-1} \\ 0 & u_3^1 & u_3^2 & \dots & u_3^{n_t-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & u_{n_x-1}^1 & \dots & \dots & u_{n_x-1}^{n_t-1} \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

PROPOSITION 2.2. *The second member matrix  $M_0$  bears the initial conditions, given for the specific value  $n = 0$ , which correspond to the initialization process when computing loops, and the boundary conditions, given for the specific values  $i = 0$ ,  $i = n_x$ .*

Denote by  $u_{exact}$  the exact solution of (1.1).  
The corresponding matrix  $U_{exact}$  will be:

$$(2.8) \quad U_{exact} = [U_{exact_i}^n]_{1 \leq i \leq n_x-1, 1 \leq n \leq n_t}$$

where:

$$(2.9) \quad U_{exact_i}^n = U_{exact}(x_i, t_n)$$

with  $x_i = i h$ ,  $t_n = n \tau$ .

DEFINITION 2.3. *We will call error matrix the matrix defined by:*

$$(2.10) \quad E = U - U_{exact}$$

Consider the matrix  $F$  defined by:

$$(2.11) \quad F = M_1 U_{exact} + U_{exact} M_2 + \mathcal{L}(U_{exact}) - M_0$$

PROPOSITION 2.4. *The error matrix  $E$  satisfies:*

$$(2.12) \quad M_1 E + E M_2 + \mathcal{L}(E) = F$$

## 2.2. The matrix equation.

**2.2.1. Theoretical formulation.** THEOREM 2.5. *Minimizing the error due to the approximation induced by the numerical scheme is equivalent to minimizing the norm of the matrices  $E$  satisfying (2.12).*

*Note:* Since the linear matricial operator  $\mathcal{L}$  appears only in the Crank-Nicolson scheme, we will restrain our study to the case  $\mathcal{L} = 0$ . The generalization to the case  $\mathcal{L} \neq 0$  can be easily deduced.

PROPOSITION 2.6.

*The problem is then the determination of the minimum norm solution of:*

$$(2.13) \quad M_1 E + E M_2 = F$$

*which is a specific form of the Sylvester equation:*

$$(2.14) \quad AX + XB = C$$

*where  $A$  and  $B$  are respectively  $m$  by  $m$  and  $n$  by  $n$  matrices,  $C$  and  $X$ ,  $m$  by  $n$  matrices.*

The solving of the Sylvester equation is generally based on Schur decomposition: for a given square  $n$  by  $n$  matrix  $A$ ,  $n$  being an even number of the form  $n = 2p$ , there exists a unitary matrix  $U$  and a upper triangular block matrix  $T$  such that:

$$(2.15) \quad A = U^* T U$$

where  $U^*$  denotes the (complex) conjugate matrix of the transposed matrix  $^T U$ . The diagonal blocks of the matrix  $T$  correspond to the complex eigenvalues  $\lambda_i$  of  $A$ :

$$(2.16) \quad T = \begin{pmatrix} T_1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & T_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & T_p \end{pmatrix}$$

where the block matrices  $T_i$ ,  $i = 1, \dots, p$  are given by:

$$(2.17) \quad \begin{pmatrix} \mathcal{R}e[\lambda_i] & \mathcal{I}m[\lambda_i] \\ -\mathcal{I}m[\lambda_i] & \mathcal{R}e[\lambda_i] \end{pmatrix}$$

$\mathcal{R}e$  being the real part of a complex number, and  $\mathcal{I}m$  the imaginary one.

Due to this decomposition, the Sylvester equation require, to be solved, that the

dimensions of the matrices be even numbers. We will therefore, in the following, restrain our study to  $n_x - 1$  and  $n_t$  being even numbers. So far, it is interesting to note that the Schur decomposition being more stable for higher order matrices, it perfectly fits finite difference problems.

Complete parametric solutions of the generalized Sylvester equation (2.13) is given in [2], [3].

As for the determination of the solution of the Sylvester equation, it is a major topic in control theory, and has been the subject of numerous works (see [1], [6], [8], [9], [10], [11], [12]).

