The functor of units of Burnside rings for p-groups

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Abstract: In this note I describe the structure of the biset functor $B^\times$ sending a p-group P to the group of units of its Burnside ring $B(P)$. In particular, I show that $B^\times$ is a rational biset functor. It follows that if P is a p-group, the structure of $B^\times(P)$ can be read from a genetic basis of P : the group $B^\times(P)$ is an elementary abelian 2-group of rank equal to the number isomorphism classes of rational irreducible representations of P whose type is trivial, cyclic of order 2, or dihedral.

1. Introduction

If G is a finite group, denote by $B(G)$ the Burnside ring of G, i.e. the Grothendieck ring of the category of finite G-sets (see e.g. [2]). The question of structure of the multiplicative group $B^\times(G)$ has been studied by T. tom Dieck ([13]), T. Matsuda ([11]), T. Matsuda and T. Miyata ([12]), T. Yoshida ([16]), by geometric and algebraic methods.

Recently, E. Yalçin wrote a very nice paper ([14]), in which he proves an induction theorem for $B^\times$ for 2-groups, which says that if P is a 2-group, then any element of $B^\times(P)$ is a sum of elements obtained by inflation and tensor induction from sections (T,S) of P, such that T/S is trivial or dihedral.

The main theorem of the present paper implies a more precise form of Yalçin's Theorem, but the proof is independent, and uses entirely different methods. In particular, the biset functor techniques developed in [1], [4] and [6], lead to a precise description of $B^\times(P)$, when P is a 2-group (actually also for arbitrary p-groups, but the case p odd is known to be rather trivial). The main ingredient consists to show that $B^\times$ is a *rational* biset functor, and this is done by showing that the functor $B^\times$ (restricted to p-groups) is a subfunctor of the functor $\mathbb{F}_2R_\mathbb{Q}^*$. This leads to a description of $B^\times(P)$ in

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terms of a *genetic basis* of P, or equivalently, in terms of rational irreducible representations of P.

The paper is organized as follows : in Section 2, I recall the main definitions and notation on biset functors. Section 3 deals with genetic subgroups and rational biset functors. Section 4 gives a natural exposition of the biset functor structure of $B^\times$. In Section 5, I state results about faithful elements in $B^\times(P)$ for some specific p-groups P. In Section 6, I introduce a natural transformation of biset functors from $B^\times$ to $\mathbb{F}_2B^*$. This transformation is injective, and in Section 7, I show that the image of its restriction to p-groups is contained in the subfunctor $\mathbb{F}_2R_\mathbb{Q}^*$ of $\mathbb{F}_2B^*$. This is the key result, leading in Section 8 to a description of the lattice of subfunctors of the restriction of $B^\times$ to p-groups : it is always a uniserial p-biset functor (even simple if p is odd). This also provides an answer to the question, raised by Yalçin ([14]), of the surjectivity of the exponential map $B(P) \to B^\times(P)$ for a 2-group P.

2. Biset functors

2.1. Notation and Definition : *Denote by $\mathcal{C}$ the following category :*

- *The objects of $\mathcal{C}$ are the finite groups.*
- *If G and H are finite p-groups, then $\mathrm{Hom}_\mathcal{C}(G,H) = B(H \times G^{op})$ is the Burnside group of finite (H,G)-bisets. An element of this group is called a* virtual *(H,G)-biset.*
- *The composition of morphisms is $\mathbb{Z}$-bilinear, and if G, H, K are finite groups, if U is a finite (H,G)-biset, and V is a finite (K,H)-biset, then the composition of (the isomorphism classes of) V and U is the (isomorphism class) of $V \times_H U$. The identity morphism Id_G of the group G is the class of the set G, with left and right action by multiplication.*

If p is a prime number, denote by $\mathcal{C}_p$ the full subcategory of $\mathcal{C}$ whose objects are finite p-groups.

Let $\mathcal{F}$ denote the category of additive functors from $\mathcal{C}$ to the category $\mathbb{Z}$-Mod of abelian groups. An object of $\mathcal{F}$ is called a biset functor*. Similarly, denote by $\mathcal{F}_p$ the category of additive functors from $\mathcal{C}_p$ to $\mathbb{Z}$-Mod. An object of $\mathcal{F}_p$ will be called a p-*biset functor*.*

If F is an object of $\mathcal{F}$, if G and H are finite groups, and if $\varphi \in \mathrm{Hom}_\mathcal{C}(G,H)$, then the image of $w \in F(G)$ by the map $F(\varphi)$ will generally be denoted by $\varphi(w)$. The composition $\psi \circ \varphi$ of morphisms $\varphi \in \mathrm{Hom}_\mathcal{C}(G,H)$ and $\psi \in \mathrm{Hom}_\mathcal{C}(H,K)$ will also be denoted by $\psi \times_H \varphi$.

2.2. Notation : *The* Burnside *biset functor (defined e.g. as the Yoneda functor* $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},-)$*), will be denoted by* B*. The functor of rational representations (see Section 1 of [4]) will be denoted by* $R_{\mathbb{Q}}$*. The restriction of* B *and* $R_{\mathbb{Q}}$ *to* $\mathcal{C}_p$ *will also be denoted by* B *and* $R_{\mathbb{Q}}$*.*

2.3. Examples : Recall that this formalism of bisets gives a single framework for the usual operations of induction, restriction, inflation, deflation, and transport by isomorphism via the following correspondences :

- If H is a subgroup of G, then let $\mathrm{Ind}_H^G \in \mathrm{Hom}_{\mathcal{C}}(H,G)$ denote the set G, with left action of G and right action of H by multiplication.
- If H is a subgroup of G, then let $\mathrm{Res}_H^G \in \mathrm{Hom}_{\mathcal{C}}(G,H)$ denote the set G, with left action of H and right action of G by multiplication.
- If $N \trianglelefteq G$, and $H = G/N$, then let $\mathrm{Inf}_H^G \in \mathrm{Hom}_{\mathcal{C}}(H,G)$ denote the set H, with left action of G by projection and multiplication, and right action of H by multiplication.
- If $N \trianglelefteq G$, and $H = G/N$, then let $\mathrm{Def}_H^G \in \mathrm{Hom}_{\mathcal{C}}(G,H)$ denote the set H, with left action of H by multiplication, and right action of G by projection and multiplication.
- If $\varphi : G \to H$ is a group isomorphism, then let $\mathrm{Iso}_G^H = \mathrm{Iso}_G^H(\varphi) \in \mathrm{Hom}_{\mathcal{C}}(G,H)$ denote the set H, with left action of H by multiplication, and right action of G by taking image by φ, and then multiplying in H.

2.4. Definition : *A* section *of the group* G *is a pair* (T,S) *of subgroups of* G *such that* $S \trianglelefteq T$*.*

2.5. Notation : *If* (T,S) *is a section of* G*, set*

$$\mathrm{Indinf}_{T/S}^G = \mathrm{Ind}_T^G \mathrm{Inf}_{T/S}^T \qquad \textit{and} \qquad \mathrm{Defres}_{T/S}^G = \mathrm{Def}_{T/S}^T \mathrm{Res}_T^G \quad .$$

Then $\mathrm{Indinf}_{T/S}^G \cong G/S$ as $(G,T/S)$-biset, and $\mathrm{Defres}_{T/S}^G \cong S\backslash G$ as $(T/S,G)$-biset.

