Singular phenomena in nonlinear elliptic problems

From blow-up boundary solutions to equations with singular nonlinearities

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Abstract. In this survey we report on some recent results related to various singular phenomena arising in the study of some classes of nonlinear elliptic equations. We establish qualitative results on the existence, nonexistence or the uniqueness of solutions and we focus on the following types of problems: (i) blow-up boundary solutions of logistic equations; (ii) Lane-Emden-Fowler equations with singular nonlinearities and subquadratic convection term. We study the combined effects of various terms involved in these problems: sublinear or superlinear nonlinearities, singular nonlinear terms, convection nonlinearities, as well as sign-changing potentials. We also take into account bifurcation nonlinear problems and we establish the precise rate decay of the solution in some concrete situations. Our approach combines standard techniques based on the maximum principle with non-standard arguments, such as the Karamata regular variation theory.

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1 Motivation and Previous Results

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$, $N \geq 2$. We are concerned in this paper with the following types of stationary singular problems:

I. The logistic equation

$$\begin{cases} \Delta u = \Phi(x, u, \nabla u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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II. The Lane-Emden-Fowler equation

$$\begin{cases} -\Delta u = \Psi(x, u, \nabla u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Φ is a smooth nonlinear function, while Ψ has one or more singularities. The solutions of (1.1) are called *large* (or *blow-up*) *solutions*.

In this work we focus on Problems (1.1) and (1.2) and we establish several recent contributions in the study of these equations. In order to illustrate the link between these problems, consider the most natural case where $\Phi(u, \nabla u) = u^p$, where $p > 1$. Then the function $v = u^{-1}$ satisfies (1.2) for $\Psi(u, \nabla v) = v^{2-p} - 2v^{-1} |\nabla v|^2$.

The study of large solutions has been initiated in 1916 by Bieberbach [12] for the particular case $\Phi(x, u, \nabla u) = \exp(u)$ and $N = 2$. He showed that there exists a unique solution of (1.1) such that $u(x) - \log(d(x)^{-2})$ is bounded as $x \to \partial\Omega$, where $d(x) := \text{dist}\,(x, \partial\Omega)$. Problems of this type arise in Riemannian geometry: if a Riemannian metric of the form $|ds|^2 = \exp(2u(x))|dx|^2$ has constant Gaussian curvature $-c^2$ then $\Delta u = c^2 \exp(2u)$. Motivated by a problem in mathematical physics, Rademacher [82] continued the study of Bieberbach on smooth bounded domains in $\mathbb{R}^3$. Lazer and McKenna [69] extended the results of Bieberbach and Rademacher for bounded domains in $\mathbb{R}^N$ satisfying a uniform external sphere condition and for nonlinearities $\Phi(x, u, \nabla u) = b(x) \exp(u)$, where b is continuous and strictly positive on $\overline{\Omega}$. Let $\Phi(x, u, \nabla u) = f(u)$ where $f \in C^1[0, \infty)$, $f'(s) \geq 0$ for $s \geq 0$, $f(0) = 0$ and $f(s) > 0$ for $s > 0$. In this case, Keller [63] and Osserman [79] proved that large solutions of (1.1) exist if and only if

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)\, ds.$$

In a celebrated paper, Loewner and Nirenberg [73] linked the uniqueness of the blow-up solution to the growth rate at the boundary. Motivated by certain geometric problems, they established the uniqueness for the case $f(u) = u^{(N+2)/(N-2)}$, $N > 2$. Bandle and Marcus [8] give results on asymptotic behaviour and uniqueness of the large solution for more general nonlinearities including $f(u) = u^p$ for any $p > 1$. We refer to Bandle [5], Bandle and M. Essèn [6], Bandle and Marcus [9], Du and Huang [40], García-Melián, Letelier-Albornoz, and Sabina de Lis [44], Lazer and McKenna [70], Le Gall [71], Marcus and Véron [75, 76], Ratto, Rigoli and Véron [83] and the references therein for several results on large solutions extended to N-dimensional domains and for other classes of nonlinearities.

Singular problems like (1.2) have been intensively studied in the last decades. Stationary problems involving singular nonlinearities, as well as the associated evolution equations, describe naturally several physical phenomena. At our best knowledge, the first study in this direction is due to Fulks and Maybee [42], who proved existence and uniqueness results by using a fixed point argument; moreover, they showed that solutions of the associated parabolic problem tend to the unique solution of the corresponding elliptic equation. A different approach (see Coclite and Palimieri [34], Crandall, Rabinowitz, and Tartar [35], Stuart [88]) consists in approximating the singular equation with a regular problem, where the standard techniques (e.g., monotonicity methods) can be applied and then passing to the limit to obtain the solution of the original equation. Nonlinear singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids (we refer for more details to Caffarelli, Hardt, and L. Simon [16], Callegari and Nachman [17, 18], Díaz [38], Díaz, Morel, and Oswald [39] and the more recent papers by Haitao [58], Hernández, Mancebo, and Vega [59, 60], Meadows [77], Shi and Yao [86, 87]). We also point out that, due to the meaning of the unknowns (concentrations, populations, etc.), only the positive solutions are relevant in most cases. For instance, problems of this type characterize some reaction-diffusion processes where $u \geq 0$ is viewed as the density of a reactant and the region where $u = 0$ is called the *dead core*, where no reaction takes place (see Aris [4] for the study of a single, irreversible steady-state reaction). Nonlinear singular elliptic equations are also encountered in glacial advance, in transport of coal slurries down conveyor belts and in several other geophysical and industrial contents (see Callegari and Nachman [18] for the case of the incompressible flow of a uniform stream past a semi-infinite flat plate at zero incidence).

In [35], Crandall, Rabinowitz and Tartar established that the boundary value problem

$$\begin{cases} -\Delta u - u^{-\alpha} = -u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution, for any $\alpha > 0$. The importance of the linear and nonlinear terms is crucial for the existence of solutions. For instance, Coclite and Palmieri studied in [34] the problem

$$\begin{cases} -\Delta u - u^{-\alpha} = \lambda u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\lambda \geq 0$ and $\alpha, p \in (0,1)$. In [34] it is proved that problem (1.3) has at least one solution for all $\lambda \geq 0$ and $0 < p < 1$. Moreover, if $p \geq 1$, then there exists λ^* such that problem (1.3) has a solution for $\lambda \in [0, \lambda^*)$ and no solution for $\lambda > \lambda^*$. In [34] it is also proved a related non–existence result. More exactly, the problem

$$\begin{cases} -\Delta u + u^{-\alpha} = u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has no solution, provided that $0 < \alpha < 1$ and $\lambda_1 \geq 1$ (that is, if Ω is "small"), where λ_1 denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$.

Problems related to multiplicity and uniqueness become difficult even in simple cases. Shi and Yao studied in [86] the existence of radial symmetric solutions of the problem

$$\begin{cases} \Delta u + \lambda(u^p - u^{-\alpha}) = 0 & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where $\alpha > 0$, $0 < p < 1$, $\lambda > 0$, and B_1 is the unit ball in $\mathbb{R}^N$. Using a bifurcation theorem of Crandall and Rabinowitz, it has been shown in [86] that there exists $\lambda_1 > \lambda_0 > 0$ such that the above problem has no solutions for $\lambda < \lambda_0$, exactly one solution for $\lambda = \lambda_0$ or $\lambda > \lambda_1$, and two solutions for $\lambda_0 < \lambda \leq \lambda_1$.

The author's interest for the study of singular problems is motivated by several stimulating discussions with Professor Haim Brezis in Spring 2001. I would like to use this opportunity to thank once again Professor Brezis for his constant scientific support during the years.

This work is organized as follows. Sections 2–5 are mainly devoted to the study of blow-up boundary solutions of logistic type equations with absorption. In the second part of this work (Sections 6–8), in connection with the previous results, we are concerned with the study of the Dirichlet boundary value problem for the singular Lane-Emden-Fowler equation. Our framework includes the presence of a convection term.

2 Large solutions of elliptic equations with absorption and sub-quadratic convection term

Consider the problem

$$\begin{cases} \Delta u + q(x)|\nabla u|^a = p(x)f(u) & \text{in } \Omega\,, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \Omega\,, \end{cases} \tag{2.4}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth domain (bounded or possibly unbounded) with compact (possibly empty) boundary. We assume that $a \leq 2$ is a positive real number, p, q are non-negative function such that $p \not\equiv 0$, $p, q \in C^{0,\alpha}(\overline{\Omega})$ if Ω is bounded, and $p, q \in C^{0,\alpha}_{\text{loc}}(\Omega)$, otherwise. Throughout this section we assume that the nonlinearity f fulfills the following conditions

$(f1)\qquad f\in C^1[0,\infty),\ f'\geq 0,\ f(0)=0$ and $f>0$ on $(0,\infty)$.

$$(f2)\qquad \int_1^{\infty}[F(t)]^{-1/2}\,dt<\infty\,,\quad \text{where}\quad F(t)=\int_0^t f(s)\,ds.$$

$$(f3)\qquad \frac{F(t)}{f^{2/a}(t)}\quad\to 0\quad \text{as}\quad t\to\infty.$$

Cf. Véron [91], f is called an absorption term. The above conditions hold provided that $f(t)=t^k$, $k>1$ and $0<a<\frac{2r}{r+1}(<2)$, or $f(t)=e^t-1$, or $f(t)=e^t-t$ and $a<2$. We observe that by $(f1)$ and $(f3)$ it follows that $f/F^{a/2}\geq\beta>0$ for t large enough, that is, $(F^{1-a/2})'\geq\beta>0$ for t large enough which yields $0<a\leq 2$. We also deduce that conditions $(f2)$ and $(f3)$ imply $\int_1^{\infty} f^{-1/a}(t)dt<\infty\,.$

We are mainly interested in finding properties of *large (explosive) solutions* of (2.4), that is solutions u satisfying $u(x)\to\infty$ as $\mathrm{dist}\,(x,\partial\Omega)\to 0$ (if $\Omega\not\equiv\mathbb{R}^N$), or $u(x)\to\infty$ as $|x|\to\infty$ (if $\Omega=\mathbb{R}^N$). In the latter case the solution is called an *entire large (explosive)* solution.

Problems of this type appear in stochastic control theory and have been first study by Lasry and Lions [67]. The corresponding parabolic equation was considered in Quittner [81] and in Galaktionov and Vázquez [43]. In terms of the dynamic programming approach, an explosive solution of (2.4) corresponds to a value function (or Bellman function) associated to an infinite exit cost (see Lasry and Lions [67]).

Bandle and Giarrusso [7] studied the existence of a large solution of problem (2.4) in the case $p\equiv 1$, $q\equiv 1$ and Ω bounded. Lair and Wood [66] studied the sublinear case corresponding to $p\equiv 1$, while Cîrstea and Rădulescu [24] proved the existence of large solutions to (2.4) in the case $q\equiv 0$.

As observed by Bandle and Giarrusso [7], the simplest case is $a=2$, which can be reduced to a problem without gradient term. Indeed, if u is a solution of (2.4) for $q\equiv 1$, then the function $v=e^u$ (Gelfand transformation) satisfies

$$\begin{cases} \Delta v=p(x)vf(\ln v) & \text{in }\Omega\,,\\ v(x)\to+\infty & \text{if }\mathrm{dist}\,(x,\partial\Omega)\to 0.\end{cases}$$

We shall therefore mainly consider the case where $0<a<2$.

The main results in this Section are due to Ghergu, Niculescu, and Rădulescu [45]. These results generalize those obtained by Cîrstea and Rădulescu [24] in the case of the presence of a convection (gradient) term.

Our first result concerns the existence of a large solution to problem (2.4) when Ω is bounded.

Theorem 2.1. *Suppose that Ω is bounded and assume that p satisfies*
$(p1)$ *for every $x_0\in\Omega$ with $p(x_0)=0$, there exists a domain $\Omega_0\ni x_0$ such that $\overline{\Omega_0}\subset\Omega$ and $p>0$ on $\partial\Omega_0$.*

Then problem (2.4) has a positive large solution.

A crucial role in the proof of the above result is played by the following auxiliary result (see Ghergu, Niculescu, and Rădulescu [45]).

Lemma 2.2. *Let Ω be a bounded domain. Assume that $p, q \in C^{0,\alpha}(\overline{\Omega})$ are non-negative functions, $0 < a < 2$ is a real number, f satisfies $(f1)$ and $g : \partial\Omega \to (0, \infty)$ is continuous. Then the boundary value problem*

$$\begin{cases} \Delta u + q(x)|\nabla u|^a = p(x)f(u), & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \\ u \geq 0, \ u \not\equiv 0, & \text{in } \Omega \end{cases} \tag{2.5}$$

has a classical solution. If p is positive, then the solution is unique.

Sketch of the proof of Theorem 2.1. By Lemma 2.2, the boundary value problem

$$\begin{cases} \Delta v_n + q(x)|\nabla v_n|^a = \left(p(x) + \dfrac{1}{n}\right) f(v_n), & \text{in } \Omega \\ v_n = n, & \text{on } \partial\Omega \\ v_n \geq 0, \ v_n \not\equiv 0, & \text{in } \Omega \end{cases}$$

has a unique positive solution, for any $n \geq 1$. Next, by the maximum principle, the sequence (v_n) is non-decreasing and is bounded from below in Ω by a positive function.

To conclude the proof, it is sufficient to show that

(a) for all $x_0 \in \Omega$ there exists an open set $\mathcal{O} \subset\subset \Omega$ which contains x_0 and $M_0 = M_0(x_0) > 0$ such that $v_n \leq M_0$ in $\mathcal{O}$ for all $n \geq 1$

(b) $\lim_{x \to \partial\Omega} v(x) = \infty$, where $v(x) = \lim_{n\to\infty} v_n(x)$.

We observe that the statement (a) shows that the sequence (v_n) is uniformly bounded on every compact subset of Ω. Standard elliptic regularity arguments (see Gilbarg and Trudinger [55]) show that v is a solution of problem (2.4). Then, by (b), it follows that v is a large solution of problem (2.4).

To prove (a) we distinguish two cases :

Case $p(x_0) > 0$. By the continuity of p, there exists a ball $B = B(x_0, r) \subset\subset \Omega$ such that

$$m_0 := \min\{p(x);\ x \in \overline{B}\} > 0.$$

Let w be a positive solution of the problem

$$\begin{cases} \Delta w + q(x)|\nabla w|^a = m_0 f(w), & \text{in } B \\ w(x) \to \infty, & \text{as } x \to \partial B. \end{cases}$$

The existence of w follows by considering the problem

$$\begin{cases} \Delta w_n + q(x)|\nabla w_n|^a = m_0 f(w_n), & \text{in } B \\ w_n = n, & \text{on } \partial B. \end{cases}$$

The maximum principle implies $w_n \le w_{n+1} \le \theta$, where

$$\begin{cases} \Delta \theta + \|q\|_{L^\infty}|\nabla \theta|^a = m_0 f(\theta), & \text{in } B \\ \theta(x) \to \infty, & \text{as } x \to \partial B. \end{cases}$$

Standard arguments show that $v_n \le w$ in B. Furthermore, w is bounded in $\overline{B(x_0, r/2)}$. Setting $M_0 = \sup_{\mathcal{O}} w$, where $\mathcal{O} = B(x_0, r/2)$, we obtain (a).

Case $p(x_0) = 0$. Our hypothesis $(p1)$ and the boundedness of Ω imply the existence of a domain $\mathcal{O} \subset\subset \Omega$ which contains x_0 such that $p > 0$ on $\partial\mathcal{O}$. The above case shows that for any $x \in \partial\mathcal{O}$ there exist a ball $B(x, r_x)$ strictly contained in Ω and a constant $M_x > 0$ such that $v_n \le M_x$ on $B(x, r_x/2)$, for any $n \ge 1$. Since $\partial\mathcal{O}$ is compact, it follows that it may be covered by a finite number of such balls, say $B(x_i, r_{x_i}/2)$, $i = 1, \cdots, k_0$. Setting $M_0 = \max\{M_{x_1}, \cdots, M_{x_{k_0}}\}$ we have $v_n \le M_0$ on $\partial\mathcal{O}$, for any $n \ge 1$. Applying the maximum principle we obtain $v_n \le M_0$ in $\mathcal{O}$ and (a) follows.

Let z be the unique function satisfying $-\Delta z = p(x)$ in Ω and $z = 0$, on $\partial\Omega$. Moreover, by the maximum principle, we have $z > 0$ in Ω. We first observe that for proving (b) it is sufficient to show that

$$\int_{v(x)}^{\infty} \frac{dt}{f(t)} \le z(x) \quad \text{for any } x \in \Omega. \tag{2.6}$$

By [24, Lemma 1], the left hand-side of (2.6) is well defined in Ω. We choose $R > 0$ so that $\overline{\Omega} \subset B(0, R)$ and fix $\varepsilon > 0$. Since $v_n = n$ on $\partial\Omega$, let $n_1 = n_1(\varepsilon)$ be such that

$$n_1 > \frac{1}{\varepsilon(N-3)(1+R^2)^{-1/2} + 3\varepsilon(1+R^2)^{-5/2}}, \tag{2.7}$$

and

$$\int_{v_n(x)}^{\infty} \frac{dt}{f(t)} \le z(x) + \varepsilon(1+|x|^2)^{-1/2} \qquad \forall\, x \in \partial\Omega, \forall\, n \ge n_1. \tag{2.8}$$

In order to prove (2.6), it is enough to show that

$$\int_{v_n(x)}^{\infty} \frac{dt}{f(t)} \le z(x) + \varepsilon(1+|x|^2)^{-1/2} \qquad \forall\, x \in \Omega,\ \forall n \ge n_1. \tag{2.9}$$

Indeed, taking $n \to \infty$ in (2.9) we deduce (2.6), since $\varepsilon > 0$ is arbitrarily chosen. Assume now, by contradiction, that (2.9) fails. Then

$$\max_{x\in\overline{\Omega}} \left\{ \int_{v_n(x)}^{\infty} \frac{dt}{f(t)} - z(x) - \varepsilon(1+|x|^2)^{-1/2} \right\} > 0.$$

Using (2.8) we see that the point where the maximum is achieved must lie in Ω. A straightforward computation shows that at this point, say x_0, we have

$$0 \geq \Delta \left(\int_{v_n(x)}^{\infty} \frac{dt}{f(t)} - z(x) - \varepsilon(1+|x|^2)^{-1/2} \right)_{|x=x_0} > 0.$$

This contradiction shows that inequality (2.8) holds and the proof of Theorem 1 is complete. □

Similar arguments based on the maximum principle and the approximation of large balls $B(0,n)$ imply the following existence result.

