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ANALYSIS OF CONTACT OF ELASTIC RODS SUBJECT TO LARGE DISPLACEMENTS

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ABSTRACT. We present a mathematical model for describing motion of two elastic rods in contact. The model allows for large displacements and is essentially based on COSSERAT's modeling of rods. Existence of a static solution is proved in the case of a unique rod using a penalty technique. The contact modeling involves unilateral constraints on the central lines of the rods. Existence is also proved for this contact problem.

1. Introduction

Studying contact between elastic rods is of great interest in particular for analyzing internal friction forces in wire ropes. We consider here the case of two rods in frictionless contact. The modeling of each rod is based on the Cosserat's model [4]. A penalized formulation of the energy in which orthonormality constraints of the director vectors is imposed by a penalty technique, is used. Mathematical results related to this formulation are given. Let us mention here that only the static case is considered in the present study.

The main issue here is to define contact constraints that take advantage of the one-dimensional feature of rod models. To obtain the desired model, we define the total energy as the sum of energies of the two bodies, the final problem consisting in the minimization of this energy under the non-penetration constraint. We express this condition on the central lines of the two thin bodies. We write the constrained optimization problem, use again a penalty formulation to impose contact constraints and then derive optimality conditions. The obtained model is then analyzed and existence of a solution is proved.

In the sequel we shall make use of the following notations : For a vector field $\boldsymbol{v}$ in $\mathbb{R}^3$, the scalar function v^i will denote its i–th contravariant component while a subscript i will denote its covariant one. Moreover, the same subscript in $\boldsymbol{v}_i$ (vector $\boldsymbol{v}_i$) will be used to denote different vectors. In addition, the summation convention of repeated indices will be adopted; the superscripts i, j will vary from 1 to 3 and α, β from 1 to 2. The spaces $L^p(0, \ell; \mathbb{R}^3)$ and $H^1(0, \ell; \mathbb{R}^3)$ will denote traditional Sobolev spaces L^p and H^1 for vector valued functions.

2. A model for elastic rods

In order to model elastic rods bodies the theory developed in [4], [1] is used. For the sake of conciseness, details that can be found in [1] will be omitted.

In the reference configuration the generating (or central) line is assumed to be straight and is then aligned with the Ox^3–axis. The reference configuration is

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defined as :

$$\Omega = \{(x^1, x^2, s);\ (x^1, x^2) \in \Lambda(s),\ 0 \le s \le \ell\},$$

where $\Lambda(s)$ is a given domain in the plane describing the cross section at location s.

The deformed configuration is defined by means of the three vectors $\boldsymbol{r}(s)$, $\boldsymbol{d}_1(s)$, $\boldsymbol{d}_2(s)$ where $\boldsymbol{r}$ is a parameterization of the deformed generating line. The vectors $\boldsymbol{d}_1(s)$, $\boldsymbol{d}_2(s)$ are orthonormal; they are orthogonal to $\boldsymbol{r}(s)$ and they span $\Lambda(s)$. We also define $\boldsymbol{d}_3 := \boldsymbol{d}_1 \times \boldsymbol{d}_2$. A material point located at $\boldsymbol{x} = (x^1, x^2, s)$ will be located in the deformed configuration at the position

$$\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{r}(s) + x^1 \boldsymbol{d}_1(s) + x^2 \boldsymbol{d}_2(s).$$

Since the triple $(\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3)$ is orthogonal, there exists (Cf. [1]) a vector field $\boldsymbol{u}$ such that

$$\boldsymbol{d}_i' = \boldsymbol{u} \times \boldsymbol{d}_i, \tag{2.1}$$

where $\boldsymbol{u}$ is given by $u_i := \boldsymbol{u} \cdot \boldsymbol{d}_i$ for $1 \le i \le 3$. We also define $v_i := \boldsymbol{r}' \cdot \boldsymbol{d}_i$ for $1 \le i \le 3$. Note here that the components u_i and v_i have the following mechanical interpretation (Cf. [1]) : u_1 (resp. u_2) measures the bending in the plane $(\boldsymbol{d}_2, \boldsymbol{d}_3)$ (resp. $(\boldsymbol{d}_3, \boldsymbol{d}_1)$), while u_3 measures the torsion of the rod. The components v_1 and v_2 measure the shear in the $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ directions respectively and v_3 represents the dilatation of the rod.

