AN UNIFORME ESTIMATE FOR SCALAR CURVATURE EQUATION ON MANIFOLDS OF DIMENSION 4.

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ABSTRACT. We give an a priori estimate for the solutions of the prescribed scalar curvature equation on manifolds of dimension 4. We have an idea on the supremum of the solutions if we control their infimum.

1. INTRODUCTION AND RESULT.

In this paper, we are on Riemannian manifold of dimension 4, (M,g) (not necessarily compact). Here we denote by $\Delta_g = -\nabla^i(\nabla_i)$ the geometric Laplacian.

Let us consider the prescribed scalar curvature equation in four dimension:

$$\Delta_g u + R_g u = V u^3, \;\; u > 0 \qquad (E)$$

where R_g is a scalar curvature of (M,g) and V the prescribed scalar curvature.

We assume:

$$0 < a \leq V(x) \leq b \;\text{ and }\; ||\nabla V||_{L^\infty(M)} \leq A \qquad (C).$$

In this paper, we want to prove an uniform estimate for the solutions of the eqution (E) with minimal conditions on the prescribed scalar curvature equation. Conditions like (C) are minimal.

Note that the equation (E) was studied when $M = \Omega$ is a open set of $\mathbb{R}^4$, see for example, [B], [C-L] and when $\Omega = \mathbb{S}_4$ the unit sphere of dimension 4 by Li, see [L].

If we suppose $V \in C^2(\Omega)$, Chen and Lin gave a $\sup \times \inf$ inequality for the solutions of the equation (E). In [L], on the fourth unit sphere, Li study the same equation with the same conditions on V, he obtains the boundedness of the energy and an upper bound for the product $\sup \times \inf$. He use the simple blow-up analysis (for the definition of simple blow up points see for exemple [L]).

In [B], we can see (on a bounded domain of $\mathbb{R}^4$) that we have an uniform estimate for the solutions of the equation (E) if we control the infimum of those functions, with only Lipschitzian assumption on the prescribed scalar curvature V.

Here we extend the result of [B], to general manifolds of dimension 4.

Note, if we assume $V \equiv 1$, Li and Zhang (see [L-Z 1]), have proved a $\sup \times \inf$ inequality for the solutions of (E) on any Riemannian manifold of dimension 4.

If we suppose M compact, the existence result for this equation when $V \equiv 1$ was proved by T. Aubin (non conformally flat case and $n \geq 6$) and R. Schoen (conformally flat case and $n = 3, 4, 5$). The previous equation with $V \equiv 1$ is called *the Yamabe equation*.

Note that, in diemsions $n = 3$ and $n \geq 5$, we have many results about prescribed scalar curvature equation, see for example [B], [C-L], [L], [L-Z 1] and [L-Zh].

For example (when M is compact), in [L-Zh], Li and Zhu have proved the compactness of the solutions of the Yamabe equation with the positive mass theorem. They also describe the blow-up points of the solutions (only simple blow-up points). In [D], [L-Z 2] and [M], Druet, Li, Zhang and Marques have obtained the same result for the dimensions 4, 5, 6 and 7.

About the compactness of the solutions of the Yamabe equation, we can find in [L-Z 2] some conditions on the Weyl tensor to have this result. In [Au 2], T. Aubin have proved recently, the compactness of the soltuions of the Yamabe problem without other assumptions.

Note that here we have no assumption on energy. There is many results if we suppose the energy bounded. In our work, we use, in particular, the moving-plane method. This strong method was developped by Gidas-Ni-Nirenberg, see [G-N-N]. This method was used by many author to obtain uniform estimates, in dimension 2, see for example [B-L-S], in diemsnion greater than 3, see for example, [B], [C-L], [L-Z 1] and [L-Z 2].

We have:

Theorem. *For all $a, b, m > 0$, $A \geq 0$ with $A \to 0$ and all compact K of M, there is a positive constant $c = c(a, b, m, A, K, M, g)$ such that:*

$$\sup_K u \leq c \ \text{ if } \ \inf_M u \geq m,$$

for all solution u of (E) relatively to V with the conditions (C).