In [1], the method is based on the reduction of the observable pair  $(A, C)$  to an observer-Hessenberg pair  $(H, D)$ ,  $H$  being a block upper Hessenberg matrix. The reduction to the observer-Hessenberg form  $(H, D)$  is achieved by means of the staircase algorithm (see [4], ...).

In [9], in the specific case of  $B$  being a companion form matrix, the authors propose a very neat general complete parametric solution, which is expressed in terms of the controllability of the matrix pair  $(A, B)$ , a symmetric matrix operator, and a parametric matrix in the Hankel form.

We recall that a companion form, or Frobenius matrix is one of the following kind:

$$(2.18) \quad B = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 & -b_0 \\ 1 & 0 & \dots & \dots & 0 & -b_1 \\ 0 & 1 & 0 & \dots & \vdots & \vdots \\ \vdots & 0 & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -b_{p-1} \end{pmatrix}$$

These results can be generalized through matrix block decomposition to a block companion form matrix:

$$(2.19) \quad M_2 = \begin{pmatrix} M_2^{B^1} & 0 & \dots & \dots & 0 \\ 0 & M_2^{B^2} & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & M_2^{B^k} \end{pmatrix}$$

the  $M_2^{B^p}$ ,  $1 \leq p \leq k$  being companion form matrices.

Another method is presented in [14], where the determination of the minimum-norm solution of a Sylvester equation is specifically developed.

The accuracy and computational stability of the solutions is examined in [5].

**2.2.2. Existence condition of the solution .** THEOREM 2.7. *In the specific cases of the Lax and Lax-Wendroff schemes, (2.14) has a unique solution.*

*Proof.* (2.14) has a unique solution if and only if  $A$  and  $B$  have no common eigenvalues.

The characteristic polynomials  $P_{M_1}$ ,  $P_{M_2}$  of  $M_1$  and  $M_2$ , can be respectively calculated as the determinants of respectively  $\frac{n_x-1}{2}$  by  $\frac{n_x-1}{2}$ ,  $\frac{n_t}{2}$  by  $\frac{n_t}{2}$  diagonal block matrices:

$$(2.20) \quad P_{M_1}(\lambda) = ((\lambda - \beta)^2 - \delta \varepsilon)^{\frac{n_x-1}{2}}, \quad P_{M_2}(\lambda) = (\lambda^2 - \alpha \gamma)^{\frac{n_t}{2}}$$

In the specific cases of the Lax and Lax-Wendroff schemes:  $\alpha\gamma = 0 \neq \pm\beta + \sqrt{\delta\varepsilon}$ . Hence, (2.12) has a unique solution, which accounts for the consistency of the given problem.

□

**COROLLARY 2.8.** *In the specific case of the Leapfrog scheme, (2.14) has a unique solution if and only if  $\tau \neq \frac{h}{c}$ .*

### 3. Specific properties of the matrices $M_1$ and $M_2$ .

**3.0.3. Inversibility of the matrix  $M_1$ .** **THEOREM 3.1.** *In the specific case of the Lax and Leapfrog schemes,  $M_1$  is invertible.*

**THEOREM 3.2.** *In the specific case of the Lax-Wendroff scheme,  $M_1$  is invertible if and only if*

$$(3.1) \quad \left(\frac{-1}{\tau} + \frac{c^2\tau}{h^2}\right)^2 \neq \frac{(\sigma^2 - 1)c^2}{4h^2}$$

*Proof.* The determinant  $D_1$  of  $M_1$  can be calculated as the determinant of a  $\frac{n_x-1}{2}$  by  $\frac{n_x-1}{2}$  diagonal block matrix:

$$(3.2) \quad D_1 = (\beta^2 - \delta\varepsilon)^{\frac{n_x-1}{2}}$$

□

**3.0.4. Nilpotent components of the matrix  $M_2$ .** **PROPOSITION 3.3.**  *$M_2$  can be written as:*

$$(3.3) \quad M_2 = \alpha {}^T N + \gamma N$$

where  $N$  is the nilpotent matrix;  ${}^T N$  denotes the corresponding transposed matrix:

$$(3.4) \quad N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \dots & 0 & 1 \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

