

Weak Gibbs property and systems of numeration

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RÉSUMÉ. Nous étudions les propriétés d'autosimilarité et la nature gibbsienne de certaines mesures définies sur l'espace produit $\Omega_r := \{0, 1, \dots, r-1\}^{\mathbb{N}}$. Cet espace peut être identifié à l'intervalle $[0, 1]$ au moyen de la numération en base r . Le dernier paragraphe concerne la convolution de Bernoulli en base $\beta = \frac{1+\sqrt{5}}{2}$, appelée mesure de Erdős, et son analogue en base $-\beta = -\frac{1+\sqrt{5}}{2}$, que nous étudions au moyen d'un système de numération approprié.

ABSTRACT. We study the selfsimilarity and the Gibbs properties of several measures defined on the product space $\Omega_r := \{0, 1, \dots, r-1\}^{\mathbb{N}}$. This space can be identified with the interval $[0, 1]$ by means of the numeration in base r . The last section is devoted to the Bernoulli convolution in base $\beta = \frac{1+\sqrt{5}}{2}$, called the Erdős measure, and its analogue in base $-\beta = -\frac{1+\sqrt{5}}{2}$, that we study by means of a suitable system of numeration.

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1. Introduction

One calls the Bernoulli convolution associated with the base $\beta > 1$ and the parameter vector $\mathbf{p} = (p_0, \dots, p_{s-1})$, the infinite product of the Dirac measures $p_0 \delta_{\frac{0}{\beta^n}} + \dots + p_{s-1} \delta_{\frac{s-1}{\beta^n}}$ for $n \geq 1$ (see [5, 19, 12, 13]). In other words, it is the distribution measure of the random variable defined by

$$X(\omega) = \sum_{n \geq 1} \frac{\omega_n}{\beta^n},$$

when $\omega = (\omega_n)_{n \in \mathbb{N}}$ has a Bernoulli distribution such that, for any $n \in \mathbb{N}$,

$$P(\omega_n = 0) = p_0, \dots, P(\omega_n = s-1) = p_{s-1}.$$

The Bernoulli convolution associated with β and \mathbf{p} is the unique measure μ with bounded support that satisfies the self-similarity relation ([17]):

$$\mu = \sum_{i=0}^{s-1} p_i \cdot \mu \circ S_i^{-1},$$

where the affine contractions $S_i : \mathbb{R} \rightarrow \mathbb{R}$ are defined by $S_i(x) := \frac{x+i}{\beta}$. The measure μ is purely singular with respect to the Lebesgue measure when \mathbf{p} is uniform and β a Pisot number, that is, the conjugates of β have modulus less than 1. The problem to know if μ has the weak Gibbs property in the sense of Yuri [21] is not simple; it is solved in case β is a multinacci number ([6, 13]), but more complicated for other Pisot numbers of degree at least 3 (for instance in [13, Example 2.4], computing the values of the Bernoulli convolution in case $\beta^3 = 3\beta^2 - 1$ requires matrices of order 8).

Section 2 recalls the definition of the weak Gibbs property, and its link with the notions of Bernoulli or Markov measure.

Section 3 is devoted to some results of Mukherjea, Nakassis and Ratti about products of i. i. d. random stochastic matrices, that we present in a slightly different way (Proposition 3.1). They have computed the density of the limit distribution, in case this distribution is the Bernoulli convolution in base $\beta = \sqrt[r]{r}$ with parameters $p_0 = \dots = p_{r-1} = \frac{1}{r}$.

The framework is different in the sections 5 to 7; we define a measure on $\Omega_r := \{0, 1, \dots, r-1\}^{\mathbb{N}}$ by giving its values on the cylinders of Ω_r , under the form of products of 2×2 matrices and vectors. Theorem 6.1 gives a condition for such a measure, to be related to a Bernoulli convolution, via the representation of the reals in the integral base r . Establishing the weak Gibbs property requires the convergence of the involved product of matrices and vectors in the projective space of dimension 2. It is proved in [6] that the uniform Bernoulli convolution in base $\beta = \frac{1+\sqrt{5}}{2}$ is weak Gibbs; here, Section 7 give analogue result in the base $-\beta = -\frac{1+\sqrt{5}}{2}$.

2. Weak Gibbs measures

One says that the probability measure μ on the product space $\Omega_r = \{0, 1, \dots, r-1\}^{\mathbb{N}}$ has the weak Gibbs property if there exists a map $\phi : \Omega_r \rightarrow \mathbb{R}$, continuous for the product topology on Ω_r , such that

$$(1) \quad \lim_{n \rightarrow \infty} \left(\frac{\mu[\omega_1 \dots \omega_n]}{e^{\phi(\omega)} e^{\phi(\sigma\omega)} \dots e^{\phi(\sigma^{n-1}\omega)}} \right)^{1/n} = 1 \quad \text{uniformly on } \omega \in \Omega_r$$

(where σ is the shift on Ω_r , and $[\omega_1 \dots \omega_n]$ is the cylinder of order n around ω that is, the set of the $\omega' \in \Omega_r$ such that $\omega'_i = \omega_i$ for $1 \leq i \leq n$). If (1) holds, ϕ is called a potential of μ .