Theorem 2.3. *Assume that $\Omega = \mathbb{R}^N$ and that problem (2.4) has at least one solution. Suppose that p satisfies the condition*

$(p1)'$ *There exists a sequence of smooth bounded domains $(\Omega_n)_{n\geq 1}$ such that $\overline{\Omega_n} \subset \Omega_{n+1}$, $\mathbb{R}^N = \cup_{n=1}^{\infty}\Omega_n$, and $(p1)$ holds in Ω_n, for any $n \geq 1$.*

Then there exists a classical solution U of (2.4) which is a maximal solution if p is positive.

Assume that p verifies the additional condition

$(p2)$ $$\int_0^{\infty} r\Phi(r)\, dr < \infty\,, \quad \textit{where } \Phi(r) = \max\{p(x) :\ |x| = r\}.$$

Then U is an entire large solution of (2.4).

We now consider the case in which $\Omega \neq \mathbb{R}^N$ and Ω is unbounded. We say that a large solution u of (2.4) is *regular* if u tends to zero at infinity. In [74, Theorem 3.1] Marcus proved for this case (and if $q = 0$) the existence of regular large solutions to problem (2.4) by assuming that there exist $\gamma > 1$ and $\beta > 0$ such that

$$\liminf_{t\to 0} f(t)t^{-\gamma} > 0 \qquad \text{and} \qquad \liminf_{|x|\to\infty} p(x)|x|^{\beta} > 0.$$

The large solution constructed in Marcus [74] is the *smallest* large solution of problem (2.4). In the next result we show that problem (2.4) admits a *maximal* classical solution U and that U blows-up at infinity if $\Omega = \mathbb{R}^N \setminus \overline{B(0,R)}$.

Theorem 2.4. *Suppose that $\Omega \neq \mathbb{R}^N$ is unbounded and that problem (2.4) has at least a solution. Assume that p satisfies condition $(p1)'$ in Ω. Then there exists a classical solution U of problem (2.4) which is maximal solution if p is positive.*

If $\Omega = \mathbb{R}^N \setminus \overline{B(0,R)}$ and p satisfies the additional condition $(p2)$, with $\Phi(r) = 0$ for $r \in [0,R]$, then the solution U of (2.4) is a large solution that blows-up at infinity.

We refer to Ghergu, Niculescu and Rădulescu [45] for complete proofs of Theorems 2.3 and 2.4.

A useful observation is given in the following

Remark 1. *Assume that $p \in C(\mathbb{R}^N)$ is a non-negative and non-trivial function which satisfies $(p2)$. Let f be a function satisfying assumption $(f1)$. Then condition*

$$\int_1^\infty \frac{dt}{f(t)} < \infty \tag{2.10}$$

is necessary for the existence of entire large solutions to (2.4).

Indeed, let u be an entire large solution of problem (2.4). Define

$$\bar{u}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} \left(\int_{a_0}^{u(x)} \frac{dt}{f(t)} \right) dS = \frac{1}{\omega_N} \int_{|\xi|=1} \left(\int_{a_0}^{u(r\xi)} \frac{dt}{f(t)} \right) dS,$$

where ω_N denotes the surface area of the unit sphere in $\mathbb{R}^N$ and a_0 is chosen such that $a_0 \in (0, u_0)$, where $u_0 = \inf_{\mathbb{R}^N} u > 0$. By the divergence theorem we have

$$\bar{u}'(r) = \frac{1}{\omega_N r^{N-1}} \int_{B(0,r)} \Delta \left(\int_{a_0}^{u(x)} \frac{dt}{f(t)} \right) dx.$$

Since u is a positive classical solution it follows that

$$|\bar{u}'(r)| \leq Cr \to 0 \qquad \text{as } r \to 0\,.$$

On the other hand

$$\omega_N \left(R^{N-1}\bar{u}'(R) - r^{N-1}\bar{u}'(r) \right) = \int_r^R \left(\int_{|x|=z} \Delta \left(\int_{a_0}^{u(x)} \frac{dt}{f(t)} \right) dS \right) dz.$$

Dividing by $R - r$ and taking $R \to r$ we find

$$\begin{aligned} \omega_N (r^{N-1}\bar{u}'(r))' &= \int_{|x|=r} \Delta \left(\int_{a_0}^{u(x)} \frac{dt}{f(t)} \right) dS = \int_{|x|=r} \operatorname{div} \left(\frac{1}{f(u(x))} \nabla u(x) \right) dS \\ &= \int_{|x|=r} \left[\left(\frac{1}{f} \right)' (u(x)) \cdot |\nabla u(x)|^2 + \frac{1}{f(u(x))} \Delta u(x) \right] dS \\ &\leq \int_{|x|=r} \frac{p(x) f(u(x))}{f(u(x))}\, dS \leq \omega_N r^{N-1} \Phi(r). \end{aligned}$$

The above inequality yields by integration

$$\bar{u}(r) \le \bar{u}(0) + \int_0^r \sigma^{1-N} \left(\int_0^\sigma \tau^{N-1} \Phi(\tau)\, d\tau \right) d\sigma \qquad \forall r \ge 0. \tag{2.11}$$

On the other hand, according to $(p2)$, for all $r > 0$ we have

$$\begin{aligned} \int_0^r \sigma^{1-N} \left(\int_0^\sigma \tau^{N-1} \Phi(\tau)\, d\tau \right) d\sigma &= \frac{1}{2-N} r^{2-N} \int_0^r \tau^{N-1} \Phi(\tau)\, d\tau - \frac{1}{2-N} \int_0^r \sigma \Phi(\sigma)\, d\sigma \\ &\le \frac{1}{N-2} \int_0^\infty r \Phi(r)\, dr < \infty. \end{aligned}$$

So, by (2.11), $\bar{u}(r) \le \bar{u}(0) + K$, for all $r \ge 0$. The last inequality implies that $\bar{u}$ is bounded and assuming that (2.10) is not fulfilled it follows that u cannot be a large solution. □

We point out that the hypothesis $(p2)$ on p is essential in the statement of Remark 1. Indeed, let us consider $f(t) = t$, $p \equiv 1$, $\alpha \in (0,1)$, $q(x) = 2^{\alpha-2} \cdot |x|^\alpha$, $a = 2 - \alpha \in (1,2)$. Then the corresponding problem has the entire large solution $u(x) = |x|^2 + 2N$, but (2.10) is not fulfilled.

3 Singular solutions with lack of the Keller-Osserman condition

We have already seen that if f is smooth and increasing on $[0,\infty)$ such that $f(0) = 0$ and $f > 0$ in $(0,\infty)$, then the problem

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega \end{cases}$$

has a solution if and only if the Keller-Osserman condition $\int_1^\infty [F(t)]^{-1/2}\, dt < \infty$ is fulfilled, where $F(t) = \int_0^t f(s)\, ds$. In particular, this implies that f must have a superlinear growth. In this section we are concerned with the problem

$$\begin{cases} \Delta u + |\nabla u| = p(x) f(u) & \text{in } \Omega, \\ u \ge 0 & \text{in } \Omega, \end{cases} \tag{3.12}$$

where $\Omega \subset \mathbb{R}^N$ $(N \ge 3)$ is either a smooth bounded domain or the whole space. Our main assumptions on f is that it has a *sublinear* growth, so we cannot expect that Problem (3.12) admits a blow-up boundary solution. Our main purpose in this section is to establish a necessary and sufficient condition on the variable potential $p(x)$ for the existence of an entire large solution.

Throughout this section we assume that p is a non-negative function such that $p \in C^{0,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ if Ω is bounded, and $p \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$, otherwise. The non-decreasing non-linearity

$f \in C^{0,\alpha}_{\text{loc}}[0,\infty)$ fulfills $f(0) = 0$ and $f > 0$ on $(0,\infty)$. We also assume that f is sublinear at infinity, in the sense that $\Lambda := \sup_{s\geq 1} \frac{f(s)}{s} < \infty$.

The main results in this section have been established by Ghergu and Rădulescu [51].

If Ω is bounded we prove the following non-existence result.

Theorem 3.1. *Suppose that $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Then problem (3.12) has no positive large solution in Ω.*

Proof. Suppose by contradiction that problem (3.12) has a positive large solution u and define $v(x) = \ln(1 + u(x))$, $x \in \Omega$. It follows that v is positive and $v(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$. We have

$$\Delta v = \frac{1}{1+u}\Delta u - \frac{1}{(1+u)^2}|\nabla u|^2 \qquad \text{in } \Omega$$

and so

$$\Delta v \leq p(x)\frac{f(u)}{1+u} \leq \|p\|_\infty \frac{f(u)}{1+u} \leq A \qquad \text{in } \Omega,$$

for some constant $A > 0$. Therefore

$$\Delta(v(x) - A|x|^2) < 0, \qquad \text{for all } x \in \Omega.$$

Let $w(x) = v(x) - A|x|^2$, $x \in \Omega$. Then $\Delta w < 0$ in Ω. Moreover, since Ω is bounded, it follows that $w(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$.

Let $M > 0$ be arbitrary. We claim that $w \geq M$ in Ω. For all $\delta > 0$, we set

$$\Omega_\delta = \{x \in \Omega\,;\ \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

Since $w(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$, we can choose $\delta > 0$ such that

$$w(x) \geq M \qquad \text{for all } x \in \Omega \setminus \Omega_\delta. \tag{3.13}$$

On the other hand,

$$\begin{aligned} -\Delta(w(x) - M) > 0 \qquad & \text{in } \Omega_\delta, \\ w(x) - M \geq 0 \qquad & \text{on } \partial\Omega_\delta. \end{aligned}$$

By the maximum principle we get $w(x) - M \geq 0$ in Ω_δ. So, by (3.13), $w \geq M$ in Ω. Since $M > 0$ is arbitrary, it follows that $w \geq n$ in Ω, for all $n \geq 1$. Obviously, this is a contradiction and the proof is now complete. □

Next, we consider the problem (3.12) when $\Omega = \mathbb{R}^N$. For all $r \geq 0$ we set

$$\phi(r) = \max_{|x|=r} p(x), \qquad \psi(r) = \min_{|x|=r} p(x), \qquad \text{and} \qquad h(r) = \phi(r) - \psi(r).$$

We suppose that

$$\int_0^\infty r h(r)\Psi(r)dr < \infty, \tag{3.14}$$

where

$$\Psi(r) = \exp\left(\Lambda_N \int_0^r s\psi(s)ds\right), \qquad \Lambda_N = \frac{\Lambda}{N-2}.$$

Obviously, if p is radial then $h \equiv 0$ and (3.14) occurs. Assumption (3.14) shows that the variable potential $p(x)$ has a slow variation. An example of non-radial potential for which (3.14) holds is $p(x) = \frac{1+|x_1|^2}{(1+|x_1|^2)(1+|x|^2)+1}$. In this case $\phi(r) = \frac{r^2+1}{(r^2+1)^2+1}$ and $\psi(r) = \frac{1}{r^2+2}$. If $\Lambda_N = 1$, by direct computation we get $rh(r)\Psi(r) = O\left(r^{-2}\right)$ as $r \to \infty$ and so (3.14) holds.

Theorem 3.2. *Assume $\Omega = \mathbb{R}^N$ and p satisfies (3.14). Then problem (3.12) has a positive entire large solution if and only if*

$$\int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\psi(s)dsdt = \infty. \tag{3.15}$$

Proof. Several times in the proof of Theorem 3.2 we shall apply the following elementary inequality:

$$\int_0^r e^{-t}t^{1-N}\int_0^t e^s s^{N-1}g(s)dsdt \le \frac{1}{N-2}\int_0^r tg(t)dt, \qquad \forall\, r>0, \tag{3.16}$$

for any continuous function $g:[0,\infty)\to[0,\infty)$. The proof follows easily by integration by parts.
Necessary condition. Suppose that (3.14) fails and the equation (3.12) has a positive entire large solution u. We claim that

$$\int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\phi(s)dsdt < \infty. \tag{3.17}$$

We first recall that $\phi = h + \psi$. Thus

$$\begin{aligned}\int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\phi(s)dsdt &= \int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\psi(s)dsdt\\ &+\int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}h(s)dsdt.\end{aligned}$$

By virtue of (3.16) we find

$$\begin{aligned}\int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\phi(s)dsdt &\le \int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\psi(s)dsdt + \frac{1}{N-2}\int_0^\infty th(t)dt\\ &\le \int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\psi(s)dsdt + \frac{1}{N-2}\int_0^\infty th(t)\Psi(t)dt.\end{aligned}$$

Since $\int\limits_1^\infty e^{-t}t^{1-N}\int\limits_0^t e^s s^{N-1}\psi(s)dsdt<\infty$, by (3.14) we deduce that (3.17) follows.

Now, let $\bar u$ be the spherical average of u, i.e.,

$$\bar u(r)=\frac{1}{\omega_N r^{N-1}}\int\limits_{|x|=r}u(x)d\sigma_x,\qquad r\ge 0,$$

where ω_N is the surface area of the unit sphere in $\mathbb{R}^N$. Since u is a positive entire large solution of (2.4) it follows that $\bar u$ is positive and $\bar u(r)\to\infty$ as $r\to\infty$. With the change of variable $x\to ry$, we have

$$\bar u(r)=\frac{1}{\omega_N}\int\limits_{|y|=1}u(ry)\,d\sigma_y,\qquad r\ge 0$$

and

$$\bar u'(r)=\frac{1}{\omega_N}\int\limits_{|y|=1}\nabla u(ry)\cdot y\,d\sigma_y,\qquad r\ge 0. \tag{3.18}$$

Hence

$$\bar u'(r)=\frac{1}{\omega_N}\int\limits_{|y|=1}\frac{\partial u}{\partial r}(ry)\,d\sigma_y=\frac{1}{\omega_N r^{N-1}}\int\limits_{|x|=r}\frac{\partial u}{\partial r}(x)\,d\sigma_x,$$

that is

$$\bar u'(r)=\frac{1}{\omega_N r^{N-1}}\int\limits_{B(0,R)}\Delta u(x)\,dx,\qquad \text{for all } r\ge 0. \tag{3.19}$$

Due to the gradient term $|\nabla u|$ in (2.4), we cannot infer that $\Delta u\ge 0$ in $\mathbb{R}^N$ and so we cannot expect that $\bar u'\ge 0$ in $[0,\infty)$. We define the auxiliary function

$$U(r)=\max_{0\le t\le r}\bar u(t),\qquad r\ge 0. \tag{3.20}$$

Then U is positive and non-decreasing. Moreover, $U\ge\bar u$ and $U(r)\to\infty$ as $r\to\infty$.

The assumptions $(f1)$ and $(f2)$ yield $f(t)\le\Lambda(1+t)$, for all $t\ge 0$. So, by (3.18) and (3.19),

$$\begin{aligned}\bar u''+\frac{N-1}{r}\bar u'+\bar u' &\le \frac{1}{\omega_N r^{N-1}}\int\limits_{|x|=r}[\Delta u(x)+|\nabla u|(x)]\,d\sigma_x=\frac{1}{\omega_N r^{N-1}}\int\limits_{|x|=r}p(r)f(u(x))d\sigma_x\\ &\le \Lambda\phi(r)\frac{1}{\omega_N r^{N-1}}\int\limits_{|x|=r}(1+u(x))\,d\sigma_x=\Lambda\phi(r)\,(1+\bar u(r))\le\Lambda\phi(r)\,(1+U(r))\,,\end{aligned}$$

for all $r\ge 0$. It follows that

$$\left(r^{N-1}e^r\bar u'\right)'\le\Lambda e^r r^{N-1}\phi(r)\,(1+U(r))\,,\qquad \text{for all } r\ge 0.$$

So, for all $r\ge r_0>0$,

$$\bar u(r)\le\bar u(r_0)+\Lambda\int_{r_0}^r e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\phi(s)(1+U(s))dsdt.$$

The monotonicity of U implies

$$\bar{u}(r) \le \bar{u}(r_0) + \Lambda(1+U(r)) \int_{r_0}^{r} e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \phi(s) ds dt, \tag{3.21}$$

for all $r \ge r_0 \ge 0$. By (3.17) we can choose $r_0 \ge 1$ such that

$$\int_{r_0}^{\infty} e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \phi(s) ds dt < \frac{1}{2\Lambda}. \tag{3.22}$$

Thus (3.21) and (3.22) yield

$$\bar{u}(r) \le \bar{u}(r_0) + \frac{1}{2}(1+U(r)), \qquad \text{for all } \ r \ge r_0. \tag{3.23}$$

By the definition of U and $\lim_{r\to\infty} \bar{u}(r) = \infty$, we find $r_1 \ge r_0$ such that

$$U(r) = \max_{r_0 \le t \le r} \bar{u}(r), \qquad \text{for all } \ r \ge r_1. \tag{3.24}$$

Considering now (3.23) and (3.24) we obtain

$$U(r) \le \bar{u}(r_0) + \frac{1}{2}(1+U(r)), \qquad \text{for all } \ r \ge r_1.$$

Hence

$$U(r) \le 2\bar{u}(r_0) + 1, \qquad \text{for all } \ r \ge r_1.$$

This means that U is bounded, so u is also bounded, a contradiction. It follows that (2.4) has no positive entire large solutions.

