Quasistatic Frictional Contact and Wear of a Beam

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Abstract. A problem of frictional contact between an elastic beam and a moving foundation and the resulting wear of the beam is considered. The process is assumed to be quasistatic, the contact is modeled with normal compliance, and the wear is described by the Archard law. Existence and uniqueness of the weak solution for the problem is proved using the theory of strongly monotone operators and the Cauchy-Lipschitz theorem. It is also shown that growth of the the wear function is at most linear. Finally, a numerical approach to the problem is considered using a time semi-discrete scheme. The existence of the unique solution for the discretized scheme is established and error estimates on the approximate solutions are derived.

1 Introduction

Wear in mechanical systems is a major factor in their proper functioning over time. Therefore, considerable effort is being directed toward understanding, predicting and controlling the process. Most of the wear is generated by dynamic contact of parts and components; as an example, consider the wear of the car tires resulting from frictional contact with the road. The subject is very important to the automotive industry, in particular, as the warranty periods for cars grow longer and the need to guarantee good vehicle performance is directly related to wear control. Indeed, concentrated efforts take place in the design of automotive parts and components aimed at minimizing the wear.

There exists a very large volume of engineering literature dealing with various aspects of wear. However, general models of frictional contact with wear have been derived only recently from thermodynamic considerations in [22, 23]. A dynamic thermoelastic frictional contact problem with normal compliance and surface wear has been analyzed in [2], and the isothermal problem with discontinuous friction coefficient was studied in [15]. Two versions of the problem of beam wear due to the frictional contact of its end with a rigid moving foundation, one dynamic and the other quasistatic, can be found in [8]. Numerical simulations of the quasistatic problem were presented in [9]. Variational analysis of weak solutions of quasistatic viscoelastic frictional contact problems with wear can be found in [19, 20, 21].

The purpose of this study is to continue the variational and numerical analysis of problems involving frictional contact with wear but in a setting where some of the mathematical difficulties associated with two- or three-dimensions are avoided. Therefore, we consider the process of frictional contact between an elastic beam and a moving foundation. The beam is acted upon by an applied force which is assumed to change slowly in time, and as a result the accelerations in the system are small. This allows us to neglect the inertial term in the equation of motion and obtain the quasistatic approximation for the process. We assume that the foundation is flexible; therefore, we model the contact by a version of the normal compliance condition, in which the interpenetration of the beam asperities into the foundation is allowed. As a result of sliding contact the wear of the beam evolves which we model with the rate version of the Archard law.

The model consists of a nonlinear time-dependent elliptic equation for the beam displacements coupled with an ordinary differential equation for the wear. In [8, 9] the contact and wear took place at the end of the beam and therefore, entered the model as boundary conditions. Here we consider the case when the beam

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can contact the obstacle lengthwise; consequently, the contact appears in the equation of motion, modifying the beam's moment of inertia. This is the main mathematical interest in the problem, as the coefficient of the equation depends on the wear and thus on the solution. We use a variational formulation of the problem and prove the existence and uniqueness of its weak solution. Then we study the evolution of the wear function and show that its growth is at most linear in time.

Finally, we investigate a numerical approximation of the problem, using a semi-discrete time scheme. We establish the unique solvability of the semi-discrete problem and derive error estimates on the approximate solutions. There exists abundant literature on the numerical treatment of variational problems arising in mechanics, see for instance the monographs [6, 7, 10, 11, 18].

The paper is organized as follows. In Section 2 we describe the model for the process. In Section 3 we list the assumptions on the problem data, present its variational formulation and state our main existence and uniqueness result in Theorem 3.1. The proof of the theorem is given in Section 4. It is based on results from the theory of time-dependent nonlinear equations involving a strongly monotone operator and on the Cauchy-Lipschitz theorem. In Section 5 we present a boundedness result for the wear function. Finally, in Section 6 we consider numerical approximations for the problem.

2 The model

In this section we construct a model for the process of quasistatic frictional contact between an elastic beam and a moving obstacle or foundation. The wear of the beam which results from the contact is taken into account. Thus, the cross section of the beam changes over time, which, in turn, affects the contact process. This makes the problem highly nonlinear. We assume that the forces acting on the system change slowly and, therefore, the evolution is slow which justifies the assumption that the process is quasistatic.

The physical setting and the process are as follows. An elastic beam of length L is clamped at its left end and the right end is free. The beam is acted upon by an applied force of (linear) density $f = f(x,t)$ which is directed downward, $f \leq 0$, where x is the spatial variable and t is time. Let $g = g(x) \leq 0$ denote the gap between the beam in its reference configuration $[0, L]$ and the moving surface S. We may consider the motion either in the x direction, say a moving conveyor belt; or into the plane. Let $v^* = v^*(t)$ denote the velocity of the foundation, which may change over time. The beam comes into contact with S only when the vertical displacement exceeds g and then the contact process is accompanied by wear. The physical setting is depicted in Fig. 1.

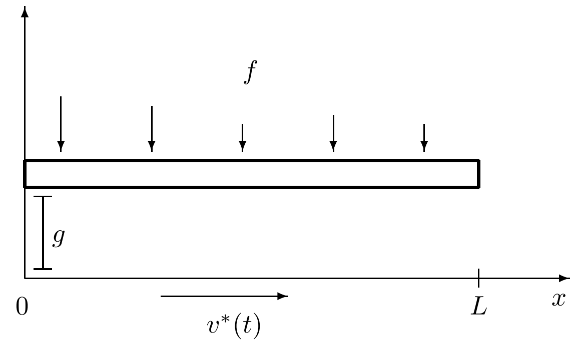

Fig. 1. The setting of the problem.