Equivalently, μ has the weak Gibbs property if and only if the measure of any cylinder $[\omega_1 \dots \omega_n]$ can be approached by a product in the following way: there exists a continuous map $\varphi : \Omega_r \rightarrow]0, +\infty[$ such that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \forall \omega \in \Omega_r \\ (\varphi(\omega) - \varepsilon) \dots (\varphi(\sigma^{n-1}\omega) - \varepsilon) \leq \mu[\omega_1 \dots \omega_n] \leq (\varphi(\omega) + \varepsilon) \dots (\varphi(\sigma^{n-1}\omega) + \varepsilon).$$

In case μ is σ -invariant, the following theorem gives an equivalent definition (see [8], [20], [15]), which involves the map ϕ_μ defined as follows:

$$(2) \quad \phi_\mu(\omega) := \lim_{n \rightarrow \infty} \log \frac{\mu[\omega_1 \dots \omega_n]}{\mu[\omega_2 \dots \omega_n]}$$

at each point $\omega \in \Omega_r$ such that this limit exists.

Theorem 2.1. *Let μ be a σ -invariant probability measure on Ω_r , and $\phi : \Omega_r \rightarrow \mathbb{R}$ a continuous map. The following assertions are equivalent:*

(i) μ is a weak Gibbs measure of potential ϕ and, for any $\omega \in \Omega_r$,

$$\sum_{a=0}^{r-1} e^{\phi(a\omega)} = 1;$$

(ii) $\phi_\mu(\omega)$ exists for any $\omega \in \Omega_r$, and $\phi_\mu = \phi$;

(iii) μ has entropy $h_\mu = -\mu(\phi)$ and, for any $\omega \in \Omega_r$, $\sum_{a=0}^{r-1} e^{\phi(a\omega)} = 1$.

This theorem can be used to prove that a σ -invariant probability measure has the weak Gibbs property, by using the implication (ii) \Rightarrow (i). Now for any probability measure μ on Ω_r , not necessarily σ -invariant, the following implication is straightforward (see [13]):

Proposition 2.2. *If ϕ_μ is defined and continuous on Ω_r , then μ is a weak Gibbs measure of potential ϕ_μ .*

The two following examples show that the Bernoulli and the Markovian measures are weak Gibbs. The third is a counterexample: the potential of the weak Gibbs measure μ_3 is not ϕ_{μ_3} .

Example. If μ_1 is a Bernoulli measure with support Ω_r , then ϕ_{μ_1} is the continuous map such that $\phi_{\mu_1}(\omega) = \log \mu_1[\omega_1]$ for any $\omega \in \Omega_r$.

Example. If μ_2 is a Markov measure with support Ω_r , then ϕ_{μ_2} is the continuous map such that $\phi_{\mu_2}(\omega) = \log \frac{\mu_2[\omega_1 \omega_2]}{\mu_2[\omega_2]}$ for any $\omega \in \Omega_r$.

Example. (see [12]) Let the probability measure μ_3 be defined on Ω_r by

$$\mu_3[\omega_1 \dots \omega_n] := \frac{1}{2 \cdot (2r)^n} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \omega'_1 & 0 \\ 1 & 1 \end{pmatrix} \dots \begin{pmatrix} \omega'_n & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with $\omega'_i = 1 + \frac{2\omega_i}{r-1}$. Then μ_3 is weak Gibbs of potential $\phi : \omega \mapsto \log \frac{\omega'_1}{2r}$, although ϕ_{μ_3} is discontinuous at any point ω such that the series

$S_\omega := \sum_{n \geq 1} \frac{1}{\omega'_1 \dots \omega'_n}$ converges:

$$\phi_{\mu_3}(\omega) = \begin{cases} \log \frac{1}{2r} + \log(1 + \frac{1}{S_\omega}) & \text{if } S_\omega < \infty \\ \log \frac{1}{2r} & \text{if } S_\omega = \infty. \end{cases}$$

The notion of weak Gibbs measure generalize the one of Gibbs measure (see for instance [1]). Let us generalize in the same way the notion of quasi-Bernoulli measure (see [3], [7]), and say that μ is weakly quasi-Bernoulli if it satisfies the following condition:

$$(3) \quad \lim_{n \rightarrow \infty} \left(\frac{\mu[\omega_1 \dots \omega_n]}{\mu[\omega_1 \dots \omega_i] \mu[\omega_{i+1} \dots \omega_n]} \right)^{1/n} = 1 \quad \text{uniformly on } \omega \in \Omega_r \text{ and } i \in \{1, \dots, n\}.$$

Then one has the following

Proposition 2.3. *If a probability measure on Ω_r has the weak Gibbs property, it satisfies (3).*

This proposition is straightforward, but can be used to prove that a probability measure do not have the weak Gibbs property:

Example. Let μ_4 be defined on Ω_2 by

$$\mu_4[\omega_1 \dots \omega_n] := \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} M_{\omega_1} \dots M_{\omega_n} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where $M_0 = \frac{1}{4} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $M_1 = \frac{1}{4} \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}$. It is not weak Gibbs

because $\left(\frac{\mu_4[1^n 0^n]}{\mu_4[1^n] \mu_4[0^n]} \right)^{1/n}$ do not converge to 1.

One can ask if the converse of Proposition 2.3 true, or if the condition (3) imply that μ is weak Bernoulli in the sense of Bowen [2].

3. Products of stochastic matrices

We consider a finite set of stochastic 2×2 matrices, let $M_k = \begin{pmatrix} x_k & 1 - x_k \\ y_k & 1 - y_k \end{pmatrix}$ for $k = 0, 1, \dots, r-1$, where $x_k, y_k \in [0, 1]$. We suppose the M_k are different from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

A. Mukherjea and al. have studied in [11] and [10] the distribution of the random matrix $\Omega_r \ni \omega \mapsto M_{\omega_1} \dots M_{\omega_n}$ when the distribution of ω is Bernoulli with positive parameters p_0, \dots, p_{r-1} . This distribution converges when $n \rightarrow \infty$, though the matrix $M_{\omega_1} \dots M_{\omega_n}$ itself do not converge (that is, its entries are – in much cases – divergent sequences). But we shall prove

the convergence of the matrix $M_{\omega_n} \dots M_{\omega_1}$ (which has of course the same distribution as $M_{\omega_1} \dots M_{\omega_n}$ when the distribution of ω is Bernoulli).