Sufficient condition. We need the following auxiliary comparison result.

Lemma 3.3. *Assume that (3.14) and (3.15) hold. Then the equations*

$$\Delta v + |\nabla v| = \phi(|x|) f(v) \qquad \Delta w + |\nabla w| = \psi(|x|) f(w) \tag{3.25}$$

have positive entire large solution such that

$$v \le w \qquad in \ \mathbb{R}^N. \tag{3.26}$$

Proof. Radial solutions of (3.25) satisfy

$$v'' + \frac{N-1}{r} v' + |v'| = \phi(r) f(v)$$

and

$$w'' + \frac{N-1}{r} w' + |w'| = \psi(r) f(w).$$

Assuming that v' and w' are non-negative, we deduce

$$\left(e^r r^{N-1} v'\right)' = e^r r^{N-1} \phi(r) f(v)$$

and

$$\left(e^r r^{N-1} w'\right)' = e^r r^{N-1} \psi(r) f(w).$$

Thus any positive solutions v and w of the integral equations

$$v(r) = 1 + \int_0^r e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \phi(s) f(v(s)) ds dt, \qquad r \geq 0, \tag{3.27}$$

$$w(r) = b + \int_0^r e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \psi(s) f(w(s)) ds dt, \qquad r \geq 0, \tag{3.28}$$

provide a solution of (3.25), for any $b > 0$. Since $w \geq b$, it follows that $f(w) \geq f(b) > 0$ which yields

$$w(r) \geq b + f(b) \int_0^r e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \psi(s) ds dt, \qquad r \geq 0.$$

By (3.15), the right hand side of this inequality goes to $+\infty$ as $r \to \infty$. Thus $w(r) \to \infty$ as $r \to \infty$. With a similar argument we find $v(r) \to \infty$ as $r \to \infty$.

Let $b > 1$ be fixed. We first show that (3.28) has a positive solution. Similarly, (3.27) has a positive solution.

Let $\{w_k\}$ be the sequence defined by $w_1 = b$ and

$$w_{k+1}(r) = b + \int_0^r e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \psi(s) f(w_k(s)) ds dt, \qquad k \geq 1. \tag{3.29}$$

We remark that $\{w_k\}$ is a non-decreasing sequence. To get the convergence of $\{w_k\}$ we will show that $\{w_k\}$ is bounded from above on bounded subsets. To this aim, we fix $R > 0$ and we prove that

$$w_k(r) \leq b e^{Mr}, \qquad \text{for any } 0 \leq r \leq R, \text{ and for all } k \geq 1, \tag{3.30}$$

where $M \equiv \Lambda_N \max_{t \in [0,R]} t\psi(t)$.

We achieve (3.30) by induction. We first notice that (3.30) is true for $k = 1$. Furthermore, the assumption $(f2)$ and the fact that $w_k \geq 1$ lead us to $f(w_k) \leq \Lambda w_k$, for all $k \geq 1$. So, by (3.29),

$$w_{k+1}(r) \leq b + \Lambda \int_0^r e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \psi(s) w_k(s) ds dt, \qquad r \geq 0.$$

Using now (3.16) (for $g(t) = \psi(t) w_k(t)$) we deduce

$$w_{k+1}(r) \leq b + \Lambda_N \int_0^r t\psi(t) w_k(t) dt, \qquad \forall\, r \in [0, R].$$

The induction hypothesis yields

$$w_{k+1}(r) \le b + bM \int_0^r e^{Mt} dt = be^{Mr}, \qquad \forall\, r \in [0, R].$$

Hence, by induction, the sequence $\{w_k\}$ is bounded in $[0, R]$, for any $R > 0$. It follows that $w(r) = \lim_{k\to\infty} w_k(r)$ is a positive solution of (3.28). In a similar way we conclude that (3.27) has a positive solution on $[0, \infty)$.

The next step is to show that the constant b may be chosen sufficiently large so that (3.26) holds. More exactly, if

$$b > 1 + K\Lambda_N \int_0^\infty sh(s)\Psi(s)ds, \tag{3.31}$$

where $K = \exp\left(\Lambda_N \int_0^\infty th(t)dt\right)$, then (3.26) occurs.

We first prove that the solution v of (3.27) satisfies

$$v(r) \le K\Psi(r), \qquad \forall\, r \ge 0. \tag{3.32}$$

Since $v \ge 1$, from $(f2)$ we have $f(v) \le \Lambda v$. We use this fact in (3.27) and then we apply the estimate (3.16) for $g = \phi$. It follows that

$$v(r) \le 1 + \Lambda_N \int_0^r s\phi(s)v(s)ds, \qquad \forall\, r \ge 0. \tag{3.33}$$

By Gronwall's inequality we obtain

$$v(r) \le \exp\left(\Lambda_N \int_0^r s\phi(s)ds\right), \qquad \forall\, r \ge 0,$$

and, by (3.33),

$$v(r) \le 1 + \Lambda_N \int_0^r s\phi(s) \exp\left(\Lambda_N \int_0^s t\phi(t)dt\right) ds, \qquad \forall\, r \ge 0.$$

Hence

$$v(r) \le 1 + \int_0^r \left(\exp\left(\Lambda_N \int_0^s t\phi(t)dt\right)\right)' ds, \qquad \forall\, r \ge 0,$$

that is

$$v(r) \le \exp\left(\Lambda_N \int_0^r t\phi(t)dt\right), \qquad \forall\, r \ge 0. \tag{3.34}$$

Inserting $\phi = h + \psi$ in (3.34) we have

$$v(r) \le e^{\Lambda_N \int_0^r th(t)dt} \Psi(r) \le K\Psi(r), \qquad \forall\, r \ge 0,$$

so (3.32) follows.

Since $b > 1$ it follows that $v(0) < w(0)$. Then there exists $R > 0$ such that $v(r) < w(r)$, for any $0 \le r \le R$. Set

$$R_\infty = \sup\{\ R > 0 \,|\, v(r) < w(r),\ \ \forall\, r \in [0, R]\,\}.$$

In order to conclude our proof, it remains to show that $R_\infty = \infty$. Suppose the contrary. Since $v \le w$ on $[0, R_\infty]$ and $\phi = h + \psi$, from (3.27) we deduce

$$v(R_\infty) = 1 + \int_0^{R_\infty} e^{-t}t^{1-N} \int_0^t e^s s^{N-1} h(s) f(v(s)) ds dt + \int_0^{R_\infty} e^{-t}t^{1-N} \int_0^t e^s s^{N-1} \psi(s) f(v(s)) ds dt.$$

So, by (3.16),

$$v(R_\infty) \le 1 + \frac{1}{N-2} \int_0^{R_\infty} th(t) f(v(t)) dt + \int_0^{R_\infty} e^{-t}t^{1-N} \int_0^t e^s s^{N-1} \psi(s) f(w(s)) ds dt.$$

Taking into account that $v \ge 1$ and the assumption $(f2)$, it follows that

$$v(R_\infty) \le 1 + K\Lambda_N \int_0^{R_\infty} th(t)\Psi(t) dt + \int_0^{R_\infty} e^{-t}t^{1-N} \int_0^t e^s s^{N-1} \psi(s) f(w(s)) ds dt.$$

Now, using (3.31) we obtain

$$v(R_\infty) < b + \int_0^{R_\infty} e^{-t}t^{1-N} \int_0^t e^s s^{N-1} \psi(s) f(w(s)) ds dt = w(R_\infty).$$

Hence $v(R_\infty) < w(R_\infty)$. Therefore, there exists $R > R_\infty$ such that $v < w$ on $[0, R]$, which contradicts the maximality of R_∞. This contradiction shows that inequality (3.26) holds and the proof of Lemma 2.2 is now complete. □

Proof of Theorem 3.2 completed. Suppose that (3.15) holds. For all $k \ge 1$ we consider the problem

$$\begin{cases} \Delta u_k + |\nabla u_k| = p(x) f(u_k) & \text{in } B(0,k), \\ u_k(x) = w(k) & \text{on } \partial B(0,k). \end{cases} \tag{3.35}$$

Then v and w defined by (3.27) and (3.28) are positive sub and super-solutions of (3.35). So this problem has at least a positive solution u_k and

$$v(|x|) \le u_k(x) \le w(|x|) \qquad \text{in } B(0,k), \text{ for all } k \ge 1.$$

By Theorem 14.3 in Gilbarg and Trudinger [55], the sequence $\{\nabla u_k\}$ is bounded on every compact set in $\mathbb{R}^N$. Hence the sequence $\{u_k\}$ is bounded and equicontinuous on compact subsets of $\mathbb{R}^N$. So, by the Arzela-Ascoli Theorem, the sequence $\{u_k\}$ has a uniform convergent subsequence, $\{u_k^1\}$ on the ball $B(0,1)$. Let $u^1 = \lim_{k\to\infty} u_k^1$. Then $\{f(u_k^1)\}$ converges uniformly to $f(u^1)$ on $B(0,1)$ and, by (3.35), the sequence $\{\Delta u_k^1 + |\nabla u_k^1|\}$ converges uniformly to $pf(u^1)$. Since the sum of the Laplace and Gradient operators is a closed operator, we deduce that u^1 satisfies (2.4) on $B(0,1)$.

Now, the sequence $\{u_k^1\}$ is bounded and equicontinuous on the ball $B(0,2)$, so it has a convergent subsequence $\{u_k^2\}$. Let $u^2 = \lim\limits_{k\to\infty} u_k^2$ on $B(0,2)$ and u^2 satisfies (2.4) on $B(0,2)$. Proceeding in the same way, we construct a sequence $\{u^n\}$ so that u^n satisfies (2.4) on $B(0,n)$ and $u^{n+1} = u^n$ on $B(0,n)$ for all n. Moreover, the sequence $\{u^n\}$ converges in $L^\infty_{\text{loc}}(\mathbb{R}^N)$ to the function u defined by

$$u(x) = u^m(x), \qquad \text{for } \ x \in B(0,m).$$

Since $v \le u^n \le w$ on $B(0,n)$ it follows that $v \le u \le w$ on $\mathbb{R}^N$, and u satisfies (2.4). From $v \le u$ we deduce that u is a positive entire large solution of (2.4). This completes the proof. □

4 Blow-up boundary solutions of the logistic equation

Consider the semilinear elliptic equation

$$\Delta u + au = b(x)f(u) \qquad \text{in } \Omega, \tag{4.36}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \ge 3$. Let a be a real parameter and $b \in C^{0,\mu}(\overline{\Omega})$, $0 < \mu < 1$, such that $b \ge 0$ and $b \not\equiv 0$ in Ω. Set

$$\Omega_0 = \text{int}\,\{x \in \Omega : \ b(x) = 0\}$$

and suppose, throughout, that $\overline{\Omega}_0 \subset \Omega$ and $b > 0$ on $\Omega \setminus \overline{\Omega}_0$. Assume that $f \in C^1[0,\infty)$ satisfies

(A_1) $\quad f \ge 0$ and $f(u)/u$ is increasing on $(0,\infty)$.

Following Alama and Tarantello [2], define by H_∞ the Dirichlet Laplacian on Ω_0 as the unique self-adjoint operator associated to the quadratic form $\psi(u) = \int_\Omega |\nabla u|^2\,dx$ with form domain

$$H^1_D(\Omega_0) = \{u \in H^1_0(\Omega) : \ u(x) = 0 \quad \text{for a.e. } x \in \Omega \setminus \Omega_0\}.$$

If $\partial\Omega_0$ satisfies the exterior cone condition then, according to [2], $H^1_D(\Omega_0)$ coincides with $H^1_0(\Omega_0)$ and H_∞ is the classical Laplace operator with Dirichlet condition on $\partial\Omega_0$.

Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of H_∞ in Ω_0. We understand $\lambda_{\infty,1} = \infty$ if $\Omega_0 = \emptyset$.

Set $\mu_0 := \lim_{u\searrow 0} \frac{f(u)}{u}$, $\mu_\infty := \lim_{u\to\infty} \frac{f(u)}{u}$, and denote by $\lambda_1(\mu_0)$ (resp., $\lambda_1(\mu_\infty)$) the first eigenvalue of the operator $H_{\mu_0} = -\Delta + \mu_0 b$ (resp., $H_{\mu_\infty} = -\Delta + \mu_\infty b$) in $H^1_0(\Omega)$. Recall that $\lambda_1(+\infty) = \lambda_{\infty,1}$.

Alama and Tarantello [2] proved that problem (4.36) subject to the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega \tag{4.37}$$

has a positive solution u_a if and only if $a \in (\lambda_1(\mu_0), \lambda_1(\mu_\infty))$. Moreover, u_a is the unique positive solution for (4.36)+(4.37) (see [2, Theorem A (bis)]). We shall refer to the combination of (4.36)+(4.37) as problem (E_a).

Our first aim in this section is to give a corresponding necessary and sufficient condition, but for the existence of *large* (or *explosive*) solutions of (4.36). An elementary argument based on the maximum principle shows that if such a solution exists, then it is *positive* even if f satisfies a weaker condition than (A_1), namely

$(A_1)'$ $f(0) = 0$, $f' \geq 0$ and $f > 0$ on $(0, \infty)$.

We recall that Keller [63] and Osserman [79] supplied a necessary and sufficient condition on f for the existence of large solutions to (1) when $a \equiv 0$, $b \equiv 1$ and f is assumed to fulfill $(A_1)'$. More precisely, f must satisfy the Keller-Osserman condition (see [63, 79]),

(A_2) $\displaystyle\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty\,, \quad \text{where } F(t) = \int_0^t f(s)\,ds.$

Typical examples of non-linearities satisfying (A_1) and (A_2) are:

(i) $f(u) = e^u - 1$; (ii) $f(u) = u^p$, $p > 1$; (iii) $f(u) = u[\ln(u+1)]^p$, $p > 2$.

Our first result gives the maximal interval for the parameter a that ensures the existence of large solutions to problem (4.36). More precisely, we prove

Theorem 4.1. *Assume that f satisfies conditions (A_1) and (A_2). Then problem (4.36) has a large solution if and only if $a \in (-\infty, \lambda_{\infty,1})$.*

We point out that our framework in the above result includes the case when b vanishes at some points on $\partial\Omega$, or even if $b \equiv 0$ on $\partial\Omega$. This later case includes the "competition" $0 \cdot \infty$ on $\partial\Omega$. We also point out that, under our hypotheses, $\mu_\infty := \lim_{u\to\infty} f(u)/u = \lim_{u\to\infty} f'(u) = \infty$. Indeed, by l'Hospital's rule, $\lim_{u\to\infty} F(u)/u^2 = \mu_\infty/2$. But, by (A_2), we deduce that $\mu_\infty = \infty$. Then, by (A_1) we find that $f'(u) \geq f(u)/u$ for any $u > 0$, which shows that $\lim_{u\to\infty} f'(u) = \infty$.

Before giving the proof of Theorem 4.1 we claim that assuming (A_1), then problem (4.36) can have large solutions only if f satisfies the Keller-Osserman condition (A_2). Indeed, suppose that problem (4.36) has a large solution u_∞. Set $\tilde{f}(u) = |a|u + \|b\|_\infty f(u)$ for $u \geq 0$. Notice that $\tilde{f} \in C^1[0,\infty)$ satisfies $(A_1)'$. For any $n \geq 1$, consider the problem

$$\begin{cases} \Delta u = \tilde{f}(u) & \text{in } \Omega\,, \\ u = n & \text{on } \partial\Omega\,, \\ u \geq 0 & \text{in } \Omega\,. \end{cases}$$

A standard argument based on the maximum principle shows that this problem has a unique solution, say u_n, which, moreover, is positive in $\overline{\Omega}$. Applying again the maximum principle we deduce that $0 < u_n \le u_{n+1} \le u_\infty$, in Ω, for all $n \ge 1$. Thus, for every $x \in \Omega$, we can define $\bar{u}(x) = \lim_{n\to\infty} u_n(x)$. Moreover, since (u_n) is uniformly bounded on every compact subset of Ω, standard elliptic regularity arguments show that $\bar{u}$ is a positive large solution of the problem $\Delta u = \tilde{f}(u)$. It follows that $\tilde{f}$ satisfies the Keller-Osserman condition (A_2). Then, by (A_1), $\mu_\infty := \lim_{u\to\infty} f(u)/u > 0$ which yields $\lim_{u\to\infty} \tilde{f}(u)/f(u) = |a|/\mu_\infty + \|b\|_\infty < \infty$. Consequently, our claim follows.

Proof of Theorem 4.1. A. NECESSARY CONDITION. Let u_∞ be a large solution of problem (4.36). Then, by the maximum principle, u_∞ is positive. Suppose $\lambda_{\infty,1}$ is finite. Arguing by contradiction, let us assume $a \ge \lambda_{\infty,1}$. Set $\lambda \in (\lambda_1(\mu_0), \lambda_{\infty,1})$ and denote by u_λ the unique positive solution of problem (E_a) with $a = \lambda$. We have

$$\begin{cases} \Delta(Mu_\infty) + \lambda_{\infty,1}(Mu_\infty) \le b(x)f(Mu_\infty) & \text{in } \Omega\,, \\ Mu_\infty = \infty & \text{on } \partial\Omega\,, \\ Mu_\infty \ge u_\lambda & \text{in } \Omega\,, \end{cases}$$

where $M := \max\left\{\max_{\overline{\Omega}} u_\lambda / \min_\Omega u_\infty;\, 1\right\}$. By the sub-super solution method we conclude that problem (E_a) with $a = \lambda_{\infty,1}$ has at least a positive solution (between u_λ and Mu_∞). But this is a contradiction. So, necessarily, $a \in (-\infty, \lambda_{\infty,1})$.