We let $T > 0$ and denote the time interval of interest by $[0, T]$, and let $\Omega_T = (0, L) \times (0, T)$. For $(x, t) \in \overline{\Omega}_T$, $u = u(x, t)$ represents the vertical displacement of the beam. When the meaning is clear, we do not indicate explicitly the dependence on x or t.

Let $A = EI$, where I is the beam's moment of inertia and E its Young modulus, and denote by $L_A(u)$ the function given by

$$L_A(u) = \frac{\partial^2}{\partial x^2}\Big(A\frac{\partial^2 u}{\partial x^2}\Big). \tag{2.1}$$

When $u > g$ there is no contact between the beam and the foundation, and we have $L_A(u) = f$, which is the classical equilibrium equation of the beam. Therefore,

$$u > g \quad \Longrightarrow \quad L_A(u) = f. \tag{2.2}$$

When $u \le g$, there is contact between the beam and the foundation which reacts with normal force ξ directed upward, $\xi \ge 0$. The equilibrium equation now is $L_A(u) = f + \xi$. We assume that the reaction ξ depends on the penetration, i.e., $\xi = p(g - u)$ where p is a given nonnegative function. This assumption represents a version of the so-called *normal compliance contact condition* used in, *e.g.*, [14, 15, 17, 20] (see also reference therein). Thus,

$$u \le g \quad \Longrightarrow \quad L_A(u) = f + p(g - u). \tag{2.3}$$

Conditions (2.2) and (2.3) may be restated in the form

$$L_A(u) = f + p(g - u) \qquad \text{in} \quad \Omega_T, \tag{2.4}$$

if we assume that $p(r) = 0$ for $r \le 0$.

This is a common formulation of a unilateral contact or obstacle problem, and leads to a standard variational inequality. The nonstandard features in our problem are related to the inclusion of the beam's wear which we now model.

Let $w = w(x, t)$, for $(x, t) \in \overline{\Omega}_T$, be the wear function, which represents the change in the beam's cross section as a result of material removal due to friction. We assume that the evolution of the surface wear is governed by a version of the Archard law (see, *e.g.* [2, 22, 23] and references therein) which we now describe.

In the region where there is no contact between the beam and the foundation the wear does not change, thus,

$$u > g \quad \Longrightarrow \quad \dot{w} = 0, \tag{2.5}$$

where here and below a dot above a variable represents the time derivative. In the region where there is contact, the rate of wear is proportional to the velocity of the foundation and to the reaction contact force $\xi = p(g - u)$. Therefore,

$$u \leq g \quad \Longrightarrow \quad \dot{w} = \alpha |v^*| p(g - u), \tag{2.6}$$

were $\alpha = \alpha(x) \geq 0$ is the wear coefficient (very small in practice). Since the function p vanishes for negative arguments, an equivalent form of the wear conditions (2.5) and (2.6) is

$$\dot{w} = \alpha |v^*| p(g - u) \qquad \text{in} \quad \Omega_T. \tag{2.7}$$

Now, $A = EI$ depends on the cross section of the beam which is decreasing in time as it wears out. Therefore, we have $A = A(w)$, where $A(\cdot)$ is a real valued function that represents the way the change in the beam's geometry, due to material removal, changes both E and I.

We note that because of the contribution of the wear to the change of the beam's cross-section we should have used $g - u - w$ as the argument of p. But, w is much smaller than u or g and in this work we neglect it and use the version above. On the other hand, if one considers the grinding process (see, *e.g.*, [3]), then w is comparable to u and g and has to be included.

To complete the model we need to prescribe appropriate initial and boundary conditions. The initial condition takes the form

$$w(x, 0) = w_0(x) \qquad \text{for} \quad x \in [0, L], \tag{2.8}$$

where w_0 represents the initial wear. Obviously $w_0 = 0$ in a new beam.

The beam is rigidly attached at its left end, thus,

$$u(0, t) = u_x(0, t) = 0 \qquad \text{for} \quad t \in [0, T]. \tag{2.9}$$

Here and below the subscripts x, xx and xxx denote the first, second and third partial x derivatives, respectively.

Finally, we assume that no moments act on the free end of the beam; thus,

$$u_{xx}(L, t) = u_{xxx}(L, t) = 0 \qquad \text{for} \quad t \in [0, T]. \tag{2.10}$$

The classical statement of the problem of *quasistatic frictional contact and wear of a beam* is the following.

Problem P. Find a displacement function $u : \overline{\Omega}_T \to$
mathbbR and a wear function $w : \overline{\Omega}_T \to \mathbb{R}$ *such that* (2.4), (2.7)–(2.10) *hold.*

We deal with a contact or an obstacle problem, and it is well known that there exists a regularity ceiling for the solutions which, generally, prevents them from having all the classical derivatives. Therefore, we proceed to derive a weak or variational formulation of the problem.

3 Variational formulation and statement of results

We restate the problem in a variational form, list the assumptions on the data and state our main existence and uniqueness result.