2. PROOF OF THE THEOREM.

Let x_0 be a point of M. We want to prove an uniform estimate around x_0.

Let $(u_i)_i$ be a sequence of solutions of:

$$\Delta u_i + R_g u_i = V_i {u_i}^3, \ u_i > 0,$$

where V_i is such that:

$$0 < a \leq V_i(x) \leq b \ \text{ and } \ ||\nabla V_i||_{L^\infty(M)} \leq A_i \ \text{ with } \ A_i \to 0.$$

We argue by contradiction, we assume that the $\sup$ is not bounded.

$\forall \ c, R > 0 \ \exists \ u_{c,R}$ solution de (E) telle que:

$$R^2 \sup_{B(x_0,R)} u_{c,R} \geq c, \qquad (H)$$

<u>Proposition 1:</u>*(blow-up analysis)*

There is a sequence of points $(y_i)_i$, $y_i \to x_0$ and two sequences of positive real numbers $(l_i)_i, (L_i)_i$, $l_i \to 0$, $L_i \to +\infty$, such that if we set $v_i(y) = \dfrac{u_i[\exp_{y_i}(y/[u_i(y_i)])]}{u_i(y_i)}$, we have:

$$0 < v_i(y) \leq \beta_i \leq 2, \ \beta_i \to 1.$$

$$v_i(y) \to \frac{1}{1+|y|^2}, \ \text{uniformly on compact sets of } \mathbb{R}^4.$$

$$l_i u_i(y_i) \to +\infty.$$

<u>Proof of the proposition 1:</u>

We use the hypothesis (H), we take two sequences $R_i > 0, R_i \to 0$ and $c_i \to +\infty$, such that,

$${R_i}^2 \sup_{B(x_0,R_i)} u_i \geq c_i \to +\infty,$$

Let, $x_i \in B(x_0, R_i)$, such that $\sup_{B(x_0,R_i)} u_i = u_i(x_i)$ and $s_i(x) = [R_i - d(x, x_i)]u_i(x), x \in B(x_i, R_i)$. Then, $x_i \to x_0$.

We have:

$$\max_{B(x_i,R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i u_i(x_i) \geq \sqrt{c_i} \to +\infty.$$

We set :

$$l_i = R_i - d(y_i, x_i),\ \bar{u}_i(y) = u_i[\exp_{y_i}(y)],\ v_i(z) = \frac{u_i[\exp_{y_i}(z/[u_i(y_i)])]}{u_i(y_i)}.$$

Clearly, we have, $y_i \to x_0$. We obtain:

$$L_i = \frac{l_i}{(c_i)^{1/4}}[u_i(y_i)] = \frac{[s_i(y_i)]}{c_i^{1/4}} \geq \frac{c_i^{1/2}}{c_i^{1/4}} = c_i^{1/4} \to +\infty.$$

If $|z| \leq L_i$, then $y = \exp_{y_i}[z/[u_i(y_i)]] \in B(y_i, \delta_i l_i)$ with $\delta_i = \dfrac{1}{(c_i)^{1/4}}$ and $d(y, y_i) < R_i - d(y_i, x_i)$, thus, $d(y, x_i) < R_i$ and, $s_i(y) \leq s_i(y_i)$. We can write,

$$u_i(y)[R_i - d(y, y_i)] \leq u_i(y_i) l_i.$$

But, $d(y, y_i) \leq \delta_i l_i$, $R_i > l_i$ and $R_i - d(y, y_i) \geq R_i - \delta_i l_i > l_i - \delta_i l_i = l_i(1 - \delta_i)$, we obtain,

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \frac{l_i}{l_i(1-\delta_i)} \leq 2.$$

We set, $\beta_i = \dfrac{1}{1-\delta_i}$, clearly $\beta_i \to 1$.

The function v_i satisfies the following equation:

$$-g^{jk}[\exp_{y_i}(y)]\partial_{jk} v_i - \partial_k \left[g^{jk}\sqrt{|g|}\right][\exp_{y_i}(y)]\partial_j v_i + \frac{R_g[\exp_{y_i}(y)]}{[u_i(y_i)]^2} v_i = \tilde{V}_i {v_i}^3,$$

with, $\tilde{V}_i(y) = V_i[\exp_{y_i}(y/[u_i(y_i)])]$. Without loss of generality, we can suppose $V(x_0) = 8$.