**PROPOSITION 3.4.**

*For either  $\alpha = 0$ , or  $\gamma = 0$ ,  $M_2$  will thus be nilpotent, of order  $n_t$ .*

**PROPOSITION 3.5.**

*For  $\gamma = 0$ , if  $M_1$  is invertible, the solution at  $t = n_t dt$  can then be immediately determined.*

*Proof.*

In such a case, multiplying (2.2) on the right side by  $M_2^{n_t-1}$  leads to:

$$(3.5) \quad M_1 U M_2^{n_t-1} + U M_2^{n_t} = M_0 M_2^{n_t-1}$$

i. e.:

$$(3.6) \quad M_1 U M_2^{n_t-1} = M_0 M_2^{n_t-1}$$

which leads to:

$$(3.7) \quad U M_2^{n_t-1} = M_1^{-1} M_0 M_2^{n_t-1}$$

Due to:

$$(3.8) \quad M_2^{n_t-1} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

we have:

$$(3.9) \quad U M_2^{n_t-1} = \begin{pmatrix} u_{1n_t} & 0 & 0 & \dots & 0 \\ u_{2n_t} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ u_{n_x n_t} & 0 & \dots & 0 & 0 \end{pmatrix}$$

The solution at  $t = n_t dt$  can then be immediately determined.

□

#### 4. Minimization of the error.

4.1. Theory. Calculation yields:

$$(4.1) \quad \begin{cases} M_1^T M_1 & = \text{diag}\left(\begin{pmatrix} \beta^2 + \delta^2 & \beta(\delta + \varepsilon) \\ \beta(\delta + \varepsilon) & \varepsilon^2 + \beta^2 \end{pmatrix}, \dots, \begin{pmatrix} \beta^2 + \delta^2 & \beta(\delta + \varepsilon) \\ \beta(\delta + \varepsilon) & \varepsilon^2 + \beta^2 \end{pmatrix}\right) \\ M_2^T M_2 & = \text{diag}\left(\begin{pmatrix} \gamma^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}, \dots, \begin{pmatrix} \gamma^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}\right) \end{cases}$$

The singular values of  $M_1$  are the singular values of the block matrix  $\begin{pmatrix} \beta^2 + \delta^2 & \beta(\delta + \varepsilon) \\ \beta(\delta + \varepsilon) & \varepsilon^2 + \beta^2 \end{pmatrix}$ ,

i. e.

$$(4.2) \quad \frac{1}{2} (2\beta^2 + \delta^2 + \varepsilon^2 - (\delta + \varepsilon) \sqrt{4\beta^2 + \delta^2 + \varepsilon^2 - 2\delta\varepsilon})$$

of order  $\frac{n_x-1}{2}$ , and

$$(4.3) \quad \frac{1}{2} (2\beta^2 + \delta^2 + \varepsilon^2 + (\delta + \varepsilon) \sqrt{4\beta^2 + \delta^2 + \varepsilon^2 - 2\delta\varepsilon})$$

of order  $\frac{n_x-1}{2}$ .

The singular values of  $M_2$  are  $\alpha^2$ , of order  $\frac{n_t}{2}$ , and  $\gamma^2$ , of order  $\frac{n_t}{2}$ .

Consider the singular value decomposition of the matrices  $M_1$  and  $M_2$ :

$$(4.4) \quad U_1^T M_1 V_1 = \begin{pmatrix} \widetilde{M}_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad U_2^T M_2 V_2 = \begin{pmatrix} \widetilde{M}_2 & 0 \\ 0 & 0 \end{pmatrix}$$

where  $U_1, V_1, U_2, V_2$ , are orthogonal matrices.  $\widetilde{M}_1, \widetilde{M}_2$  are diagonal matrices, the diagonal terms of which are respectively the nonzero eigenvalues of the symmetric matrices  $M_1^T M_1, M_2^T M_2$ .