B. SUFFICIENT CONDITION. This will be proved with the aid of several results.

Lemma 4.2. *Let ω be a smooth bounded domain in $\mathbb{R}^N$. Assume p, q, r are $C^{0,\mu}$-functions on $\overline{\omega}$ such that $r \ge 0$ and $p > 0$ in $\overline{\omega}$. Then for any non-negative function $0 \not\equiv \Phi \in C^{0,\mu}(\partial\omega)$ the boundary value problem*

$$\begin{cases} \Delta u + q(x)u = p(x)f(u) - r(x) & \text{in } \omega, \\ u > 0 & \text{in } \omega, \\ u = \Phi & \text{on } \partial\omega, \end{cases} \tag{4.38}$$

has a unique solution.

We refer to Cîrstea and Rădulescu [27, Lemma 3.1] for the proof of the above result.

Under the assumptions of Lemma 4.2 we obtain the following result which generalizes [75, Lemma 1.3].

Corollary 4.3. *There exists a positive large solution of the problem*

$$\Delta u + q(x)u = p(x)f(u) - r(x) \qquad \text{in } \omega. \tag{4.39}$$

Proof. Set $\Phi = n$ and let u_n be the unique solution of (4.38). By the maximum principle, $u_n \le u_{n+1} \le \overline{u}$ in ω, where $\overline{u}$ denotes a large solution of

$$\Delta u + \|q\|_\infty u = p_0 f(u) - \bar{r} \quad \text{in } \omega.$$

Thus $\lim_{n\to\infty} u_n(x) = u_\infty(x)$ exists and is a positive large solution of (4.39). Furthermore, every positive large solution of (4.39) dominates u_∞, i.e., the solution u_∞ is the *minimal large solution*. This follows from the definition of u_∞ and the maximum principle. ☐

Lemma 4.4. *If* $0 \not\equiv \Phi \in C^{0,\mu}(\partial\Omega)$ *is a non-negative function and* $b > 0$ *on* $\partial\Omega$*, then the boundary value problem*

$$\begin{cases} \Delta u + au = b(x) f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Phi & \text{on } \partial\Omega, \end{cases} \tag{4.40}$$

has a solution if and only if $a \in (-\infty, \lambda_{\infty,1})$*. Moreover, in this case, the solution is unique.*

Proof. The first part follows exactly in the same way as the proof of Theorem 4.1 (necessary condition).

For the sufficient condition, fix $a < \lambda_{\infty,1}$ and let $\lambda_{\infty,1} > \lambda_* > \max\{a, \lambda_1(\mu_0)\}$. Let u_* be the unique positive solution of (E_a) with $a = \lambda_*$.

Let Ω_i $(i = 1, 2)$ be subdomains of Ω such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ and $\Omega \setminus \overline{\Omega}_1$ is smooth. We define $u_+ \in C^2(\Omega)$ as a positive function in Ω such that $u_+ \equiv u_\infty$ on $\Omega \setminus \Omega_2$ and $u_+ \equiv u_*$ on Ω_1. Here u_∞ denotes a positive large solution of (4.39) for $p(x) = b(x)$, $r(x) = 0$, $q(x) = a$ and $\omega = \Omega \setminus \overline{\Omega}_1$. So, since $b_0 := \inf_{\Omega_2\setminus\Omega_1} b$ is positive, it is easy to check that if $C > 0$ is large enough then $\overline{v}_\Phi = Cu_+$ satisfies

$$\begin{cases} \Delta \overline{v}_\Phi + a\overline{v}_\Phi \le b(x) f(\overline{v}_\Phi) & \text{in } \Omega\,, \\ \overline{v}_\Phi = \infty & \text{on } \partial\Omega\,. \\ \overline{v}_\Phi \ge \max\limits_{\partial\Omega} \Phi & \text{in } \Omega\,. \end{cases}$$

Let $\underline{v}_\Phi$ be the unique classical solution of the problem

$$\begin{cases} \Delta \underline{v}_\Phi = |a|\underline{v}_\Phi + \|b\|_\infty f(\underline{v}_\Phi) & \text{in } \Omega, \\ \underline{v}_\Phi > 0 & \text{in } \Omega, \\ \underline{v}_\Phi = \Phi & \text{on } \partial\Omega\,. \end{cases}$$

It is clear that $\underline{v}_\Phi$ is a positive sub-solution of (4.40) and $\underline{v}_\Phi \le \max_{\partial\Omega} \Phi \le \overline{v}_\Phi$ in Ω. Therefore, by the sub-super solution method, problem (4.40) has at least a solution v_Φ between $\underline{v}_\Phi$ and $\overline{v}_\Phi$.

Next, the uniqueness of solution to (4.40) can be obtained by using essentially the same technique as in [15, Theorem 1] or [14, Appendix II]. □

Proof of Theorem 4.1 completed. Fix $a \in (-\infty, \lambda_{\infty,1})$. Two cases may occur:

CASE 1: $b > 0$ on $\partial\Omega$. Denote by v_n the unique solution of (4.40) with $\Phi \equiv n$. For $\Phi \equiv 1$, set $v := \underline{v}_\Phi$ and $V := \overline{v}_\Phi$, where $\underline{v}_\Phi$ and $\overline{v}_\Phi$ are defined in the proof of Lemma 4.4. The sub and super-solutions method combined with the uniqueness of solution of (4.40) shows that $v \leq v_n \leq v_{n+1} \leq V$ in Ω. Hence $v_\infty(x) := \lim_{n\to\infty} v_n(x)$ exists and is a positive large solution of (4.36).

CASE 2: $b \geq 0$ on $\partial\Omega$. Let z_n $(n \geq 1)$ be the unique solution of (4.38) for $p \equiv b + 1/n$, $r \equiv 0$, $q \equiv a$, $\Phi \equiv n$ and $\omega = \Omega$. By the maximum principle, (z_n) is non-decreasing. Moreover, (z_n) is uniformly bounded on every compact subdomain of Ω. Indeed, if $K \subset \Omega$ is an arbitrary compact set, then $d := \operatorname{dist}(K, \partial\Omega) > 0$. Choose $\delta \in (0, d)$ small enough so that $\overline{\Omega}_0 \subset C_\delta$, where $C_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$. Since $b > 0$ on ∂C_δ, Case 1 allows us to define z_+ as a positive large solution of (4.36) for $\Omega = C_\delta$. Using A standard argument based on the maximum principle implies that $z_n \leq z_+$ in C_δ, for all $n \geq 1$. So, (z_n) is uniformly bounded on K. By the monotonicity of (z_n), we conclude that $z_n \to \underline{z}$ in $L^\infty_{\mathrm{loc}}(\Omega)$. Finally, standard elliptic regularity arguments lead to $z_n \to \underline{z}$ in $C^{2,\mu}(\Omega)$. This completes the proof of Theorem 4.1. □

Denote by $\mathcal{D}$ and $\mathcal{R}$ the boundary operators

$$\mathcal{D}u := u \qquad \text{and} \qquad \mathcal{R}u := \partial_\nu u + \beta(x)u,$$

where ν is the unit outward normal to $\partial\Omega$, and $\beta \in C^{1,\mu}(\partial\Omega)$ is non-negative. Hence, $\mathcal{D}$ is the *Dirichlet* boundary operator and $\mathcal{R}$ is either the *Neumann* boundary operator, if $\beta \equiv 0$, or the *Robin* boundary operator, if $\beta \not\equiv 0$. Throughout this work, $\mathcal{B}$ can define any of these boundary operators.

Note that the Robin condition $\mathcal{R} = 0$ relies essentially to heat flow problems in a body with constant temperature in the surrounding medium. More generally, if α and β are smooth functions on $\partial\Omega$ such that $\alpha, \beta \geq 0$, $\alpha + \beta > 0$, then the boundary condition $Bu = \alpha\partial_\nu u + \beta u = 0$ represents the exchange of heat at the surface of the reactant by Newtonian cooling. Moreover, the boundary condition $Bu = 0$ is called isothermal (Dirichlet) condition if $\alpha \equiv 0$, and it becomes an adiabatic (Neumann) condition if $\beta \equiv 0$. An intuitive meaning of the condition $\alpha + \beta > 0$ on $\partial\Omega$ is that, for the diffusion process described by problem (4.36), either the reflection phenomenon or the absorption phenomenon may occur at each point of the boundary.

We are now concerned with the following boundary blow-up problem

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega \setminus \overline{\Omega}_0\,, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega\,, \\ u = \infty & \text{on } \partial\Omega_0\,, \end{cases} \tag{4.41}$$

where $b > 0$ on $\partial\Omega$, while $\overline{\Omega}_0$ is non-empty, connected and with smooth boundary. Here, $u = \infty$ on $\partial\Omega_0$ means that $u(x) \to \infty$ as $x \in \Omega \setminus \overline{\Omega}_0$ and $d(x) := \text{dist}\,(x, \Omega_0) \to 0$.

The question of existence and uniqueness of positive solutions for problem (4.41) in the case of pure superlinear power in the non-linearity is treated by Du-Huang [40]. Our next results extend their previous paper to the case of much more general non-linearities of Keller-Osserman type.

In the following, by $(\tilde{A}_1)$ we mean that (A_1) is fulfilled and there exists $\lim_{u\to\infty} (F/f)'(u) := \gamma$. Then, $\gamma \geq 0$.

We prove

Theorem 4.5. *Let $(\tilde{A}_1)$ and (A_2) hold. Then, for any $a \in \mathbb{R}$, problem (4.41) has a minimal (resp., maximal) positive solution $\underline{U}_a$ (resp., $\overline{U}_a$).*

Proof. In proving Theorem 4.5 we rely on an appropriate comparison principle which allows us to prove that $(u_n)_{n\geq 1}$ is non-decreasing, where u_n is the unique positive solution of problem (4.43) with $\Phi \equiv n$. The minimal positive solution of (4.41) will be obtained as the limit of the sequence $(u_n)_{n\geq 1}$. Note that, since $b = 0$ on $\partial\Omega_0$, the main difficulty is related to the construction of an upper bound of this sequence which must fit to our general framework. Next, we deduce the maximal positive solution of (4.41) as the limit of the non-increasing sequence $(v_m)_{m\geq m_1}$ provided m_1 is large so that $\Omega_{m_1} \subset\subset \Omega$. We denoted by v_m the minimal positive solution of (4.41) with Ω_0 replaced by

$$\Omega_m := \{x \in \Omega : \; d(x) < 1/m\}, \qquad m \geq m_1. \tag{4.42}$$

We start with the following auxiliary result (see Cîrstea and Rădulescu [27]).

Lemma 4.6. *Assume $b > 0$ on $\partial\Omega$. If (A_1) and (A_2) hold, then for any positive function $\Phi \in C^{2,\mu}(\partial\Omega_0)$ and $a \in \mathbb{R}$ the problem*

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega \setminus \overline{\Omega}_0\,, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega\,, \\ u = \Phi & \text{on } \partial\Omega_0\,, \end{cases} \tag{4.43}$$

has a unique positive solution.

We now come back to the proof of Theorem 4.5, that will be divided into two steps:

Step 1. Existence of the minimal positive solution for problem (4.41).

For any $n \geq 1$, let u_n be the unique positive solution of problem (4.43) with $\Phi \equiv n$. By the maximum principle, $u_n(x)$ increases with n for all $x \in \overline{\Omega} \setminus \overline{\Omega}_0$. Moreover, we prove

Lemma 4.7. *The sequence $(u_n(x))_n$ is bounded from above by some function $V(x)$ which is uniformly bounded on all compact subsets of $\overline{\Omega} \setminus \overline{\Omega}_0$.*

Proof. Let b^* be a C^2-function on $\overline{\Omega} \setminus \Omega_0$ such that

$$0 < b^*(x) \le b(x) \quad \forall x \in \overline{\Omega} \setminus \overline{\Omega}_0.$$

For x bounded away from $\partial\Omega_0$ is not a problem to find such a function b^*. For x satisfying $0 < d(x) < \delta$ with $\delta > 0$ small such that $x \to d(x)$ is a C^2-function, we can take

$$b^*(x) = \int_0^{d(x)} \int_0^t [\min_{d(z)\ge s} b(z)]\, ds\, dt.$$

Let $g \in \mathcal{G}$ be a function such that (A_g) holds. Since $b^*(x) \to 0$ as $d(x) \searrow 0$, we deduce, by (A_1), the existence of some $\delta > 0$ such that for all $x \in \Omega$ with $0 < d(x) < \delta$ and $\xi > 1$

$$\frac{b^*(x) f(g(b^*(x))\xi)}{g''(b^*(x))\, \xi} > \sup_{\overline{\Omega}\setminus\Omega_0} |\nabla b^*|^2 + \frac{g'(b^*(x))}{g''(b^*(x))} \inf_{\overline{\Omega}\setminus\Omega_0} (\Delta b^*) + a\, \frac{g(b^*(x))}{g''(b^*(x))}.$$

Here, $\delta > 0$ is taken sufficiently small so that $g'(b^*(x)) < 0$ and $g''(b^*(x)) > 0$ for all x with $0 < d(x) < \delta$.

For $n_0 \ge 1$ fixed, define V^* as follows

(i) $\quad V^*(x) = u_{n_0}(x) + 1$ for $x \in \overline{\Omega}$ and near $\partial\Omega$;

(ii) $\quad V^*(x) = g(b^*(x))$ for x satisfying $0 < d(x) < \delta$;

(iii) $\quad V^* \in C^2(\overline{\Omega} \setminus \overline{\Omega}_0)$ is positive on $\overline{\Omega} \setminus \overline{\Omega}_0$.

We show that for $\xi > 1$ large enough the upper bound of the sequence $(u_n(x))_n$ can be taken as $V(x) = \xi V^*(x)$. Since

$$\mathcal{B}V(x) = \xi\, \mathcal{B}V^*(x) \ge \xi \min\{1, \beta(x)\} \ge 0, \quad \forall x \in \partial\Omega \quad \text{and} \quad \lim_{d(x)\searrow 0} [u_n(x) - V(x)] = -\infty < 0,$$

to conclude that $u_n(x) \le V(x)$ for all $x \in \overline{\Omega} \setminus \overline{\Omega}_0$ it is sufficient to show that

$$-\Delta V(x) \ge aV(x) - b(x) f(V(x)), \qquad \forall x \in \Omega \setminus \overline{\Omega}_0. \tag{4.44}$$

For $x \in \Omega$ satisfying $0 < d(x) < \delta$ and $\xi > 1$ we have

$$\begin{aligned} -\Delta V(x) - aV(x) + b(x) f(V(x)) &= -\xi \Delta g(b^*(x)) - a\,\xi g(b^*(x)) + b(x) f(g(b^*(x))\xi) \\ &\ge \xi g''(b^*(x)) \left(-\frac{g'(b^*(x))}{g''(b^*(x))}\, \Delta b^*(x) - |\nabla b^*(x)|^2 - a\, \frac{g(b^*(x))}{g''(b^*(x))} + b^*(x)\, \frac{f(g(b^*(x))\xi)}{g''(b^*(x))\,\xi} \right) > 0. \end{aligned}$$

For $x \in \Omega$ satisfying $d(x) \ge \delta$,

$$-\Delta V(x) - aV(x) + b(x) f(V(x)) = \xi \left(-\Delta V^*(x) - aV^*(x) + b(x)\, \frac{f(\xi V^*(x))}{\xi} \right) \ge 0$$

for ξ sufficiently large. It follows that (4.44) is fulfilled provided ξ is large enough. This finishes the proof of the lemma. □

By Lemma 4.7, $\underline{U}_a(x) \equiv \lim_{n\to\infty} u_n(x)$ exists, for any $x \in \overline{\Omega} \setminus \overline{\Omega}_0$. Moreover, $\underline{U}_a$ is a positive solution of (4.41). Using the maximum principle once more, we find that any positive solution u

of (4.41) satisfies $u \geq u_n$ on $\overline{\Omega} \setminus \overline{\Omega}_0$, for all $n \geq 1$. Hence $\underline{U}_a$ is the minimal positive solution of (4.41).

Proof of Theorem 4.5 completed.

Step 2. Existence of the maximal positive solution for problem (4.41).

Lemma 4.8. *If Ω_0 is replaced by Ω_m defined in (4.42), then problem (4.41) has a minimal positive solution provided that (A_1) and (A_2) are fulfilled.*

Proof. The argument used here (more easier, since $b > 0$ on $\overline{\Omega} \setminus \Omega_m$) is similar to that in Step 1. The only difference which appears in the proof (except the replacement of Ω_0 by Ω_m) is related to the construction of $V^*(x)$ for x near $\partial\Omega_m$. Here, we use our Theorem 4.1 which says that, for any $a \in \mathbb{R}$, there exists a positive large solution $u_{a,\infty}$ of problem (4.36) in the domain $\Omega \setminus \overline{\Omega}_m$. We define $V^*(x) = u_{a,\infty}(x)$ for $x \in \Omega \setminus \overline{\Omega}_m$ and near $\partial\Omega_m$. For $\xi > 1$ and $x \in \Omega \setminus \overline{\Omega}_m$ near $\partial\Omega_m$ we have

$$
\begin{aligned}
-\Delta V(x) - aV(x) + b(x)f(V(x)) &= -\xi\Delta V^*(x) - a\xi V^*(x) + b(x)f(\xi V^*(x)) \\
&= b(x)[f(\xi V^*(x)) - \xi f(V^*(x)] \geq 0.
\end{aligned}
$$

This completes the proof. □

Let v_m be the minimal positive solution for the problem considered in the statement of Lemma 4.8. By the maximum principle, $v_m \geq v_{m+1} \geq u$ on $\overline{\Omega} \setminus \overline{\Omega}_m$, where u is any positive solution of (4.41). Hence $\overline{U}_a(x) := \lim_{m\to\infty} v_m(x) \geq u(x)$. A regularity and compactness argument shows that $\overline{U}_a$ is a positive solution of (4.41). Consequently, $\overline{U}_a$ is the maximal positive solution. This concludes the proof of Theorem 4.5. □

The next question is whether one can conclude the uniqueness of positive solutions of problem (4.41). We recall first what is already known in this direction. When $f(u) = u^p$, $p > 1$, Du-Huang [40] proved the uniqueness of solution to problem (4.41) and established its behaviour near $\partial\Omega_0$, under the assumption

$$
\lim_{d(x)\searrow 0} \frac{b(x)}{[d(x)]^\tau} = c \quad \text{for some positive constants } \tau, c > 0. \tag{4.45}
$$

We shall give a general uniqueness result provided that b and f satisfy the following assumptions:

(B_1) $\displaystyle\lim_{d(x)\searrow 0} \frac{b(x)}{k(d(x))} = c$ for some constant $c > 0$, where $0 < k \in C^1(0, \delta_0)$ is increasing and satisfies

(B_2) $\displaystyle K(t) = \frac{\int_0^t \sqrt{k(s)}\, ds}{\sqrt{k(t)}} \in C^1[0, \delta_0)$, for some $\delta_0 > 0$.