2.1. **The equations.** Balance equations for a rod can be written in the following way :

$$\begin{aligned} &-\boldsymbol{n}'(s) = \boldsymbol{f}_3(s) && 0 \le s \le \ell, \\ &-\boldsymbol{m}'(s) + \boldsymbol{r}'(s) \times \boldsymbol{n}(s) + \boldsymbol{d}_\alpha(s) \times \boldsymbol{f}_\alpha(s) = 0 && 0 \le s \le \ell, \end{aligned}$$

where $\boldsymbol{n}$ and $\boldsymbol{m}$ denote respectively internal forces and torque of internal moments.

Concerning constitutive laws, we shall consider hyperelastic material, *i.e.* material such that the following relationships hold :

$$\begin{aligned} \boldsymbol{m}(\boldsymbol{u}(s), \boldsymbol{v}(s), s) &= \frac{\partial W}{\partial u_i}(\boldsymbol{u}, \boldsymbol{v}, s)\boldsymbol{d}_i(s) && 0 \le s \le \ell, \\ \boldsymbol{n}(\boldsymbol{u}(s), \boldsymbol{v}(s), s) &= \frac{\partial W}{\partial v_i}(\boldsymbol{u}, \boldsymbol{v}, s)\boldsymbol{d}_i(s) && 0 \le s \le \ell \end{aligned}$$

where W is a given energy potential. As in [3], we choose a quadratic energy potential given by

$$W(\boldsymbol{u}, \boldsymbol{v}, s) = \frac{EI(s)}{2}(u_1^2 + u_2^2) + GI(s)u_3^2,$$

where E is the Young's modulus, A is the section area, G is the shear modulus and I is the principal momentum of inertia for an assumed circular cross section of the rod.

Note that we have neglected shear and volume change effects ($v_1 = v_2 = 0$, $\boldsymbol{r}' = \boldsymbol{d}_3$).

The equilibrium state for a single rod under the action of force $(\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$ is therefore a minimum of the energy functional :

$$J(\boldsymbol{r}, (\boldsymbol{d}_i)) := \int_0^\ell W(\boldsymbol{u}, \boldsymbol{v}, \cdot)\, ds - \int_0^\ell (\boldsymbol{f}_3 \cdot \boldsymbol{r} + \boldsymbol{f}_\alpha \cdot \boldsymbol{d}_\alpha)\ ds$$

In what follows, for the sake of simplicity we shall restrict ourselves to the cases where

$$\int_0^\ell \boldsymbol{f}_\alpha \cdot \boldsymbol{d}_\alpha \, ds = 0.$$

Owing to the fact that W depends no more on v, the notation $W(\boldsymbol{u}, \cdot)$ will replace $W(\boldsymbol{u}, \boldsymbol{v}, \cdot)$. Now, using identity (2.1), we obtain

$$\sum_{j \neq i} u_j^2(\boldsymbol{d}_k) = |\boldsymbol{d}_i'|^2 \qquad \text{for } 1 \leq i \leq 3.$$

We finally obtain

$$J(\boldsymbol{r}, (\boldsymbol{d}_i)) := \frac{1}{2} \int_0^\ell \left(GI \left(|\boldsymbol{d}_1'|^2 + |\boldsymbol{d}_2'|^2 \right) + (E - G) I |\boldsymbol{r}''|^2 \right) ds - \int_0^\ell \boldsymbol{f}_3 \cdot \boldsymbol{r} \, ds.$$