We use Ascoli and Ladyzenskaya theorems to obtain the uniform convergence (on each compact set of $\mathbb{R}^4$) of $(v_i)_i$ to v solution on $\mathbb{R}^4$ of:

$$\Delta v = 8v^3,\ v(0) = 1,\ 0 \leq v \leq 1 \leq 2,$$

By the maximum principle, we have $v > 0$ on $\mathbb{R}^n$. I we use the Caffarelli-Gidas-Spruck result (see [C-G-S]), we have,$v(y) = \dfrac{1}{1+|y|^2}$.

Polar Geodesic Coordinates

Let u be a function on M. We set $\bar{u}(r,\theta) = u[\exp_x(r\theta)]$. We denote $g_{x,ij}$ the local expression of the metric g in the exponential chart centered in x.

We set,

$$w_i(t,\theta) = e^t \bar{u}_i(e^t,\theta) = e^t u_i[\exp_{y_i}(e^t\theta)] \ \text{ and } \ \bar{V}_i(t,\theta) = V_i[\exp_{y_i}(e^t\theta)].$$

$$a(y_i, t, \theta) = \log J(y_i, e^t, \theta) = \log[\sqrt{det(g_{y_i,ij})}].$$

We can write the Laplacian in the geodesic polar coordinates:

$$-\Delta u = \partial_{rr}\bar{u} + \frac{3}{r}\partial_r \bar{u} + \partial_r[\log J(x,r,\theta)]\partial_r \bar{u} - \frac{1}{r^2}\Delta_\theta \bar{u}.$$

We deduce the two following lemmas:

Lemma 1:

The function w_i is a solution of:

$$-\partial_{tt} w_i - \partial_t a \partial_t w_i - \Delta_\theta w_i + c w_i = V_i w_i^3,$$

avec,

$$c = c(y_i, t, \theta) = 1 + \partial_t a + R_g e^{2t},$$

Proof of the Lemma 1:

We write:

$$\partial_t w_i = e^{2t}\partial_r \bar{u}_i + w_i,\ \ \partial_{tt} w_i = e^{3t}\left[\partial_{rr}\bar{u}_i + \frac{3}{e^t}\partial_r \bar{u}_i\right] + w_i.$$

$$\partial_t a = e^t \partial_r \log J(y_i, e^t, \theta), \partial_t a \partial_t w_i = e^{3t}\left[\partial_r \log J \partial_r \bar{u}_i\right] + \partial_t a w_i.$$

Le lemma 1 follows.

Let $b_1(y_i, t, \theta) = J(y_i, e^t, \theta) > 0$. We can write:

$$-\frac{1}{\sqrt{b_1}}\partial_{tt}(\sqrt{b_1} w_i) - \Delta_\theta w_i + [c(t) + b_1^{-1/2} b_2(t,\theta)]w_i = \bar{V}_i {w_i}^3,$$

where, $b_2(t,\theta) = \partial_{tt}(\sqrt{b_1}) = \dfrac{1}{2\sqrt{b_1}}\partial_{tt} b_1 - \dfrac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2$.

We set,

$$\tilde{w}_i = \sqrt{b_1} w_i.$$

Lemma 2:

The function $\tilde{w}_i$ is solution of:

$$-\partial_{tt}\tilde{w}_i + \Delta_\theta(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i).\nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2} b_2 - c_2)\tilde{w}_i =$$

$$= \bar{V}_i \left(\frac{1}{b_1}\right)^{1/2} \tilde{w}_i^3,$$

where, c_2 is a function to be deterined.