2.6. Notation : *Let* G *and* H *be groups, let* U *be an* (H,G)*-biset, and let* $u \in U$*. If* T *is a subgroup of* H*, set*

$$T^u = \{g \in G \mid \exists t \in T,\ tu = ug\} \quad .$$

This is a subgroup of G*. Similarly, if* S *is a subgroup of* G*, set*

$${}^uS = \{h \in H \mid \exists s \in S,\ us = hu\} \quad .$$

This is a subgroup of H*.*

2.7. Lemma : *Let G and H be groups, let U be an (H,G)-biset, and let S be a subgroup of G. Then there is an isomorphism of H-sets*

$$U/G = \bigsqcup_{u\in[H\backslash U/S]} H/{}^uS \quad ,$$

where $[H\backslash U/S]$ is a set of representatives of (H,S)-orbits on U.

Proof: Indeed $H\backslash U/S$ is the set of orbits of H on U/S, and ${}^uS$ is the stabilizer of uS in H.

2.8. Opposite bisets : If G and H are finite groups, and if U is a finite (H,G)-biset, then let U^{op} denote the opposite biset : as a set, it is equal to U, and it is a (G,H)-biset for the following action

$$\forall h \in H, \forall u \in U, \forall g \in G,\ g.u.h \text{ (in } U^{op}) = h^{-1}ug^{-1} \text{ (in } U) \quad .$$

This definition can be extended by linearity, to give an isomorphism

$$\varphi \mapsto \varphi^{op} : \mathrm{Hom}_{\mathcal{C}}(G,H) \to \mathrm{Hom}_{\mathcal{C}}(H,G) \quad .$$

It is easy to check that $(\varphi\circ\psi)^{op} = \psi^{op}\circ\varphi^{op}$, with obvious notation, and the functor

$$\left\{\begin{array}{l} G \mapsto G \\ \varphi \mapsto \varphi^{op} \end{array}\right.$$

is an equivalence of categories from $\mathcal{C}$ to the dual category, which restricts to an equivalence of $\mathcal{C}_p$ to its dual category.

2.9. Example : if G is a finite group, and (T,S) is a section of G, then

$$(\mathrm{Indinf}_{T/S}^G)^{op} \cong \mathrm{Defres}_{T/S}^G$$

as $(T/S,G)$-bisets.

2.10. Definition and Notation : *If F is a biset functor, the* dual *biset functor F^* is defined by*

$$F^*(G) = \mathrm{Hom}_{\mathbb{Z}}(F(G),\mathbb{Z}) \quad ,$$

for a finite group G, and by

$$F^*(\varphi)(\alpha) = \alpha\circ F(\varphi^{op}) \quad ,$$

for any $\alpha \in F^*(G)$, any finite group H, and any $\varphi \in \mathrm{Hom}_{\mathcal{C}}(G,H)$.

2.11. Some idempotents in $\mathrm{End}_{\mathcal{C}}(G)$ **:** Let G be a finite group, and let $N \trianglelefteq G$. Then it is clear from the definitions that

$$\mathrm{Def}^G_{G/N} \circ \mathrm{Inf}^G_{G/N} = (G/N) \times_G (G/N) = \mathrm{Id}_{G/N} \quad .$$

It follows that the composition $e^G_N = \mathrm{Inf}^G_{G/N} \circ \mathrm{Def}^G_{G/N}$ is an idempotent in $\mathrm{End}_{\mathcal{C}}(G)$. Moreover, if M and N are normal subgroups of G, then $e^G_N \circ e^G_M = e^G_{NM}$. Moreover $e^G_1 = \mathrm{Id}_G$.

2.12. Lemma : ([6] Lemma 2.5) *If $N \trianglelefteq G$, define $f^G_N \in \mathrm{End}_{\mathcal{C}}(G)$ by*

$$f^G_N = \sum_{\substack{M \trianglelefteq G \\ N \subseteq M}} \mu_{\trianglelefteq G}(N, M) e^G_M \quad ,$$

where $\mu_{\trianglelefteq G}$ denotes the Möbius function of the poset of normal subgroups of G. Then the elements f^G_N, for $N \trianglelefteq G$, are orthogonal idempotents of $\mathrm{End}_{\mathcal{C}}(G)$, and their sum is equal to Id_G.

Moreover, it is easy to check from the definition that for $N \trianglelefteq G$,

$$f^G_N = \mathrm{Inf}^G_{G/N} \circ f^{G/N}_{\mathbf{1}} \circ \mathrm{Def}^G_{G/N} \quad , \tag{2.13}$$

and

$$e^G_N = \mathrm{Inf}^G_{G/N} \circ \mathrm{Def}^G_{G/N} = \sum_{\substack{M \trianglelefteq G \\ M \supseteq N}} f^G_M \quad .$$

2.14. Lemma : *If N is a non trivial normal subgroup of G, then*

$$f^G_{\mathbf{1}} \circ \mathrm{Inf}^G_{G/N} = 0 \quad \textit{and} \quad \mathrm{Def}^G_{G/N} \circ f^G_{\mathbf{1}} = 0 \quad .$$

Proof: Indeed by 2.13

$$\begin{aligned} f^G_{\mathbf{1}} \circ \mathrm{Inf}^G_{G/N} &= f^G_{\mathbf{1}} \circ \mathrm{Inf}^G_{G/N} \circ \mathrm{Def}^G_{G/N} \circ \mathrm{Inf}^G_{G/N} \\ &= \sum_{\substack{M \trianglelefteq N \\ M \supseteq N}} f^G_{\mathbf{1}} f^G_M \mathrm{Inf}^G_{G/N} = 0 \quad , \end{aligned}$$

since $M \neq \mathbf{1}$ when $M \supseteq N$. The other equality of the lemma follows by taking opposite bisets. □

2.15. Remark : It was also shown in Section 2.7 of [6] that if P is a p-group, then

$$f^P_{\mathbf{1}} = \sum_{N \subseteq \Omega_1 Z(P)} \mu(\mathbf{1}, N) P/N \quad ,$$

where μ is the Möbius function of the poset of subgroups of N, and $\Omega_1 Z(P)$ is the subgroup of the centre of P consisting of elements of order at most p.