We start with formal calculus which is made under the assumption that all the involved functions are sufficiently regular. We proceed in a standard way which may be found, *e.g.*, in [5] or [13]. Let $v = v(x)$ be a test function such that $v(0) = v_x(0) = 0$ and let $t \in [0,T]$. Using (2.4) and the fact that $A = A(w)$ we obtain

$$\int_0^L L_{A(w(t))}(u(t))v\,dx = \int_0^L f(t)v\,dx + \int_0^L p(g-u(t))v\,dx. \tag{3.1}$$

Using now (2.1), performing two integrations by parts and keeping in mind the boundary conditions (2.9) and (2.10) we obtain

$$\int_0^L L_{A(w(t))}(u(t))v\,dx = \int_0^L A(w(t))u_{xx}(t)v_{xx}\,dx. \tag{3.2}$$

Thus, (3.1) and (3.2) yield

$$\int_0^L A(w(t))u_{xx}(t)v_{xx}\,dx = \int_0^L f(t)v\,dx + \int_0^L p(g-u(t))v\,dx. \tag{3.3}$$

This equality is the basis for the variational formulation of the mechanical Problem P.

We need the following functional notation. We use standard notation for L^p and Sobolev spaces (see, *e.g.*, [1, 12, 16]) and introduce the closed subspace of $H^2(0,L)$ given by

$$V = \{\ v \in H^2(0,L) \mid v(0) = v_x(0) = 0\ \}. \tag{3.4}$$

In addition, if $(X, |\cdot|_X)$ is a real normed space, we denote by $C(0,T;X)$ and $C^1(0,T;X)$ the spaces of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with norms

$$|u|_{C(0,T;X)} = \max_{t\in[0,T]} |u(t)|_X, \quad |u|_{C^1(0,T;V)} = \max_{t\in[0,T]} |u(t)|_X + \max_{t\in[0,T]} |\dot{u}(t)|_X.$$

In our study of Problem P we assume the following on the data:

$$f \in C(0,T;L^2(0,L)), \tag{3.5}$$

$$g \in L^\infty(0,L), \qquad g \le 0 \quad \text{a.e. on} \quad [0,L], \tag{3.6}$$

$$\alpha \in L^\infty(0,L), \qquad \alpha \ge 0 \quad \text{a.e. on} \quad [0,L], \tag{3.7}$$

$$v^* \in C(0,T), \tag{3.8}$$

$$w_0 \in L^\infty(0,L), \qquad w_0 \ge 0 \quad \text{a.e. on} \quad [0,L]. \tag{3.9}$$

We also assume that the function $A : [0,L] \times \mathbb{R} \to \mathbb{R}$ satisfies the properties:

$$\left\{\begin{array}{l} \text{(a) There exists an } M > 0 \text{ such that} \\ \qquad |A(x,w_1) - A(x,w_2)| \le M\,|w_1 - w_2| \quad \forall\, w_1,\, w_2 \in \mathbb{R},\ x \in [0,L]. \\ \text{(b) The mapping } x \mapsto A(x,w) \text{ is Lebesgue measurable on } [0,L] \\ \qquad \text{for any } w \in \mathbb{R}. \\ \text{(c) There exist } m_1 > 0 \text{ and } m_2 > 0 \text{ such that} \\ \qquad m_1 \le A(x,w) \le m_2 \quad \forall\, w \in \mathbb{R}, \quad x \in [0,L]. \end{array}\right. \tag{3.10}$$

Here we assume that $m_1 \leq A$ since our interest lies in the wear of the beam. In a real system when the material removal reaches the state $A = 0$ at a point the cross-section vanishes and the beam will break. However, from the applied point of view our model for the beam will break down long before such a state is reached because the assumptions which underlie the Euler-Bernoulli equation loose their validity. Condition (2.7) implies that the function $w(x, \cdot)$ is non-decreasing and then the assumption $A \geq m_1$ is fulfilled when A is a decreasing function of w, $A(\cdot, w_0(\cdot)) > m_1$ on $[0, L]$, and T is sufficiently small.

The normal compliance function $p : [0, L] \to \mathbb{R}$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists an } L_p > 0 \text{ such that} \\ \qquad |p(x, r_1) - p(x, r_2)| \leq L_p\,|r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R},\, x \in [0, L]. \\ \text{(b) } (p(x, r_1) - p(x, r_2)) \cdot (r_1 - r_2) \geq 0 \quad \forall\, r_1,\, r_2 \in \mathbb{R},\, x \in [0, L]. \\ \text{(c) The mapping } x \mapsto p(x, r) \text{ is Lebesgue measurable on } [0, L] \\ \qquad \text{for any } r \in \mathbb{R}. \\ \text{(d) } p(x, r) = 0 \quad \forall\, r \leq 0, \quad x \in [0, L]. \end{array}\right. \tag{3.11}$$

We observe that the assumptions (3.11) on $p(x, \cdot)$ are fairly general. The only restriction is in condition (a) which requires that asymptotically the function grows at most linearly. We need this condition in Section 4, in order to apply the Cauchy-Lipschitz theorem. From the mechanical point of view, conditions (b) and (d) express the fact that at each point of the beam, the reaction force increases with the penetration and vanishes when there is no contact with the foundation. We remark that assumptions (3.11) are satisfied for the function p given by

$$p(x, r) = \beta(x) r_+ \tag{3.12}$$

where $\beta \in L^\infty(0, L)$, $\beta \geq 0$ a.e. on $[0, L]$, and $r_+ = \max\{0, r\}$. In this case the upward reaction force is proportional to the penetration, the proportionality coefficient β is a large positive material constant related to the hardness of the foundation. A power law was used in [14, 17] and more general expressions in [2, 15]. However, from the applications point of view the behavior of p at infinity is of little interest, because the model is not valid for large penetrations.