Multiplying respectively 2.13 on the left side by  ${}^T U_1$ , on the right side by  $V_2$ , yields:

$$(4.5) \quad U_1^T M_1 E V_2 + U_1^T E M_2 V_2 = U_1^T F V_2$$

which can also be taken as:

$$(4.6) \quad {}^T U_1 M_1 V_1 {}^T V_1 E V_2 + {}^T U_1 E {}^T U_2 {}^T U_2 M_2 V_2 = U_1^T F V_2$$

Set:

$$(4.7) \quad {}^T V_1 E V_2 = \begin{pmatrix} \widetilde{E}_{11} & \widetilde{E}_{12} \\ \widetilde{E}_{21} & \widetilde{E}_{22} \end{pmatrix}, \quad {}^T U_1 E {}^T U_2 = \begin{pmatrix} \widetilde{\widetilde{E}}_{11} & \widetilde{\widetilde{E}}_{12} \\ \widetilde{\widetilde{E}}_{21} & \widetilde{\widetilde{E}}_{22} \end{pmatrix}$$

$$(4.8) \quad {}^T U_1 F V_2 = \begin{pmatrix} \widetilde{F}_{11} & \widetilde{F}_{12} \\ \widetilde{F}_{21} & \widetilde{F}_{22} \end{pmatrix}$$

We have thus:

$$(4.9) \quad \begin{pmatrix} \widetilde{M}_1 \widetilde{E}_{11} & \widetilde{M}_1 \widetilde{E}_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \widetilde{\widetilde{E}}_{11} \widetilde{M}_2 & 0 \\ \widetilde{\widetilde{E}}_{21} \widetilde{M}_2 & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{F}_{11} & \widetilde{F}_{12} \\ \widetilde{F}_{21} & \widetilde{F}_{22} \end{pmatrix}$$

It yields:

$$(4.10) \quad \begin{cases} \widetilde{M}_1 \widetilde{E}_{11} + \widetilde{\widetilde{E}}_{11} \widetilde{M}_2 & = \widetilde{F}_{11} \\ \widetilde{M}_1 \widetilde{E}_{12} & = \widetilde{F}_{12} \\ \widetilde{\widetilde{E}}_{21} \widetilde{M}_2 & = \widetilde{F}_{21} \end{cases}$$

One easily deduces:

$$(4.11) \quad \begin{cases} \widetilde{E}_{12} & = \widetilde{M}_1^{-1} \widetilde{F}_{12} \\ \widetilde{E}_{21} & = \widetilde{F}_{21} \widetilde{M}_2^{-1} \end{cases}$$

The problem is then the determination of the  $\widetilde{E}_{11}$  and  $\widetilde{\widetilde{E}}_{11}$  satisfying:

$$(4.12) \quad \widetilde{M}_1 \widetilde{E}_{11} + \widetilde{\widetilde{E}}_{11} \widetilde{M}_2 = \widetilde{F}_{11}$$

Denote respectively by  $\widetilde{e}_{ij}$ ,  $\widetilde{\widetilde{e}}_{ij}$  the components of the matrices  $\widetilde{E}$ ,  $\widetilde{\widetilde{E}}$ .  
The problem 4.12 uncouples into the independent problems:  
minimize

$$(4.13) \quad \sum_{i,j} \widetilde{e}_{ij}^2 + \widetilde{\widetilde{e}}_{ij}^2$$

under the constraint

$$(4.14) \quad \widetilde{M}_{1ii} \widetilde{e}_{ij} + \widetilde{M}_{2ii} \widetilde{\widetilde{e}}_{ij} = \widetilde{F}_{11ij}$$

This latter problem has the solution:

$$(4.15) \quad \begin{cases} \widetilde{e}_{ij} = \frac{\widetilde{M}_{1ii} \widetilde{F}_{11ij}}{\widetilde{M}_{1ii} + \widetilde{M}_{2jj}} \\ \widetilde{\widetilde{e}}_{ij} = \frac{\widetilde{M}_{2jj} \widetilde{F}_{11ij}}{\widetilde{M}_{1ii} + \widetilde{M}_{2jj}} \end{cases}$$

The minimum norm solution of 2.13 will then be obtained when the norm of the matrix  $\widetilde{F}_{11}$  is minimum.