The equilibrium problem is finally described by the following minimization formulation :

$$\text{(2.2)} \qquad \begin{cases} \text{Find } (\boldsymbol{r}, (\boldsymbol{d}_i)) \in \mathcal{V} \text{ such that} \\ J(\boldsymbol{r}, (\boldsymbol{d}_i)) \leq J(\boldsymbol{p}, (\boldsymbol{g}_i)) \qquad \text{for } (\boldsymbol{p}, (\boldsymbol{g}_i)) \in \mathcal{V} \end{cases}$$

the set $\mathcal{V}$ being given by :

$$\begin{aligned} \mathcal{V} := \{ & (\boldsymbol{p}, (\boldsymbol{g}_i)) \in H^1(0, \ell; \mathbb{R}^{12}); \ \boldsymbol{p}' = \boldsymbol{g}_3, \ \boldsymbol{p}(0) = 0, \\ & \boldsymbol{g}_i(0) = \boldsymbol{d}_i^0, \ \boldsymbol{g}_i(\ell) = \boldsymbol{d}_i^\ell, \ \boldsymbol{g}_i \cdot \boldsymbol{g}_j = \delta_{ij}, \ (\boldsymbol{g}_1 \times \boldsymbol{g}_2) \cdot \boldsymbol{g}_3 > 0 \}, \end{aligned}$$

where $\boldsymbol{d}_i^\ell \in \mathbb{R}^3$, $1 \leq i \leq 3$ are given. Notice that we have prescribed Dirichlet boundary conditions in the set $\mathcal{V}$ corresponding to the example case of a clamped rod.

2.2. **A Penalized energy formulation.** In order to impose the constraints contained in $\mathcal{V}$ we develop here and analyze an exterior penalty method. For a large positive number $\theta \gg 1$ we define the functional :

$$H(\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3) := \int_0^\ell \left((\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2 + (|\boldsymbol{d}_1| - 1)^2 + (|\boldsymbol{d}_2| - 1)^2 + |\boldsymbol{d}_3 - \boldsymbol{d}_1 \times \boldsymbol{d}_2|^2 \right) ds$$

and the penalized energy :

$$J^\theta(\boldsymbol{r}, \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3) := J(\boldsymbol{r}, \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3) + \frac{\theta}{2} H(\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3).$$

The minimization problem (2.2) is then approximated by the following one :

$$\text{(2.3)} \qquad \begin{cases} \text{Find } (\boldsymbol{r}^\theta, (\boldsymbol{d}_i^\theta)) \in \mathcal{V}_0 \text{ such that} \\ J^\theta(\boldsymbol{r}^\theta, (\boldsymbol{d}_i^\theta)) \leq J^\theta(\boldsymbol{p}, (\boldsymbol{g}_i)) \qquad \text{for } (\boldsymbol{p}, (\boldsymbol{g}_i)) \in \mathcal{V}_0 \end{cases}$$

where

$$\mathcal{V}_0 := \{ (\boldsymbol{p}, (\boldsymbol{g}_i)) \in H^1(0, \ell; \mathbb{R}^{12}); \ \boldsymbol{p}' = \boldsymbol{g}_3, \ \boldsymbol{p}(0) = 0, \ \boldsymbol{g}_i(0) = \boldsymbol{d}_i^0, \ \boldsymbol{g}_i(\ell) = \boldsymbol{d}_i^\ell \}.$$

Theorem 2.1. *Let us assume that $\boldsymbol{f}_3 \in L^2(0, \ell; \mathbb{R}^3)$, then for each $\theta > 0$, Problem* (2.3) *has at least one solution.*

Proof. To simplify the notations we introduce, for $\boldsymbol{g} \in L^2(0,\ell;\mathbb{R}^3)$, the function

$$\boldsymbol{K}(\boldsymbol{g})(s) := \int_0^s \boldsymbol{g}(t)\,dt$$

and the functionals :

$$\widetilde{J}(\boldsymbol{g}_1,\boldsymbol{g}_2,\boldsymbol{g}_3) := J(K(\boldsymbol{g}_3),\boldsymbol{g}_1,\boldsymbol{g}_2,\boldsymbol{g}_3)$$
$$\widetilde{J}^\theta(\boldsymbol{g}_1,\boldsymbol{g}_2,\boldsymbol{g}_3) := J^\theta(K(\boldsymbol{g}_3),\boldsymbol{g}_1,\boldsymbol{g}_2,\boldsymbol{g}_3).$$

Let us first prove that $\widetilde{J}^\theta$ is weakly lower semi–continuous (l.s.c.). We denote by $(\boldsymbol{d}_1^n,\boldsymbol{d}_2^n,\boldsymbol{d}_3^n)$ a sequence of $H^1(0,\ell;\mathbb{R}^9)$ that converges weakly in this space.