Proof of the Lemma 2:

We have:

$$-\partial_{tt}\tilde{w}_i - \sqrt{b_1}\Delta_\theta w_i + (c + b_2)\tilde{w}_i = \bar{V}_i \left(\frac{1}{b_1}\right)^{1/2} \tilde{w}_i^3,$$

But,

$$\Delta_\theta(\sqrt{b_1} w_i) = \sqrt{b_1}\Delta_\theta w_i - 2\nabla_\theta w_i.\nabla_\theta \sqrt{b_1} + w_i \Delta_\theta(\sqrt{b_1}),$$

and,

$$\nabla_\theta(\sqrt{b_1} w_i) = w_i \nabla_\theta \sqrt{b_1} + \sqrt{b_1}\nabla_\theta w_i,$$

we can write,

$$\nabla_\theta w_i.\nabla_\theta \sqrt{b_1} = \nabla_\theta(\tilde{w}_i).\nabla_\theta \log(\sqrt{b_1}) - \tilde{w}_i |\nabla_\theta \log(\sqrt{b_1})|^2,$$

we deduce,

$$\sqrt{b_1}\Delta_\theta w_i = \Delta_\theta(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i).\nabla_\theta \log(\sqrt{b_1}) - c_2 \tilde{w}_i,$$

with $c_2 = [\dfrac{1}{\sqrt{b_1}}\Delta_\theta(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2]$. The lemma 2 is proved.

The moving-plane method:

Let ξ_i be a real number, we assume $\xi_i \leq t$. We set $t^{\xi_i} = 2\xi_i - t$ and $\tilde{w}_i^{\xi_i}(t,\theta) = \tilde{w}_i(t^{\xi_i}, \theta)$.

Proposition 2:

We have:

$$1)\ \tilde w_i(\lambda_i,\theta)-\tilde w_i(\lambda_i+4,\theta)\geq \tilde k>0,\ \forall\ \theta\in \mathbb{S}_3.$$

For all $\beta>0$, there exists $c_\beta>0$ such that:

$$2)\ \frac{1}{c_\beta}e^t\leq \tilde w_i(\lambda_i+t,\theta)\leq c_\beta e^t,\ \forall\ t\leq\beta,\ \forall\ \theta\in\mathbb{S}_3.$$

Proof of the Proposition 2:

Like in [B], we have, $w_i(\lambda_i,\theta)-w_i(\lambda_i+4,\theta)\geq k>0$ for i large, $\forall\ \theta$. We can remark that $b_1(y_i,\lambda_i,\theta)\to 1$ and $b_1(y_i,\lambda_i+4,\theta)\to 1$ uniformly in θ, we obtain 1) of the proposition 2. For 2) we use the previous lemma 2, see also [B].

We set:

$$\bar Z_i=-\partial_{tt}(...)+\Delta_\theta(...)+2\nabla_\theta(...).\nabla_\theta\log(\sqrt{b_1})+(c+b_1^{-1/2}b_2-c_2)(...)$$

Remark : In the operator $\bar Z_i$, we can remark that:

$$c+b_1^{-1/2}b_2-c_2\geq k'>0,\ \text{for}\ t<<0,$$

it is fundamental if we want to apply the Hopf maximum principle.

Goal:

Like in [B], we have elliptic second order operator. Here it is $\bar Z_i$, the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:

$$\bar Z_i(\tilde w_i^{\xi_i}-\tilde w_i)\leq 0,\ \text{if}\ \tilde w_i^{\xi_i}-\tilde w_i\leq 0.$$

We write, $\Delta_\theta=\Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}}$. We obtain:

$$\bar Z_i(\tilde w_i^{\xi_i}-\tilde w_i)=(\Delta_{g_{y_i,e^{t^{\xi_i}},\mathbb{S}_3}}-\Delta_{g_{y_i,e^t,\mathbb{S}_3}})(\tilde w_i^{\xi_i})+$$

$$+2(\nabla_{\theta,e^{t^{\xi_i}}}-\nabla_{\theta,e^t})(w_i^{\xi_i}).\nabla_{\theta,e^{t^{\xi_i}}}\log(\sqrt{b_1^{\xi_i}})+2\nabla_{\theta,e^t}(\tilde w_i^{\xi_i}).\nabla_{\theta,e^{t^{\xi_i}}}[\log(\sqrt{b_1^{\xi_i}})-\log\sqrt{b_1}]+$$

$$+2\nabla_{\theta,e^t}w_i^{\xi_i}.(\nabla_{\theta,e^{t^{\xi_i}}}-\nabla_{\theta,e^t})\log\sqrt{b_1}-[(c+b_1^{-1/2}b_2-c_2)^{\xi_i}-(c+b_1^{-1/2}b_2-c_2)]\tilde w_i^{\xi_i}+$$

$$+\bar V_i^{\xi_i}\left(\frac{1}{b_1^{\xi_i}}\right)^{1/2}(\tilde w_i^{\xi_i})^3-\bar V_i\left(\frac{1}{b_1}\right)^{1/2}\tilde w_i^3.\qquad(***1)$$