2.16. Notation and Definition : *If F is a a biset functor, and if G is a finite group, then the idempotent $f_{\mathbf{1}}^G$ of $\mathrm{End}_{\mathcal{C}}(G)$ acts on $F(G)$. Its image*

$$\partial F(G) = f_{\mathbf{1}}^G F(G)$$

is a direct summand of $F(G)$ as $\mathbb{Z}$-module : it will be called the set of faithful *elements of $F(G)$.*

The reason for this name is that any element $u \in F(G)$ which is inflated from a proper quotient of G is such that $F(f_{\mathbf{1}}^G)u = 0$. From Lemma 2.14, it is also clear that

$$\partial F(G) = \bigcap_{\mathbf{1} \neq N \trianglelefteq G} \mathrm{Ker}\ \mathrm{Def}_{G/N}^G \quad .$$

3. Genetic subgroups and rational p-biset functors

The following definitions are essentially taken from Section 2 of [7] :

3.1. Definition and Notation : *Let P be a finite p-group. If Q is a subgroup of P, denote by $Z_P(Q)$ the subgroup of P defined by*

$$Z_P(Q)/Q = Z(N_P(Q)/Q) \quad .$$

A subgroup Q of P is called genetic *if it satisfies the following two conditions :*

1. *The group $N_P(Q)/Q$ has normal p-rank 1.*
2. *If $x \in P$, then $Q^x \cap Z_P(Q) \subseteq Q$ if and only if $Q^x = Q$.*

Two genetic subgroups Q and R are said to be linked modulo P *(notation $Q \frac{\quad}{}_P R$), if there exist elements x and y in P such that $Q^x \cap Z_P(R) \subseteq R$ and $R^y \cap Z_P(Q) \subseteq Q$.*

This relation is an equivalence relation on the set of genetic subgroups of P. The set of equivalence classes is in one to one correspondence with the set of isomorphism classes of rational irreducible representations of P. A genetic basis *of P is a set of representatives of these equivalences classes.*

If V is an irreducible representation of P, then the *type* of V is the isomorphism class of the group $N_P(Q)/Q$, where Q is a genetic subgroup of P in the equivalence class corresponding to V by the above bijection.

3.2. Remark : The definition of the relation $\frac{\quad}{}_P$ given here is different from Definition 2.9 of [7], but it is equivalent to it, by Lemma 4.5 of [6].

The following is Theorem 3.2 of [6], in a slightly different form :

3.3. Theorem : *Let P be a finite p-group, and $\mathcal{G}$ be a genetic basis of P. Let F be a p-biset functor. Then the map*

$$\mathcal{I}_{\mathcal{G}} = \oplus_{Q\in\mathcal{G}}\mathrm{Indinf}_{N_P(Q)/Q}^{P} : \oplus_{Q\in\mathcal{G}}\partial F\big(N_P(Q)/Q\big) \to F(P)$$

is split injective.

3.4. Remark : There are two differences with the initial statement of Theorem 3.2 of [6] : here I use genetic *subgroups* instead of genetic *sections*, because these two notions are equivalent by Proposition 4.4 of [6]. Also the definition of the map $\mathcal{I}_{\mathcal{G}}$ is apparently different : with the notation of [6], the map $\mathcal{I}_{\mathcal{G}}$ is the sum of the maps $F(a_Q)$, where a_Q is the trivial $(P, P/P)$-biset if $Q = P$, and a_Q is the virtual $(P, N_P(Q)/Q)$-biset $P/Q - P/\hat{Q}$ if $Q \neq P$, where $\hat{Q}$ is the unique subgroup of $Z_P(Q)$ containing Q, and such that $|\hat{Q} : Q| = p$. But it is easy to see that the restriction of the map $F(P/\hat{Q})$ to $\partial F(N_P(Q)/Q)$ is actually 0. Moreover, the map $F(a_Q)$ is equal to $\mathrm{Indinf}_{N_P(Q)/Q}^{P}$. So in fact, the above map $\mathcal{I}_{\mathcal{G}}$ is the same as the one defined in Theorem 3.2 of [6].

3.5. Definition : *A p-biset functor F is called* rational *if for any finite p-group P and any genetic basis $\mathcal{G}$ of P, the map $\mathcal{I}_{\mathcal{G}}$ is an isomorphism.*

It was shown in Proposition 7.4 of [6] that subfunctors, quotient functors, and dual functors of rational p-biset functors are rational.

4. The functor of units of the Burnside ring

4.1. Notation : *If G is a finite group, let $B^\times(G)$ denote the group of units of the Burnside ring $B(G)$.*

If G and H are finite groups, if U is a finite (H,G)-biset, recall that U^{op} denotes the (G,H)-biset obtained from U by reversing the actions. If X is a finite G-set, then $T_U(X) = \mathrm{Hom}_G(U^{op}, X)$ is a finite H-set. The correspondence $X \mapsto T_U(X)$ can be extended to a correspondence $T_U : B(G) \to B(H)$, which is multiplicative (i.e. $T_U(ab) = T_U(a)T_U(b)$ for any $a, b \in B(G)$), and preserves identity elements (i.e. $T_U(G/G) = H/H$). This extension to $B(G)$ can be built by different means, and the following is described in Section 4.1 of [3] : if a is an element of $B(G)$, then there exists a finite G-poset X such that a is equal to the Lefschetz invariant Λ_X. Now $\mathrm{Hom}_G(U^{op}, X)$ has a natural structure of H-poset, and one can set $T_U(a) = \Lambda_{\mathrm{Hom}_G(U^{op},X)}$. It is an element of $B(H)$, which does not depend of the choice of the poset X

such that $a = \Lambda_X$, because with Notation 2.6 and Lemma 2.7, for any subgroup T of H the Euler-Poincaré characteristics $\chi\big(\mathrm{Hom}_G(U^{op}, X)^T\big)$ can be computed by

$$\chi\big(\mathrm{Hom}_G(U^{op}, X)^T\big) = \prod_{u\in T\backslash U/G} \chi(X^{T^u}) \quad ,$$

and the latter only depends on the element Λ_X of $B(G)$. As a consequence, one has that

$$|T_U(a)^T| = \prod_{u\in T\backslash U/G} |a^{T^u}| \quad .$$

It follows in particular that $T_U\big(B^\times(G)\big) \subseteq B^\times(H)$. Moreover, it is easy to check that $T_U = T_{U'}$ if U and U' are isomorphic (H,G)-bisets, that $T_{U_1\sqcup U_2}(a) = T_{U_1}(a)T_{U_2}(a)$ for any (H,G)-bisets U_1 and U_2, and any $a \in B(G)$.

It follows that there is a well defined bilinear pairing

$$B(H \times G^{op}) \times B^\times(G) \to B^\times(H) \quad ,$$

extending the correspondence $(U, a) \mapsto T_U(a)$. If $f \in B(H \times G^{op})$ (i.e. if f is a virtual (H,G)-biset), the corresponding group homomorphism $B^\times(G) \to B^\times(H)$ will be denoted by $B^\times(f)$.

Now let K be a third group, and V be a finite (K,H)-set. If X is a finite G-set, there is a canonical isomorphism of K-sets

$$\mathrm{Hom}_H\big(V^{op}, \mathrm{Hom}_G(U^{op}, X)\big) \cong \mathrm{Hom}_G\big((V \times_H U)^{op}, X\big) \quad ,$$

showing that $T_V \circ T_U = T_{V\times_H U}$.

It follows more generally that $B^\times(g) \circ B^\times(f) = B^\times(g \times_H f)$ for any $g \in B(K \times H^{op})$ and any $f \in B(H \times G^{op})$. Finally this shows :

4.2. Proposition : *The correspondence sending a finite group G to $B^\times(G)$, and an homomorphism f in $\mathcal{C}$ to $B^\times(f)$, is a biset functor.*

4.3. Remark and Notation : The restriction and inflation maps for the functor $B^\times$ are the usual ones for the functor B. The deflation map $\mathrm{Def}^G_{G/N}$ corresponds to taking fixed points under N (so it *does not coincide* with the usual deflation map for B, which consist in taking *orbits under* N).