Proposition 3.1. *The product matrix $P_n^\omega := M_{\omega_n} \dots M_{\omega_1}$ converges uniformly on $\omega \in \Omega_r$ to the matrix $\begin{pmatrix} x^\omega & 1 - x^\omega \\ x^\omega & 1 - x^\omega \end{pmatrix}$, where $x^\omega := \sum_{i=1}^{\infty} y_{\omega_i} \det P_{i-1}^\omega$ and – by convention – P_0^ω is the unit-matrix.*

Proof. Setting $x_n^\omega := y_n^\omega + \det P_n^\omega$ with $y_n^\omega := \sum_{i=1}^n y_{\omega_i} \det P_{i-1}^\omega$ one check easily by induction that

$$P_n^\omega = \begin{pmatrix} x_n^\omega & 1 - x_n^\omega \\ y_n^\omega & 1 - y_n^\omega \end{pmatrix}.$$

The uniform convergence of the sequences x_n^ω and y_n^ω is due to the fact that each matrix M_k has – from the hypotheses – a determinant less than 1 in absolute value. □

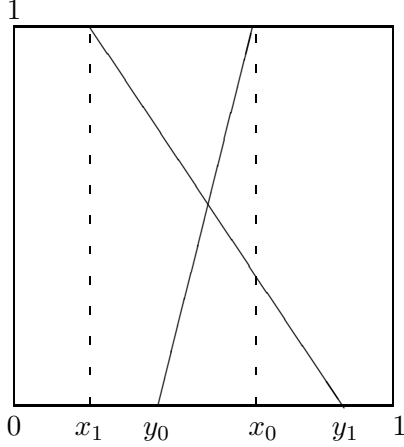
Theorem 3.2. ([11, Section 2]) *The distribution of $\omega \mapsto x^\omega$ is*
– *discrete if at least one of the matrices M_k is non invertible;*
– *singular continuous if the product $\left(\frac{|\det M_0|}{p_0}\right)^{p_0} \dots \left(\frac{|\det M_{r-1}|}{p_{r-1}}\right)^{p_{r-1}}$ belongs to $]0, 1[$ and at least one of its factors is different from 1.*

Selfsimilarity relation: The random variable $\omega \mapsto x^\omega$ takes its values in $[0, 1]$ because $\begin{pmatrix} x^\omega & 1 - x^\omega \\ x^\omega & 1 - x^\omega \end{pmatrix}$ is the limit of nonnegative matrices. Let λ be the probability distribution of $\omega \mapsto x^\omega$. If all the matrices M_k are invertible, then λ is selfsimilar in the sense that, for any borelian $B \subset [0, 1]$,

$$\lambda(B) = \sum_{k=0}^{r-1} p_k \lambda\left(\frac{B - y_k}{x_k - y_k}\right)$$

(see [11, equation (2.6)] for the proof).

Let us represent, for instance in the case $r = 2$ with $(x_0 - y_0)(x_1 - y_1) < 0$, the two maps $x \mapsto \frac{x - y_k}{x_k - y_k}$ involved in the selfsimilarity relation:



Example. The probability distribution λ of $\omega \mapsto x^\omega$ is related to the numeration in a given base $\beta > 1$ if we suppose that $x_k = y_k + \frac{1}{\beta}$ and that y_0, \dots, y_{r-1} are in arithmetic progression. Since we want that x_k and y_k belong to $[0, 1]$, the good choice is

$$y_k = \frac{k}{r-1} \left(1 - \frac{1}{\beta}\right) \quad \text{for } k = 0, \dots, r-1.$$

Then $x_\omega = \frac{\beta-1}{r-1} \sum_{n \geq 1} \frac{\omega_n}{\beta^n}$ and $\lambda\left(\frac{\beta-1}{r-1} \cdot\right)$ is the convolution of the measures $p_0 \delta_{\frac{0}{\beta^n}} + \dots + p_{r-1} \delta_{\frac{r-1}{\beta^n}}$ for $n = 1, 2, \dots$.

In case $\beta = \sqrt[r]{r}$ with $m \in \mathbb{N}$, if the distribution is uniform ($p_0 = \dots = p_{r-1} = \frac{1}{r}$), it is proved in [11, Proposition 1] that the density of the (absolutely continuous) distribution of $\omega \mapsto x^\omega$ is a piece-wise polynomial of degree at most m .

4. Uniform convergence (in direction) of the sequence of vectors $n \mapsto M_{\omega_1} \dots M_{\omega_n} V$

In this section $\mathcal{M} = \{M_0, \dots, M_{r-1}\}$ is a finite set of 2×2 matrices, where each matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}$ has nonnegative entries and each of the columns $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ is distinct from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. One denotes by \mathcal{M}^2 the set of matrices MM' for M and M' in \mathcal{M} , and \mathcal{M}_1 the set of matrices $M \in \mathcal{M}$ with $a = 0$;
 \mathcal{M}_2 the set of matrices in $M \in \mathcal{M}^2$ with $b = 0$;
 \mathcal{M}_3 the set of matrices in $M \in \mathcal{M}^2$ with $c = 0$;
 \mathcal{M}_4 the set of matrices in $M \in \mathcal{M}$ with $d = 0$.

