Self–inductance coefficient for toroidal thin conductors

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Abstract

We consider the inductance coefficient for a thin toroidal inductor whose thickness depends on a small parameter $\varepsilon > 0$. An explicit form of the singular part of the corresponding potential u^ε is given. This allows to construct the limit potential u (as $\varepsilon \to 0$) and an approximation of the inductance coefficient L^ε. We establish some estimates of the deviation $u^\varepsilon - u$ and of the error of approximation of the inductance. The main result shows that L^ε behaves asymptotically as $\ln \varepsilon$, when $\varepsilon \to 0$.

Key words: Asymptotic behaviour, self inductance, eddy currents, thin domain

1 Introduction

In electrotechnical engineering, eddy current devices often involve thick conductors in which a magnetic field is induced, and circuits made of thin wires or coils, as inductors, connected to a power source generator. Mathematical modelling of such devices has then to take into account the simultaneous presence of thick conductors and thin inductors. For a two–dimensional configuration where the magnetic field has only one nonvanishing component, it was shown that the eddy current equation has the Kirchhoff circuit equation as a limit problem, as the thickness of the inductor tends to zero, see [1]. For the three–dimensional case, eddy current models require the use of a relevant quantity that is the self–inductance of the inductor, see [2], [3]. This number has to be evaluated *a priori* as a part of problem data. It is the purpose of the present

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paper to study the asymptotic behaviour of this number when the thickness of the inductor goes to zero.

Let us consider a toroidal domain of $\mathbb{R}^3$, denoted by Ω_ε, whose thickness depends on a small parameter $\varepsilon > 0$. The geometry of Ω_ε will be described in the next section. We denote by Γ_ε the boundary of Ω_ε, by n_ε the outward unit normal to Γ_ε, and by Ω'_ε the complement of its closure, that is $\Omega'_\varepsilon = \mathbb{R}^3 \setminus \overline{\Omega}_\varepsilon$. We denote by Σ a cut in the domain Ω'_ε, that is, Σ is a smooth orientable surface such that, for any $\varepsilon > 0$, $\Omega'_\varepsilon \setminus \Sigma$ is simply connected.

Let now $\boldsymbol{h}^\varepsilon$ denote the time–harmonic and complex–valued magnetic field. Neglecting the displacement currents, it follows from Maxwell's equations that

$$\mathbf{curl}\, \boldsymbol{h}^\varepsilon = 0,\ \operatorname{div} \boldsymbol{h}^\varepsilon = 0 \quad \text{in } \Omega'_\varepsilon.$$

Then, by a result in [4], p. 265, $\boldsymbol{h}^\varepsilon$ may be written in the form

$$\boldsymbol{h}^\varepsilon_{|\Omega'_\varepsilon} = \nabla \varphi^\varepsilon + I^\varepsilon \nabla u^\varepsilon, \tag{1}$$

where I^ε is a complex number, $\varphi^\varepsilon \in W^1(\Omega'_\varepsilon)$ and satisfies

$$\Delta \varphi^\varepsilon = 0 \qquad \text{in } \Omega'_\varepsilon,$$

and u^ε is solution of :

$$\begin{cases} \Delta u^\varepsilon = 0 & \text{in } \Omega'_\varepsilon \setminus \Sigma, \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Gamma_\varepsilon, \\ [u^\varepsilon]_\Sigma = 1, & \\ \left[\dfrac{\partial u^\varepsilon}{\partial n}\right]_\Sigma = 0. & \end{cases} \tag{2}$$