Assume there exist $\zeta > 0$ and $t_0 \geq 1$ such that

(A_3) $f(\xi t) \leq \xi^{1+\zeta} f(t), \quad \forall \xi \in (0, 1), \quad \forall t \geq t_0/\xi$

(A_4) the mapping $(0,1] \ni \xi \longmapsto A(\xi) = \lim_{u\to\infty} \dfrac{f(\xi u)}{\xi f(u)}$ is a continuous positive function.

Our uniqueness result is

Theorem 4.9. *Assume the conditions $(\tilde{A}_1)$ with $\gamma \neq 0$, (A_3), (A_4), (B_1) and (B_2) hold. Then, for any $a \in \mathbb{R}$, problem (4.41) has a unique positive solution U_a. Moreover,*

$$\lim_{d(x)\searrow 0} \frac{U_a(x)}{h(d(x))} = \xi_0,$$

where h is defined by

$$\int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t \sqrt{k(s)}\, ds, \quad \forall t \in (0, \delta_0) \tag{4.46}$$

and ξ_0 is the unique positive solution of $A(\xi) = \dfrac{K'(0)(1-2\gamma)+2\gamma}{c}$.

Remark 2. *(a)* $(A_1) + (A_3) \Rightarrow (A_2)$. Indeed, $\lim_{u\to\infty} \frac{f(u)}{u^{1+\zeta}} > 0$ since $\frac{f(t)}{t^{1+\zeta}}$ is non-decreasing for $t \geq t_0$.
(b) $K'(0)(1-2\gamma)+2\gamma \in (0,1]$ *when $(\tilde{A}_1)$ with $\gamma \neq 0$, (A_2), (B_1) and (B_2) hold.*
(c) The function $(0,\infty) \ni \xi \longmapsto A(\xi) \in (0,\infty)$ is bijective when (A_3) and (A_4) hold (see Lemma 4.10).

Among the non-linearities f that satisfy the assumptions of Theorem 4.9 we note: (i) $f(u) = u^p$, $p > 1$; (ii) $f(u) = u^p \ln(u+1)$, $p > 1$; (iii) $f(u) = u^p \arctan u$, $p > 1$.

Proof of Theorem 4.9. By (A_4) we deduce that the mapping $(0,\infty) \ni \xi \longmapsto A(\xi) = \lim\limits_{u\to\infty} \dfrac{f(\xi u)}{\xi f(u)}$ is a continuous positive function, since $A(1/\xi) = 1/A(\xi)$ for any $\xi \in (0,1)$. Moreover, we claim

Lemma 4.10. *The function $A : (0,\infty) \to (0,\infty)$ is bijective, provided that (A_3) and (A_4) are fulfilled.*

Proof. By the continuity of A, we see that the surjectivity of A follows if we prove that $\lim_{\xi\searrow 0} A(\xi) = 0$. To this aim, let $\xi \in (0,1)$ be fixed. Using (A_3) we find

$$\frac{f(\xi u)}{\xi f(u)} \leq \xi^{\zeta}, \quad \forall u \geq \frac{t_0}{\xi}$$

which yields $A(\xi) \leq \xi^{\zeta}$. Since $\xi \in (0,1)$ is arbitrary, it follows that $\lim_{\xi\searrow 0} A(\xi) = 0$.

We now prove that the function $\xi \longmapsto A(\xi)$ is increasing on $(0,\infty)$ which concludes our lemma. Let $0 < \xi_1 < \xi_2 < \infty$ be chosen arbitrarily. Using assumption (A_3) once more, we obtain

$$f(\xi_1 u) = f\left(\frac{\xi_1}{\xi_2}\xi_2 u\right) \leq \left(\frac{\xi_1}{\xi_2}\right)^{1+\zeta} f(\xi_2 u), \quad \forall u \geq t_0 \frac{\xi_2}{\xi_1}.$$

It follows that

$$\frac{f(\xi_1 u)}{\xi_1 f(u)} \leq \left(\frac{\xi_1}{\xi_2}\right)^{\zeta} \frac{f(\xi_2 u)}{\xi_2 f(u)}, \quad \forall u \geq t_0 \frac{\xi_2}{\xi_1}.$$

Passing to the limit as $u \to \infty$ we find

$$A(\xi_1) \leq \left(\frac{\xi_1}{\xi_2}\right)^{\zeta} A(\xi_2) < A(\xi_2),$$

which finishes the proof. $\square$

Proof of Theorem 4.9 completed. Set $\Pi(\xi) = \lim_{d(x)\searrow 0} b(x)\,\frac{f(h(d(x))\xi)}{h''(d(x))\,\xi}$, for any $\xi > 0$. Using (B_1) we find

$$\begin{aligned}\Pi(\xi) &= \lim_{d(x)\searrow 0} \frac{b(x)}{k(d(x))}\,\frac{k(d(x))f(h(d(x)))}{h''(d(x))}\,\frac{f(h(d(x))\xi)}{\xi f(h(d(x)))} = c\,\lim_{t\searrow 0}\frac{k(t)f(h(t))}{h''(t)}\,\lim_{u\to\infty}\frac{f(\xi u)}{\xi f(u)}\\ &= \frac{c}{K'(0)(1-2\gamma)+2\gamma}\,A(\xi).\end{aligned}$$

This and Lemma 4.10 imply that the function $\Pi : (0,\infty) \to (0,\infty)$ is bijective. Let ξ_0 be the unique positive solution of $\Pi(\xi) = 1$, that is $A(\xi_0) = \dfrac{K'(0)(1-2\gamma)+2\gamma}{c}$.

For $\varepsilon \in (0, 1/4)$ arbitrary, we denote $\xi_1 = \Pi^{-1}(1-4\varepsilon)$, respectively $\xi_2 = \Pi^{-1}(1+4\varepsilon)$.

We choose $\delta > 0$ small enough such that

(i) $\operatorname{dist}(x, \partial\Omega_0)$ is a C^2 function on the set $\{x \in \Omega :\ \operatorname{dist}(x, \partial\Omega_0) \leq 2\delta\}$;

(ii) $\left|\dfrac{h'(s)}{h''(s)}\,\Delta d(x) + a\,\dfrac{h(s)}{h''(s)}\right| < \varepsilon$ and $h''(s) > 0$ for all $s \in (0, 2\delta)$ and x satisfying $0 < d(x) < 2\delta$;

(iii) $(\Pi(\xi_2)-\varepsilon)\,\dfrac{h''(d(x))\,\xi_2}{f(h(d(x))\xi_2)} \leq b(x) \leq (\Pi(\xi_1)+\varepsilon)\,\dfrac{h''(d(x))\,\xi_1}{f(h(d(x))\xi_1)}$, for every x with $0 < d(x) < 2\delta$.

(iv) $b(y) < (1+\varepsilon)b(x)$, for every x, y with $0 < d(y) < d(x) < 2\delta$.

Let $\sigma \in (0, \delta)$ be arbitrary. We define $\underline{v}_\sigma(x) = h(d(x)+\sigma)\xi_1$, for any x with $d(x)+\sigma < 2\delta$, respectively $\overline{v}_\sigma(x) = h(d(x)-\sigma)\xi_2$ for any x with $\sigma < d(x) < 2\delta$.

Using (ii), (iv) and the first inequality in (iii), when $\sigma < d(x) < 2\delta$, we obtain (since $|\nabla d(x)| \equiv 1$)

$$\begin{aligned}&-\Delta\overline{v}_\sigma(x) - a\overline{v}_\sigma(x) + b(x)f(\overline{v}_\sigma(x))\\ &\quad= \xi_2\left(-h'(d(x)-\sigma)\,\Delta d(x) - h''(d(x)-\sigma) - a\,h(d(x)-\sigma) + \frac{b(x)f(h(d(x)-\sigma)\xi_2)}{\xi_2}\right)\\ &\quad= \xi_2\,h''(d(x)-\sigma)\left(-\frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)}\,\Delta d(x) - a\,\frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} - 1 + \frac{b(x)f(h(d(x)-\sigma)\xi_2)}{h''(d(x)-\sigma)\,\xi_2}\right)\\ &\quad\geq \xi_2\,h''(d(x)-\sigma)\left(-\frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)}\,\Delta d(x) - a\,\frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} - 1 + \frac{\Pi(\xi_2)-\varepsilon}{1+\varepsilon}\right) \geq 0\end{aligned}$$

for all x satisfying $\sigma < d(x) < 2\delta$.

Similarly, using (ii), (iv) and the second inequality in (iii), when $d(x)+\sigma<2\delta$ we find

$$\begin{aligned}
&-\Delta\underline{v}_\sigma(x)-a\underline{v}_\sigma(x)+b(x)f(\underline{v}_\sigma(x))\\
&\quad=\xi_1\,h''(d(x)+\sigma)\left(-\frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)}\,\Delta d(x)-a\,\frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)}-1+\frac{b(x)f(h(d(x)+\sigma)\xi_1)}{h''(d(x)+\sigma)\,\xi_1}\right)\\
&\quad\le\xi_1\,h''(d(x)+\sigma)\left(-\frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)}\,\Delta d(x)-a\,\frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)}-1+(1+\varepsilon)(\Pi(\xi_1)+\varepsilon)\right)\le 0,
\end{aligned}$$

for all x satisfying $d(x)+\sigma<2\delta$.

Define $\Omega_\delta\equiv\{x\in\Omega:\ d(x)<\delta\}$. Let $\omega\subset\subset\Omega_0$ be such that the first Dirichlet eigenvalue of $(-\Delta)$ in the smooth domain $\Omega_0\setminus\overline{\omega}$ is strictly greater than a. Denote by w a positive large solution to the following problem

$$-\Delta w=aw-p(x)f(w)\qquad\text{in }\Omega_\delta\,,$$

where $p\in C^{0,\mu}(\overline{\Omega}_\delta)$ satisfies $0<p(x)\le b(x)$ for $x\in\overline{\Omega}_\delta\setminus\overline{\Omega}_0$, $p(x)=0$ on $\overline{\Omega}_0\setminus\omega$ and $p(x)>0$ for $x\in\omega$. The existence of w is guaranteed by our Theorem 4.1.

Suppose that u is an arbitrary solution of (4.41) and let $v:=u+w$. Then v satisfies

$$-\Delta v\ge av-b(x)f(v)\quad\text{in }\Omega_\delta\setminus\overline{\Omega}_0.$$

Since

$$v|_{\partial\Omega_0}=\infty>\underline{v}_\sigma|_{\partial\Omega_0}\quad\text{and}\quad v|_{\partial\Omega_\delta}=\infty>\underline{v}_\sigma|_{\partial\Omega_\delta},$$

we find

$$u+w\ge\underline{v}_\sigma\quad\text{on }\Omega_\delta\setminus\overline{\Omega}_0.\tag{4.47}$$

Similarly

$$\overline{v}_\sigma+w\ge u\quad\text{on }\Omega_\delta\setminus\overline{\Omega}_\sigma.\tag{4.48}$$

Letting $\sigma\to 0$ in (4.47) and (4.48), we deduce

$$h(d(x))\,\xi_2+2w\ge u+w\ge h(d(x))\,\xi_1,\quad\forall x\in\Omega_\delta\setminus\overline{\Omega}_0.$$

Since w is uniformly bounded on $\partial\Omega_0$, it follows that

$$\xi_1\le\liminf_{d(x)\searrow 0}\frac{u(x)}{h(d(x))}\le\limsup_{d(x)\searrow 0}\frac{u(x)}{h(d(x))}\le\xi_2.\tag{4.49}$$

Letting $\varepsilon\to 0$ in (4.49) and looking at the definition of ξ_1 respectively ξ_2 we find

$$\lim_{d(x)\searrow 0}\frac{u(x)}{h(d(x))}=\xi_0.\tag{4.50}$$

This behavior of the solution will be speculated in order to prove that problem (4.41) has a unique solution. Indeed, let u_1, u_2 be two positive solutions of (4.41). For any $\varepsilon>0$, denote $\tilde{u}_i=(1+\varepsilon)\,u_i$, $i=1,2$. By virtue of (4.50) we get

$$\lim_{d(x)\searrow 0}\frac{u_1(x)-\tilde{u}_2(x)}{h(d(x))}=\lim_{d(x)\searrow 0}\frac{u_2(x)-\tilde{u}_1(x)}{h(d(x))}=-\varepsilon\,\xi_0<0$$

which implies

$$\lim_{d(x)\searrow 0}[u_1(x) - \tilde{u}_2(x)] = \lim_{d(x)\searrow 0}[u_2(x) - \tilde{u}_1(x)] = -\infty.$$

On the other hand, since $\frac{f(u)}{u}$ is increasing for $u > 0$, we obtain

$$-\Delta\tilde{u}_i = -(1+\varepsilon)\,\Delta u_i = (1+\varepsilon)\,(a\,u_i - b(x)f(u_i)) \geq a\,\tilde{u}_i - b(x)f(\tilde{u}_i) \quad \text{in } \Omega\setminus\overline{\Omega}_0,$$

$$\mathcal{B}\tilde{u}_i = \mathcal{B}u_i = 0 \quad \text{on } \partial\Omega.$$

So, by the maximum principle,

$$u_1(x) \leq \tilde{u}_2(x), \quad u_2(x) \leq \tilde{u}_1(x), \quad \forall x \in \Omega\setminus\overline{\Omega}_0.$$

Letting $\varepsilon \to 0$, we obtain $u_1 \equiv u_2$. The proof of Theorem 4.9 is complete. □

The above results have been established by Cîrstea and Rădulescu [27, 29].

4.1 Uniqueness and asymptotic behaviour of the large solution. A Karamata regular variation theory approach

The major purpose in this section is to advance innovative methods to study the uniqueness and asymptotic behavior of large solutions of (4.36). This approach is due to Cîrstea and Rădulescu [25, 28, 30, 31, 32] and it relies essentially on the *regular variation theory* introduces by Karamata (see Bingham, Goldie, and Teugels [13], Karamata [72]), not only in the statement but in the proof as well. This enables us to obtain significant information about the qualitative behavior of the large solution to (4.36) in a general framework that removes previous restrictions in the literature.

Definition 4.11. *A positive measurable function R defined on $[D,\infty)$, for some $D > 0$, is called regularly varying (at infinity) with index $q \in \mathbb{R}$ (written $R \in RV_q$) if for all $\xi > 0$*

$$\lim_{u\to\infty} R(\xi u)/R(u) = \xi^q.$$

When the index of regular variation q is zero, we say that the function is slowly varying.

We remark that any function $R \in RV_q$ can be written in terms of a slowly varying function. Indeed, set $R(u) = u^q L(u)$. From the above definition we easily deduce that L varies slowly.

The canonical q-varying function is u^q. The functions $\ln(1+u)$, $\ln\ln(e+u)$, $\exp\{(\ln u)^\alpha\}$, $\alpha \in (0,1)$ vary slowly, as well as any measurable function on $[D,\infty)$ with positive limit at infinity.

In what follows L denotes an arbitrary slowly varying function and $D > 0$ a positive number. For details on the below properties, we refer to Seneta [85].

Proposition 4.12. *(i) For any $m > 0$, $u^m L(u) \to \infty$, $\;u^{-m}L(u) \to 0$ as $u \to \infty$.*

(ii) Any positive C^1-function on $[D,\infty)$ satisfying $uL_1'(u)/L_1(u) \to 0$ as $u \to \infty$ is slowly varying. Moreover, if the above limit is $q \in \mathbb{R}$, then $L_1 \in RV_q$.

(iii) Assume $R : [D, \infty) \to (0, \infty)$ is measurable and Lebesgue integrable on each finite subinterval of $[D, \infty)$. Then R varies regularly iff there exists $j \in \mathbb{R}$ such that

$$\lim_{u \to \infty} \frac{u^{j+1} R(u)}{\int_D^u x^j R(x)\, dx} \tag{4.51}$$

exists and is a positive number, say $a_j + 1$. In this case, $R \in RV_q$ with $q = a_j - j$.

(iv) (Karamata Theorem, 1933). If $R \in RV_q$ is Lebesgue integrable on each finite subinterval of $[D, \infty)$, then the limit defined by (4.51) is $q + j + 1$, for every $j > -q - 1$.

Lemma 4.13. *Assume (A_1) holds. Then we have the equivalence*

$$a)\ f' \in RV_\rho \Longleftrightarrow b)\ \lim_{u \to \infty} u f'(u)/f(u) := \vartheta < \infty \Longleftrightarrow c)\ \lim_{u \to \infty} (F/f)'(u) := \gamma > 0.$$

Remark 3. *Let a) of Lemma 4.13 be fulfilled. Then the following assertions hold*

(i) ρ is non-negative;

(ii) $\gamma = 1/(\rho + 2) = 1/(\vartheta + 1)$;

*(iii) If $\rho \neq 0$, then (A_2) holds (*use $\lim_{u \to \infty} f(u)/u^p = \infty$, $\forall p \in (1, 1 + \rho)$*). The converse implication is not necessarily true (take $f(u) = u \ln^4(u + 1)$). However, there are cases when $\rho = 0$ and (A_2) fails so that (4.36) has* **no** *large solutions. This is illustrated by $f(u) = u$ or $f(u) = u \ln(u + 1)$.*

Inspired by the definition of γ, we denote by $\mathcal{K}$ the set of all positive, increasing C^1-functions k defined on $(0, \nu)$, for some $\nu > 0$, which satisfy $\lim_{t \to 0^+} \left(\frac{\int_0^t k(s)\, ds}{k(t)} \right)^{(i)} := \ell_i,\ i = \overline{0, 1}$.