Similarly, if H is a subgroup of G, the induction map from H to G for the functor $B^\times$ is sometimes called *multiplicative induction*. I will call it *tensor induction*, and denote it by Ten_H^G. If (T,S) is a section of G, I will also set $\mathrm{Teninf}^P_{T/S} = \mathrm{Ten}^P_T\mathrm{Inf}^T_{T/S}$.

5. Faithful elements in $B^\times(G)$

5.1. Notation and definition : *Let G be a finite group. Denote by $[s_G]$ a set of representatives of conjugacy classes of subgroups of G. Then the elements G/L, for $L \in [s_G]$, form a basis of $B(G)$ over $\mathbb{Z}$, called the* canonical basis *of $B(G)$.*

The primitive idempotents of $\mathbb{Q}B(G)$ are also indexed by $[s_G]$: if $H \in [s_G]$, the correspondent idempotent e_H^G is equal to

$$e_H^G = \frac{1}{|N_G(H)|} \sum_{K \subseteq H} |K|\mu(K,H)G/K \quad ,$$

where $\mu(K,H)$ denotes the Möbius function of the poset of subgroups of G, ordered by inclusion (see [10], [15], or [2]).

Recall that if $a \in B(G)$, then $a \cdot e_H^G = |a^H| e_H^G$ so that a can be written as

$$a = \sum_{H \in [s_G]} |a^H| e_H^G \quad .$$

Now $a \in B^\times(G)$ if and only if $a \in B(G)$ and $|a^H| \in \{\pm 1\}$ for any $H \in [s_G]$, or equivalently if $a^2 = G/G$. If now P is a p-group, and if $p \neq 2$, since $|a^H| \equiv |a| \ (p)$ for any subgroup $|H|$ of P, it follows that $|a^H| = |a|$ for any H, thus $a = \pm P/P$. This shows the following well know

5.2. Lemma : *If P is an odd order p-group, then $B^\times(P) = \{\pm P/P\}$.*

5.3. Remark : So in the sequel, when considering p-groups, the only really non-trivial case will occur for $p = 2$. However, some statements will be given for arbitrary p-groups.

5.4. Notation : *If G is a finite group, denote by F_G the set of subgroups H of G such that $H \cap Z(G) = \mathbf{1}$, and set $[F_G] = F_G \cap [s_G]$.*

5.5. Lemma : *Let G be a finite group. If $|Z(G)| > 2$, then $\partial B^\times(G)$ is trivial.*

Proof: Indeed let $a \in \partial B^\times(G)$. Then $\mathrm{Def}_{G/N}^G a$ is the identity element of $B^\times(G/N)$, for any non-trivial normal subgroup N of G. Now suppose that H is a subgroup of G containing N. Then

$$|a^H| = |\mathrm{Defres}_{N_G(H)/H}^G a| = |\mathrm{Iso}_{N_{G/N}(H/N)/(H/N)}^{N_G(H)/H} \mathrm{Defres}_{N_{G/N}(H/N)}^{G/N} \mathrm{Def}_{G/N}^G a| = 1 \quad .$$

In particular $|a^H| = 1$ if $H \cap Z(G) \neq \mathbf{1}$. It follows that there exists a subset A of $[F_G]$ such that

$$a = G/G - 2 \sum_{H \in A} e_H^G \quad .$$

If $A \neq \emptyset$, i.e. if $a \neq G/G$, let L be a maximal element of A. Then $L \neq G$, because $Z(G) \neq \mathbf{1}$. The coefficient of G/L in the expression of a in the canonical basis of $B(G)$ is equal to

$$-2\frac{|L|\mu(L,L)}{|N_G(L)|} = -2\frac{|L|}{|N_G(L)|} \quad .$$

This is moreover an integer, since $a \in B^\times(G)$. It follows that $|N_G(L):L|$ is equal to 1 or 2. But since $L \cap Z(G) = \mathbf{1}$, the group $Z(G)$ embeds into the group $N_G(L)/L$. Hence $|N_G(L):L| \geq 3$, and this contradiction shows that $A = \emptyset$, thus $a = G/G$. □

5.6. Lemma : *Let P be a finite 2-group, of order at least 4, and suppose that the maximal elements of F_P have order 2. If $|P| \geq 2|F_P|$, then $\partial B^\times(P)$ is trivial.*

Proof: Let $a \in \partial B^\times(P)$. By the argument of the previous proof, there exists a subset A of $[F_P]$ such that

$$a = P/P - 2\sum_{H\in A} e_H^P \quad .$$

The hypothesis implies that $\mu(\mathbf{1},H) = -1$ for any non-trivial element H of $[F_P]$. Now if $\mathbf{1} \in A$, the coefficient of $P/1$ in the expression of a in the canonical basis of $B(P)$ is equal to

$$-2\frac{1}{|P|} + 2\sum_{H\in A-\{\mathbf{1}\}} \frac{1}{|N_P(H)|} = -2\frac{1}{|P|} + 2\sum_{H\in \overline{A}-\{\mathbf{1}\}} \frac{1}{|P|} = \frac{-4+2|\overline{A}|}{|P|} \quad ,$$

where $\overline{A}$ is the set of subgroups of P which are conjugate to some element of A. This coefficient is an integer if $a \in B(P)$, so $|P|$ divides $2|\overline{A}|-4$. But $|\overline{A}|$ is always odd, since the trivial subgroup is the only normal subgroup of P which is in $\overline{A}$ in this case. Thus $2|\overline{A}|-4$ is congruent to 2 modulo 4, and cannot be divisible by $|P|$, since $|P| \geq 4$.

So $\mathbf{1} \notin A$, and the coefficient of $P/1$ in the expression of a is equal to

$$2\sum_{H\in A} \frac{1}{|N_P(H)|} = \frac{2|\overline{A}|}{|P|} \quad .$$

Now this is an integer, so $2|\overline{A}|$ is congruent to 0 or 1 modulo the order of P, which is even since $|P| \geq 2|F_P| \geq 2$. Thus $\mathbf{1} \notin A$, and $2|\overline{A}|$ is a multiple of $|P|$. But $2|\overline{A}| < 2|F_P|$ since $\mathbf{1} \notin A$. So if $2|F_P| \leq |P|$, it follows that $\overline{A}$ is empty, and A is empty. Hence $a = P/P$, as was to be shown. □

5.7. Corollary : *Let P be a finite 2-group. Then the group $\partial B^\times(P)$ is trivial in each of the following cases :*

1. *P is abelian of order at least 3.*
2. *P is generalized quaternion or semi-dihedral.*

5.8. Remark : Case 1 follows easily from Matsuda's Theorem ([11]). Case 2 follows from Lemma 4.6 of Yalçin ([14]).

Proof: Case 1 follows from Lemma 5.5. In Case 2, if P is generalized quaternion, then $F_P = \{\mathbf{1}\}$, thus $|P| \geq 2|F_P|$. And if P is semidihedral, then there is a unique conjugacy class of non-trivial subgroups H of P such that $H \cap Z(P) = \mathbf{1}$. Such a group has order 2, and $N_P(H) = HZ(P)$ has order 4. Thus $|F_P| = 1 + \frac{|P|}{4}$, and $|P| \geq 2|F_P|$ also in this case. □

5.9. Corollary : [Yalçin [14] Lemma 4.6 and Lemma 5.2] *Let P be a p-group of normal p-rank 1. Then $\partial B^\times(P)$ is trivial, except if P is*

- *the trivial group, and $\partial B^\times(P)$ is the group of order 2 generated by $\upsilon_P = -P/P$.*
- *cyclic of order 2, and $\partial B^\times(P)$ is the group of order 2 generated by*
$$\upsilon_P = P/P - P/\mathbf{1} \quad .$$
- *dihedral of order at least 16, and then $\partial B^\times(P)$ is the group of order 2 generated by the element*
$$\upsilon_P = P/P + P/1 - P/I - P/J \quad ,$$
where I and J are non-central subgroups of order 2 of P, not conjugate in P.