Here $W^1(\Omega'_\varepsilon)$ is the Sobolev space

$$W^1(\Omega'_\varepsilon) = \left\{ v;\ \rho v \in L^2(\Omega'_\varepsilon),\ \nabla v \in \boldsymbol{L}^2(\Omega'_\varepsilon) \right\},$$

equipped with the norm

$$\|v\|_{W^1(\Omega'_\varepsilon)} = \left(\|\rho v\|^2_{L^2(\Omega'_\varepsilon)} + \|\nabla v\|^2_{\boldsymbol{L}^2(\Omega'_\varepsilon)} \right)^{\frac{1}{2}}, \tag{3}$$

where $\boldsymbol{L}^p(\Omega'_\varepsilon)$ denotes the space $L^p(\Omega'_\varepsilon)^3$ and ρ is the weight function $\rho(\boldsymbol{x}) = (1 + |\boldsymbol{x}|^2)^{-\frac{1}{2}}$. Let us note here, see [4], pp. 649–651, that

$$|v|_{W^1(\Omega'_\varepsilon)} = \left(\int_{\Omega'_\varepsilon} |\nabla v|^2 \, d\boldsymbol{x} \right)^{\frac{1}{2}}$$

is a norm on $W^1(\Omega'_\varepsilon)$, equivalent to (3). In (2), n is the unit normal on Σ, and $[u^\varepsilon]_\Sigma$ (resp. $\left[\dfrac{\partial u^\varepsilon}{\partial n}\right]_\Sigma$) denotes the jump of u^ε (resp. $\dfrac{\partial u^\varepsilon}{\partial n}$) across Σ.

In (1), the number I^ε can be interpreted as the total current flowing in the inductor, see [3].

The inductance coefficient is then defined by the expression

$$L^\varepsilon = \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla u^\varepsilon|^2 \, d\boldsymbol{x}.$$

Our goal is to study the asymptotic behaviour of u^ε and L^ε as ε goes to zero. We first give an explicit form of the singular part of the potential u^ε which allows to construct the limit potential u (as $\varepsilon \to 0$) and an approximation of the inductance L^ε. We then prove that the deviation $\|u^\varepsilon - u\|_{W^1(\Omega'_\varepsilon)}$ and the error of approximation of L^ε are of order $O(\varepsilon^{\frac{5}{6}-\eta})$ for every $\eta > 0$. Finally we show that the inductance coefficient L^ε behaves asymptotically as $\ln \varepsilon$, when $\varepsilon \to 0$, and we thus recover the result stated (without proof) in [5], p. 137.

Let us outline the organization of this paper. In Section 2 we specify the geometry of the inductor, assuming that it is a toroidal neighborhood of a closed curve, the internal radius of the torus being proportional to a small positive number ε. Section 3 states the main result and gives the main steps in its proof. Let us note here that an extended version of this paper with detailed proofs can be consulted in [6].

2 Geometry of the domain

We consider a toroidal domain, with a small cross section. This domain may be defined as a tubular neighborhood of a closed curve. Let γ denote a closed Jordan arc of class $\mathcal{C}^3$ in $\mathbb{R}^3$, with a parametric representation defined by a function $\boldsymbol{g} : [0,1] \to \mathbb{R}^3$ satisfying

$$\boldsymbol{g}(0) = \boldsymbol{g}(1),\ \boldsymbol{g}'(0) = \boldsymbol{g}'(1),\ |\boldsymbol{g}'(s)| \geq C_0 > 0. \tag{4}$$

For each $s \in (0,1]$ we denote by $(\boldsymbol{t}(s), \boldsymbol{\nu}(s), \boldsymbol{b}(s))$ the Serret–Frénet coordinates at the point $\boldsymbol{g}(s)$, *i.e.*, $\boldsymbol{t}(s), \boldsymbol{\nu}(s), \boldsymbol{b}(s)$ are respectively the unit tangent vector to γ, the principal normal and the binormal, given by

$$\boldsymbol{t} = \frac{\boldsymbol{g}'}{|\boldsymbol{g}'|},\ \boldsymbol{\nu} = \frac{\boldsymbol{t}'}{|\boldsymbol{t}'|},\ \boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{\nu},$$

and by κ and τ respectively the curvature and the torsion of the arc γ.