The sequences $(\boldsymbol{d}_i^n)'$, $i=1,2,3$ are then weakly convergent in $L^2(0,\ell;\mathbb{R}^3)$ and consequently the $(\boldsymbol{d}_i^n)$ are strongly convergent in $\mathcal{C}^0(0,\ell;\mathbb{R}^3)$. From this we deduce the weak convergence of the products $(\boldsymbol{d}_k^n)'{\cdot}\boldsymbol{d}_\ell^n$ in $L^2(0,\ell)$ for $k,\ell=1,2,3$. Therefore the mappings

$$(\boldsymbol{d}_1,\boldsymbol{d}_2,\boldsymbol{d}_3)\in H^1(0,\ell;\mathbb{R}^9)\mapsto u_j=\frac{1}{2}\varepsilon_{jkl}\boldsymbol{d}_k'\cdot\boldsymbol{d}_l\in L^2(0,\ell) \qquad j,k,l=1,2,3$$

are weakly continuous.

Since the energy potential W is convex (as a function of u) and quadratic, then by integration, the mapping

$$\boldsymbol{u}\in L^2(0,\ell;\mathbb{R}^3)\mapsto\int_0^\ell W(\boldsymbol{u},s)\,ds\in\mathbb{R}$$

is convex and continuous and consequently l.s.c. for the weak topology. Using the weak continuity of the mappings

$$(\boldsymbol{d}_1,\boldsymbol{d}_2,\boldsymbol{d}_3)\in H^1(0,\ell;\mathbb{R}^9)\mapsto u\in H^1(0,\ell;\mathbb{R}^3)$$

we obtain the weak l.s.c. of the mapping

$$(\boldsymbol{d}_1,\boldsymbol{d}_2,\boldsymbol{d}_3)\in H^1(0,\ell;\mathbb{R}^9)\mapsto\int_0^\ell W(\boldsymbol{u},s)\,ds\in L^1(0,\ell).$$

Finally, the mapping

$$\boldsymbol{p}\in H^1(0,\ell;\mathbb{R}^3)\mapsto\int_0^\ell \boldsymbol{f}_3(s)\cdot\boldsymbol{K}(\boldsymbol{p})(s)\,ds\in\mathbb{R}$$

is convex and l.s.c.

In [5], it is proved that the functional $\widetilde{J}$ is sequentially weakly l.s.c. Let us prove that the penalty term has the same property.

The compact imbedding of $H^1(0,\ell;\mathbb{R}^3)$ into $\mathcal{C}^0([0,\ell];\mathbb{R}^3)$ implies that the sequences $(\boldsymbol{d}_1^n\cdot\boldsymbol{d}_2^n)$, $(|\boldsymbol{d}_1^n|^2-1)$, $(|\boldsymbol{d}_2^n|^2-1)$ and $(\boldsymbol{d}_3^n-\boldsymbol{d}_1^n\times\boldsymbol{d}_2^n)$ are weakly convergent in $L^2(0,\ell)$ for the first three ones and in $L^2(0,\ell;\mathbb{R}^3)$ for the last one. Moreover, the weak l.s.c. of the mappings :

$$(\boldsymbol{d}_1,\boldsymbol{d}_2)\in H^1(0,\ell;\mathbb{R}^6)\mapsto\int_0^\ell(\boldsymbol{d}_1\cdot\boldsymbol{d}_2)^2\,ds\in\mathbb{R},$$

$$\boldsymbol{d}_\alpha\in H^1(0,\ell;\mathbb{R}^3)\mapsto\int_0^\ell(|\boldsymbol{d}_\alpha|^2-1)^2\,ds\in\mathbb{R}\quad\text{for }\alpha=1,2,$$

$$(\boldsymbol{d}_1,\boldsymbol{d}_2,\boldsymbol{d}_3)\in H^1(0,\ell;\mathbb{R}^9)\mapsto\int_0^\ell|\boldsymbol{d}_3-\boldsymbol{d}_1\times\boldsymbol{d}_2|^2\,ds\in\mathbb{R}.$$

Therefore the functional $\widetilde{J}^\theta$ (or equivalently J^θ) is weakly l.s.c. in $H^1(0,\ell;\mathbb{R}^9)$.