Proof: Lemma 5.2 and Lemma 5.5 show that $\partial B^\times(P)$ is trivial, when P has normal p-rank 1, and P is not trivial, cyclic of order 2, or dihedral : indeed then, the group P is cyclic of order at least 3, or generalized quaternion, or semi-dihedral.

Now if P is trivial, then obviously $B(P) = \mathbb{Z}$, so $B^\times(P) = \partial B^\times(P) = \{\pm P/P\}$. If P has order 2, then clearly $B^\times(P)$ consists of $\pm P/P$ and $\pm(P/P - P/\mathbf{1})$, and $\partial B^\times(P) = \{P/P, P/P - P/\mathbf{1}\}$. Finally, if P is dihedral, the set F_P consists of the trivial group, and of two conjugacy classes of subgroups H of order 2 of P, and $N_P(H) = HZ$ for each of these, where Z is the centre of P. Thus
$$|F_P| = 1 + 2\frac{|P|}{4} = 1 + \frac{|P|}{2} \quad .$$

Now with the notation of the proof of Lemma 5.6, one has that $2|\overline{A}| \equiv 0\ (|P|)$, and $2|\overline{A}| < |F_P| = 2 + |P|$. So either $A = \emptyset$, and in this case $a = P/P$, or $2|\overline{A}| = |P|$, which means that $\overline{A}$ is the whole set of non-trivial elements of F_P. In this case

$$a = P/P - 2(e_I^P + e_J^P) \quad ,$$

where I and J are non-central subgroups of order 2 of P, not conjugate in P. It is then easy to check that

$$a = P/P + P/\mathbf{1} - (P/I + P/J) \quad ,$$

so a is indeed in $B(P)$, hence in $B^\times(P)$. Moreover $\mathrm{Def}_{P/Z}^P a$ is the identity element of $B^\times(P/Z)$, so $a = f_{\mathbf{1}}^P a$, and $a \in \partial B^\times(P)$. This completes the proof. □

6. A morphism of biset functors

If k is any commutative ring, there is an obvious isomorphism of biset functor from $kB^* = k \otimes_{\mathbb{Z}} B^*$ to $\mathrm{Hom}(B, k)$, which is defined for a group G by sending the element $\alpha = \sum_i \alpha_i \otimes \psi_i$, where $\alpha_i \in k$ and $\psi_i \in B^*(G)$, to the linear form $\tilde{\alpha} : B(G) \to k$ defined by $\tilde{\alpha}(G/H) = \sum_i \psi_i(G/H)\alpha_i$.

6.1. Notation : *Let $\{\pm 1\} = \mathbb{Z}^\times$ be the group of units of the ring $\mathbb{Z}$. The unique group isomorphism from $\{\pm 1\}$ to $\mathbb{Z}/2\mathbb{Z}$ will be denoted by $u \mapsto u_+$.*

If G is a finite group, and if $a \in B^\times(G)$, then recall that for each subgroup S of G, the integer $|a^S|$ is equal to ± 1. Define a map $\epsilon_G : B^\times(G) \to \mathbb{F}_2 B^*(G)$ by setting $\epsilon_G(a)(G/S) = |a^S|_+$, for any $a \in B^\times(G)$ and any subgroup S of G.

6.2. Proposition : *The maps ϵ_G define a injective morphism of biset functors*

$$\epsilon : B^\times \to \mathbb{F}_2 B^* \quad .$$

Proof: The injectivity of the map ϵ_G is obvious. Now let G and H be finite groups, and let U be a finite (H,G)-biset. Also denote by U the corresponding element of $B(H \times G^{op})$. If $a \in B^\times(G)$, and if T is a subgroup of H, then

$$|B^\times(U)(a)^T| = \prod_{u \in T\backslash U/G} |a^{T^u}| \quad .$$

Thus

$$\begin{aligned}
\epsilon_H\big(B^\times(U)(a)\big)(H/T) &= \Big(\prod_{u\in T\backslash U/G}|a^{T^u}|\Big)_+\\
&= \sum_{u\in T\backslash U/G}|a^{T^u}|_+\\
&= \sum_{u\in T\backslash U/G}\epsilon_G(a)(G/T^u)\\
&= \epsilon_G(a)(U^{op}/T)\\
&= \epsilon_G(a)(U^{op}\times_H H/T)\\
&= \mathbb{F}_2B^*(U)\big(\epsilon_G(a)\big)(H/T)
\end{aligned}$$

thus $\epsilon_H\circ B^\times(U)=\mathbb{F}_2B^*(U)\circ\epsilon_G$. Since both sides are additive with respect to U, the same equality holds when U is an arbitrary element of $B(H\times G^{op})$, completing the proof. □

7. Restriction to p-groups

The additional result that holds for finite p-groups (and not for arbitrary finite groups) is the Ritter-Segal theorem, which says that the natural transformation $B\to R_{\mathbb{Q}}$ of biset functors for p-groups, is surjective. By duality, it follows that the natural transformation $i: kR_{\mathbb{Q}}^*\to kB^*$ is injective, for any commutative ring k. The following gives a characterization of the image $i(kR_{\mathbb{Q}}^*)$ inside kB^* :

7.1. Proposition : *Let p be a prime number, let P be a p-group, let k be a commutative ring. Then the element $\varphi\in kB^*(P)$ lies in $i\big(kR_{\mathbb{Q}}^*(P)\big)$ if and only if the element $\mathrm{Defres}^P_{T/S}\varphi$ lies in $i\big(kR_{\mathbb{Q}}^*(T/S)\big)$, for any section T/S of P which is*

- *elementary abelian of rank 2, or non-abelian of order p^3 and exponent p, if $p\neq 2$.*
- *elementary abelian of rank 2, or dihedral of order at least 8, if $p=2$.*

Proof: Since the image of $kR_{\mathbb{Q}}^*$ is a subfunctor of kB^*, if $\varphi\in i\big(kR_{\mathbb{Q}}^*(P)\big)$, then $\mathrm{Defres}^P_{T/S}\varphi\in i\big(kR_{\mathbb{Q}}^*(T/S)\big)$, for any section (T,S) of P.