Let $\widehat{\Omega} = (0,1)^2 \times (0, 2\pi)$ and let δ denote a positive number to be chosen in a convenient way. We define, for any ε, $0 \le \varepsilon < \delta$, the mapping $\boldsymbol{F}_\varepsilon : \widehat{\Omega} \to \mathbb{R}^3$ by

$$\boldsymbol{F}_\varepsilon(s, \xi, \theta) = \boldsymbol{g}(s) + r_\varepsilon(\xi)(\cos\theta\, \boldsymbol{\nu}(s) + \sin\theta\, \boldsymbol{b}(s)),$$

where $r_\varepsilon(\xi) = (\delta - \varepsilon)\xi + \varepsilon$. The jacobian of $\boldsymbol{F}_\varepsilon$ is therefore given by

$$J_\varepsilon(s, \xi, \theta) = (\delta - \varepsilon) a_\varepsilon(s, \xi, \theta) r_\varepsilon(\xi),$$

where

$$a_\varepsilon(s, \xi, \theta) = |\boldsymbol{g}'(s)| - r_\varepsilon(\xi)\kappa(s)\cos\theta.$$

According to (4), if δ is chosen such that

$$\delta|\kappa(s)| < |\boldsymbol{g}'(s)|, \qquad 0 \le s \le 1,$$

then

$$0 < C_1 \le a_\varepsilon \le C_2,$$

and the mapping $\boldsymbol{F}_\varepsilon$ is a $\mathcal{C}^1$–diffeomorphism from $\widehat{\Omega}$ into $\Lambda^\delta_\varepsilon = \boldsymbol{F}_\varepsilon(\widehat{\Omega})$.

Here and in the sequel, the quantities $C, C_1, C_2, \ldots$ denote generic positive numbers that do not depend on ε.

We now set, for any $0 < \varepsilon < \delta$,

$$\Omega_\delta = \Lambda^\delta_0 = \boldsymbol{F}_0(\widehat{\Omega}),\ \Omega'_\delta = \mathbb{R}^3 \setminus \overline{\Omega}_\delta,\ \Omega'_\varepsilon = \mathrm{Int}(\overline{\Omega}'_\delta \cup \overline{\Lambda}^\delta_\varepsilon),\ \Omega_\varepsilon = \mathbb{R}^3 \setminus \overline{\Omega}'_\varepsilon.$$

For technical reasons, we choose in the sequel $0 < \varepsilon \le \frac{\delta}{2}$.

Given a function v on $\Lambda^\delta_\varepsilon$, we define the function $\widehat{v}$ on $\widehat{\Omega}$ by $\widehat{v} = v \circ \boldsymbol{F}_\varepsilon$. If $v \in L^p(\Lambda^\delta_\varepsilon)$, $1 \le p \le \infty$, then $\widehat{v} \in L^p(\widehat{\Omega})$ and we have

$$\int_{\Lambda^\delta_\varepsilon} v \, d\boldsymbol{x} = \int_{\widehat{\Omega}} \widehat{v}\,(\delta - \varepsilon)\, a_\varepsilon r_\varepsilon \, d\widehat{\boldsymbol{x}}.$$

Moreover, for u and v in $H^1(\Lambda^\delta_\varepsilon)$, we have

$$\begin{aligned}\int_{\Lambda^\delta_\varepsilon} \nabla u . \nabla v \, d\boldsymbol{x} = (\delta - \varepsilon) \int_{\widehat{\Omega}} \Bigg(&\frac{r_\varepsilon}{a_\varepsilon} \frac{\partial \widehat{u}}{\partial s} \frac{\partial \widehat{v}}{\partial s} + \frac{r_\varepsilon a_\varepsilon}{(\delta - \varepsilon)^2} \frac{\partial \widehat{u}}{\partial \xi} \frac{\partial \widehat{v}}{\partial \xi} \\ &+ \left(\frac{a_\varepsilon}{r_\varepsilon} + \frac{\tau^2 r_\varepsilon}{a_\varepsilon} \right) \frac{\partial \widehat{u}}{\partial \theta} \frac{\partial \widehat{v}}{\partial \theta} \\ &- \frac{r_\varepsilon \tau}{a_\varepsilon} \left(\frac{\partial \widehat{u}}{\partial s} \frac{\partial \widehat{v}}{\partial \theta} + \frac{\partial \widehat{u}}{\partial \theta} \frac{\partial \widehat{v}}{\partial s} \right) \Bigg) \, d\widehat{\boldsymbol{x}}.\end{aligned}$$