It is easy to see that $\ell_0 = 0$ and $\ell_1 \in [0, 1]$, for every $k \in \mathcal{K}$. Our next result gives examples of functions $k \in \mathcal{K}$ with $\lim_{t \to 0^+} k(t) = 0$, for *every* $\ell_1 \in [0, 1]$.

Lemma 4.14. *Let $S \in C^1[D, \infty)$ be such that $S' \in RV_q$ with $q > -1$. Hence the following hold:*

a) If $k(t) = \exp\{-S(1/t)\}$ $\forall t \leq 1/D$, then $k \in \mathcal{K}$ with $\ell_1 = 0$.

b) If $k(t) = 1/S(1/t)$ $\forall t \leq 1/D$, then $k \in \mathcal{K}$ with $\ell_1 = 1/(q + 2) \in (0, 1)$.

c) If $k(t) = 1/\ln S(1/t)$ $\forall t \leq 1/D$, then $k \in \mathcal{K}$ with $\ell_1 = 1$.

Remark 4. *If $S \in C^1[D, \infty)$, then $S' \in RV_q$ with $q > -1$ iff for some $m > 0$, $C > 0$ and $B > D$ we have $S(u) = C u^m \exp\left\{ \int_B^u \frac{y(t)}{t}\, dt \right\}$, $\forall u \geq B$, where $y \in C[B, \infty)$ satisfies $\lim_{u \to \infty} y(u) = 0$. In this case, $S' \in RV_q$ with $q = m - 1$. (This is a consequence of Property 4.12 (iii) and (iv).*

Our main result is

Theorem 4.15. *Let (A_1) hold and $f' \in RV_\rho$ with $\rho > 0$. Assume $b \equiv 0$ on $\partial\Omega$ satisfies*

(B) $b(x) = c\, k^2(d(x)) + o(k^2(d(x)))$ as $d(x) \to 0$, for some constant $c > 0$ and $k \in \mathcal{K}$.

Then, for any $a \in (-\infty, \lambda_{\infty,1})$, Eq. (4.36) admits a unique large solution u_a. Moreover,

$$\lim_{d(x) \to 0} \frac{u_a(x)}{h(d(x))} = \xi_0, \tag{4.52}$$

where $\xi_0 = \left(\dfrac{2+\ell_1\rho}{c(2+\rho)}\right)^{1/\rho}$ *and* h *is defined by*

$$\int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t k(s)\,ds, \quad \forall t \in (0,\nu). \tag{4.53}$$

By Remark 4, the assumption $f' \in RV_\rho$ with $\rho > 0$ holds if and only if there exist $p > 1$ and $B > 0$ such that $f(u) = Cu^p \exp\left\{\int_B^u \frac{y(t)}{t}\,dt\right\}$, for all $u \geq B$ (y as before and $p = \rho+1$). If B is large enough ($y > -\rho$ on $[B,\infty)$), then $f(u)/u$ is increasing on $[B,\infty)$. Thus, to get the whole range of functions f for which our Theorem 4.15 applies we have only to "paste" a suitable smooth function on $[0,B]$ in accordance with (A_1). A simple way to do this is to define $f(u) = u^p \exp\{\int_0^u \frac{z(t)}{t}\,dt\}$, for all $u \geq 0$, where $z \in C[0,\infty)$ is non-negative such that $\lim_{t\to 0^+} z(t)/t \in [0,\infty)$ and $\lim_{u\to\infty} z(u) = 0$. Clearly, $f(u) = u^p$, $f(u) = u^p \ln(u+1)$, and $f(u) = u^p \arctan u$ ($p > 1$) fall into this category.

Lemma 4.14 provides a practical method to find functions k which can be considered in the statement of Theorem 4.15. Here are some examples: $k(t) = -1/\ln t$, $k(t) = t^\alpha$, $k(t) = \exp\{-1/t^\alpha\}$, $k(t) = \exp\left\{-\ln(1+\frac{1}{t})/t^\alpha\right\}$, $k(t) = \exp\left\{-\left[\arctan\left(\frac{1}{t}\right)\right]/t^\alpha\right\}$, $k(t) = t^\alpha/\ln(1+\frac{1}{t})$, for some $\alpha > 0$.

As we shall see, the uniqueness lies upon the crucial observation (4.52), which shows that all explosive solutions have the same boundary behaviour. Note that the only case of Theorem 4.15 studied so far is $f(u) = u^p$ ($p > 1$) and $k(t) = t^\alpha$ ($\alpha > 0$) (see García-Melián, Letelier-Albornoz, and Sabina de Lis [44]). For related results on the uniqueness of explosive solutions (mainly in the cases $b \equiv 1$ and $a = 0$) we refer to Bandle and Marcus [8], Loewner and Nirenberg [73], Marcus and Véron [75].

Proof of Lemma 4.13. From Property 4.12 (iv) and Remark 3 (i) we deduce $a) \Longrightarrow b)$ and $\vartheta = \rho + 1$. Conversely, $b) \Longrightarrow a)$ follows by 4.12 (iii) since $\vartheta \geq 1$ cf. (A_1).

$b) \Longrightarrow c)$. Indeed, $\lim_{u\to\infty} \frac{uf(u)}{F(u)} = 1+\vartheta$, which yields $\frac{\vartheta}{1+\vartheta} = \lim_{u\to\infty}\left[1 - \left(\frac{F}{f}\right)'(u)\right] = 1-\gamma$.

$c) \Longrightarrow b)$. Choose $s_1 > 0$ such that $\left(\frac{F}{f}\right)'(u) \geq \frac{\gamma}{2}$, $\forall u \geq s_1$. So, $\left(\frac{F}{f}\right)(u) \geq \frac{(u-s_1)\gamma}{2} + \left(\frac{F}{f}\right)(s_1)$, $\forall u \geq s_1$. Passing to the limit $u \to \infty$, we find $\lim_{u\to\infty} \frac{F(u)}{f(u)} = \infty$. Thus, $\lim_{u\to\infty}\frac{uf(u)}{F(u)} = \frac{1}{\gamma}$. Since $1-\gamma := \lim_{u\to\infty}\frac{F(u)f'(u)}{f^2(u)}$, we obtain $\lim_{u\to\infty}\frac{uf'(u)}{f(u)} = \frac{1-\gamma}{\gamma}$. □

Proof of Lemma 4.14. Since $\lim_{u\to\infty} uS'(u) = \infty$ (cf. Property 4.12 (i)), from Karamata Theorem we deduce $\lim_{u\to\infty}\frac{uS'(u)}{S(u)} = q+1 > 0$. Therefore, in any of the cases $a)$, $b)$, $c)$, $\lim_{t\to 0^+} k(t) = 0$ and k is an increasing C^1-function on $(0,\nu)$, for $\nu > 0$ sufficiently small.

$a)$ It is clear that $\lim_{t\to 0^+}\frac{tk'(t)}{k(t)\ln k(t)} = \lim_{t\to 0^+}\frac{-S'(1/t)}{tS(1/t)} = -(q+1)$. By l'Hospital's rule, $\ell_0 = \lim_{t\to 0^+}\frac{k(t)}{k'(t)} = 0$ and $\lim_{t\to 0^+}\frac{\left(\int_0^t k(s)\,ds\right)\ln k(t)}{tk(t)} = -\frac{1}{q+1}$. So, $1-\ell_1 := \lim_{t\to 0^+}\frac{\left(\int_0^t k(s)\,ds\right)k'(t)}{k^2(t)} = 1$.

$b)$ We see that $\lim_{t\to 0^+}\frac{tk'(t)}{k(t)} = \lim_{t\to 0^+}\frac{S'(1/t)}{tS(1/t)} = q+1$. By l'Hospital's rule, $\ell_0 = 0$ and $\lim_{t\to 0^+}\frac{\int_0^t k(s)\,ds}{tk(t)} = \frac{1}{q+2}$. So, $\ell_1 = 1 - \lim_{t\to 0^+}\frac{\int_0^t k(s)\,ds}{tk(t)}\,\frac{tk'(t)}{k(t)} = \frac{1}{q+2}$.

c) We have $\lim_{t\to 0^+}\frac{tk'(t)}{k^2(t)}=\lim_{t\to 0^+}\frac{S'(1/t)}{tS(1/t)}=q+1$. By l'Hospital's rule, $\lim_{t\to 0^+}\frac{\int_0^t k(s)\,ds}{tk(t)}=1$. Thus, $\ell_0=0$ and $\ell_1=1-\lim_{t\to 0^+}\frac{\int_0^t k(s)\,ds}{t}\frac{tk'(t)}{k^2(t)}=1$. □

Proof of Theorem 4.15. Fix $a\in(-\infty,\lambda_{\infty,1})$. By Theorem 4.1, problem (4.36) has at least a large solution.

If we prove that (4.52) holds for an *arbitrary* large solution u_a of (4.36), then the uniqueness follows easily. Indeed, if u_1 and u_2 are two arbitrary large solutions of (4.36), then (4.52) yields $\lim_{d(x)\to 0^+}\frac{u_1(x)}{u_2(x)}=1$. Hence, for any $\varepsilon\in(0,1)$, there exists $\delta=\delta(\varepsilon)>0$ such that

$$(1-\varepsilon)u_2(x)\le u_1(x)\le(1+\varepsilon)u_2(x),\quad \forall x\in\Omega \text{ with } 0<d(x)\le\delta. \tag{4.54}$$

Choosing eventually a smaller $\delta>0$, we can assume that $\overline{\Omega}_0\subset C_\delta$, where $C_\delta:=\{x\in\Omega:\ d(x)>\delta\}$.

It is clear that u_1 is a positive solution of the boundary value problem

$$\Delta\phi+a\phi=b(x)f(\phi)\quad\text{in } C_\delta,\qquad \phi=u_1\quad\text{on }\partial C_\delta. \tag{4.55}$$

By (A_1) and (4.54), we see that $\phi^-=(1-\varepsilon)u_2$ (resp., $\phi^+=(1+\varepsilon)u_2$) is a positive sub-solution (resp., super-solution) of (4.55). By the sub and super-solutions method, (4.55) has a positive solution ϕ_1 satisfying $\phi^-\le\phi_1\le\phi^+$ in C_δ. Since $b>0$ on $\overline{C}_\delta\setminus\overline{\Omega}_0$, we deduce that (4.55) has a *unique* positive solution, that is, $u_1\equiv\phi_1$ in C_δ. This yields $(1-\varepsilon)u_2(x)\le u_1(x)\le(1+\varepsilon)u_2(x)$ in C_δ, so that (4.54) holds in Ω. Passing to the limit $\varepsilon\to 0^+$, we conclude that $u_1\equiv u_2$.

In order to prove (4.52) we state some useful properties about h:

(h_1) $h\in C^2(0,\nu)$, $\lim_{t\to 0^+}h(t)=\infty$ (straightforward from (4.53)).

(h_2) $\lim_{t\to 0^+}\dfrac{h''(t)}{k^2(t)f(h(t)\xi)}=\dfrac{1}{\xi^{\rho+1}}\dfrac{2+\rho\ell_1}{2+\rho}$, $\forall\xi>0$ (so, $h''>0$ on $(0,2\delta)$, for $\delta>0$ small enough).

(h_3) $\lim_{t\to 0^+}h(t)/h''(t)=\lim_{t\to 0^+}h'(t)/h''(t)=0$.

We check (h_2) for $\xi=1$ only, since $f\in RV_{\rho+1}$. Clearly, $h'(t)=-k(t)\sqrt{2F(h(t))}$ and

$$h''(t)=k^2(t)f(h(t))\left(1-2\frac{k'(t)\left(\int_0^t k(s)\,ds\right)}{k^2(t)}\frac{\sqrt{F(h(t))}}{f(h(t))\int_{h(t)}^\infty[F(s)]^{-1/2}ds}\right)\qquad\forall t\in(0,\nu). \tag{4.56}$$

We see that $\lim_{u\to\infty}\sqrt{F(u)}/f(u)=0$. Thus, from l'Hospital's rule and Lemma 4.13 we infer that

$$\lim_{u\to\infty}\frac{\sqrt{F(u)}}{f(u)\int_u^\infty[F(s)]^{-1/2}ds}=\frac{1}{2}-\gamma=\frac{\rho}{2(\rho+2)}. \tag{4.57}$$

Using (4.56) and (4.57) we derive (h_2) and also

$$\lim_{t\to 0^+}\frac{h'(t)}{h''(t)}=\frac{-2(2+\rho)}{2+\ell_1\rho}\lim_{t\to 0^+}\frac{\int_0^t k(s)\,ds}{k(t)}\lim_{u\to\infty}\frac{\sqrt{F(u)}}{f(u)\int_u^\infty[F(s)]^{-1/2}ds}=\frac{-\rho\ell_0}{2+\ell_1\rho}=0. \tag{4.58}$$

From (h_1) and (h_2), $\lim_{t\to 0^+}h'(t)=-\infty$. So, l'Hospital's rule and (4.58) yield $\lim_{t\to 0^+}\frac{h(t)}{h'(t)}=0$. This and (4.58) lead to $\lim_{t\to 0^+}\frac{h(t)}{h''(t)}=0$ which proves (h_3).

Proof of (4.52). Fix $\varepsilon \in (0, c/2)$. Since $b \equiv 0$ on $\partial\Omega$ and (B) holds, we take $\delta > 0$ so that
(i) $d(x)$ is a C^2-function on the set $\{x \in \mathbb{R}^N : \ d(x) < 2\delta\}$;
(ii) k^2 is increasing on $(0, 2\delta)$;
(iii) $(c-\varepsilon)k^2(d(x)) < b(x) < (c+\varepsilon)k^2(d(x))$, $\forall x \in \Omega$ with $0 < d(x) < 2\delta$;
(iv) $h''(t) > 0$ $\forall t \in (0, 2\delta)$ (from (h_2)).

Let $\sigma \in (0, \delta)$ be arbitrary. We define $\xi^{\pm} = \left[\frac{2+\ell_1\rho}{(c\mp 2\varepsilon)(2+\rho)}\right]^{1/\rho}$ and $v_\sigma^-(x) = h(d(x)+\sigma)\xi^-$, for all x with $d(x)+\sigma < 2\delta$ resp., $v_\sigma^+(x) = h(d(x)-\sigma)\xi^+$, for all x with $\sigma < d(x) < 2\delta$.

Using (i)-(iv), when $\sigma < d(x) < 2\delta$ we obtain (since $|\nabla d(x)| \equiv 1$)

$$\Delta v_\sigma^+ + a v_\sigma^+ - b(x) f(v_\sigma^+) \leq \xi^+ h''(d(x)-\sigma)\left(\frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)}\Delta d(x) + a\frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} + 1 \right.$$
$$\left. -(c-\varepsilon)\frac{k^2(d(x)-\sigma) f(h(d(x)-\sigma)\xi^+)}{h''(d(x)-\sigma)\xi^+}\right).$$

Similarly, when $d(x)+\sigma < 2\delta$ we find

$$\Delta v_\sigma^- + a v_\sigma^- - b(x) f(v_\sigma^-) \geq \xi^- h''(d(x)+\sigma)\left(\frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)}\Delta d(x) + a\frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)} + 1 \right.$$
$$\left. -(c+\varepsilon)\frac{k^2(d(x)+\sigma) f(h(d(x)+\sigma)\xi^-)}{h''(d(x)+\sigma)\xi^-}\right).$$

Using (h_2) and (h_3) we see that, by diminishing δ, we can assume

$$\Delta v_\sigma^+(x) + a v_\sigma^+(x) - b(x) f(v_\sigma^+(x)) \leq 0 \quad \forall x \text{ with } \sigma < d(x) < 2\delta;$$

$$\Delta v_\sigma^-(x) + a v_\sigma^-(x) - b(x) f(v_\sigma^-(x)) \geq 0 \quad \forall x \text{ with } d(x)+\sigma < 2\delta.$$

Let Ω_1 and Ω_2 be smooth bounded domains such that $\Omega \subset\subset \Omega_1 \subset\subset \Omega_2$ and the first Dirichlet eigenvalue of $(-\Delta)$ in the domain $\Omega_1 \setminus \overline{\Omega}$ is greater than a. Let $p \in C^{0,\mu}(\overline{\Omega}_2)$ satisfy $0 < p(x) \leq b(x)$ for $x \in \Omega \setminus C_{2\delta}$, $p = 0$ on $\overline{\Omega}_1 \setminus \Omega$ and $p > 0$ on $\Omega_2 \setminus \overline{\Omega}_1$. Denote by w a positive large solution of

$$\Delta w + a w = p(x) f(w) \quad \text{in } \Omega_2 \setminus \overline{C}_{2\delta}.$$

The existence of w is ensured by Theorem 4.1.