Conversely, consider the exact sequence of biset functors over p-groups

$$0\to K\to B\to R_Q\to 0\quad .$$

Every evaluation of this sequence at a particular p-group is a split exact sequence of (free) abelian groups. Hence by duality, for any ring k, there is an exact sequence

$$0 \to kR_{\mathbb{Q}}^* \to kB^* \to kK^* \to 0 \quad .$$

With the identification $kB^* \cong \mathrm{Hom}_{\mathbb{Z}}(B,k)$, this means that if P is a p-group, the element $\varphi \in RB^*(P)$ lies in $i\big(kR_{\mathbb{Q}}^*(P)\big)$ if and only if $\varphi\big(K(P)\big) = 0$. Now by Corollary 6.16 of [7], the group $K(P)$ is the set of linear combinations of elements of the form $\mathrm{Indinf}_{T/S}^P\theta(\kappa)$, where T/S is a section of P, and θ is a group isomorphism from one of the group listed in the proposition to T/S, and κ is a specific element of $K(T/S)$ in each case. The proposition follows, because

$$\varphi\big(\mathrm{Indinf}_{T/S}^P\theta(\kappa)\big) = (\mathrm{Defres}_{T/S}^P\varphi)\big(\theta(\kappa)\big) \quad ,$$

and this is zero if $\mathrm{Defres}_{T/S}^P\varphi$ lies in $i\big(kR_{\mathbb{Q}}^*(T/S)\big)$. □

7.2. Theorem : *Let p be a prime number, and P be a finite p-group. The image of the map ϵ_P is contained in $i\big(\mathbb{F}_2R_{\mathbb{Q}}^*(P)\big)$.*

Proof: Let $a \in B^\times(P)$, and let T/S be any section of P. Since

$$\mathrm{Defres}_{T/S}^P i_P(a) = i_{T/S}\mathrm{Defres}_{T/S}^P a \quad ,$$

by Proposition 7.1, it is enough to check that the image of ϵ_P is contained in $i\big(\mathbb{F}_2R_{\mathbb{Q}}^*(P)\big)$, when P is elementary abelian of rank 2 or non-abelian of order p^3 and exponent p if p is odd, or when P is elementary abelian of rank 2 or dihedral if $p = 2$.

Now if N is a normal subgroup of P, one has that

$$f_N^P i_P(a) = \mathrm{Inf}_{P/N}^P\left(i_{P/N}(f_{\mathbf{1}}^{P/N}\mathrm{Def}_{P/N}^P a)\right) \quad .$$

Thus by induction on the order of P, one can suppose $a \in \partial B^\times(P)$. But if P is elementary abelian of rank 2, or if P has odd order, then $\partial B^\times(P)$ is trivial, by Lemma 5.2 and Corollary 5.7. Hence there is nothing more to prove if p is odd. And for $p = 2$, the only case left is when P is dihedral. In that case by Corollary 5.9, the group $\partial B^\times(P)$ has order 2, generated by the element

$$\upsilon_P = \sum_{H\in[s_P]-\{I,J\}} e_H^P - (e_I^P + e_J^P) \quad ,$$

where $[s_P]$ is a set of representatives of conjugacy classes of subgroups of P, and where I and J are the elements of $[s_P]$ which have order 2, and are non central in P. Moreover the element $\theta(\kappa)$ mentioned above is equal to

$$(P/I' - P/I'Z) - (P/J' - P/J'Z) \quad ,$$

where Z is the centre of P, and I' and J' are non-central subgroups of order 2 of P, not conjugate in P. Hence up to sign $\theta(\kappa)$ is equal to

$$\delta_P = (P/I - P/IZ) - (P/J - P/JZ) \quad .$$

Since $\epsilon_P(\upsilon_P)(P/H)$ is equal to zero, except if H is conjugate to I or J, and then $\epsilon_P(\upsilon_P)(P/H) = 1$, it follows that $\epsilon_P(\upsilon_P)(\delta_P) = 1 - 1 = 0$, as was to be shown. This completes the proof. □

7.3. Corollary : *The p-biset functor $B^\times$ is rational.*

Proof: Indeed, it is isomorphic to a subfunctor of $\mathbb{F}_2 R_{\mathbb{Q}}^* \cong \mathrm{Hom}_{\mathbb{Z}}(R_{\mathbb{Q}}, \mathbb{F}_2)$, which is rational by Proposition 7.4 of [6]. □

7.4. Theorem : *Let P be a p-group. Then $B^\times(P)$ is an elementary abelian 2-group of rank equal to the number of isomorphism classes of rational irreducible representations of P whose type is trivial, cyclic of order 2, or dihedral. More precisely :*

1. *If $p \neq 2$, then $B^\times(P) = \{\pm 1\}$.*
2. *If $p = 2$, then let $\mathcal{G}$ be a genetic basis of P, and let $\mathcal{H}$ be the subset of $\mathcal{G}$ consisting of elements Q such that $N_P(Q)/Q$ is trivial, cyclic of order 2, or dihedral. If $Q \in \mathcal{H}$, then $\partial B^\times\big(N_P(Q)/Q\big)$ has order 2, generated by $\upsilon_{N_P(Q)/Q}$. Then the set*

$$\{\mathrm{Teninf}_{N_P(Q)/Q}^P \upsilon_{N_P(Q)/Q} \mid Q \in \mathcal{H}\}$$

is an $\mathbb{F}_2$-basis of $B^\times(P)$.

Proof: This follows from the definition of a rational biset functor, and from Corollary 5.9. □

7.5. Remark : If P is abelian, then there is a unique genetic basis of P, consisting of subgroups Q such that P/Q is cyclic. So in that case, the rank of $B^\times(P)$ is equal 1 plus the number of subgroups of index 2 in P : this gives a new proof of Matsuda's Theorem ([11]).

8. The functorial structure of $B^\times$ for p-groups

In this section, I will describe the lattice of subfunctors of the p-biset functor $B^\times$.

8.1. The case $p \neq 2$. If $p \neq 2$, there is not much to say, since $B^\times(P) \cong \mathbb{F}_2$ for any p-group P. In this case, the functor $B^\times$ is the constant functor $\Gamma_{\mathbb{F}_2}$ introduced in Corollary 8.4 of [8]. It is also isomorphic to the simple functor $S_{\mathbf{1},\mathbb{F}_2}$. In this case, the results of [6] and [7] lead to the following remarkable version of Theorem 11.2 of [8]:

8.2. Proposition : *If $p \neq 2$, the inclusion $B^\times \to \mathbb{F}_2R^*_\mathbb{Q}$ leads to a short exact sequence of p-biset functors*

$$0 \to B^\times \to \mathbb{F}_2R^*_\mathbb{Q} \to D_{tors} \to 0 \quad ,$$

where D_{tors} is the torsion part of the Dade p-biset functor.

8.3. The case $p = 2$. There is a bilinear pairing

$$\langle\ ,\ \rangle : \mathbb{F}_2R^*_\mathbb{Q} \times \mathbb{F}_2R_\mathbb{Q} \to \mathbb{F}_2 \quad .$$

This means that for each 2-group P, there is a bilinear form

$$\langle\ ,\ \rangle_P : \mathbb{F}_2R^*_\mathbb{Q}(P) \times \mathbb{F}_2R_\mathbb{Q}(P) \to \mathbb{F}_2 \quad ,$$

with the property that for any 2-group Q, for any $f \in \mathrm{Hom}_{\mathcal{C}_p}(P,Q)$, for any $a \in \mathbb{F}_2R^*_\mathbb{Q}(P)$ and any $b \in \mathbb{F}_2R_\mathbb{Q}(Q)$, one has that

$$\langle \mathbb{F}_2R^*_\mathbb{Q}(f)(a), b\rangle_Q = \langle a, \mathbb{F}_2R_\mathbb{Q}(f^{op})(b)\rangle_P \quad .$$

Moreover this pairing is non-degenerate : this means that for any 2-group P, the pairing $\langle\ ,\ \rangle_P$ is non-degenerate. In particular, each subfunctor F of $\mathbb{F}_2R^*_\mathbb{Q}$ is isomorphic to $\mathbb{F}_2R_\mathbb{Q}/F^\perp$, where $F^\perp$ is the orthogonal of F for the pairing $\langle\ ,\ \rangle$.