We also define the set $\widehat{\Gamma} = (0,1) \times (0, 2\pi)$ and the mapping $\boldsymbol{G}_\varepsilon : \widehat{\Gamma} \to \mathbb{R}^3$ by

$$\boldsymbol{G}_\varepsilon(s, \theta) = \boldsymbol{g}(s) + \varepsilon(\cos\theta\, \boldsymbol{\nu}(s) + \sin\theta\, \boldsymbol{b}(s)).$$

The boundary of Ω'_ε is then represented by $\Gamma_\varepsilon = \overline{\boldsymbol{G}_\varepsilon(\widehat{\Gamma})}$. If $w \in L^2(\Gamma_\varepsilon)$, we define $\widehat{w} \in L^2(\widehat{\Gamma})$ by $\widehat{w} = w \circ \boldsymbol{G}_\varepsilon$, and we have

$$\int_{\Gamma_\varepsilon} w \, d\sigma = \int_{\widehat{\Gamma}} \widehat{w} \, \varepsilon(|\boldsymbol{g}'| - \varepsilon\kappa\cos\theta) \, d\widehat{\sigma}.$$

Clearly, Ω_ε and its complement Ω'_ε are connected domains but they are not simply connected. To define a cut in Ω'_ε, we denote by Σ_0 the set $\boldsymbol{F}_0((0,1)^2 \times \{0\})$ and $\partial\Sigma_0 = \boldsymbol{F}_0((0,1) \times \{1\} \times \{0\})$. Let Σ' denote a smooth simple surface that has $\partial\Sigma_0$ as a boundary and such that the surface $\Sigma = \Sigma' \cup \Sigma_0$ is oriented and of class $\mathcal{C}^1$ (cf. [7]). We denote by Σ^+ (*resp.* Σ^-) the oriented surface with positive (*resp.* negative) orientation, and by n the unit normal on Σ directed from Σ^+ to Σ^-. If $w \in W^1(\mathbb{R}^3 \setminus \Sigma)$, we denote by $[w]_\Sigma$ the jump of w across Σ in the direction of n, *i.e.*

$$[w]_\Sigma = w_{|\Sigma^+} - w_{|\Sigma^-}.$$

3 Formulation of the problem and statement of the result

We consider the boundary value problem

$$\begin{cases} \Delta u^\varepsilon = 0 & \text{in } \Omega'_\varepsilon \setminus \Sigma, \\ \dfrac{\partial u^\varepsilon}{\partial n_\varepsilon} = 0 & \text{on } \Gamma_\varepsilon, \\ [u^\varepsilon]_\Sigma = 1, & \\ \left[\dfrac{\partial u^\varepsilon}{\partial n}\right]_\Sigma = 0, & \end{cases} \tag{5}$$

where n_ε denotes the unit normal on Γ_ε pointing outward Ω'_ε and $\boldsymbol{n}$ is the unit normal on Σ oriented from Σ^+ toward Σ^-. The inductance coefficient is defined by

$$L^\varepsilon = \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla u^\varepsilon|^2 \, d\boldsymbol{x}. \tag{6}$$

We want to describe the asymptotic behaviour of u^ε and L^ε as $\varepsilon \to 0$.

We first exhibit a function that has the same singularity as might be expected for the solution of Problem (5) (as $\varepsilon \to 0$). Let us define

$$\widehat{v}(s,\xi,\theta) = \frac{\theta}{2\pi}\,\widehat{\varphi}(\xi), \qquad (s,\xi,\theta) \in \widehat{\Omega},$$

where $\widehat{\varphi} \in \mathcal{C}^2(\mathbb{R})$ and such that

$$\widehat{\varphi}(\xi) = 1 \text{ for } 0 \le \xi \le \frac{1}{2}, \quad \widehat{\varphi}(\xi) = 0 \text{ for } \xi \ge \frac{3}{4}.$$