Keeping in mind (3.3), we denote by $a(w; \cdot, \cdot)$ the bilinear form on V given by

$$a(w; u, v) = \int_0^L A(w) u_{xx} v_{xx} dx \qquad \forall u,\ v \in V \tag{3.13}$$

for all $w \in L^\infty(0, L)$, $w \geq 0$. By condition (3.10) the function $A(w)$ lies in $L^\infty(0, L)$, and the integral in (3.13) is well defined, for all $w \in L^\infty(0, L)$.

We define the functional $j : V \times V \to \mathbb{R}$ by

$$j(u, v) = -\int_0^L p(g - u) v\, dx \qquad \forall u,\ v \in V, \tag{3.14}$$

which is well defined by (3.6) and (3.11).

Using now (3.3), (3.4), (3.13) and (3.14) we obtain that if $\{u, w\}$ is a sufficiently regular solution of Problem P then $u(t) \in V$ and satisfies

$$a(w(t); u(t), v) + j(u(t), v) = \langle f(t), v) \rangle_{L^2(0,L)} \qquad \forall v \in V, \tag{3.15}$$

$$\dot{w}(t) = \alpha |v^*(t)| p(g - u(t)), \tag{3.16}$$

for all $t \in [0, T]$, and

$$w(0) = w_0. \tag{3.17}$$

This leads us to the following variational formulation of Problem P.

Problem P_V. Find the displacement function $u : [0,T] \to V$ and the wear function $w : [0,T] \to L^\infty(0,L)$ such that (3.15)–(3.17) *are satisfied.*

Our main result, which we establish in the next section, is the following.

Theorem 3.1. *Assume that* (3.5)–(3.11) *hold. Then there exists a unique solution $\{u, w\}$ of problem P_V. Moreover, the solution satisfies*

$$u \in C(0,T;V), \qquad w \in C^1(0,T;L^\infty(0,L)). \tag{3.18}$$

We conclude that, under the assumptions (3.5)–(3.11), the mechanical problem (2.4), (2.7)–(2.10) has a unique weak solution $\{u, w\}$.

4 Proof of the Theorem

The proof of Theorem 3.1 is based on arguments from the theory of time-dependent nonlinear equations with strongly monotone operators and the classical Cauchy-Lipschitz theorem. It will be carried out in several steps. To simplify the notation we shall not indicate explicitly the dependence of various variables on t. In this section C will denote a strictly positive generic constant which may depend on L, A, T, p and on the input data f, g, α, v^*, but does not depend on time t, on the wear w or on the initial data w_0 and whose value may change from line to line.

We start by defining an appropriate inner product on V. To this end we observe that there exists $C > 0$ such that $C|v|_{L^2(0,L)} \le |v_x|_{L^2(0,L)}$ for all $v \in H^1(0,L)$ satisfying $v(0) = 0$; thus,

$$C|v|_{H^2(0,L)} \le |v_{xx}|_{L^2(0,L)} \qquad \forall v \in V. \tag{4.1}$$

We use the following inner product on V

$$\langle u, v\rangle_V = \langle u_{xx}, v_{xx}\rangle_{L^2(0,L)} \tag{4.2}$$

and let $|\cdot|_V$ be the associated norm. By using (4.1) we find that $|\cdot|_{H^2(0,L)}$ and $|\cdot|_V$ are equivalent norms on V and, therefore, $(V, \langle\cdot,\cdot\rangle_V)$ is a real Hilbert space.

In the first step of the proof we assume that the wear on the contact surface is known and we focus our attention on the corresponding contact problem. More precisely, we solve the variational equation (3.15) when w and t are fixed.

Lemma 4.1. *Let $t \in [0,T]$ and $w \in L^\infty(0,L)$. Then there exists a unique function $u \in V$ such that*

$$a(w;u,v) + j(u,v) = \langle f(t), v\rangle_{L^2(0,L)} \qquad \forall v \in V. \tag{4.3}$$

Moreover,

$$|u|_V \le C. \tag{4.4}$$

Proof. Using (3.10) and (4.2) we find that $a(w;\cdot,\cdot)$ is a bilinear continuous and coercive form on V, that is

$$|a(w;u,v)| \le C|u|_V|v|_V \qquad \forall u, v \in V, \tag{4.5}$$

$$a(w;v,v) \geq C|v|_V^2 \qquad \forall v \in V. \tag{4.6}$$

Moreover, it follows from (3.11) that

$$j(u_1, u_1 - u_2) - j(u_2, u_1 - u_2) \geq 0 \qquad \forall u_1,\, u_2 \in V, \tag{4.7}$$

$$j(u_1, v) - j(u_2, v) \leq C|u_1 - u_2|_V |v|_V \qquad \forall u_1,\, u_2,\, v \in V. \tag{4.8}$$

Using now the Riesz's representation theorem we define the operator $B(w;\cdot) : V \to V$ by

$$\langle B(w;z), v\rangle_V = a(w;z,v) + j(z,v) \qquad \forall v,\, z \in V. \tag{4.9}$$