Proposition 4.1. ([12, theorem A]) Let $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be a column matrix with positive entries. The sequence $n \mapsto \frac{M_{\omega_1} \dots M_{\omega_n} V}{\|M_{\omega_1} \dots M_{\omega_n} V\|}$ converges uniformly on $\omega \in \Omega_r$ if and only if at least one of the following conditions holds:

- (i) $\exists M \in \mathcal{M}_2$ such that $a > d$ and $\exists M \in \mathcal{M}_3$ such that $a < d$ and $\mathcal{M}_2 \cap \mathcal{M}_3 = \emptyset$
- (ii) $\exists M \in \mathcal{M}_2$ such that $a \leq d$, and $\exists M \in \mathcal{M}_3$ such that $a \geq d$
- (iii) $\exists M \in \mathcal{M}_2$ such that $a \leq d$ and $\exists M \in \mathcal{M}_3$ such that $a < d$ and $\mathcal{M}_1 = \emptyset$
- (iv) $\exists M \in \mathcal{M}_2$ such that $a > d$ and $\exists M \in \mathcal{M}_3$ such that $a \geq d$ and $\mathcal{M}_4 = \emptyset$
- (v) V is an eigenvector of all the matrices in \mathcal{M} .

5. Application to the measures defined by products of matrices

Let $\mathcal{M} = \{M_0, \dots, M_{r-1}\}$ be a finite set of 2×2 matrices whose columns are distinct from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and let L (resp. V) be a positive row matrix (resp., a positive column matrix). If V is an eigenvector of $\sum_i M_i$ for the eigenvalue 1, one can define some measure η on Ω_r by setting

$$\eta[\omega_1 \dots \omega_n] = LM_{\omega_1} \dots M_{\omega_n} V.$$

Proposition 5.1. The map ϕ_η defined in (2), exists and is continuous if and only if \mathcal{M} satisfies at least one of the above conditions (i), ..., (v), or the following:

- (vi) L is an eigenvector of all the matrices in \mathcal{M} .

Proof. The map ϕ_η is related to the map $\psi_{\mathcal{M}} : \omega \mapsto \lim_{n \rightarrow \infty} \frac{M_{\omega_1} \dots M_{\omega_n} V}{\|M_{\omega_1} \dots M_{\omega_n} V\|}$.

Indeed

$$\phi_\eta(\omega) = \frac{LM_{\omega_1} \psi_{\mathcal{M}} \circ \sigma(\omega)}{L \psi_{\mathcal{M}} \circ \sigma(\omega)}$$

for any $\omega \in \Omega_r$ such that $\psi_{\mathcal{M}} \circ \sigma(\omega)$ exists. Moreover if (vi) does not hold, the domains of definition of ϕ_η and $\psi_{\mathcal{M}} \circ \sigma$ are the same. \square

Now this proposition gives a sufficient condition for η to have the weak Gibbs property (by using Proposition 2.2). This condition is of course not necessary (see Example 1.5).

6. Measures associated with the numeration in integral base r .

Let the map $X_{q,r} : \Omega_q \mapsto \left[0, \frac{q-1}{r-1}\right]$ be defined by

$$X_{q,r}(\omega) = \sum_{n \geq 1} \frac{\omega_n}{r^n}.$$

In particular $X_{r,r}$ is one-to-one except on a countable set because, if ω is not eventually $r-1$, the real $X_{r,r}(\omega)$ has expansion ω in base r . In the present section we identify the set of sequences Ω_r with the interval $[0, 1]$, by means the map $X_{r,r}$.

The following theorem gives a condition for a measure defined by products of 2×2 matrices, to be related to some Bernoulli convolution in base r :

Theorem 6.1. ([12], Theorem 4.25) *Let ν be a σ -invariant probability measure on Ω_r ; the following assertions are equivalent:*

(i) *there exists a nonnegative row matrix L , a column matrix V and some square matrices M_0, \dots, M_{r-1} such that*

$$\forall \omega \in \Omega_r, \forall n \in \mathbb{N}, \quad \nu[\omega_1 \dots \omega_n] = LM_{\omega_1} \dots M_{\omega_n} V,$$

where the matrices $M_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ satisfy the conditions

$$b_0 = 0 \text{ and } \begin{pmatrix} a_k \\ c_k \end{pmatrix} = \begin{cases} \begin{pmatrix} b_{k+1} \\ d_{k+1} \end{pmatrix} & \text{if } 0 \leq k \leq r-2 \\ \begin{pmatrix} d_0 \\ b_0 \end{pmatrix} & \text{if } k = r-1 \end{cases}$$

(ii) *there exists a nonnegative parameter vector $\mathbf{p} = (p_0, \dots, p_{2r-2})$ such that ν is the probability distribution $\nu_{\mathbf{p}}$ of the fractional part of $X_{2r-1,r}(\omega)$, when $\omega \in \Omega_{2r-1}$ has a Bernoulli distribution with parameter \mathbf{p} .*

The relations between the matrices M_k and the parameter \mathbf{p} are

$$p_0 = a_0, \dots, p_{r-1} = a_{r-1}, p_r = c_0, \dots, p_{2r-2} = c_{r-2}$$

and thus $\nu_{\mathbf{p}}$ is weak Gibbs from Proposition 5.1 in certain cases, for instance if the p_k are positive.