We then define $v : \mathbb{R}^3 \to \mathbb{R}$ by :

$$v(\boldsymbol{x}) = \begin{cases} \widehat{v}(\boldsymbol{F}_0^{-1}(\boldsymbol{x})) & \text{if } \boldsymbol{x} \in \Omega_\delta, \\ 0 & \text{if } \boldsymbol{x} \in \Omega'_\delta. \end{cases}$$

Let us also define

$$\widehat{f}(s,\xi,\theta) = \frac{1}{2\pi a_0}\left(\frac{\kappa\,\sin\theta}{\delta\xi} - \frac{\tau^2\,\delta\,\xi\,\kappa\,\sin\theta}{a_0^2} - \frac{\partial}{\partial s}\left(\frac{\tau}{a_0}\right)\right)\widehat{\varphi} + \frac{\theta}{2\pi\,a_0\,\delta^2\,\xi}(2a_0 - |\boldsymbol{g}'|)\,\widehat{\varphi}' + \frac{\theta}{2\pi\,\delta^2}\,\widehat{\varphi}'', \qquad (s,\xi,\theta) \in \widehat{\Omega},$$

$$f(\boldsymbol{x}) = \begin{cases} \widehat{f}(\boldsymbol{F}_0^{-1}(\boldsymbol{x})) & \text{if } \boldsymbol{x} \in \Omega_\delta, \\ 0 & \text{if } \boldsymbol{x} \in \Omega'_\delta, \end{cases}$$

$$\varphi(\boldsymbol{x}) = \begin{cases} \widehat{\varphi}(\xi) & \text{if } \boldsymbol{x} \in \Omega_\delta, \text{ with } (s,\xi,\theta) = \boldsymbol{F}_0^{-1}(\boldsymbol{x}), \\ 0 & \text{if } \boldsymbol{x} \in \Omega'_\delta. \end{cases}$$

By straightforward calculations, we see that function v is solution of

$$\begin{cases} \Delta v = f & \text{in } \mathbb{R}^3 \setminus \Sigma, \\ [v]_\Sigma = \varphi, \\ \left[\dfrac{\partial v}{\partial n}\right]_\Sigma = 0. \end{cases} \tag{7}$$

Moreover, it satisfies

$$\frac{\partial v}{\partial n_\varepsilon} = 0 \qquad \text{on } \Gamma_\varepsilon.$$

Furthermore, we have for any $1 \le p < 2$,

$$f \in L^p(\mathbb{R}^3),\ v \in L^\infty(\mathbb{R}^3) \cap W^{1,p}(\mathbb{R}^3 \setminus \Sigma).$$

We note here that $v \notin H^1(\mathbb{R}^3 \setminus \Sigma)$. However, for any ε, $v \in H^1(\Omega'_\varepsilon \setminus \Sigma)$.

Let us now set $w^\varepsilon = u^\varepsilon - v$. We have by subtracting (7) from (5),

$$\begin{cases} -\Delta w^\varepsilon = f & \text{in } \Omega'_\varepsilon \setminus \Sigma, \\ \dfrac{\partial w^\varepsilon}{\partial n_\varepsilon} = 0 & \text{on } \Gamma_\varepsilon, \\ [w^\varepsilon]_\Sigma = 1 - \varphi, \\ \left[\dfrac{\partial w^\varepsilon}{\partial n}\right]_\Sigma = 0. \end{cases} \tag{8}$$

We note here that Problem (8) differs from (5) by the value of the jump of the solution across Σ and by the presence of a right-hand side f. However, we notice that $1-\varphi$ vanishes in a neighborhood of $\partial\Sigma$ and then, for Problem (8), the jump of w^ε vanishes in a neighborhood of $\partial\Sigma$.