The coercivity of $\widetilde{J}^\theta$ results from the fact that this one is the sum of a coercive functional $\widetilde{J}$ (See [3]) and a positive term.

In addition, we have $\widetilde{J}^\theta(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) < +\infty$. Therefore, the domain of $\widetilde{J}^\theta$ is nonempty and its definition implies that it is proper.

The weak closure of the set $\mathcal{V}_0$ is proved in [3].

Invoking the Weierstrass theorem we conclude that Problem (2.3) has at least one solution. □

We can now prove the convergence of the penalized problem.

Theorem 2.2. *There is a subsequence of $(\boldsymbol{r}^\theta, (\boldsymbol{d}_i^\theta))$ that converges to a solution $(\boldsymbol{r}, (\boldsymbol{d}_i))$ of Problem* (2.2) *when $\theta \to \infty$.*

Proof. Since $\mathcal{V} \subset \mathcal{V}_0$ we have the inequalities

$$\widetilde{J}(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_3^\theta) \le \widetilde{J}^\theta(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_3^\theta) \le \widetilde{J}(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) \qquad \text{for } (\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) \in \mathcal{V}.$$

The functional $\widetilde{J}^\theta$ is therefore uniformly bounded. The coercivity of $\widetilde{J}^\theta$ implies then that $(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_1^\theta)$ is bounded. Therefore, we can extract from this sequence a subsequence still denoted $(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_1^\theta)$ that converges weakly to a triple $(\boldsymbol{d}_1^*, \boldsymbol{d}_2^*, \boldsymbol{d}_3^*)$ in $H^1(0, \ell; \mathbb{R}^9)$. In addition, for all $\theta > 0$ we have

$$\widetilde{J}(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_3^\theta) + \frac{\theta}{2} H(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_3^\theta) \le \widetilde{J}(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) \qquad \text{for } (\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) \in \mathcal{V}.$$

Thus

$$H(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_3^\theta) \le \frac{2}{\theta} \Big(\widetilde{J}(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) - \widetilde{J}(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_3^\theta) \Big) \qquad \text{for } (\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) \in \mathcal{V}.$$

The functional $\widetilde{J}$ is weakly l.s.c. and coercive in $\mathcal{V}_0$ (cf. [5]) which is weakly bounded. Therefore, by the generalized Weierstrass theorem (cf. [1]) $\widetilde{J}$ possesses at least one minimum in $\mathcal{V}_0$. Then, there exists a real number M such that

$$H(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_3^\theta) \le \frac{2}{\theta} \Big(\widetilde{J}(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) - M \Big) \qquad \text{for } \theta > 0, \quad (\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) \in \mathcal{V}.$$

Letting $\theta \to +\infty$ we have $H(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_3^\theta) \to 0$. Moreover, H is l.s.c. (from the proof of the preceding lemma) so that

$$H(\boldsymbol{d}_1^*, \boldsymbol{d}_2^*, \boldsymbol{d}_3^*) \le \varliminf_{\theta \to \infty} H(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_3^\theta).$$

We then deduce that $H(\boldsymbol{d}_1^*, \boldsymbol{d}_2^*, \boldsymbol{d}_3^*) \le 0$. Since H is nonnegative we conclude that $H(\boldsymbol{d}_1^*, \boldsymbol{d}_2^*, \boldsymbol{d}_3^*) = 0$. Therefore $(\boldsymbol{d}_1^*, \boldsymbol{d}_2^*, \boldsymbol{d}_3^*) \in \mathcal{V}$.