Suppose that u_a is an arbitrary large solution of (4.36) and let $v := u_a + w$. Then v satisfies

$$\Delta v + a v - b(x) f(v) \leq 0 \quad \text{in } \Omega \setminus \overline{C}_{2\delta}.$$

Since $v_{|\partial\Omega} = \infty > v^-_{\sigma|\partial\Omega}$ and $v_{|\partial C_{2\delta}} = \infty > v^-_{\sigma|\partial C_{2\delta}}$, the maximum principle implies

$$u_a + w \geq v_\sigma^- \quad \text{on } \Omega \setminus \overline{C}_{2\delta}. \tag{4.59}$$

Similarly,

$$v_\sigma^+ + w \geq u_a \quad \text{on } C_\sigma \setminus \overline{C}_{2\delta}. \tag{4.60}$$

Letting $\sigma \to 0$ in (4.59) and (4.60), we deduce $h(d(x))\xi^+ + 2w \geq u_a + w \geq h(d(x))\xi^-$, for all $x \in \Omega \setminus \overline{C}_{2\delta}$. Since w is uniformly bounded on $\partial\Omega$, we have $\xi^- \leq \liminf_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} \leq \limsup_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} \leq \xi^+$. Letting $\varepsilon \to 0^+$ we obtain (4.52). This concludes the proof of Theorem 4.15. □

Bandle and Marcus proved in [9] that the blow-up rate of the unique large solution of (4.36) depends on the curvature of the boundary of Ω. Our purpose in what follows is to refine the blow-up rate of u_a near $\partial\Omega$ by giving the second term in its expansion near the boundary. This is a more subtle question which represents the goal of more recent literature (see García-Melián, Letelier-Albornoz, and Sabina de Lis [44] and the references therein). The following is very general and, as a novelty, it relies on the Karamata regular variation theory.

Recall that $\mathcal{K}$ denotes the set of all positive increasing C^1-functions k defined on $(0,\nu)$, for some $\nu > 0$, which satisfy $\lim_{t\searrow 0}(\int_0^t k(s)\,ds/k(t))^{(i)} := \ell_i$, $i \in \overline{0,1}$. We also recall that RV_q ($q \in \mathbb{R}$) is the set of all positive measurable functions $Z : [A,\infty) \to \mathbb{R}$ (for some $A > 0$) satisfying $\lim_{u\to\infty} Z(\xi u)/Z(u) = \xi^q$, $\forall \xi > 0$. Define by NRV_q the class of functions f in the form $f(u) = Cu^q \exp\left\{\int_B^u \phi(t)/t\,dt\right\}$, $\forall u \geq B > 0$, where $C > 0$ is a constant and $\phi \in C[B,\infty)$ satisfies $\lim_{t\to\infty}\phi(t) = 0$. The Karamata Representation Theorem shows that $NRV_q \subset RV_q$.

For any $\zeta > 0$, set $\mathcal{K}_{0,\zeta}$ the subset of $\mathcal{K}$ with $\ell_1 = 0$ and $\lim_{t\searrow 0} t^{-\zeta}(\int_0^t k(s)\,ds/k(t))' := L_\star \in \mathbb{R}$. It can be proven that $\mathcal{K}_{0,\zeta} \equiv \mathcal{R}_{0,\zeta}$, where

$$\mathcal{R}_{0,\zeta} = \left\{ \begin{array}{l} k: \quad k(u^{-1}) = d_0 u\,[\Lambda(u)]^{-1} \exp\left[-\int_{d_1}^u (s\Lambda(s))^{-1}\,ds\right] \ (u \geq d_1),\ 0 < \Lambda \in C^1[d_1,\infty), \\ \qquad \lim_{u\to\infty}\Lambda(u) = \lim_{u\to\infty} u\Lambda'(u) = 0,\ \lim_{u\to\infty} u^{\zeta+1}\Lambda'(u) = \ell_\star \in \mathbb{R},\ d_0,\, d_1 > 0 \end{array} \right\}.$$

Define

$$\mathcal{F}_{\rho\eta} = \{f \in NRV_{\rho+1}\ (\rho > 0):\ \phi \in RV_\eta \text{ or } -\phi \in RV_\eta\},\ \eta \in (-\rho-2, 0];$$
$$\mathcal{F}_{\rho 0,\tau} = \{f \in \mathcal{F}_{\rho 0}:\ \lim_{u\to\infty}(\ln u)^\tau \phi(u) = \ell^\star \in \mathbb{R}\},\ \tau \in (0,\infty).$$

The following result establishes a precise asymptotic estimate in the neighbourhood of the boundary.

Theorem 4.16. *Assume that*

$$b(x) = k^2(d)(1 + \tilde{c}d^\theta + o(d^\theta)) \quad \text{if } d(x) \to 0, \quad \text{where } k \in \mathcal{R}_{0,\zeta},\ \theta > 0,\ \tilde{c} \in \mathbb{R}. \tag{4.61}$$

Suppose that f fulfills (A_1) and one of the following growth conditions at infinity:
(i) *$f(u) = Cu^{\rho+1}$ in a neighbourhood of infinity;*
(ii) *$f \in \mathcal{F}_{\rho\eta}$ with $\eta \neq 0$;*
(iii) *$f \in \mathcal{F}_{\rho 0,\tau_1}$ with $\tau_1 = \varpi/\zeta$, where $\varpi = \min\{\theta,\zeta\}$.*
Then, for any $a \in (-\infty, \lambda_{\infty,1})$, the unique positive solution u_a of (4.36) satisfies

$$u_a(x) = \xi_0 h(d)(1 + \chi d^\varpi + o(d^\varpi)) \quad \text{if } d(x) \to 0, \quad \text{where } \xi_0 = [2(2+\rho)^{-1}]^{1/\rho} \tag{4.62}$$

and h is defined by $\int_{h(t)}^{\infty}[2F(s)]^{-1/2}ds = \int_0^t k(s)\,ds$, *for* $t > 0$ *small enough. The expression of* χ *is*

$$\chi = \begin{cases} -(1+\zeta)\ell_\star(2\zeta)^{-1}\,\text{Heaviside}\,(\theta-\zeta) - \tilde{c}\rho^{-1}\,\text{Heaviside}\,(\zeta-\theta) := \chi_1 \quad \textit{if } \text{(i)} \textit{ or } \text{(ii)} \textit{ holds} \\ \chi_1 - \ell^\star\rho^{-1}(-\rho\ell_\star/2)^{\tau_1}[1/(\rho+2)+\ln\xi_0] \quad \textit{if } f \textit{ obeys } \text{(iii)}. \end{cases}$$

Note that the only case related, in same way, to our Theorem 4.16 corresponds to $\Omega_0 = \emptyset$, $f(u) = u^{\rho+1}$ on $[0,\infty)$, $k(t) = ct^\alpha \in \mathcal{K}$ (where $c, \alpha > 0$), $\theta = 1$ in (4.61), being studied in [44]. There, the two-term asymptotic expansion of u_a near $\partial\Omega$ ($a \in \mathbb{R}$ since $\lambda_{\infty,1} = \infty$) involves both the distance function $d(x)$ and the mean curvature H of $\partial\Omega$. However, the blow-up rate of u_a we present in Theorem 4.16 is of a different nature since the class $\mathcal{R}_{0,\zeta}$ does not include $k(t) = ct^\alpha$.

Our main result contributes to the knowledge in some new directions. More precisely, the blow-up rate of the unique positive solution u_a of (4.36) is refined as follows in the above result:

(a) on the maximal interval $(-\infty, \lambda_{\infty,1})$ for the parameter a, which is in connection with an appropriate semilinear eigenvalue problem; thus, the condition $b > 0$ in Ω is removed by defining the set Ω_0, but we maintain $b \equiv 0$ on $\partial\Omega$ since this is a *natural* restriction inherited from the logistic problem.

(b) when b satisfies (4.61), where θ is *any* positive number and k belongs to a very rich class of functions, namely $\mathcal{R}_{0,\zeta}$. The equivalence $\mathcal{R}_{0,\zeta} \equiv \mathcal{K}_{0,\zeta}$ shows the connection to the larger class $\mathcal{K}$ for which the uniqueness of u_a holds. In addition, the explicit form of $k \in \mathcal{R}_{0,\zeta}$ shows us how to built $k \in \mathcal{K}_{0,\zeta}$.

(c) for a wide class of functions $f \in NRV_{\rho+1}$ where either $\phi \equiv 0$ (case (i)) or ϕ (resp., $-\phi$) belongs to RV_η with $\eta \in (-\rho-2, 0]$ (cases (ii) and (iii)). Therefore, the theory of regular variation plays a key role in understanding the general framework and the approach as well.

Proof of Theorem 4.16. We first state two auxiliary results. Their proofs are straightforward and we shall omit them.

Lemma 4.17. *Assume (4.61) and* $f \in NRV_{\rho+1}$ *satisfies* (A_1). *Then* h *has the following properties:*

(i) $h \in C^2(0,\nu)$, $\lim_{t\searrow 0} h(t) = \infty$ *and* $\lim_{t\searrow 0} h'(t) = -\infty$;

(ii) $\lim_{t\searrow 0} h''(t)/[k^2(t)f(h(t)\xi)] = (2+\rho\ell_1)/[\xi^{\rho+1}(2+\rho)]$, $\forall \xi > 0$;

(iii) $\lim_{t\searrow 0} h(t)/h''(t) = \lim_{t\searrow 0} h'(t)/h''(t) = \lim_{t\searrow 0} h(t)/h'(t) = 0$;

(iv) $\lim_{t\searrow 0} h'(t)/[th''(t)] = -\rho\ell_1/(2+\rho\ell_1)$ *and* $\lim_{t\searrow 0} h(t)/[t^2h''(t)] = \rho^2\ell_1^2/[2(2+\rho\ell_1)]$;

(v) $\lim_{t\searrow 0} h(t)/[th'(t)] = \lim_{t\searrow 0}[\ln t]/[\ln h(t)] = -\rho\ell_1/2$;

(vi) *If* $\ell_1 = 0$, *then* $\lim_{t\searrow 0} t^j h(t) = \infty$, *for all* $j > 0$;

(vii) $\lim_{t\searrow 0} 1/[t^\zeta \ln h(t)] = -\rho\ell_\star/2$ *and* $\lim_{t\searrow 0} h'(t)/[t^{\zeta+1}h''(t)] = \rho\ell_\star/(2\zeta)$, $\forall k \in \mathcal{R}_{0,\zeta}$.

Let $\tau > 0$ be arbitrary. For any $u > 0$, define $T_{1,\tau}(u) = \{\rho/[2(\rho+2)] - \Xi(u)\}(\ln u)^\tau$ and $T_{2,\tau}(u) = \{f(\xi_0 u)/[\xi_0 f(u)] - \xi_0^\rho\}(\ln u)^\tau$. Note that if $f(u) = Cu^{\rho+1}$, for u in a neighbourhood V_∞ of infinity, then $T_{1,\tau}(u) = T_{2,\tau}(u) = 0$ for each $u \in V_\infty$.

Lemma 4.18. *Assume* (A_1) *and* $f\in\mathcal{F}_{\rho\eta}$. *The following hold:*

(i) *If* $f\in\mathcal{F}_{\rho 0,\tau}$, *then* $\lim_{u\to\infty}T_{1,\tau}(u)=-\ell^\star/(\rho+2)^2$ *and* $\lim_{u\to\infty}T_{2,\tau}(u)=\xi_0^\rho\ell^\star\ln\xi_0$.

(ii) *If* $f\in\mathcal{F}_{\rho\eta}$ *with* $\eta\neq 0$, *then* $\lim_{u\to\infty}T_{1,\tau}(u)=\lim_{u\to\infty}T_{2,\tau}(u)=0$.

Fix $\varepsilon\in(0,1/2)$. We can find $\delta>0$ such that $d(x)$ is of class C^2 on $\{x\in\mathbb{R}^N:\ d(x)<\delta\}$, k is nondecreasing on $(0,\delta)$, and $h'(t)<0<h''(t)$ for all $t\in(0,\delta)$. A straightforward computation shows that $\lim_{t\searrow 0}t^{1-\theta}k'(t)/k(t)=\infty$, for every $\theta>0$. Using now (4.61), it follows that we can diminish $\delta>0$ such that $k^2(t)\left[1+(\tilde c-\varepsilon)t^\theta\right]$ is increasing on $(0,\delta)$ and

$$1+(\tilde c-\varepsilon)d^\theta<b(x)/k^2(d)<1+(\tilde c+\varepsilon)d^\theta,\quad \forall x\in\Omega \text{ with } d\in(0,\delta). \tag{4.63}$$

We define $u^\pm(x)=\xi_0 h(d)(1+\chi_\varepsilon^\pm d^\varpi)$, with $d\in(0,\delta)$, where $\chi_\varepsilon^\pm=\chi\pm\varepsilon\,[1+\text{Heaviside}\,(\zeta-\theta)]/\rho$. Take $\delta>0$ small enough such that $u^\pm(x)>0$, for each $x\in\Omega$ with $d\in(0,\delta)$. By the Lagrange mean value theorem, we obtain $f(u^\pm(x))=f(\xi_0h(d))+\xi_0\chi_\varepsilon^\pm d^\varpi h(d)f'(\Upsilon^\pm(d))$, where $\Upsilon^\pm(d)=\xi_0h(d)(1+\lambda^\pm(d)\chi_\varepsilon^\pm d^\varpi)$, for some $\lambda^\pm(d)\in[0,1]$. We claim that

$$\lim_{d\searrow 0}f(\Upsilon^\pm(d))/f(\xi_0h(d))=1. \tag{4.64}$$

Fix $\sigma\in(0,1)$ and $M>0$ such that $|\chi_\varepsilon^\pm|<M$. Choose $\mu^\star>0$ so that $|(1\pm Mt)^{\rho+1}-1|<\sigma/2$, for all $t\in(0,2\mu^\star)$. Let $\mu_\star\in(0,(\mu^\star)^{1/\varpi})$ be such that, for every $x\in\Omega$ with $d\in(0,\mu_\star)$

$$\left|f(\xi_0h(d)(1\pm M\mu^\star))/f(\xi_0h(d))-(1\pm M\mu^\star)^{\rho+1}\right|<\sigma/2.$$

Hence, $1-\sigma<(1-M\mu^\star)^{\rho+1}-\sigma/2<f(\Upsilon^\pm(d))/f(\xi_0h(d))<(1+M\mu^\star)^{\rho+1}+\sigma/2<1+\sigma$, for every $x\in\Omega$ with $d\in(0,\mu_\star)$. This proves (4.64).

Step 1. There exists $\delta_1\in(0,\delta)$ so that $\Delta u^++au^+-k^2(d)[1+(\tilde c-\varepsilon)d^\theta]f(u^+)\le 0$, $\forall x\in\Omega$ with $d\in(0,\delta_1)$ and $\Delta u^-+au^--k^2(d)[1+(\tilde c+\varepsilon)d^\theta]f(u^-)\ge 0$, $\forall x\in\Omega$ with $d\in(0,\delta_1)$.

Indeed, for every $x\in\Omega$ with $d\in(0,\delta)$, we have

$$\begin{aligned}
&\Delta u^\pm+au^\pm-k^2(d)\left[1+(\tilde c\mp\varepsilon)d^\theta\right]f(u^\pm)\\
=\xi_0d^\varpi h''(d)\quad&\Big[a\chi_\varepsilon^\pm\tfrac{h(d)}{h''(d)}+\chi_\varepsilon^\pm\Delta d\,\tfrac{h'(d)}{h''(d)}+2\varpi\chi_\varepsilon^\pm\,\tfrac{h'(d)}{dh''(d)}+\varpi\chi_\varepsilon^\pm\Delta d\,\tfrac{h(d)}{dh''(d)}\\
&+\varpi(\varpi-1)\chi_\varepsilon^\pm\,\tfrac{h(d)}{d^2h''(d)}+\Delta d\,\tfrac{h'(d)}{d^\varpi h''(d)}+\tfrac{a\,h(d)}{d^\varpi h''(d)}+\textstyle\sum_{j=1}^4\mathcal{S}_j^\pm(d)\Big]
\end{aligned}$$

where, for any $t\in(0,\delta)$, we denote

$$\begin{aligned}
&\mathcal{S}_1^\pm(t)=(-\tilde c\pm\varepsilon)t^{\theta-\varpi}\,k^2(t)f(\xi_0h(t))/[\xi_0h''(t)],\ \ \mathcal{S}_2^\pm(t)=\chi_\varepsilon^\pm(1-k^2(t)h(t)f'(\Upsilon^\pm(t))/h''(t)),\\
&\mathcal{S}_3^\pm(t)=(-\tilde c\pm\varepsilon)\chi_\varepsilon^\pm t^\theta\,k^2(t)h(t)f'(\Upsilon^\pm(t))/h''(t),\ \ \mathcal{S}_4^\pm(t)=t^{-\varpi}(1-k^2(t)f(\xi_0h(t))/[\xi_0h''(t)]).
\end{aligned}$$

By Lemma 4.17 (ii), we find $\lim_{t\searrow 0}k^2(t)f(\xi_0h(t))[\xi_0h''(t)]^{-1}=1$, which yields $\lim_{t\searrow 0}\mathcal{S}_1^\pm(t)=(-\tilde c\pm\varepsilon)\text{Heaviside}\,(\zeta-\theta)$. Using (4.64), we obtain $\lim_{t\searrow 0}k^2(t)h(t)f'(\Upsilon^\pm(t))/h''(t)=\rho+1$. Hence, $\lim_{t\searrow 0}\mathcal{S}_2^\pm(t)=-\rho\chi_\varepsilon^\pm$ and $\lim_{t\searrow 0}\mathcal{S}_3^\pm(t)=0$.

Using the expression of h'', we derive $\mathcal{S}_4^{\pm}(t) = \frac{k^2(t)f(h(t))}{h''(t)} \sum_{i=1}^3 \mathcal{S}_{4,i}(t)$, $\forall t \in (0,\delta)$, where we denote $\mathcal{S}_{4,1}(t) = 2\frac{\Xi(h(t))}{t^{\varpi}}(\int_0^t k(s)\,ds/k(t))'$, $\mathcal{S}_{4,2}(t) = 2\frac{T_{1,\tau_1}(h(t))}{[t^{\zeta}\ln h(t)]^{\tau_1}}$ and $\mathcal{S}_{4,3}(t) = -\frac{T_{2,\tau_1}(h(t))}{[t^{\zeta}\ln h(t)]^{\tau_1}}$.