Let $u_1,\, u_2 \in V$, then

$$\langle B(w;u_1) - B(w;u_2), u_1 - u_2\rangle_V = a(w;u_1 - u_2, u_1 - u_2) + j(u_1, u_1 - u_2) - j(u_2, u_1 - u_2),$$

and using (4.6), (4.7) we deduce

$$\langle B(w;u_1) - B(w;u_2), u_1 - u_2\rangle_V \geq C|u_1 - u_2|_V^2. \tag{4.10}$$

Using (4.9) again we get

$$\langle B(w;u_1) - B(w;u_2), v\rangle_V = a(w;u_1 - u_2, v) + j(u_1, v) - j(u_2, v)$$

for all $v \in V$ and, keeping in mind (4.5) and (4.8), we obtain

$$\langle B(w;u_1) - B(w;u_2), v\rangle_V \leq C|u_1 - u_2|_V |v|_V \qquad \forall v \in V,$$

thus,

$$|B(w;u_1) - B(w;u_2)|_V \leq C|u_1 - u_2|_V. \tag{4.11}$$

The inequalities (4.10) and (4.11) show that $B(w;\cdot)$ is a strongly monotone Lipschitz continuous operator on V. The existence and uniqueness part in Lemma 4.1 follows now from standard results for nonlinear equations (see, *e.g.*, [4], Corollary 15). Choosing $v = u$ in (4.3) we find

$$a(w;u,u) + j(u,u) = \langle f(t), u\rangle_{L^2(0,L)},$$

and since (3.11) and (3.6) imply that $j(u,u) \geq 0$, we obtain

$$a(w;u,u) \leq \langle f(t), u\rangle_{L^2(0,L)}. \tag{4.12}$$

Inequality (4.4) is now a consequence of (3.5), (4.6) and (4.12).

We denote by $u(t,w)$ the element $u \in V$ which solves (4.3) for all $t \in [0,T]$ and $w \in L^\infty(0,L)$. The dependence of this element on t and w is described in the following result.

Lemma 4.2. *Let $t_i \in [0,T]$, $w_i \in L^\infty(0,L)$ and let $u_i = u(t_i, w_i)$, $i = 1,2$. Then there exists a constant $C > 0$ such that*

$$C|u_1 - u_2|_V \leq |f(t_1) - f(t_2)|_{L^2(0,L)} + |w_1 - w_2|_{L^\infty(0,L)}. \tag{4.13}$$

Proof. Using (4.3) we find

$$a(w_1; u_1, v) + j(u_1, v) = \langle f(t_1), v\rangle_{L^2(0,L)},$$
$$a(w_2; u_2, v) + j(u_2, v) = \langle f(t_2), v\rangle_{L^2(0,L)},$$

for all $v \in V$. We choose $v = u_1 - u_2$ in the first equality, $v = u_2 - u_1$ in the second, add them up, use (4.7) and obtain

$$a(w_1; u_1, u_1 - u_2) + a(w_2; u_2, u_2 - u_1) \leq \langle f(t_1) - f(t_2), u_1 - u_2\rangle_{L^2(0,L)},$$

and thus,

$$\begin{aligned} a(w_1; u_1 - u_2, u_1 - u_2) \leq\ & a(w_2; u_2, u_1 - u_2) - a(w_1; u_2, u_1 - u_2) \\ & + \langle f(t_1) - f(t_2), u_1 - u_2\rangle_{L^2(0,L)}. \end{aligned} \tag{4.14}$$

Moreover, using (3.10) and (4.4) we deduce

$$a(w_2; u_2, u_1 - u_2) - a(w_1; u_2, u_1 - u_2) \leq C|w_1 - w_2|_{L^\infty(0,L)}|u_1 - u_2|_V, \tag{4.15}$$

and, clearly,

$$\langle f(t_1) - f(t_2), u_1 - u_2\rangle_{L^2(0,L)} \leq |f(t_1) - f(t_2)|_{L^2(0,L)}|u_1 - u_2|_V. \tag{4.16}$$

The inequality (4.13) is now a consequence of (4.14)–(4.16) and (4.6). □

We denote by $F(t, w)$ the function

$$F(t, w) = \alpha|v^*(t)|p(g - u(t, w)) \qquad \text{a.e. on} \quad [0, L] \tag{4.17}$$

for all $t \in [0, T]$ and $w \in L^\infty(0, L)$. Using a Sobolev embedding theorem (see, *e.g.*, [1]), we obtain that $H^2(0, L) \subset L^\infty(0, L)$ with continuous embedding. Thus,

$$|v|_{L^\infty(0,L)} \leq C|v|_V \qquad \forall v \in V, \tag{4.18}$$

and moreover, $u(t, w) \in L^\infty(0, L)$ since $u(t, w) \in V$. Therefore, keeping in mind (3.11) and the assumptions (3.6)–(3.8) on the data, we find that $F : [0, T] \times L^\infty(0, L) \to L^\infty(0, L)$. In addition, we have the following result.

Lemma 4.3. *The operator* $F : [0, T] \times L^\infty(0, L) \to L^\infty(0, L)$ *defined by* (4.17) *is continuous and there exists a constant* $C > 0$ *such that*

$$|F(t, w_1) - F(t, w_2)|_{L^\infty(0,L)} \leq C|w_1 - w_2|_{L^\infty(0,L)} \tag{4.19}$$

for all $w_1,\ w_2 \in L^\infty(0, L), \quad t \in [0, T]$.