Finally, since for each $(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_2) \in \mathcal{V}$ we have $\widetilde{J}(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_3^\theta) \le \widetilde{J}(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3)$, the weak l.s.c. of $\widetilde{J}$ yields

$$\widetilde{J}(\boldsymbol{d}_1^*, \boldsymbol{d}_2^*, \boldsymbol{d}_3^*) \le \varliminf_{\theta \to \infty} \widetilde{J}(\boldsymbol{d}_1^\theta, \boldsymbol{d}_2^\theta, \boldsymbol{d}_3^\theta) \le \widetilde{J}(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) \qquad \text{for } (\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) \in \mathcal{V}.$$

Therefore $(\boldsymbol{d}_1^*, \boldsymbol{d}_2^*, \boldsymbol{d}_3^*) \in \mathcal{V}$ is a solution of Problem (2.2). □

3. Frictionless contact of two rods

We consider in this section two elastic rods that may be in contact. The reference bodies of these rods are respectively denoted by Ω_1 and Ω_2 with respective boundaries Γ_1 and Γ_2. We formulate the problem as a constrained optimization one, and give a procedure to determine contact points.

To simplify the presentation, we consider two elastic rods of equal length ℓ and equal thickness ε. The motion of each rod is defined by the triples

$$\boldsymbol{r}_\alpha(s_\alpha), \boldsymbol{d}_{1\alpha}(s_\alpha), \boldsymbol{d}_{2\alpha}(s_\alpha) \qquad \alpha = 1, 2.$$

The position of a displaced point $\boldsymbol{x}_\alpha = (x^1_\alpha, x^2_\alpha, s_\alpha)$ of the rod α is given by

$$\boldsymbol{p}_\alpha(\boldsymbol{x}_\alpha) = \boldsymbol{r}_\alpha(s_\alpha) + x^1_\alpha \boldsymbol{d}_{1\alpha}(s_\alpha) + x^2_\alpha \boldsymbol{d}_{2\alpha}(s_\alpha).$$

We are interested in the frictionless contact process. For this, we shall introduce a contact distance taking advantage of the one-dimensional character of the problem. In a classical approach of the contact, we consider any point $\boldsymbol{p}_1$ of the boundary of the rod $\alpha = 1$, called *master rod* :

$$\boldsymbol{p}_1(\boldsymbol{x}_1) = \boldsymbol{r}_1(s_1) + x^1_1 \boldsymbol{d}_{11}(s_1) + x^2_1 \boldsymbol{d}_{21}(s_1),$$

with $(x^1_1(s_1))^2 + (x^2_1(s_1))^2 = \varepsilon^2$. To this point we associate at least one point in the boundary of the rod $\alpha = 2$, called *slave rod* $\boldsymbol{p}_2(\boldsymbol{x}^p_2)$ with

$$\boldsymbol{x}^p_2 = \arg\min_{\boldsymbol{x}\in\Gamma_2} |\boldsymbol{p}_2(\boldsymbol{x}) - \boldsymbol{p}_1(\boldsymbol{x}_1)|.$$

In classical modeling of contact the norm of the vector $\boldsymbol{p}_2(\boldsymbol{x}^p_2) - \boldsymbol{p}_1(\boldsymbol{x}_1)$ is called *contact distance* and its use enables prescribing a non-penetration constraint. Here, in the case of two thin rods, we wish to formulate an approximation of this constraint invoking the central lines of the rods rather than their actual boundaries. Clearly, for a small thickness ε the vector $\boldsymbol{p}_2(\boldsymbol{x}^p_2) - \boldsymbol{p}_1(\boldsymbol{x}_1)$ is close to the vector $\boldsymbol{r}_2(s^p_2) - \boldsymbol{r}_1(s_1)$. It is then natural to adopt the following approach : For each point $\boldsymbol{r}_1(s_1)$ on the central line of rod 1 we seek a point $\boldsymbol{r}_2(s^p_2)$ on the central line of rod 2 such that

$$s^p_2 := \arg\min_{s_2\in[0,\ell]} |\boldsymbol{r}_2(s^p_2) - \boldsymbol{r}_1(s_1)|.$$

We then define the signed distance :

$$\begin{aligned} d(\boldsymbol{r}_1, \boldsymbol{r}_2, s_1) &= 2\varepsilon - |\boldsymbol{r}_2(s^p_2) - \boldsymbol{r}_1(s_1)| \\ &= 2\varepsilon - \min_{s_2\in[0,\ell]} |\boldsymbol{r}_2(s_2) - \boldsymbol{r}_1(s_1)|. \end{aligned}$$