Clearly, we have:

Lemma 3 :

$$b_1(y_i,t,\theta)=1-\frac{1}{3}Ricci_{y_i}(\theta,\theta)e^{2t}+\dots,$$

$$R_g(e^t\theta)=R_g(y_i)+<\nabla R_g(y_i)|\theta>e^t+\dots.$$

According to proposition 1 and lemma 3,

Propostion 3 :

$$\bar Z_i(\tilde w_i^{\xi_i}-\tilde w_i)\leq|\bar V_i^{\xi_i}-\bar V_i|(b_1^{\xi_i})^{-1/2}(w_i^{\xi_i})^3+\bar V_i(b_1^{\xi_i})^{-1/2}[(\tilde w_i^{\xi_i})^3-\tilde w_i^3]+$$

$$+C|e^{2t}-e^{2t^{\xi_i}}|\left[|\nabla_\theta\tilde w_i^{\xi_i}|+|\nabla_\theta^2(\tilde w_i^{\xi_i})|+|Ricci_{y_i}|[\tilde w_i^{\xi_i}+(\tilde w_i^{\xi_i})^3]+|R_g(y_i)|\tilde w_i^{\xi_i}\right]+C'w_i^{\xi_i}|e^{3t^{\xi_i}}-e^{3t}|.$$

Proof of the proposition 3:

In polar geodesic coordinates (and the Gauss lemma):

$$g = dt^2 + r^2 \tilde g_{ij}^k d\theta^i d\theta^j \text{ et } \sqrt{|\tilde g^k|} = \alpha^k(\theta)\sqrt{[det(g_{x,ij})]},$$

where α^k is the volume element of the unit sphere for the open set U^k.
We can write (with the lemma 3):

$$|\partial_t b_1(t)| + |\partial_{tt} b_1(t)| + |\partial_{tt} a(t)| \leq Ce^{2t},$$

and,

$$|\partial_{\theta_j} b_1| + |\partial_{\theta_j,\theta_k} b_1| + \partial_{t,\theta_j} b_1| + |\partial_{t,\theta_j,\theta_k} b_1| \leq Ce^{2t},$$

But,

$$\Delta_\theta = \Delta_{g_{y_i,e^t,\mathbb{S}_3}} = -\frac{\partial_{\theta^l}[\tilde g^{\theta^l\theta^j}(e^t,\theta)\sqrt{|\tilde g^k(e^t,\theta)|}\partial_{\theta^j}]}{\sqrt{|\tilde g^k(e^t,\theta)|}}.$$

Then,

$$A_i := \left[\left[\frac{\partial_{\theta^l}(\tilde g^{\theta^l\theta^j}\sqrt{|\tilde g^k|}\partial_{\theta^j})}{\sqrt{|\tilde g^k|}}\right]^{\xi_i} - \left[\frac{\partial_{\theta^l}(\tilde g^{\theta^l\theta^j}\sqrt{|\tilde g^k|}\partial_{\theta^j})}{\sqrt{|\tilde g^k|}}\right]\right](\tilde w_i^{\xi_i}) = B_i + D_i$$

where,

$$B_i = \left[\tilde g^{\theta^l\theta^j}(e^{t^{\xi_i}},\theta) - \tilde g^{\theta^l\theta^j}(e^t,\theta)\right]\partial_{\theta^l\theta^j}\tilde w_i^{\xi_i},$$

and,

$$D_i = \left[\frac{\partial_{\theta^l}[\tilde g^{\theta^l\theta^j}(e^{t^{\xi_i}},\theta)\sqrt{|\tilde g^k|}(e^{t^{\xi_i}},\theta)]}{\sqrt{|\tilde g^k|}(e^{t^{\xi_i}},\theta)} - \frac{\partial_{\theta^l}[\tilde g^{\theta^l\theta^j}(e^t,\theta)\sqrt{|\tilde g^k|}(e^t,\theta)]}{\sqrt{|\tilde g^k|}(e^t,\theta)}\right]\partial_{\theta^j}\tilde w_i^{\xi_i}.$$