Proof. Let $t_i \in [0, T]$, $w_i \in L^\infty(0, L)$ and denote $u_i = u(t_i, w_i)$, $i = 1, 2$. Using (4.17), (3.7) and (3.11) we obtain

$$\begin{aligned} |F(t_1, w_1) - F(t_2, w_2)|_{L^\infty(0,L)} &\leq C\Big||v^*(t_1)|p(g - u_1) - |v^*(t_2)|p(g - u_2)\Big|_{L^\infty(0,L)} \\ \leq C\Big||v^*(t_1)|p(g - u_1) - |v^*(t_2)|p(g - u_1)\Big|_{L^\infty(0,L)} &+ C|v^*(t_2)|\Big|p(g - u_1) - p(g - u_2)\Big|_{L^\infty(0,L)} \\ \leq C|v^*(t_1) - v^*(t_2)|\ |g - u_1|_{L^\infty(0,L)} &+ C|v^*(t_2)|\ |u_1 - u_2|_{L^\infty(0,L)}. \end{aligned}$$

Using now (4.18), (4.4) and (3.8) we get

$$|F(t_1, w_1) - F(t_2, w_2)|_{L^\infty(0,L)} \leq C|v^*(t_1) - v^*(t_2)| + C|u_1 - u_2|_V. \tag{4.20}$$

Combining now (4.20) and (4.13) we find

$$|F(t_1, w_1) - F(t_2, w_2)|_{L^\infty(0,L)} \tag{4.21}$$
$$\le C\Big(|v^*(t_1) - v^*(t_2)| + |f(t_1) - f(t_2)|_{L^2(0,L)} + |w_1 - w_2|_{L^\infty(0,L)}\Big).$$

The continuity of $F : [0, T] \times L^\infty(0, L) \to L^\infty(0, L)$ results now from (4.21), (3.5) and (3.8). Moreover, by choosing $t_1 = t_2$ in (4.21) we obtain (4.19). □

We have now all the ingredients needed to prove the theorem.

Proof of Theorem 3.1. We start by considering the Cauchy problem

$$\dot{w}(t) = F(t, w(t)) \qquad \forall t \in [0, T], \tag{4.22}$$

$$w(0) = w_0, \tag{4.23}$$

where F is defined by (4.17). Lemma 4.3 and condition (3.9) allow us to apply the well-known Cauchy-Lipschitz theorem. It follows that there exists a unique function $w \in C^1(0, T, L^\infty(0, L))$ which satisfies (4.22) and (4.23). We denote by $u : [0, T] \to V$ the function given by

$$u(t) = u(t, w(t)) \qquad \forall t \in [0, T]. \tag{4.24}$$

Let $t_i \in [0, T]$ and let $w_i = w(t_i)$, $i = 1, 2$. It follows from (4.24) that $u(t_i) = u(t_i, w_i)$ and therefore (4.13) implies

$$C|u(t_1) - u(t_2)|_V \le |f(t_1) - f(t_2)|_{L^2(0,L)}| + |w_1 - w_2|_{L^\infty(0,L)}.$$

Now, $f \in C(0, T; L^2(0, L))$ and $w \in C^1(0, T; L^\infty(0, L))$, thus, $u \in C(0, T; V)$.

Using (4.3) and (4.24) we obtain that the pair $\{u, w\}$ satisfies (3.15) for all $t \in [0, T]$. Moreover, it follows from (4.17), (4.22) and (4.24) that (3.16) holds, for all $t \in [0, T]$. Finally, the initial condition (3.17) is satisfied, keeping in mind (4.23). Therefore, $\{u, w\}$ is a solution of Problem P_V such that (3.18) holds. This concludes the existence part in Theorem 3.1. The uniqueness part is a consequence of the uniqueness of the solution for the Cauchy problem (4.22) and (4.23). □

5 Evolution of wear

In this section we study the evolution of the wear and establish the following result.

Theorem 5.1. *Assume that* (3.5)–(3.11) *hold and let* $\{u, w\}$ *denote the solution of Problem* P_V *given by Theorem* 3.1. *Then, there exists a positive constant* C *which depends on* L, A, T, p *and on the data* f, g, α *and* v^* *but does not depend on time or* w_0, *such that*

$$|w(t)|_{L^\infty(0,L)} \le |w_0|_{L^\infty(0,L)} + Ct \qquad t \in [0, T]. \tag{5.1}$$

This result has important consequences in applications. Indeed, when in a given setting it is known that the beam will break or will need to be replaced when the wear reaches the limit w^*, then we can estimate the time interval over which it may be used safely. Assume that the model is valid as long as

$$|w(t)|_{L^\infty(0,L)} < w^*, \tag{5.2}$$

where $w^* > 0$ is a given wear limit. Assume also that initially $|w_0|_{L^\infty(0,L)} < w^*$. Using (5.1) we find that condition (5.2) is satisfied for all $t \in [0, T_0)$ where

$$T_0 = \frac{1}{C}(w^* - |w_0|_{L^\infty(0,L)}). \tag{5.3}$$

Thus, (5.3) provides an estimate of the time interval for which the solution of the Problem P_V represents a reasonable behavior of the mechanical Problem P. We conclude from Theorem 3.1 that in this case problem P has a unique weak solution on the time interval $[0, T_0)$, which represents a local existence and uniqueness result. Equality (5.3) also shows, which is to be expected, that the time interval where the local weak solution exists is maximal when $w_0 = 0$, i.e., for a new beam.