Now, to study the asymptotic behaviour of w^ε and L^ε as $\varepsilon \to 0$ we consider the following decomposition. Let w_1 denote the solution of

$$\begin{cases} \Delta w_1 = 0 & \text{in } \mathbb{R}^3 \setminus \Sigma, \\ [w_1]_\Sigma = 1 - \varphi, & \\ \left[\dfrac{\partial w_1}{\partial n}\right]_\Sigma = 0, & \\ w_1(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}) & |\boldsymbol{x}| \to \infty. \end{cases} \tag{9}$$

Using [4], p. 654, and the fact that $1 - \varphi$ vanishes in a neighborhood of $\partial\Sigma$, we see that Problem (9) has a unique solution in $W^1(\mathbb{R}^3 \setminus \Sigma)$ given by

$$w_1(\boldsymbol{x}) = \frac{1}{4\pi} \int_\Sigma (1 - \varphi(\boldsymbol{y})) \frac{\boldsymbol{n}(\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^3} \, d\sigma(\boldsymbol{y}), \qquad \boldsymbol{x} \in \mathbb{R}^3 \setminus \Sigma.$$

Then we write $w^\varepsilon = w_1 + w_2^\varepsilon$, where the function w_2^ε is the unique solution, in $W^1(\Omega'_\varepsilon)$, of the exterior Neumann problem

$$\begin{cases} -\Delta w_2^\varepsilon = f & \text{in } \Omega'_\varepsilon, \\ \dfrac{\partial w_2^\varepsilon}{\partial n_\varepsilon} = -\dfrac{\partial w_1}{\partial n_\varepsilon} & \text{on } \Gamma_\varepsilon, \\ w_2^\varepsilon(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}) & |\boldsymbol{x}| \to \infty. \end{cases} \tag{10}$$

Finally, let w_2 denote the unique solution in $W^1(\mathbb{R}^3)$ of

$$\begin{cases} -\Delta w_2 = f & \text{in } \mathbb{R}^3, \\ w_2(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}), & |\boldsymbol{x}| \to \infty. \end{cases} \tag{11}$$

As it is classical (see [8] for instance) the function w_2 is given by

$$w_2(\boldsymbol{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} \, d\boldsymbol{y}, \qquad \boldsymbol{x} \in \mathbb{R}^3.$$

Summarizing the decomposition process of the solution to Problem (5), we have

$$u^\varepsilon = v + w_1 + w_2^\varepsilon \qquad \text{in } \Omega'_\varepsilon \setminus \Sigma,$$

where v, w_1 and w_2^ε are solutions of (7), (9) and (10) respectively.

We now state our main result.

Theorem 3.1 *Let u^ε be the solution of Problem* (5) *and let L^ε be the inductance coefficient defined by* (6). *Let u be the function defined in $\mathbb{R}^3 \setminus \Sigma$ by $u = v + w_1 + w_2$, where v, w_1 and w_2 are solutions of* (7), (9) *and* (11) *respectively. Then, for every* $\eta > 0$,

$$\begin{aligned}
&\|u - u^\varepsilon\|_{W^1(\Omega'_\varepsilon)} = O(\varepsilon^{\frac{5}{6}-\eta}),\\
&L^\varepsilon = -\frac{\ell_\gamma}{2\pi}\ln\varepsilon + L' - \int_{\mathbb{R}^3} f(w_1 + w_2)\,d\boldsymbol{x}\\
&\qquad + \int_\Sigma (1-\varphi)\left(\frac{\partial w_1}{\partial n} + \frac{\partial w_2}{\partial n} + 2\frac{\partial v}{\partial n}\right) d\sigma + O(\varepsilon^{\frac{5}{6}-\eta}),
\end{aligned}$$

where ℓ_γ *is the length of the curve* γ *and*

$$L' = \frac{\ell_\gamma}{2\pi}\ln\frac{\delta}{2} + \frac{1}{4\pi^2}\int_{\widehat{\Omega}}\left(a_0\xi\theta^2(\widehat{\varphi}')^2 + \frac{\delta^2\xi\tau^2}{a_0}\widehat{\varphi}^2\right) d\widehat{\boldsymbol{x}} + \ell_\gamma\int_{\frac{1}{2}}^1 \frac{\widehat{\varphi}^2}{2\pi\xi}\,d\xi.$$

The next section is devoted to the proof of this result.