Selfsimilarity relation Let $\mu_{\mathbf{p}}$ and $\nu_{\mathbf{p}}$ be the probability distributions of $X_{2r-1,r}$ and the fractional part of $X_{2r-1,r}$, respectively. Their respective supports are $[0, 2]$ and $[0, 1]$ and, for any borelian $B \subset [0, 1]$,

$$\nu_{\mathbf{p}}(B) = \mu_{\mathbf{p}}(B) + \mu_{\mathbf{p}}(B+1).$$

Theorem 6.1 is a consequence of the selfsimilarity relation

$$(4) \quad \mu_{\mathbf{p}}(B) = \sum_{k=0}^{2(r-1)} p_k \mu_{\mathbf{p}}(rB - k)$$

which allows to compute the column matrix $\begin{pmatrix} \mu_{\mathbf{p}}(B) \\ \mu_{\mathbf{p}}(B+1) \end{pmatrix}$.

The measure $\nu_{\mathbf{p}}$ has support $[0, 1]$, while the measure $\nu_{\mathbf{p}}^*$ defined for any borelian $B \subset \mathbb{R}$, by

$$\nu_{\mathbf{p}}^*(B) = \mu_{\mathbf{p}}(B) + \mu_{\mathbf{p}}(B+1)$$

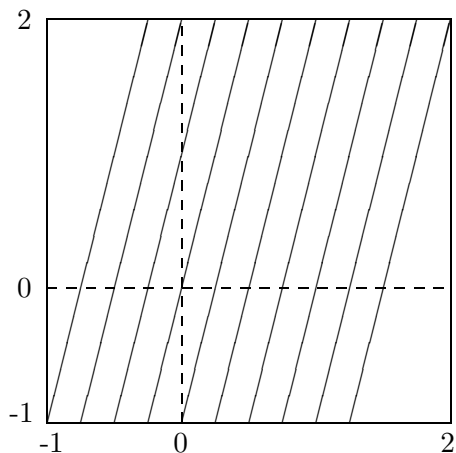
has support $[-1, 2]$, and coincide with $\nu_{\mathbf{p}}$ on $[0, 1]$. The selfsimilarity relation for $\nu_{\mathbf{p}}^*$ can be deduced from (4):

$$(5) \quad \nu_{\mathbf{p}}^*(B) = \sum_{k=-(r-1)}^{2(r-1)} p_k^* \nu_{\mathbf{p}}^*(rB - k)$$

where $p_k^* = \sum_{j \geq 0} (p_{k+2j} - p_{k+2j+1} + p_{k+2j+b} - p_{k+2j+b+1})$.

Both measures $\mu_{\mathbf{p}}$ and $\nu_{\mathbf{p}}^*$ are Bernoulli convolutions: they are – respectively – the infinite product of the measures $p_0 \delta_{\frac{0}{r^n}} + \dots + p_{2r-2} \delta_{\frac{2r-2}{r^n}}$ and the one of the measures $p_{-r+1}^* \delta_{\frac{-r+1}{r^n}} + \dots + p_{2r-2}^* \delta_{\frac{2r-2}{r^n}}$, for $n \geq 1$.

We represent below the maps $x \mapsto rx - k$ involved in (4) and (5), in the case $r = 4$:



7. The bases $\beta = \frac{1+\sqrt{5}}{2}$ and $-\beta = -\frac{1+\sqrt{5}}{2}$

We consider in this section the measures μ and μ_* which are respectively the distributions of the random variables X and Y , defined by

$$X(\omega) = \sum_{n \geq 1} \frac{\omega_n}{\beta^{n+1}} \quad \text{and} \quad Y(\omega) = \frac{1}{\beta} - \sum_{n \geq 1} \frac{\omega_n}{(-\beta)^{n+1}},$$

when the distribution of $\omega \in \Omega_2$ is Bernoulli with positive parameter vector $\mathbf{p} = (p, q)$. We use consecutively two systems of numeration (see for

instance [16], [14] and [4]): any real $x \in [0, 1[$ can be represented in an unique way on the form

$$x = \sum_{n \geq 1} \frac{\varepsilon_n}{\beta^n} \quad (\text{Parry expansion}) \quad \text{and} \quad x = \frac{1}{\beta} - \sum_{n \geq 1} \frac{\alpha_n}{(-\beta)^{n+1}},$$

where $(\varepsilon_n)_{n \geq 1} =: \varepsilon(x)$ and $(\alpha_n)_{n \geq 1} =: \alpha(x)$ are two sequences with terms in $\{0, 1\}$, without two consecutive terms 1, such that $\sigma^n \varepsilon(x)$ and $\sigma^{2n+1} \alpha(x)$ differ from the periodic sequence 1010... for any $n \geq 0$. For any word $w = \omega_1 \dots \omega_n$ on the alphabet $\{0, 1\}$ and without factor 11, we denote

$$\begin{aligned} \llbracket w \rrbracket &:= \{x \in [0, 1[; \varepsilon(x) \in [\omega_1, \dots, \omega_n]\} \\ \llbracket w \rrbracket_\star &:= \{x \in [0, 1[; \alpha(x) \in [\omega_1, \dots, \omega_n]\}. \end{aligned}$$