Proof. Let $t \in [0, T]$. Using (3.16) and (3.17) we obtain

$$w(t) = w_0 + \int_0^t \alpha |v^*(s)| p(g - u(s)) ds \qquad \text{a.e. on } [0, L],$$

and then (3.11) and (3.6)–(3.8) imply

$$|w(t)|_{L^\infty(0,L)} \leq |w_0|_{L^\infty(0,L)} + C \int_0^t \Big(|g|_{L^\infty(0,L)} + |u(s)|_{L^\infty(0,L)} \Big) ds. \tag{5.4}$$

Using now (4.4) and (4.18) we obtain

$$|u(s)|_{L^\infty(0,L)} \leq C|u(s)|_V \leq C \qquad \forall s \in [0, T]. \tag{5.5}$$

The inequality (5.1) is now a consequence of (5.4) and (5.5). □

6 Semi-discrete time approximation

In this section we present and analyze a semi-discrete time approximation scheme for numerical solutions of the contact problem P_V. To this end we consider a partition of the time interval $[0, T]$:

$$0 = t_0 < t_1 < \cdots < t_N = T.$$

We denote the step-size by $k_n = t_n - t_{n-1}$ for $n = 1, \ldots, N$. We allow a non-uniform partition of the time interval, and let

$$k = \max_{1 \leq n \leq N} k_n$$

denote the maximal step-size. For a continuous function $t \mapsto \eta(t)$, we use the notation $\eta_n = \eta(t_n)$. No summation is implied over the repeated index n.

We assume that conditions (3.5)–(3.11) hold and C denotes a strictly positive constant which may depend on the solution and on the data but is independent of the semi-discretization parameter k and whose value may change from line to line.

We use the following semi-discrete approximation based on a forward Euler scheme of the variational Problem P_V.

Problem P_V^k. Find the displacement field $u^k = \{u_n^k\}_{n=0}^N \subset V$ and the wear function $w^k = \{w_n^k\}_{n=0}^N \subset L^\infty(0, L)$ such that

$$w_0^k = w_0, \tag{6.1}$$

and for $n = 1, \dots, N$,

$$a(w_n^k; u_n^k, v) + j(u_n^k, v) = \langle f_n, v \rangle_{L^2(0,L)} \qquad \forall v \in V, \tag{6.2}$$

$$\frac{w_{n+1}^k - w_n^k}{k_n} = \alpha |v_n^*| p(g - u_n^k). \tag{6.3}$$

We next show the unique solvability of Problem P_V^k. Using arguments similar to those in Lemma 4.1 we conclude that for each n the nonlinear variational equation (6.2) has a unique solution $u_n^k \in V$, when $w_n^k \in V$ is known, and then w_{n+1}^k is found from (6.3). Therefore, the problem P_V^k has a unique solution for each fixed k.

Next, we establish error estimates on the approximate solutions, under an additional regularity assumption on the solution of P_V.

Theorem 6.1. *Let $\{u, w\}$ be the solution of the problem P_V, and let $\{u_n^k, w_n^k\}_{n=0}^N$ be the solution of the time semi-discrete problem P_V^k. Assume $w \in W^{2,\infty}(0, T, L^\infty(0, L))$. Then the error estimate*

$$\max_{0 \le n \le N} \Big(|u_n - u_n^k|_V + |w_n - w_n^k|_{L^\infty(0,L)} \Big) \le Ck \tag{6.4}$$

holds.

Inequality (6.4) forms the basis for convergence and error analysis of the time semi-discrete solutions.

Proof. Let $0 \le n \le N - 1$. Using Taylor's formula we have

$$w_{n+1} = w_n + k_n \dot{w}_n + \int_{t_n}^{t_{n+1}} (t^{n+1} - s) \ddot{w}(s) ds,$$

where $\ddot{w}$ denotes the second time derivative of w. Using now (3.16) at $t = t_n$ we obtain

$$\dot{w}_n = \alpha |v_n^*| p(g - u_n),$$

and thus,

$$w_{n+1} = w_n + k_n \alpha |v_n^*| p(g - u_n) + \int_{t_n}^{t_{n+1}} (t^{n+1} - s) \ddot{w}(s)\, ds. \tag{6.5}$$

Moreover, we deduce from (6.3) that

$$w_{n+1}^k = w_n^k + k_n \alpha |v_n^*| p(g - u_n^k). \tag{6.6}$$

Subtracting (6.6) from (6.5) we obtain

$$w_{n+1} - w_{n+1}^k = w_n - w_n^k + k_n \alpha |v_n^*| (p(g - u_n) - p(g - u_n^k)) + \int_{t_n}^{t_{n+1}} (t^{n+1} - s) \ddot{w}(s)\, ds.$$

Using now (3.7), (3.8) and (3.11) we find

$$\begin{aligned} |w_{n+1} - w_{n+1}^k|_{L^\infty(0,L)} \le\ & |w_n - w_n^k|_{L^\infty(0,L)} + k_n C |u_n - u_n^k|_{L^\infty(0,L)} \\ & + \int_{t_n}^{t_{n+1}} (t^{n+1} - s) |\ddot{w}(s)|_{L^\infty(0,L)}\, ds. \end{aligned} \tag{6.7}$$