3.1 Proof of error estimate

Let $\widetilde{w}_2^\varepsilon = w_2^\varepsilon - w_2$. Clearly $\widetilde{w}_2^\varepsilon = u^\varepsilon - u$, $\widetilde{w}_2^\varepsilon \in W^1(\Omega'_\varepsilon)$ and it satisfies

$$\begin{cases}
\Delta\widetilde{w}_2^\varepsilon = 0 & \text{in } \Omega'_\varepsilon,\\
\dfrac{\partial\widetilde{w}_2^\varepsilon}{\partial n_\varepsilon} = -\dfrac{\partial w_1}{\partial n_\varepsilon} - \dfrac{\partial w_2}{\partial n_\varepsilon} & \text{on } \Gamma_\varepsilon,\\
\widetilde{w}_2^\varepsilon(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}), & |\boldsymbol{x}| \to +\infty.
\end{cases} \tag{12}$$

To estimate the solution of Problem (12), we need the following result.

Lemma 3.1 *There is a constant* C*, independent of* ε*, such that :*

$$\|\psi\|_{L^2(\Gamma_\varepsilon)} \le C\varepsilon^{\frac{1}{2}}\,|\ln\varepsilon|^{\frac{1}{2}}\,\|\psi\|_{W^1(\Omega'_\varepsilon)} \qquad \textit{for all } \psi \in W^1(\Omega'_\varepsilon), \tag{13}$$

$$\|\psi\|_{L^2(\Gamma_\varepsilon)} \le C\left(\varepsilon^{\frac{1}{2}}\,\|\psi\|_{W^{1,p}(\Omega'_\varepsilon)} + \varepsilon^{\frac{4}{3}-\frac{2}{p}}\,\|\nabla\psi\|_{\boldsymbol{L}^p(\Lambda_\varepsilon^\delta)}\right)$$
$$\textit{for all } \psi \in W^{1,p}(\Omega'_\varepsilon) \textit{ with compact support, } \frac{3}{2} < p < 2. \tag{14}$$

For the proof we refer to [6].

Using the variational formulation associated with (12), Cauchy–Schwarz inequality and Estimate (13), we deduce

$$\begin{aligned}\int_{\Omega'_\varepsilon} |\nabla \widetilde{w}_2^\varepsilon|^2\, d\boldsymbol{x} &= \int_{\Gamma_\varepsilon} \left(\frac{\partial w_1}{\partial n_\varepsilon} + \frac{\partial w_2}{\partial n_\varepsilon}\right) \widetilde{w}_2^\varepsilon\, d\sigma \\ &\le \left\| \frac{\partial w_1}{\partial n_\varepsilon} + \frac{\partial w_2}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \|\widetilde{w}_2^\varepsilon\|_{L^2(\Gamma_\varepsilon)} \\ &\le C\, \varepsilon^{\frac12}\, |\ln \varepsilon|^{\frac12} \left(\left\|\frac{\partial w_1}{\partial n_\varepsilon}\right\|_{L^2(\Gamma_\varepsilon)} + \left\|\frac{\partial w_2}{\partial n_\varepsilon}\right\|_{L^2(\Gamma_\varepsilon)} \right) \|\nabla \widetilde{w}_2^\varepsilon\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)}.\end{aligned} \tag{15}$$

Using the integral representation of w_1, we easily check that

$$\left\|\frac{\partial w_1}{\partial n_\varepsilon}\right\|_{L^\infty(\Gamma_\varepsilon)} \le C.$$

Therefore

$$\left\|\frac{\partial w_1}{\partial n_\varepsilon}\right\|_{L^2(\Gamma_\varepsilon)} \le C\, (\operatorname{meas} \Gamma_\varepsilon)^{\frac12} \le C_1 \varepsilon^{\frac12}. \tag{16}$$