In case $\omega_n = 0$ we may compute $\mu \llbracket w \rrbracket$ and $\mu_\star \llbracket w \rrbracket_\star$ by the following formulas:

$$(6) \quad \begin{pmatrix} \mu(\frac{1}{\beta} \llbracket w \rrbracket) \\ \mu(\frac{1}{\beta} + \frac{1}{\beta} \llbracket w \rrbracket) \\ \mu(\frac{1}{\beta^2} + \frac{1}{\beta} \llbracket w \rrbracket) \end{pmatrix} = M_{\omega_1} \dots M_{\omega_n} \begin{pmatrix} \frac{p}{p+q^2} \\ \frac{q^2}{p+q^2} \\ \frac{q}{p+q^2} \end{pmatrix}$$

$$(7) \quad \begin{pmatrix} \mu_\star(\llbracket w \rrbracket_\star) \\ \mu_\star(-\frac{1}{\beta} + \llbracket w \rrbracket_\star) \\ \mu_\star(\frac{1}{\beta^2} + \llbracket w \rrbracket_\star) \end{pmatrix} = A_{\omega_1} \dots A_{\omega_n} \begin{pmatrix} 1 \\ \frac{q}{1+q} \\ \frac{1}{1+q} \end{pmatrix}$$

where

$$M_0 = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & q \\ q & p & 0 \end{pmatrix} \quad M_1 = \begin{pmatrix} q & p & 0 \\ 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix} \quad A_0 = \begin{pmatrix} p & q & 0 \\ 0 & 0 & q \\ 0 & p & 0 \end{pmatrix} \quad A_1 = \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & 0 \\ p & q & 0 \end{pmatrix}.$$

The formula (6) – and its extension to the multinacci bases – is proved in [13]. Let us sketch the proof of (7), which is equivalent to the following (assuming again that the word w do not have two consecutive letters 1 and ends by the letter 0):

$$(8) \quad \begin{pmatrix} \mu_\star(\llbracket w \rrbracket_\star) \\ \mu_\star(-\frac{1}{\beta} + \llbracket w \rrbracket_\star) \\ \mu_\star(\frac{1}{\beta^2} + \llbracket w \rrbracket_\star) \end{pmatrix} = A_{\omega_1} \begin{pmatrix} \mu_\star(\llbracket w' \rrbracket_\star) \\ \mu_\star(-\frac{1}{\beta} + \llbracket w' \rrbracket_\star) \\ \mu_\star(\frac{1}{\beta^2} + \llbracket w' \rrbracket_\star) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mu([0, 1[) \\ \mu(-\frac{1}{\beta} + [0, 1[) \\ \mu(\frac{1}{\beta^2} + [0, 1[) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{q}{1+q} \\ \frac{1}{1+q} \end{pmatrix}$$

where $w' = \omega_2 \dots \omega_n$ for $n \geq 2$ and, by convention, if $n = 1$ the word w' is empty and $\llbracket w' \rrbracket_\star = [0, 1[$. We first compute $\mu_\star(\llbracket w \rrbracket_\star)$: it is the probability of the event $Y(\xi) \in \llbracket w \rrbracket_\star$. This event is equivalent to $Y(\sigma\xi) \in \frac{\omega_1 - \xi_1}{\beta} + \llbracket w' \rrbracket_\star$ hence

- in case $\omega_1 = 0$, it is also equivalent to

$$\begin{cases} \xi_1 = 0 \\ Y(\sigma\xi) \in \llbracket w' \rrbracket_\star \end{cases} \quad \text{or} \quad \begin{cases} \xi_1 = 1 \\ Y(\sigma\xi) \in -\frac{1}{\beta} + \llbracket w' \rrbracket_\star \end{cases}$$

and this explain why the first row in A_0 is $(p \ q \ 0)$;

- in case $\omega_1 = 1$ we have necessarily $n \geq 2$ and $\omega_2 = 0$, and the event $Y(\sigma\xi) \in \frac{1-\xi_1}{\beta} + \llbracket w' \rrbracket_\star$ can occur only if $\xi_1 = 1$ and $Y(\sigma\xi) \in \llbracket w' \rrbracket_\star$; so the first row in A_1 is $(q \ 0 \ 0)$.

- We compute in the same way $\mu_\star(-\frac{1}{\beta} + \llbracket w \rrbracket_\star)$ and $\mu_\star(\frac{1}{\beta^2} + \llbracket w \rrbracket_\star)$ and we conclude that the first equality in (8) is true.

- The second equality in (8) can be deduced from the first, by making $n = 1$ and $\omega_1 = 0$.

7.1. Bernoulli convolution in base $\beta = \frac{1+\sqrt{5}}{2}$ ([5]). The Gibbs properties of μ have been studied in [13] in the following sense: let be the words

$$(9) \quad w(0) := 00, \quad w(1) = 010 \quad \text{and} \quad w(2) = 10;$$

then for any $x \in [0, 1[$, there exists a unique sequence $\xi(x) = (\xi_n)_{n \geq 1} \in \Omega_3$ such that the Parry expansion $\varepsilon(x)$ belongs for any $n \geq 1$ to the cylinder $[w(\xi_1 \dots \xi_n)]$, where $w(\xi_1 \dots \xi_n)$ is the concatenation of the words $w(\xi_1), \dots, w(\xi_n)$. The measure $\mu \circ \xi^{-1}$ is weak Gibbs on Ω_3 if and only if $p = q$ (this case is studied more in details in [6]); nevertheless $\phi_{\mu \circ \xi^{-1}}(100\dots) = \infty$ in this case.

7.2. Bernoulli convolution in base $-\beta = -\frac{1+\sqrt{5}}{2}$. The measure μ_\star has better Gibbs properties than μ : let us consider now – for any $x \in [0, 1[$ – the sequence $\xi_\star(x) = (\xi_n^\star)_{n \geq 1} \in \Omega_3$ such that $\alpha(x) \in [w(\xi_1^\star \dots \xi_n^\star)]_\star$ for all $n \geq 1$, we have the following

Theorem 7.1. (i) If $p \geq q$ the measure $\mu_\star \circ \xi_\star^{-1}$ is weak Gibbs on Ω_3 .