Clearly, we can choose $\epsilon_1 > 0$ such that:

$$|\partial_r \tilde g_{ij}^k(x,r,\theta)| + |\partial_r\partial_{\theta^m}\tilde g_{ij}^k(x,r,\theta)| \leq Cr,\ x \in B(x_0,\epsilon_1)\ r \in [0,\epsilon_1],\ \theta \in U^k.$$

finally,

$$A_i \leq C_k|e^{2t} - e^{2t^{\xi_i}}|\left[|\nabla_\theta \tilde w_i^{\xi_i}| + |\nabla_\theta^2(\tilde w_i^{\xi_i})|\right],$$

It is easy to see that:

$$\frac{|\nabla_\theta(\tilde w_i^{\xi_i})|}{\tilde w_i^{\xi_i}} \leq K \text{ and } \frac{|\nabla_\theta^2(\tilde w_i^{\xi_i})|}{\tilde w_i^{\xi_i}} \leq K'.$$

We take, $C = \max\{C_i, 1 \leq i \leq q\}$ and we use $(***1)$. The proposition 3 is proved.

We have,

$$c(y_i,t,\theta) = 1 + \partial_t a + R_g e^{2t}, \qquad (\alpha_1)$$

$$b_2(t,\theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}}\partial_{tt}b_1 - \frac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2, \qquad (\alpha_2)$$

$$c_2 = [\frac{1}{\sqrt{b_1}}\Delta_\theta(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2], \qquad (\alpha_3)$$

We do a conformal change of the metric such that:

$$Ricci_{x_0} = R_{\tilde g}(x_0) = 0, \sqrt{det(\tilde g_{x_0,jk})} = 1 + O(r^s), s \geq 4,$$

it is given by T. Aubin [Au 1], (see also Lee et Parker, [L,P]).

Without loss of generality, we can assume:

$$g = \tilde g,\ \ R_{\tilde g}(y_i) \to 0 \text{ and } Ricci_{y_i} \to 0.$$

We assume that $\lambda \leq \lambda_i + 2 = -\log u_i(y_i) + 2$.

We work on $[\lambda, t_i] \times \mathbb{S}_3$ with $t_i = \log \sqrt{l_i} \to -\infty$, l_i as in the proposition 1. For i large $\log \sqrt{l_i} >> \lambda_i + 2$.

The functions v_i tend to radially symetric function, then, $\partial_{\theta_j} w_i^\lambda \to 0$ if $i \to +\infty$ and,

$$\frac{\partial_{\theta_j} w_i^\lambda(t,\theta)}{w_i^\lambda} = \frac{e^{[(\lambda-\lambda_i)+(\xi_i-t)]} e^{[(\lambda-\lambda_i)+(\xi_i-t)]} (\partial_{\theta_j} v_i)(e^{[(\lambda-\lambda_i)+(\lambda-t)]}\theta)}{e^{[(\lambda-\lambda_i)+(\lambda-t)]} v_i[e^{(\lambda-\lambda_i)+(\lambda-t)}\theta]} \leq \bar{C}_i,$$

where $\bar{C}_i$ does not depend on λ and tend to 0. We have also,

$$|\partial_\theta w_i^\lambda(t,\theta)| + |\partial_{\theta,\theta} w_i^\lambda(t,\theta)| \leq \tilde{C}_i w_i^\lambda(t,\theta),\ \tilde{C}_i \to 0.$$

and,

$$|\partial_\theta \tilde{w}_i^\lambda(t,\theta)| + |\partial_{\theta,\theta} \tilde{w}_i^\lambda(t,\theta)| \leq \tilde{C}_i \tilde{w}_i^\lambda(t,\theta),\ \tilde{C}_i \to 0.$$

$\tilde{C}_i$ does not depend on λ.