In the following, the euclidean norm will be considered.

Due to (4.8):

$$(4.16) \quad \|\widetilde{F}_{11}\| \leq \|\widetilde{F}\| \leq \|U_1\| \|F\| \|V_2\| \leq \|U_1\| \|V_2\| \|M_1 U_{exact} + U_{exact} M_2 - M_0\|$$

$U_1$  and  $V_2$  being orthogonal matrices, respectively  $n_x - 1$  by  $n_x - 1$ ,  $n_t$  by  $n_t$ , we have:

$$(4.17) \quad \|U_1\|^2 = n_x - 1, \quad \|V_2\|^2 = n_t$$

Also:

$$(4.18) \quad \|M_1\|^2 = \frac{n_x - 1}{2} (2\beta^2 + \delta^2 + \varepsilon^2), \quad \|M_2\|^2 = \frac{n_t}{2} (\alpha^2 + \gamma^2)$$

The norm of  $M_0$  is obtained thanks to relation (2.4).

This results in:

$$(4.19) \quad \|\widetilde{F}_{11}\| \leq \sqrt{n_t(n_x - 1)} \left\{ \|U_{exact}\| \left( \sqrt{\frac{n_x - 1}{2}} \sqrt{2\beta^2 + \delta^2 + \varepsilon^2} + \sqrt{\frac{n_t}{2}} \sqrt{\alpha^2 + \gamma^2} \right) + \|M_0\| \right\}$$

$\|\widetilde{F}_{11}\|$  can be minimized through the minimization of the right-side member of (4.19), which is function of the scheme parameters.

**4.2. Numerical example: the specific case of the Lax scheme.** For the Lax scheme:  $\gamma = 0$ . The remaining coefficients (see Table 1.1) can be normalized through the following change of variables:

$$(4.20) \quad \begin{cases} \bar{\alpha} = h\alpha \\ \bar{\beta} = h\beta \\ \bar{\delta} = h\delta \\ \bar{\varepsilon} = h\varepsilon \end{cases}$$

Set:

$$(4.21) \quad \overline{M}_0 = h M_0$$

Thus:

$$(4.22) \quad \|\overline{M}_0\|^2 = \overline{\delta}^2 \sum_{n=1}^{n_t} u_{n_x}^2 + \overline{\varepsilon}^2 \sum_{n=1}^{n_t} u_0^2$$

Advect a sinusoidal signal

$$(4.23) \quad u = \cos\left[\frac{2\pi}{\lambda}(x - ct)\right]$$

through the Lax scheme, with Dirichlet boundary conditions.

The right-side member of (4.19) is then:

$$(4.24) \quad \left(\frac{1}{2} + \frac{1}{2cfl}\right)^2 n_t^2 u_0^2 + \left(\frac{1}{2} - \frac{1}{2cfl}\right)^2 n_t^2 u_L^2 + \sqrt{\frac{n_x - 1}{2}} \sqrt{\left(\frac{1}{2} + \frac{1}{2cfl}\right)^2 + \left(\frac{1}{2} - \frac{1}{2cfl}\right)^2} + \frac{\sqrt{n_t}}{\sqrt{2}cfl}$$

where  $u_0, u_L$  respectively denote the Dirichlet boundary values at  $x = 0, x = L$ .

It is minimal for the admissible value  $cfl = 1$ .

The value of the  $L_2$  norm of the error, for two significant values of the  $cfl$  number (Case 1:  $cfl = 0.9$ ; Case 2:  $cfl = 0.7$ ), is displayed in Figure 4.1. The error curve corresponding to the value of the  $cfl$  number closest to 1 is the minimal one.