(ii) if $p \leq q$ the measure $\mu_\star \circ S \circ \xi_\star^{-1}$ is weak Gibbs on Ω_3 , where $S(x) = 1 - x$ for any $x \in [0, 1]$.

Proof. (ii) can be deduced from (i) by using the symmetry relation

$$Y(\omega_1 \omega_2 \dots) = 1 - Y((1 - \omega_1)(1 - \omega_2) \dots),$$

which implies $\mu_\star^{(p,q)} \circ S = \mu_\star^{(q,p)}$.

In order to prove (i), we don't use the matrices A_k but the product matrices associated to the three words defined in (9): setting $\alpha = \frac{p}{q}$ we have

$$A_0^\star := A_0^2 = pq \begin{pmatrix} \alpha & 1 & \frac{1}{\alpha} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1^\star := A_0 A_1 A_0 = pq^2 \begin{pmatrix} \alpha & 1 & 0 \\ \alpha & 1 & \frac{1}{\alpha} \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_2^\star := A_1 A_0 = pq \begin{pmatrix} 1 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 \\ \alpha & 1 & \frac{1}{\alpha} \end{pmatrix}.$$

Let us prove (i) by means of Proposition 2.2: more precisely we shall prove the uniform convergence of the (continuous) n -step potential $\phi_n : \Omega_3 \rightarrow \mathbb{R}$ defined by

$$(10) \quad \phi_n(\omega) := \log \frac{\mu_\star \circ \xi_\star^{-1}[\omega_1 \dots \omega_n]}{\mu_\star \circ \xi_\star^{-1}[\omega_2 \dots \omega_n]} = \log \frac{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} A_{\omega_1}^* \dots A_{\omega_n}^* \begin{pmatrix} 1 \\ \frac{q}{1+q} \\ \frac{1}{1+q} \end{pmatrix}}{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} A_{\omega_2}^* \dots A_{\omega_n}^* \begin{pmatrix} 1 \\ \frac{q}{1+q} \\ \frac{1}{1+q} \end{pmatrix}}.$$

Notice that

$$(11) \quad A_0^{*n} = (pq)^n \begin{pmatrix} v_n(\alpha) & \alpha u_n(\alpha) & u_n(\alpha) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2^{*n} = (pq)^n \begin{pmatrix} 1 & 1/\alpha & 0 \\ 0 & 0 & 0 \\ u_n(1/\alpha) & u_n(1/\alpha)/\alpha & v_n(1/\alpha) \end{pmatrix}$$

where $u_n(x) := x^{-1} + x^0 + \dots + x^{n-2}$ and $v_n(x) := x^n$ for any positive real x .

From now on we use the formalism of continued fractions ([18]) in a same way as in [13]: given n (odd) and $a_0 \geq 0, a_1 > 0, \dots, a_n > 0$ we put

$$\begin{pmatrix} p_{-1} \\ q_{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} u_0 \\ 1 \end{pmatrix} \quad \text{and, for } 1 \leq k \leq n, \quad \begin{pmatrix} p_k \\ q_k \end{pmatrix} = u_k \begin{pmatrix} p_{k-1} \\ q_{k-1} \end{pmatrix} + v_{k-1} \begin{pmatrix} p_{k-2} \\ q_{k-2} \end{pmatrix}$$

where, for our purpose, $\begin{cases} u_i := u_{a_i}(\alpha) & (i \text{ even}) \\ u_i := u_{a_i}(1/\alpha) & (i \text{ odd}) \end{cases}$ and $\begin{cases} v_i := v_{a_i}(\alpha) & (i \text{ even}) \\ v_i := v_{a_i}(1/\alpha) & (i \text{ odd}). \end{cases}$

We have

$$(12) \quad A_0^{*a_0} A_2^{*a_1} A_0^{*a_2} \dots A_2^{*a_n} = (pq)^{a_0 + \dots + a_n} \begin{pmatrix} p_n & p_n/\alpha & v_n p_{n-1} \\ 0 & 0 & 0 \\ q_n & q_n/\alpha & v_n q_{n-1} \end{pmatrix}.$$

The difference $\delta_k = \left| \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} \right|$ is known to be at most $\frac{1}{a_1 + \dots + a_k}$ in the case of the regular continued fractions ([9]) that is – with our notations – in the case $\alpha = 1$. We complete by the following

Lemma 7.2. *If $\alpha > 1$, then*

- (i) for $k \geq 1$, $\delta_k \leq \frac{v_{k-1}}{u_k u_{k-1} + v_{k-1}} \delta_{k-1}$;
- (ii) for $k \geq 1$ even, $\delta_k \leq \alpha^{1-(a_{k-1}+a_k)} \delta_{k-1}$;
- (iii) for $k \geq 1$ even, $\delta_k \leq \alpha^{a_0 - (a_1 + \dots + a_k)/2}$.

Proof. (i) By the definition of p_k and q_k ,

$$\begin{aligned} \frac{p_k}{q_k} &= \frac{u_k p_{k-1} + v_{k-1} p_{k-2}}{u_k q_{k-1} + v_{k-1} q_{k-2}} \\ &= \frac{u_k q_{k-1}}{u_k q_{k-1} + v_{k-1} q_{k-2}} \cdot \frac{p_{k-1}}{q_{k-1}} + \frac{v_{k-1} q_{k-2}}{u_k q_{k-1} + v_{k-1} q_{k-2}} \cdot \frac{p_{k-2}}{q_{k-2}} \end{aligned}$$

hence

$$\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{v_{k-1} q_{k-2}}{u_k q_{k-1} + v_{k-1} q_{k-2}} \cdot \left(\frac{p_{k-2}}{q_{k-2}} - \frac{p_{k-1}}{q_{k-1}} \right)$$

and, since $q_{k-1} \geq u_{k-1} q_{k-2}$, we are done.