The non-penetration condition reads then :

$$d(\boldsymbol{r}_1, \boldsymbol{r}_2, s_1) \le 0 \qquad s_1 \in [0, \ell].$$

Consider now the sets :

$$\begin{aligned} \mathcal{U}_\alpha :=& \{(\boldsymbol{r}, (\boldsymbol{d}_i)) \in H^1(0, \ell; \mathbb{R}^{12});\ \boldsymbol{r}' = \boldsymbol{d}_3,\ \boldsymbol{r}(0) = a\delta_{2\alpha},\ \boldsymbol{d}_i \cdot \boldsymbol{d}_j = \delta_{ij}, \\ & \boldsymbol{d}_1 \times \boldsymbol{d}_2 \cdot \boldsymbol{d}_3 = 1,\ \boldsymbol{d}_i(0) = \boldsymbol{d}^0_i,\ \boldsymbol{d}_i(\ell) = \boldsymbol{d}^\ell_i\}, \qquad \alpha = 1, 2, \\ \mathcal{U}_c :=& \{(\boldsymbol{r}_1, (\boldsymbol{d}_{i1}); \boldsymbol{r}_2, (\boldsymbol{d}_{i2})) \in \mathcal{U}_1 \times \mathcal{U}_2;\ d(\boldsymbol{r}_1, \boldsymbol{r}_2, s_1) \le 0 \text{ for } s_1 \in [0, \ell]\}. \end{aligned}$$

The total energy functional of the problem is defined by

$$J_T(\boldsymbol{r}_1, \boldsymbol{r}_2, (\boldsymbol{d}_{i1}), (\boldsymbol{d}_{i2})) = \sum_{\beta=1}^{2} J_\beta(\boldsymbol{r}_\beta, (\boldsymbol{d}_{i\beta})),$$

where

$$J_\alpha(\boldsymbol{r},(\boldsymbol{d}_i)) = \frac{1}{2}\int_0^\ell \left(GI\left(|\boldsymbol{d}_1'|^2+|\boldsymbol{d}_2'|^2\right)+(E-G)I|\boldsymbol{r}''|^2\right)\,ds - \int_0^\ell \boldsymbol{f}_{3\alpha}\cdot\boldsymbol{r}\,ds.$$

Note that we have assumed, in order to simplify the notations, that the two rods have the same mechanical properties. The frictionless contact problem can therefore be stated as it follows :

$$\text{(3.1)}\qquad \begin{cases} \text{Find } (\boldsymbol{r}_1,(\boldsymbol{d}_{i1});\boldsymbol{r}_2,(\boldsymbol{d}_{i2}))\in\mathcal{U}_c \text{ such that} \\ J_T(\boldsymbol{r}_1,(\boldsymbol{d}_{i1});\boldsymbol{r}_2,(\boldsymbol{d}_{i2})) \le J(\boldsymbol{p}_1,(\boldsymbol{g}_{i1});\boldsymbol{p}_2,(\boldsymbol{g}_{i2})) \\ \qquad\qquad\qquad \text{for } (\boldsymbol{p}_1,(\boldsymbol{g}_{i1});\boldsymbol{p}_2,(\boldsymbol{g}_{i2}))\in\mathcal{U}_c \end{cases}$$

Theorem 3.1. *Assume that the force fields* $(\boldsymbol{f}_{3\alpha})$ *belong to the space* $L^2(0,\ell;\mathbb{R}^3)$ *then Problem* (3.1) *has at least one solution.*

Proof. By proceeding as in Theorem 2.1 we easily show that the functionals J_α, $\alpha=1,2$ are proper, weakly sequentially l.s.c. and coercive. Their sum has then the same properties. In order to use the Weierstrass theorem it remains to prove that the set $\mathcal{U}_c$ is weakly sequentially closed in the space $H^1(0,\ell;\mathbb{R}^{12})^2$.