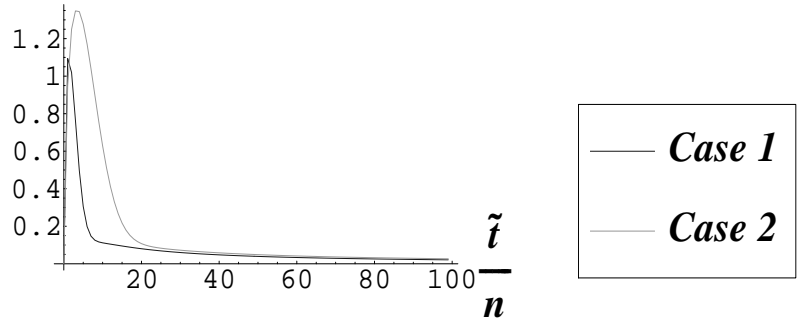


FIG. 4.1. Value of the  $L_2$  norm of the error for different values of the  $cfl$  number.

**5. Conclusion.** Thanks to the above results, we here propose:

1. to study the intrinsic properties of various schemes through those of the related matrices  $M_1$  and  $M_2$ ;
2. to optimize finite difference problems through minimization of the symbolic expression of the error as a function of the scheme parameters.

## REFERENCES

- [1] P. VAN DOOREN, *Reduced order observers: A new algorithm and proof*, Systems Control Lett., 4(1984), pp. 243-251.
- [2] A. BERMAN AND R. J. PLEMMONS, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia, PA (1994).
- [3] H. R. GAIL, S.L. HANTLER AND B. A. TAYLOR, *Spectral Analysis of M/G/1 and G/M/1 type Markov chains*, Adv. Appl. Probab., 28(1996), pp. 114-165.
- [4] D. L. BOLEY, *Computing the Controllability algorithm / Observability Decomposition of a Linear Time-Invariant Dynamic System, A Numerical Approach*, PhD. thesis, Report STAN-CS-81-860, Dept. Comp. i, Sci., Stanford University, 1981.
- [5] A. S. DEIF, N. P. SEIF, AND S. A. HUSSEIN, *Sylvester's equation: accuracy and computational stability*, Journal of Computational and Applied Mathematics, 61(1995), pp. 1-11.
- [6] J. Z. HEARON, *Nonsingular solutions of  $TA - BT = C$* , Linear Algebra and its applications, 16(1977), pp. 57-63.
- [7] C. H. HUO, *Efficient methods for solving a nonsymmetric algebraic equation arising in stochastic fluid models*, Journal of Computational and Applied Mathematics, pp. 1-21 (2004).
- [8] C. C. TSUI, *A complete analytical solution to the equation  $TA - FT = LC$  and its applications*, IEEE Trans. Automat. Control AC, 32(1987), pp. 742-744.
- [9] B. ZHOU AND G. R. DUAN, *An explicit solution to the matrix equation  $AX - XF = BY$* , Linear Algebra and its applications, 402(2005), pp. 345-366.
- [10] G. R. DUAN, *Solution to matrix equation  $AV + BW = EVF$  and eigenstructure assignment for descriptor systems*, Automatica, 28(1992), pp. 639-643.
- [11] G. R. DUAN, *On the solution to Sylvester matrix equation  $AV + BW = EVF$  and eigenstructure assignment for descriptor systems*, IEEE Trans. Automat. Control AC, 41(1996), pp. 276-280.
- [12] P. KIRRINNIS, *Fast algorithms for the Sylvester equation  $AX - XB^T = C$* , Theoretical Computer Science, 259(2000), pp. 623-638.
- [13] M. KONSTANTINOV, V. MEHRMANN, V. AND P. PETKOV, *On properties of Sylvester and Lyapunov operators*, Linear Algebra and its applications, 312(2000), pp. 35-71.
- [14] A. VARGA, *TA numerically reliable approach to robust pole assignment for descriptor systems*, Future Generation Computer Systems, 19(2003), pp. 1221-1230.
- [15] G. B. WITHAM, *Linear and Nonlinear Wave*, Wiley-Interscience (1974).
- [16] S. WOLFRAM, *The Mathematica book*, Cambridge University Press (1999).