Keeping in mind (6.2) and (3.15) at $t = t_n$, we deduce from Lemma 4.2 that

$$|u_n - u_n^k|_V \le C|w_n - w_n^k|_{L^\infty(0,L)}, \tag{6.8}$$

and, using (4.18), we obtain

$$|u_n - u_n^k|_{L^\infty(0,L)} \le C|w_n - w_n^k|_{L^\infty(0,L)}. \tag{6.9}$$

Moreover,

$$\int_{t_n}^{t_{n+1}} (t^{n+1} - s)|\ddot{w}(s)|_{L^\infty(0,L)}\, ds \le \frac{1}{2}k_n^2|\ddot{w}(s)|_{L^\infty(\Omega_T)},$$

which implies

$$\int_{t_n}^{t_{n+1}} (t^{n+1} - s)|\ddot{w}(s)|_{L^\infty(0,L)}\, ds \le k_n^2 C. \tag{6.10}$$

We substitute (6.9) and (6.10) in (6.7) and obtain

$$|w_{n+1} - w_{n+1}^k|_{L^\infty(0,L)} \le (1 + k_n C)|w_n - w_n^k|_{L^\infty(0,L)} + k_n^2 C. \tag{6.11}$$

Using now the inequality $1 + x \le e^x$, which holds for all $x \ge 0$, and by a simple induction argument in (6.11) we find

$$|w_n - w_n^k|_{L^\infty(0,L)} \le |w_0 - w_0^k|_{L^\infty(0,L)} e^{C(t_n - t_0)} + C\sum_{i=0}^{n-1} k_n^2 e^{C(t_n - t_{i+1})} \tag{6.12}$$

for all $n = 1, \ldots, N$. Since

$$\sum_{i=0}^{n-1} k_n^2 e^{C(t_n - t_{i+1})} \le \sum_{i=0}^{n-1} k_n^2 e^{CT} \le Ck,$$

from (6.1) and (6.12) we obtain

$$|w_n - w_n^k|_{L^\infty(0,L)} \le Ck \qquad \forall n = 0, \ldots, N. \tag{6.13}$$

Using now (6.8) and (6.13) it follows that

$$|u_n - u_n^k|_V \le Ck \qquad \forall n = 0, \ldots, N. \tag{6.14}$$

Inequality (6.4) is now a consequence of (6.13) and (6.14). □

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] K. T. Andrews, A. Klarbring, M. Shillor and S. Wright, "A dynamic contact problem with friction and wear," *Int. J. Engng. Sci.* **35**(14)(1997), 1291–1309.

[3] K. T. Andrews, M. Shillor and S. Wright, "A dynamic thermoviscoelastic model for grinding," preprint, 1998.

[4] H. Brezis, "Equations et inéquations non linéaires dans les espaces vectoriels en dualité," *Ann. Inst. Fourier* **18** (1968), 115–175.

[5] G. Duvaut and J. L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[6] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.

[7] R. Glowinski, J.-L. Lions and R. Trémolières, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981.

[8] R. J. Gu, K. L. Kuttler and M. Shillor, "Frictional wear of a thermoelastic beam," preprint, 1998.

[9] R. J. Gu and M. Shillor, "Thermal and wear analysis of an elastic beam in sliding contact," preprint, 1998.

[10] W. Han and B.D. Reddy, "Computational plasticity: the variational basis and numerical analysis," *Computational Mechanics Advances* **2** (1995), 283–400.

[11] I. Hlaváček, J. Haslinger, J. Necǎs and J. Lovíšek, *Solution of Variational Inequalities in Mechanics*, Springer-Verlag, New York, 1988.

[12] I. R. Ionescu and M. Sofonea, *Functional and Numerical Methods in Viscoplasticity*, Oxford University Press, Oxford, 1993.

[13] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[14] A. Klarbring, A. Mikelić and M. Shillor, "Frictional contact problems with normal compliance," *Int. J. Engng. Sci.* **26** (1988), 811–832.

[15] K. L. Kuttler and M. Shillor, "Dynamic contact with normal compliance wear and discontinuous friction coefficient," preprint 1998.

[16] J. L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems and Applications*, Vol II, Springer-Verlag, Berlin, 1972.

[17] J. T. Martins and J. T. Oden, "Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws," *Nonlin. Anal.* **11** (1987), 407–428.

[18] J. T. Oden and N. Kikuchi, *Contact Problems in Elasticity*, SIAM, Philadelphia, 1988.

[19] M. Rochdi, M. Shillor and M. Sofonea, "A quasistatic contact problem with directional friction and damped response," *Applicable Analysis* **68** (1998), 409-422.

[20] M. Rochdi, M. Shillor and M. Sofonea, "A quasistatic viscoelastic contact problem with normal compliance and friction," *Journal of Elasticity* **51** (1998), 105–126.

[21] M. Shillor and M. Sofonea, "A quasistatic viscoelastic contact problem with friction," preprint, 1997.

[22] N. Strömberg, *Continuum Thermodynamics of Contact, Friction and Wear*, Ph.D. Thesis, Linköping University, Sweden, 1995.

[23] N. Strömberg, L. Johansson and A. Klarbring, "Derivation and analysis of a generalized standard model for contact friction and wear," *Int. J. Solids Structures* **33** (1996), 1817–1836.