Now, we set:

$$\bar{w}_i = \tilde{w}_i - \frac{\tilde{m}}{2} e^t$$

Like in [B], we have,

<u>Lemma 4:</u>

There is $\nu < 0$ such that for $\lambda \leq \nu$:

$$\bar{w}_i^\lambda(t,\theta) - \bar{w}_i(t,\theta) \leq 0,\ \forall\ (t,\theta) \in [\lambda, t_i] \times \mathbb{S}_3.$$

Let ξ_i be the following real number,

$$\xi_i = \sup\{\lambda \leq \lambda_i + 2, \bar{w}_i^{\xi_i}(t,\theta) - \bar{w}_i(t,\theta) \leq 0,\ \forall\ (t,\theta) \in [\xi_i, t_i] \times \mathbb{S}_3\}.$$

Like in [B], we use the previous lemma to show:

$$\bar{w}_i^{\xi_i} - \bar{w}_i \leq 0 \Rightarrow \bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq 0.$$

If we use $(\alpha_1), (\alpha_2)$ and (α_3), we have,

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq 2A_i(e^t - e^{t^{\xi_i}})(\tilde{w}_i^{\xi_i})^3 + V_i(b_1^{\xi_i})^{-1/2}[(\tilde{w}_i^{\xi_i})^3 - \tilde{w}_i^3] + o(1)e^{2t}(e^t - e^{t^{\xi_i}}) + o(1)\tilde{w}_i^{\xi_i}(e^{2t} - e^{2t^{\xi_i}}).$$

We can write,

$$e^{2t} - e^{2t^{\xi_i}} = (e^t - e^{t^{\xi_i}})(e^t + e^{t^{\xi_i}}) \leq 2e^t(e^t - e^{t^{\xi_i}}).$$

Thus,

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq 4eA_i(e^t - e^{t^{\xi_i}})(\tilde{w}_i^{\xi_i})^2 + V_i(b_1^{\xi_i})^{-1/2}[(\tilde{w}_i^{\xi_i})^3 - \tilde{w}_i^3] + o(1)e^{2t}(e^t - e^{t^{\xi_i}}) + o(1)e^t\tilde{w}_i^{\xi_i}(e^t - e^{t^{\xi_i}}).$$

But,

$$0 < \tilde{w}_i^{\xi_i} \leq 2e,\ \ \tilde{w}_i \geq \frac{m}{2}e^t \ \text{ and }\ \tilde{w}_i^{\xi_i} - \tilde{w}_i \leq \frac{m}{2}(e^{t^{\xi_i}} - e^t),$$

and,

$$(\tilde{w}_i^{\xi_i})^3 - \tilde{w}_i^3 = (\tilde{w}_i^{\xi_i} - \tilde{w}_i)[(\tilde{w}_i^{\xi_i})^2 + \tilde{w}_i^{\xi_i}\tilde{w}_i + \tilde{w}_i^2] \leq (\tilde{w}_i^{\xi_i} - \tilde{w}_i)(\tilde{w}_i^{\xi_i})^2 + (\tilde{w}_i^{\xi_i} - \tilde{w}_i)\frac{m^2e^{2t}}{4} + (\tilde{w}_i^{\xi_i} - \tilde{w}_i)\frac{m}{2}e^t\tilde{w}_i^{\xi_i},$$

then,

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq \left[(\tilde{w}_i^{\xi_i})^2[\frac{am}{4} - 4eA_i] + [\frac{am^3}{16} - o(1)] + [\frac{am^2}{8} - o(1)]e^t\tilde{w}_i^{\xi_i}\right](e^{t^{\xi_i}} - e^t) \leq 0.$$

I fwe use the Hopf maximum principle, we obtain (like in [B]):

$$\max_{\theta \in \mathbb{S}_3} w_i(t_i, \theta) \leq \min_{\theta \in \mathbb{S}_3} w_i(2\xi_i - t_i),$$

we can write (by using the proposition 2):

$$l_i u_i(y_i) \leq c,$$

Contradiction.

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