(ii) If k is even one has $\frac{v_{k-1}}{u_k u_{k-1}} \leq \frac{\alpha^{-a_{k-1}}}{\alpha^{a_k-2} \alpha}$, hence (i) implies (ii).

(iii) The inequalities (i) and (ii) imply respectively that the sequence (δ_k) is non-increasing and, if k is even, $\delta_k \leq \alpha^{-(a_{k-1}+a_k)/2} \delta_{k-1}$; hence $\delta_2 \leq \alpha^{-(a_1+a_2)/2} \delta_1$ and, by induction $\delta_k \leq \alpha^{-(a_1+\dots+a_k)/2} \delta_1$ for any k even. Now $\delta_1 = \frac{v_0}{u_1} \leq \alpha^{a_0}$.

□

Notice that this lemma implies $\delta_k \leq \alpha^{a_0-(k-1)/2}$ for any $k \geq 1$, hence the sequence $k \mapsto \frac{p_k}{q_k}$ converges. Now we can prove the following

Lemma 7.3. *Suppose $\alpha \geq 1$ and let $\omega \in \Omega_3$.*

(i) *At least one of the followings assertions is true:*

$\exists N \geq 0$ such that $\omega_{N+1} \dots \omega_{N+n} \in \{0, 2\}^{n-1} \times \{2\}$ for infinitely many $n \geq 1$;

$\exists N \geq 0$ such that $\omega_{N+1} \dots \omega_{N+n} \in \{0\}^n$ for all $n \geq 1$;

$\exists N \geq 0$ and $n \geq 2$ such that $\omega_{N+1} \dots \omega_{N+n} \in \{1, 2\} \times \{0\}^{n-2} \times \{1\}$.

(ii) *In all cases there exists $N \geq 0$ and $n \geq 1$ such that*

$$h, h' \geq N + n, \omega' \in [\omega_1 \dots \omega_{N+n}] \Rightarrow |\phi_{h'}(\omega') - \phi_h(\omega)| \leq \varepsilon.$$

Proof. (i) If there exists $N \geq 0$ such that $\sigma^N \omega \in \{0, 2\}^{\mathbb{N}}$, we are in the two first cases. If not, the digit 1 occurs infinitely many times in the sequence ω . The second occurrence of 1 is necessarily preceded by a word in $\{1, 2\} \times \{0\}^k$ for some $k \geq 0$, hence we are in the third case.

(ii) Let N and n be as in (i). From (10), for any $h \geq N + n$ and $\omega' \in [\omega_1 \dots \omega_{N+n}]$ there exists some reals $a, b, c, a', b', c', x, y, z$ such that

$$(13) \quad \phi_h(\omega') = \log \frac{\begin{pmatrix} a & b & c \end{pmatrix} A_{\omega_{N+1}}^* \dots A_{\omega_{N+n}}^* \begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\begin{pmatrix} a' & b' & c' \end{pmatrix} A_{\omega_{N+1}}^* \dots A_{\omega_{N+n}}^* \begin{pmatrix} x \\ y \\ z \end{pmatrix}},$$

where only x, y and z depend on h and ω' .

Suppose the first assertion in (i) is true. We deduce from the expression of $A_{\omega_{N+1}}^* \dots A_{\omega_{N+n}}^*$ in (12) that

$$\phi_h(\omega') = \log \frac{(ap_n + cq_n)(x + \frac{y}{\alpha}) + v_n(ap_{n-1} + cq_{n-1})z}{(a'p_n + c'q_n)(x + \frac{y}{\alpha}) + v_n(a'p_{n-1} + c'q_{n-1})z}.$$

This ratio lies between $\log \frac{ap_n + cq_n}{a'p_n + c'q_n}$ and $\log \frac{ap_{n-1} + cq_{n-1}}{a'p_{n-1} + c'q_{n-1}}$. These bounds

do not depend on h nor ω' , and converge – for $n \rightarrow \infty$ – to $\log \frac{a\theta + c}{a'\theta + c'}$ when $n \rightarrow \infty$, where $\theta := \lim_{n \rightarrow \infty} \frac{p_n}{q_n}$. We deduce that (ii) is true in this case, by choosing n large enough.

The proof is similar when the second assertion in (i) is true, by using the expression of $A_{\omega_{N+1}}^* \dots A_{\omega_{N+n}}^*$ in (11).

If the third assertion in (i) is true, the matrix $A_{\omega_{N+1}}^* \dots A_{\omega_{N+n}}^*$ has rank 1; whence it maps \mathbb{R}^3 into a space of dimension 1, and the ratio in (13) do not depend on h nor ω' so that

$$h, h' \geq N + n, \omega' \in [\omega_1 \dots \omega_{N+n}] \Rightarrow |\phi_{h'}(\omega') - \phi_h(\omega)| = 0.$$

□

End of the proof of Theorem 7.1. Notice that Lemma 7.3(ii) implies – by making $\omega = \omega'$ – that the sequence $h \mapsto \phi_h(\omega)$ is Cauchy; let $\phi(\omega)$ be its limit.

Now we make $h, h' \rightarrow \infty$ in Lemma 7.3(ii): we obtain $|\phi(\omega') - \phi(\omega)| \leq \varepsilon$ for any ω' in the neighborhood $[\omega_1 \dots \omega_{N+n}]$ of ω , and this prove the continuity of ϕ so, by Proposition 2.2 $\mu_* \circ \xi_*^{-1}$ is weak Gibbs.

□

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