In particular, the lattice of subfunctors of $\mathbb{F}_2R^*_\mathbb{Q}$ is isomorphic to the opposite lattice of subfunctors of $\mathbb{F}_2R_\mathbb{Q}$. Now since $B^\times$ is isomorphic to a subfunctor of $\mathbb{F}_2R_\mathbb{Q}$, its lattice of subfunctors is isomorphic to the opposite lattice of subfunctors of $\mathbb{F}_2R_\mathbb{Q}$ *containing* $B^\sharp = (B^\times)^\perp$. By Theorem 4.4 of [4], any subfunctor L of $\mathbb{F}_2R_\mathbb{Q}$ is equal to the sum of subfunctors H_Q it contains, where Q is a 2-group of normal 2-rank 1, and H_Q is the subfunctor of $\mathbb{F}_2R_\mathbb{Q}$ generated by the image $\overline{\Phi}_Q$ of the unique (up to isomorphism) irreducible rational faithful $\mathbb{Q}Q$-module Φ_Q in $\mathbb{F}_2R_\mathbb{Q}$.

In particular $B^\sharp$ is the sum of the subfunctors H_Q, where Q is a 2-group of normal 2-rank 1 such that $\overline{\Phi}_Q \in B^\sharp(Q)$. This means that $\langle a, \overline{\Phi}_Q\rangle_Q = 0$, for any $a \in B^\times(Q)$. Now $\Phi_Q = f_\mathbf{1}\Phi_Q$ since Φ_Q is faithful, so

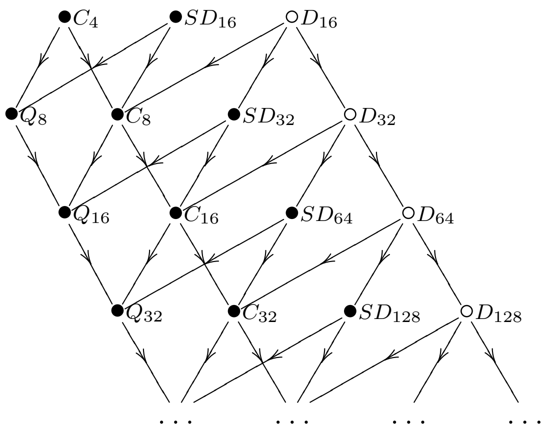
$$\langle a, \overline{\Phi}_Q\rangle_Q = \langle a, f_\mathbf{1}^Q\overline{\Phi}_Q\rangle_Q = \langle f_\mathbf{1}^Q a, \overline{\Phi}_Q\rangle_Q \quad ,$$

because $f_{\mathbf{1}}^Q = (f_{\mathbf{1}}^Q)^{op}$. Thus $\overline{\Phi}_Q \in B^\sharp(Q)$ if and only if $\overline{\Phi}_Q$ is orthogonal to $\partial B^\times(Q)$. Since Q has normal 2-rank 1, this is always the case by Corollary 5.9, except maybe if Q is trivial, cyclic of order 2, or dihedral (of order at least 16). Now $H_{\mathbf{1}} = H_{C_2} = \mathbb{F}_2 R_{\mathbb{Q}}$ by Theorem 5.6 of [4]. Since $B^\times$ is not the zero subfunctor of $\mathbb{F}_2 R_{\mathbb{Q}}$, it follows that $H_Q \not\subseteq B^\sharp$, if Q is trivial or cyclic of order 2. Now if Q is dihedral, then Φ_Q is equal to $\mathbb{Q}Q/I - \mathbb{Q}Q/IZ$, where I is a non-central subgroup of order 2 of Q, and Z is the centre of Q. Now

$$\epsilon_Q(\upsilon_Q)\big(i(\overline{\Phi}_Q)\big) = \epsilon_Q(\upsilon_Q)(Q/I - Q/IZ) = 1 - 0 = 1 \quad ,$$

It follows that $H_Q \not\subseteq B^\sharp$ if Q is dihedral. Finally $B^\sharp$ is the sum of all subfunctors H_Q, when Q is cyclic of order at least 4, or generalized quaternion, or semi-dihedral.

Recall from Theorem 6.2 of [4] that the poset of proper subfunctors of $\mathbb{F}_2 R_{\mathbb{Q}}$ is isomorphic to the poset of closed subsets of the following graph :



The vertices of this graph are the isomorphism classes of groups of normal 2-rank 1 and order at least 4, and there is an arrow from vertex Q to vertex R if and only if $H_R \subseteq H_Q$. The vertices with a filled $\bullet$ are exactly labelled by the groups Q for which $H_Q \subseteq B^\sharp$, and the vertices with a $\circ$ are labelled by dihedral groups.

By the above remarks, the lattice of subobjects of $B^\times$ is isomorphic to the opposite lattice of subfunctors of $\mathbb{F}_2 R_{\mathbb{Q}}$ containing $B^\sharp$. Thus :

8.4. Theorem : *The p-biset functor $B^\times$ is uniserial. It has an infinite strictly increasing series of proper subfunctors*

$$0 \subset L_0 \subset L_1 \cdots \subset L_n \subset \cdots$$

where L_0 is generated by the element $\upsilon_{\mathbf{1}}$, and L_i, for $i > 0$, is generated by the element $\upsilon_{D_{2^{i+3}}}$ of $B^\times(D_{2^{i+3}})$. The functor L_0 is isomorphic to the simple *functor $S_{\mathbf{1},\mathbb{F}_2}$, and the quotient L_i/L_{i-1}, for $i \geq 1$, is isomorphic to the simple functor $S_{D_{2^{i+3}},\mathbb{F}_2}$.*

Proof: Indeed $L_0^\perp = B^\sharp + H_{D_{16}}$ is the unique maximal proper subfunctor of $\mathbb{F}_2 R_\mathbb{Q}$. Thus L_0 is isomorphic to the unique simple quotient of $\mathbb{F}_2 R_\mathbb{Q}$, which is $S_{\mathbf{1},\mathbb{F}_2}$ by Proposition 5.1 of [4]. Similarly for $i \geq 1$, the simple quotient L_i/L_{i-1} is isomorphic to the quotient

$$(B^\sharp + H_{D_{2^{i+3}}})/B^\sharp + H_{D_{2^{i+4}}}) \quad ,$$

which is a quotient of

$$(B^\sharp + H_{D_{2^{i+3}}})/B^\sharp \cong H_{D_{2^{i+3}}}/(B^\sharp \cap H_{D_{2^{i+3}}}) \quad .$$

But the only simple quotient of $H_{D_{2^{i+3}}}$ is $S_{D_{2^{i+3}},\mathbb{F}_2}$, by Proposition 5.1 of [4] again. □

8.5. Remark : Let P be a 2-group. By Theorem 5.12 of [4], the $\mathbb{F}_2$-dimension of $S_{\mathbf{1},\mathbb{F}_2}(P)$ is equal to the number of isomorphism classes of rational irreducible representations of P whose type is $\mathbf{1}$ or C_2, whereas the $\mathbb{F}_2$-dimension of $S_{D_{2^{i+3}},\mathbb{F}_2}(P)$ is the number of isomorphism classes of rational irreducible representations of P whose type is isomorphic to $D_{2^{i+3}}$. This gives a way to recover Theorem 7.4 : the $\mathbb{F}_2$-dimension of $B^\times(P)$ is equal to the number of isomorphism classes of rational irreducible representations of P whose type is trivial, cyclic of order 2, or dihedral.