Consider a sequence $(\boldsymbol{r}_1^n,(\boldsymbol{d}_{i1}^n);\boldsymbol{r}_2^n,(\boldsymbol{d}_{i2}^n))\in\mathcal{U}_c$ that converges weakly to $(\boldsymbol{r}_1,(\boldsymbol{d}_{i1});\boldsymbol{r}_2,(\boldsymbol{d}_{i2}))$ in $H^1(0,\ell;\mathbb{R}^{12})^2$. We have that $(\boldsymbol{r}_\alpha^n,(\boldsymbol{d}_{i\alpha}^n))\in\mathcal{U}_\alpha$ for $\alpha=1,2$. The proof of Theorem 2.1 has shown that $\mathcal{U}_\alpha$ is weakly closed in $H^1(0,\ell;\mathbb{R}^{12})$, thus $(\boldsymbol{r}_\alpha,(\boldsymbol{d}_{i\alpha}))\in\mathcal{U}_\alpha$.

Therefore, it remains to prove that $(\boldsymbol{r}_1,(\boldsymbol{d}_{i1});\boldsymbol{r}_2,(\boldsymbol{d}_{i2}))$ satisfies the non-penetration constraint. Since $[0,\ell]$ is compact, the minimum

$$\min_{s_2\in[0,\ell]}|\boldsymbol{r}_2^n(s_2)-\boldsymbol{r}_1^n(s_1)|$$

exists. Now, since the imbedding of $H^1(0,\ell;\mathbb{R}^6)$ into $\mathcal{C}^0(0,\ell];\mathbb{R}^6)$ is compact we deduce that

$$|\boldsymbol{r}_\alpha^n(s_\alpha)-\boldsymbol{r}_\alpha(s_\alpha)|\to 0 \qquad \text{for } s_1\in[0,\ell] \qquad \alpha=1,2.$$

Let us assume that

$$\text{(3.2)}\qquad \min_{s_2\in[0,\ell]}|\boldsymbol{r}_2(s_2)-\boldsymbol{r}_1(s_1)| = \lambda < 2\varepsilon.$$

We then get the existence of two integers n_1 and n_2 such that :

$$\text{for } n\ge n_1,\quad s_1\in[0,\ell]\quad |\boldsymbol{r}_1^n(s_1)-\boldsymbol{r}_1(s_1)|\le\frac{2\varepsilon-\lambda}{4},$$
$$\text{for } n\ge n_2,\quad s_2\in[0,\ell]\quad |\boldsymbol{r}_2^n(s_2)-\boldsymbol{r}_2(s_2)|\le\frac{2\varepsilon-\lambda}{4}.$$

Therefore, for $n\ge\max(n_1,n_2)$ we deduce that for all $s_1,s_2\in[0,\ell]$:

$$\begin{aligned}|\boldsymbol{r}_2(s_2)-\boldsymbol{r}_1(s_1)| &> |\boldsymbol{r}_2^n(s_2)-\boldsymbol{r}_1^n(s_1)|-|\boldsymbol{r}_2^n(s_2)-\boldsymbol{r}_2(s_2)|-|\boldsymbol{r}_1(s_1)-\boldsymbol{r}_1^n(s_1)|\\ &> \lambda.\end{aligned}$$

This is in contradiction with the assumption (3.2). Therefore

$$|\boldsymbol{r}_2(s_2)-\boldsymbol{r}_1(s_1)|\ge 2\varepsilon \qquad \text{for } s_1,s_2\in[0,\ell].$$

We conclude that the set $\mathcal{U}_c$ is weakly closed so that the Weierstrass theorem applies. □

References

[1] S. Antman, *Nonlinear Problems of Elasticity*, Springer Verlag, Applied Mathematical Sciences, New York (1995).

[2] P. Béal, *Modélisation du contact entre tiges élastiques en grands déplacements*, PhD Thesis Université Blaise Pascal (1998).

[3] J.F. Bourgat, P. Le Tallec, S. Mani, *Modélisation et calcul des grands déplacements de tuyaux élastiques en flexion-torsion*, Journal de Mécanique théorique et appliquée, Vol. 7 (1988) 1–30.

[4] E. Cosserat, F. Cosserat, *Théorie des corps déformables*, Hermann, Paris (1909).

[5] P. Le Tallec, S. Mani, F.A. Rochina, *Finite element computation of hyperelastic rods in large displacements*, Math. Mod. Num. Anal., Vol. 26 (1992) 595–625.

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