To estimate $\dfrac{\partial w_2}{\partial n_\varepsilon}$, we use standard regularity results for elliptic problems, see [9], p. 343, to deduce, since $f \in L^p(\mathbb{R}^3)$ for $p < 2$, that $w_2 \in W^{2,p}_{\text{loc}}(\mathbb{R}^3)$. Then we apply Estimate (14) to the function $u = \dfrac{\partial w_2}{\partial x_i}$, $1 \le i \le 3$ with $p = 2 - \eta$, $0 < \eta < \frac12$,

$$\left\|\frac{\partial w_2}{\partial x_i}\right\|_{L^2(\Gamma_\varepsilon)} \le C \left(\varepsilon^{\frac12} \left\|\frac{\partial w_2}{\partial x_i}\right\|_{W^{1,p}(\Omega'_\varepsilon)} + \varepsilon^{\frac13 - \frac{\eta}{2-\eta}} \left\| \frac{\partial}{\partial x_i} \nabla w_2 \right\|_{\boldsymbol{L}^p(\Lambda^\delta_\varepsilon)} \right).$$

Since both norms on the right–hand side of the above inequality are uniformly bounded and since the outward unit normal n_ε is uniformly bounded we obtain

$$\left\|\frac{\partial w_2}{\partial n_\varepsilon}\right\|_{L^2(\Gamma_\varepsilon)} \le C \varepsilon^{\frac13 - \frac{\eta}{2-\eta}}. \tag{17}$$

Substituting (16) and (17) into (15) and using the inequality $|\ln \varepsilon| \le C\varepsilon^{-2\eta}$, we get

$$\int_{\Omega'_\varepsilon} |\nabla \widetilde{w}_2^\varepsilon|^2\, d\boldsymbol{x} \le C_1\, \varepsilon^{\frac56 - \frac{\eta}{2-\eta} - \eta}\, \|\nabla \widetilde{w}_2^\varepsilon\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)}.$$

Therefore

$$\|\nabla \widetilde{w}_2^\varepsilon\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)} \le C_2 \varepsilon^{\frac56 - \eta} \qquad \text{for all } \eta > 0.$$

□

Using the decomposition $u^\varepsilon = v + w^\varepsilon = v + w_1 + w_2^\varepsilon$ it follows

$$L^\varepsilon = \int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla v|^2\,d\boldsymbol{x} + \int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla w^\varepsilon|^2\,d\boldsymbol{x} + 2\int_{\Omega'_\varepsilon\setminus\Sigma} \nabla v\cdot\nabla w^\varepsilon\,d\boldsymbol{x}.$$

Using Green's formulae, we can write L^ε in the form

$$L^\varepsilon = \int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla v|^2\,d\boldsymbol{x} - \int_{\Omega'_\varepsilon} f w^\varepsilon\,d\boldsymbol{x} + \int_\Sigma (1-\varphi)\left(\frac{\partial w^\varepsilon}{\partial n} + 2\frac{\partial v}{\partial n}\right)d\sigma. \tag{18}$$

Using the previous estimate for w_2^ε and some regularity results ($w_2 \in W^{2,p}(\Omega_\delta)$, $w_1 \in H^2(\Omega_{\frac{\delta}{2}})$), we can estimate each term in (18), to obtain for all $\eta > 0$,

$$L^\varepsilon = \int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla v|^2\,d\boldsymbol{x} - \int_{\mathbb{R}^3} f w\,d\boldsymbol{x} + \int_\Sigma (1-\varphi)\left(\frac{\partial w}{\partial n} + 2\frac{\partial v}{\partial n}\right)d\sigma + O(\varepsilon^{\frac{5}{6}-\eta}),$$

where $w = w_1 + w_2$.

To complete the result, an explicit calculation yields

$$\int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla v|^2\,d\boldsymbol{x} = -\frac{\ell_\gamma}{2\pi}\ln\varepsilon + L' + O(\varepsilon),$$

where ℓ_γ is the length of the curve γ and

$$L' = \frac{\ell_\gamma}{2\pi}\ln\frac{\delta}{2} + \frac{1}{4\pi^2}\int_{\widehat{\Omega}}\left(a_0\xi\theta^2(\widehat{\varphi}')^2 + \frac{\delta^2\xi\tau^2}{a_0}\widehat{\varphi}^2\right)d\widehat{\boldsymbol{x}} + \frac{\ell_\gamma}{2\pi}\int_{\frac{1}{2}}^1 \frac{\widehat{\varphi}^2}{\xi}\,d\xi.$$

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