8.6. The surjectivity of the exponential map. Let G be a finite group. The exponential map $\exp_G : B(G) \to B^\times(G)$ is defined in Section 7 of Yalçin's paper ([14]) by

$$\exp_G(x) = (-1) \uparrow x \quad ,$$

where $-1 = -\mathbf{1}/\mathbf{1} \in B^\times(\mathbf{1})$, and where the exponentiation

$$(y, x) \in B^\times(G) \times B(G) \to B^\times(G)$$

is defined by extending the usual exponential map $(Y, X) \mapsto Y^X$, where X and Y are G-sets, and Y^X is the set of maps from X to Y, with G-action given by $(g \cdot f)(x) = gf(g^{-1}x)$.

Its possible to give another interpretation of this map : indeed $B(G)$ is naturally isomorphic to $\mathrm{Hom}_\mathcal{C}(\mathbf{1}, G)$, by considering any G-set as a $(G, \mathbf{1})$-biset. It is clear that if X is a finite G-set, and Y is a finite set, then

$$T_X(Y) = Y^X \quad .$$

This can be extended by linearity, to show that for any $x \in B(G)$

$$(-1)^x = B^\times(x)(-1) \quad .$$

In particular the image $\mathrm{Im}(\exp_G)$ of the exponential map $\exp_G$ is equal to $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, G)(-1)$. Denoting by I the sub-biset functor of $B^\times$ generated by $-1 \in B^\times(\mathbf{1})$, it it now clear that $\mathrm{Im}(\exp_G) = I(G)$ for any finite group G.

Now the restriction of the functor I to the category $\mathcal{C}_2$ is equal to L_0, which is isomorphic to the simple functor $S_{\mathbf{1},\mathbb{F}_2}$. Using Remark 5.13 of [4], this shows finally the following :

8.7. Proposition : *Let P be a finite 2-group. Then :*

1. *The $\mathbb{F}_2$-dimension of the image of the exponential map*

$$\exp_P : B(P) \to B^\times(P)$$

   is equal to the number of isomorphism classes of absolutely irreducible rational representations of P.

2. *The map $\exp_P$ is surjective if and only if the group P has no irreducible rational representation of dihedral type, or equivalently, no genetic subgroup Q such that $N_P(Q)/Q$ is dihedral.*

8.8. Proposition : *Let p be a prime number. There is an exact sequence of p-biset functors :*

$$0 \to B^\times \to \mathbb{F}_2 R_{\mathbb{Q}}^* \to \mathbb{F}_2 D_{tors}^{\Omega} \to 0 \quad ,$$

where D_{tors}^{Ω} is the torsion part of the functor D^{Ω} of relative syzygies in the Dade group.

Proof: In the case $p \neq 2$, this proposition is equivalent to Proposition 8.2, because $\mathbb{F}_2 D_{tors}^{\Omega} = \mathbb{F}_2 D_{tors} \cong D_{tors}$ in this case. And for $p = 2$, the 2-functor D_{tors}^{Ω} is a quotient of the functor $R_{\mathbb{Q}}^*$, by Corollary 7.5 of [6] : there is a surjective map $\pi : R_{\mathbb{Q}}^* \to D_{tors}^{\Omega}$, which is the restriction to $R_{\mathbb{Q}}^*$ of the surjection $\Theta : B^* \to D^{\Omega}$ introduced in Theorem 1.7 of [5]. The $\mathbb{F}_2$-reduction of π is a surjective map

$$\mathbb{F}_2\pi : \mathbb{F}_2 R_{\mathbb{Q}}^* \to \mathbb{F}_2 D_{tors}^{\Omega} \quad .$$

To prove the proposition in this case, is is enough to show that the image of $B^\times$ in $\mathbb{F}_2 R_{\mathbb{Q}}^*$ is contained in the kernel of $\mathbb{F}_2\pi$, and that for any 2-group P, the $\mathbb{F}_2$-dimension of $\mathbb{F}_2 R_{\mathbb{Q}}^*(P)$ is equal to the sum of the $\mathbb{F}_2$-dimensions of

$B^\times(P)$ and $\mathbb{F}_2D^\Omega_{tors}(P)$: but by Corollary 7.6 of [6], there is a group isomorphism

$$D^\Omega_{tors}(P) \cong (\mathbb{Z}/4\mathbb{Z})^{a_P} \oplus (\mathbb{Z}/2\mathbb{Z})^{b_P} \quad ,$$

where a_P is equal to the number of isomorphism classes of rational irreducible representations of P whose type is generalized quaternion, and b_P equal to the number of isomorphism classes of rational irreducible representations of P whose type is cyclic of order at least 3, or semi-dihedral. Thus

$$\dim_{\mathbb{F}_2} \mathbb{F}_2D^\Omega_{tors}(P) = a_P + b_P \quad .$$

Now since $\dim_{\mathbb{F}_2} B^\times(P)$ is equal to the number of isomorphism classes of rational irreducible representations of P whose type is cyclic of order at most 2, or dihedral, it follows that $\dim_{\mathbb{F}_2} \mathbb{F}_2D^\Omega_{tors}(P) + \dim_{\mathbb{F}_2} B^\times(P)$ is equal to the number of isomorphism classes of rational irreducible representations of P, i.e. to $\dim_{\mathbb{F}_2} \mathbb{F}_2R^*_\mathbb{Q}(P)$.

So the only thing to check to complete the proof, is that the image of $B^\times$ in $\mathbb{F}_2R^*_\mathbb{Q}$ is contained in the kernel of $\mathbb{F}_2\pi$. Since $B^\times$, $\mathbb{F}_2R^*_\mathbb{Q}$ and $\mathbb{F}_2D^\Omega_{tors}$ are rational 2-biset functors, it suffices to check that if P is a 2-group of normal 2-rank 1, and $a \in \partial B^\times(P)$, then the image of a in $\partial\mathbb{F}_2R^*_\mathbb{Q}(P)$ lies in the kernel of $\mathbb{F}_2\pi$. There is nothing to do if P is generalized quaternion, or semi-dihedral, or cyclic of order at least 3, for in this case $\partial B^\times(P) = 0$ by Corollary 5.7. Now if P is cyclic of order at most 2, then $D^\Omega(P) = \{0\}$, and the result follows. And if P is dihedral, then $D^\Omega(P)$ is torsion free by Theorem 10.3 of [9], so $D^\Omega_{tors}(P) = \{0\}$ again. □

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