Since $\mathcal{R}_{0,\zeta} \equiv \mathcal{K}_{0,\zeta}$, we find $\lim_{t\searrow 0} \mathcal{S}_{4,1}(t) = -(1+\zeta)\rho\ell_\star\zeta^{-1}(\rho+2)^{-1}\,\text{Heaviside}\,(\theta-\zeta)$.

Cases (i), (ii). By Lemma 4.17 (vii) and Lemma 4.18 (ii), we find $\lim_{t\searrow 0} S_{4,2}(t) = \lim_{t\searrow 0} S_{4,3}(t) = 0$. In view of Lemma 4.17 (ii), we derive that $\lim_{t\searrow 0} S_4^{\pm}(t) = -(1+\zeta)\rho\ell_\star(2\zeta)^{-1}\,\text{Heaviside}\,(\theta-\zeta)$.

Case (iii). By Lemma 4.17 (vii) and Lemma 4.18 (i), $\lim_{t\searrow 0} S_{4,2}(t) = -2\ell^\star(\rho+2)^{-2}(-\rho\ell_\star/2)^{\tau_1}$ and $\lim_{t\searrow 0} S_{4,3}(t) = -2\ell^\star(\rho+2)^{-1}(-\rho\ell_\star/2)^{\tau_1}\ln\xi_0$. Using Lemma 4.17 (ii) once more, we arrive at $\lim_{t\searrow 0} S_4^{\pm}(t) = -(1+\zeta)\rho\ell_\star(2\zeta)^{-1}\,\text{Heaviside}\,(\theta-\zeta) - \ell^\star(-\rho\ell_\star/2)^{\tau_1}[1/(\rho+2)+\ln\xi_0]$.

Note that in each of the cases (i)–(iii), the definition of $\chi_\varepsilon^{\pm}$ yields $\lim_{t\searrow 0}\sum_{j=1}^4 \mathcal{S}_j^+(t) = -\varepsilon < 0$ and $\lim_{t\searrow 0}\sum_{j=1}^4 \mathcal{S}_j^-(t) = \varepsilon > 0$. By Lemma 4.17 (vii), $\lim_{t\searrow 0}\frac{h'(t)}{t^{\varpi}h''(t)} = 0$. But $\lim_{t\searrow 0}\frac{h(t)}{h'(t)} = 0$, so $\lim_{t\searrow 0}\frac{h(t)}{t^{\varpi}h''(t)} = 0$. Thus, using Lemma 4.17 [(iii), (iv)], relation (4.65) concludes our Step 1.

Step 2. There exists M^+, $\delta^+ > 0$ such that $u_a(x) \le u^+(x) + M^+$, for all $x \in \Omega$ with $0 < d < \delta^+$.

Define $(0,\infty) \ni u \longmapsto \Psi_x(u) = au - b(x)f(u)$, $\forall x$ with $d \in (0,\delta_1)$. Clearly, $\Psi_x(u)$ is decreasing when $a \le 0$. Suppose $a \in (0,\lambda_{\infty,1})$. Obviously, $f(t)/t : (0,\infty) \to (f'(0),\infty)$ is bijective. Let $\delta_2 \in (0,\delta_1)$ be such that $b(x) < 1$, $\forall x$ with $d \in (0,\delta_2)$. Let u_x define the unique positive solution of $b(x)f(u)/u = a + f'(0)$, $\forall x$ with $d \in (0,\delta_2)$. Hence, for any x with $d \in (0,\delta_2)$, $u \to \Psi_x(u)$ is decreasing on (u_x,∞). But $\lim_{d(x)\searrow 0}\frac{b(x)f(u^+(x))}{u^+(x)} = +\infty$ (use $\lim_{d(x)\searrow 0} u^+(x)/h(d) = \xi_0$, (A_1) and Lemma 4.17 [(ii) and (iii)]). So, for δ_2 small enough, $u^+(x) > u_x$, $\forall x$ with $d \in (0,\delta_2)$.

Fix $\sigma \in (0,\delta_2/4)$ and set $\mathcal{N}_\sigma := \{x \in \Omega :\ \sigma < d(x) < \delta_2/2\}$. We define $u_\sigma^*(x) = u^+(d-\sigma,s) + M^+$, where (d,s) are the local coordinates of $x \in \mathcal{N}_\sigma$. We choose $M^+ > 0$ large enough to have $u_\sigma^*(\delta_2/2,s) \ge u_a(\delta_2/2,s)$, $\forall \sigma \in (0,\delta_2/4)$ and $\forall s \in \partial\Omega$. Using (4.63) and Step 1, we find

$$\begin{aligned}
-\Delta u_\sigma^*(x) &\ge au^+(d-\sigma,s) - [1+(\tilde{c}-\varepsilon)(d-\sigma)^\theta]k^2(d-\sigma)f(u^+(d-\sigma,s))\\
&\ge au^+(d-\sigma,s) - [1+(\tilde{c}-\varepsilon)d^\theta]k^2(d)f(u^+(d-\sigma,s)) \ge \Psi_x(u^+(d-\sigma,s))\\
&\ge \Psi_x(u_\sigma^*) = au_\sigma^*(x) - b(x)f(u_\sigma^*(x)) \quad \text{in } \mathcal{N}_\sigma.
\end{aligned}$$

Thus, by the maximum principle, $u_a \le u_\sigma^*$ in $\mathcal{N}_\sigma$, $\forall \sigma \in (0,\delta_2/4)$. Letting $\sigma \to 0$, we have proved Step 2.

Step 3. There exists M^-, $\delta^- > 0$ such that $u_a(x) \ge u^-(x) - M^-$, for all $x \in \Omega$ with $0 < d < \delta^-$.

For every $r \in (0,\delta)$, define $\Omega_r = \{x \in \Omega :\ 0 < d(x) < r\}$. We will prove that for $\lambda > 0$ sufficiently small, $\lambda u^-(x) \le u_a(x)$, $\forall x \in \Omega_{\delta_2/4}$. Indeed, fix arbitrarily $\sigma \in (0,\delta_2/4)$. Define $v_\sigma^*(x) = \lambda u^-(d+\sigma,s)$, for $x = (d,s) \in \Omega_{\delta_2/2}$. We choose $\lambda \in (0,1)$ small enough such that $v_\sigma^*(\delta_2/4,s) \le u_a(\delta_2/4,s)$, $\forall \sigma \in (0,\delta_2/4)$, $\forall s \in \partial\Omega$. Using (4.63), Step 1 and (A_1), we find

$$\begin{aligned}
\Delta v_\sigma^*(x) + av_\sigma^*(x) &\ge \lambda k^2(d+\sigma)[1+(\tilde{c}+\varepsilon)(d+\sigma)^\theta]f(u^-(d+\sigma,s))\\
&\ge k^2(d)[1+(\tilde{c}+\varepsilon)d^\theta]f(\lambda u^-(d+\sigma,s)) \ge bf(v_\sigma^*),
\end{aligned}$$

for all $x = (d,s) \in \Omega_{\delta_2/4}$, that is v_σ^* is a sub-solution of $\Delta u + au = b(x)f(u)$ in $\Omega_{\delta_2/4}$. By the maximum principle, we conclude that $v_\sigma^* \le u_a$ in $\Omega_{\delta_2/4}$. Letting $\sigma \to 0$, we find $\lambda u^-(x) \le u_a(x)$, $\forall x \in \Omega_{\delta_2/4}$.

Since $\lim_{d\searrow 0} u^-(x)/h(d) = \xi_0$, by using (A_1) and Lemma 4.17 [(ii), (iii)], we can easily obtain $\lim_{d\searrow 0} k^2(d)f(\lambda^2 u^-(x))/u^-(x) = \infty$. So, there exists $\tilde{\delta} \in (0, \delta_2/4)$ such that

$$k^2(d)[1 + (\tilde{c} + \varepsilon)d^\theta]f(\lambda^2 u^-)/u^- \geq \lambda^2|a|, \quad \forall x \in \Omega \text{ with } 0 < d \leq \tilde{\delta}. \tag{4.65}$$

By Lemma 4.17 [(i) and (v)], we deduce that $u^-(x)$ decreases with d when $d \in (0, \tilde{\delta})$ (if necessary, $\tilde{\delta} > 0$ is diminished). Choose $\delta_* \in (0, \tilde{\delta})$, close enough to $\tilde{\delta}$, such that

$$h(\delta_*)(1 + \chi_\varepsilon^- \delta_*^\varpi)/[h(\tilde{\delta})(1 + \chi_\varepsilon^- \tilde{\delta}^\varpi)] < 1 + \lambda. \tag{4.66}$$

For each $\sigma \in (0, \tilde{\delta} - \delta_*)$, we define $z_\sigma(x) = u^-(d + \sigma, s) - (1 - \lambda)u^-(\delta_*, s)$. We prove that z_σ is a sub-solution of $\Delta u + au = b(x)f(u)$ in Ω_{δ_*}. Using (4.66), $z_\sigma(x) \geq u^-(\tilde{\delta}, s) - (1 - \lambda)u^-(\delta_*, s) > 0$ $\forall x = (d, s) \in \Omega_{\delta_*}$. By (4.63) and Step 1, z_σ is a sub-solution of $\Delta u + au = b(x)f(u)$ in Ω_{δ_*} if

$$k^2(d + \sigma)[1 + (\tilde{c} + \varepsilon)(d + \sigma)^\theta]\left[f(u^-(d + \sigma, s)) - f(z_\sigma(d, s))\right] \geq a(1 - \lambda)u^-(\delta_*, s), \tag{4.67}$$

for all $(d, s) \in \Omega_{\delta_*}$. Applying the Lagrange mean value theorem and (A_1), we infer that (4.67) is a consequence of $k^2(d + \sigma)[1 + (\tilde{c} + \varepsilon)(d + \sigma)^\theta]\, f(z_\sigma(d, s))/z_\sigma(d, s) \geq |a|, \quad \forall (d, s) \in \Omega_{\delta_*}$. This inequality holds by virtue of (4.65), (4.66) and the decreasing character of u^- with d.

On the other hand, $z_\sigma(\delta_*, s) \leq \lambda u^-(\delta_*, s) \leq u_a(x)$, $\forall x = (\delta_*, s) \in \Omega$. Clearly, $\limsup_{d\to 0}(z_\sigma - u_a)(x) = -\infty$ and $b > 0$ in Ω_{δ_*}. Thus, by the maximum principle, $z_\sigma \leq u_a$ in Ω_{δ_*}, $\forall \sigma \in (0, \tilde{\delta} - \delta_*)$. Letting $\sigma \to 0$, we conclude the assertion of Step 3.

By Steps 2 and 3, $\chi_\varepsilon^+ \geq \{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} - M^+/[\xi_0 d^\varpi h(d)]$ $\forall x \in \Omega$ with $d \in (0, \delta^+)$ and $\chi_\varepsilon^- \leq \{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} + M^-/[\xi_0 d^\varpi h(d)]$ $\forall x \in \Omega$ with $d \in (0, \delta^-)$. Passing to the limit as $d \to 0$ and using Lemma 4.17 (vi), we obtain $\chi_\varepsilon^- \leq \liminf_{d\to 0}\{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi}$ and $\limsup_{d\to 0}\{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} \leq \chi_\varepsilon^+$. Letting $\varepsilon \to 0$, we conclude our proof. $\square$

5 Entire solutions blowing up at infinity of semilinear elliptic systems

In this section we are concerned with the existence of solutions that blow up at infinity for a class of semilinear elliptic systems defined on the whole space.

Consider the following semilinear elliptic system

$$\begin{cases} \Delta u = p(x)g(v) & \text{in } \mathbb{R}^N, \\ \Delta v = q(x)f(u) & \text{in } \mathbb{R}^N, \end{cases} \tag{5.68}$$

where $N \geq 3$ and $p, q \in C^{0,\alpha}_{\rm loc}(\mathbb{R}^N)$ $(0 < \alpha < 1)$ are non-negative and radially symmetric functions. Throughout this paper we assume that $f, g \in C^{0,\beta}_{\rm loc}[0, \infty)$ $(0 < \beta < 1)$ are positive and non-decreasing on $(0, \infty)$.

We are concerned here with the existence of positive *entire large solutions* of (5.68), that is positive classical solutions which satisfy $u(x) \to \infty$ and $v(x) \to \infty$ as $|x| \to \infty$. Set $\mathbb{R}^+ = (0, \infty)$ and define

$$\mathcal{G} = \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+;\ (\exists) \text{ an entire radial solution of (5.68) so that } (u(0), v(0)) = (a,b)\}.$$

The case of pure powers in the non-linearities was treated by Lair and Shaker in [65]. They proved that $\mathcal{G} = \mathbb{R}^+ \times \mathbb{R}^+$ if $f(t) = t^\gamma$ and $g(t) = t^\theta$ for $t \geq 0$ with $0 < \gamma, \theta \leq 1$. Moreover, they established that all positive entire radial solutions of (5.68) are *large* provided that

$$\int_0^\infty tp(t)\,dt = \infty, \qquad \int_0^\infty tq(t)\,dt = \infty. \tag{5.69}$$

If, in turn

$$\int_0^\infty tp(t)\,dt < \infty, \qquad \int_0^\infty tq(t)\,dt < \infty \tag{5.70}$$

then all positive entire radial solutions of (5.68) are *bounded*.

In what follows we generalize the above results to a larger class of systems. Theorems 5.1 and 5.4 are due to Cîrstea and Rădulescu [26].

Theorem 5.1. *Assume that*

$$\lim_{t\to\infty} \frac{g(cf(t))}{t} = 0 \quad \textit{for all } c > 0. \tag{5.71}$$

Then $\mathcal{G} = \mathbb{R}^+ \times \mathbb{R}^+$. Moreover, the following hold:

(i) If p and q satisfy (5.69), then all positive entire radial solutions of (5.68) are large.

(ii) If p and q satisfy (5.70), then all positive entire radial solutions of (5.68) are bounded. Furthermore, if f, g are locally Lipschitz continuous on $(0, \infty)$ and (u, v), $(\tilde{u}, \tilde{v})$ denote two positive entire radial solutions of (5.68), then there exists a positive constant C such that for all $r \in [0, \infty)$

$$\max\{|u(r) - \tilde{u}(r)|, |v(r) - \tilde{v}(r)|\} \leq C \max\{|u(0) - \tilde{u}(0)|, |v(0) - \tilde{v}(0)|\}.$$

Proof. We start with the following auxiliary results.

Lemma 5.2. *Condition (5.69) holds if and only if $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = \infty$ where*

$$A(r) \equiv \int_0^r t^{1-N} \int_0^t s^{N-1} p(s)\,ds\,dt, \qquad B(r) \equiv \int_0^r t^{1-N} \int_0^t s^{N-1} q(s)\,ds\,dt, \qquad \forall r > 0.$$

Proof. Indeed, for any $r > 0$

$$A(r) = \frac{1}{N-2}\left[\int_0^r tp(t)\,dt - \frac{1}{r^{N-2}}\int_0^r t^{N-1}p(t)\,dt\right] \leq \frac{1}{N-2}\int_0^r tp(t)\,dt. \tag{5.72}$$

On the other hand,

$$\begin{aligned}\int_0^r tp(t)\,dt - \frac{1}{r^{N-2}}\int_0^r t^{N-1}p(t)\,dt &= \frac{1}{r^{N-2}}\int_0^r \left(r^{N-2} - t^{N-2}\right) tp(t)\,dt \\ &\geq \frac{1}{r^{N-2}}\left[r^{N-2} - \left(\frac{r}{2}\right)^{N-2}\right]\int_0^{\frac{r}{2}} tp(t)\,dt.\end{aligned}$$

This combined with (5.72) yields

$$\frac{1}{N-2}\int_0^r tp(t)\,dt \ge A(r) \ge \frac{1}{N-2}\left[1-\left(\frac{1}{2}\right)^{N-2}\right]\int_0^{\frac{r}{2}} tp(t)\,dt.$$

Our conclusion follows now by letting $r \to \infty$. □

Lemma 5.3. *Assume that condition (5.70) holds. Let f and g be locally Lipschitz continuous functions on $(0,\infty)$. If (u,v) and $(\tilde{u},\tilde{v})$ denote two bounded positive entire radial solutions of (5.68), then there exists a positive constant C such that for all $r\in[0,\infty)$*

$$\max\{|u(r)-\tilde{u}(r)|,|v(r)-\tilde{v}(r)|\} \le C\max\{|u(0)-\tilde{u}(0)|,|v(0)-\tilde{v}(0)|\}.$$

Proof. We first see that radial solutions of (5.68) are solutions of the ordinary differential equations system

$$\begin{cases} u''(r)+\dfrac{N-1}{r}u'(r)=p(r)\,g(v(r)), & r>0\\ v''(r)+\dfrac{N-1}{r}v'(r)=q(r)\,f(u(r)), & r>0. \end{cases} \tag{5.73}$$

Define $K=\max\{|u(0)-\tilde{u}(0)|,|v(0)-\tilde{v}(0)|\}$. Integrating the first equation of (5.73), we get

$$u'(r)-\tilde{u}'(r)=r^{1-N}\int_0^r s^{N-1}p(s)(g(v(s))-g(\tilde{v}(s)))\,ds.$$

Hence

$$|u(r)-\tilde{u}(r)| \le K+\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)|g(v(s))-g(\tilde{v}(s))|\,ds\,dt. \tag{5.74}$$

Since (u,v) and $(\tilde{u},\tilde{v})$ are bounded entire radial solutions of (5.68) we have

$$|g(v(r))-g(\tilde{v}(r))| \le m|v(r)-\tilde{v}(r)| \qquad \text{for any } r\in[0,\infty)$$
$$|f(u(r))-f(\tilde{u}(r))| \le m|u(r)-\tilde{u}(r)| \qquad \text{for any } r\in[0,\infty),$$

where m denotes a Lipschitz constant for both functions f and g. Therefore, using (5.74) we find

$$|u(r)-\tilde{u}(r)| \le K+m\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)|v(s)-\tilde{v}(s)|\,ds\,dt. \tag{5.75}$$

Arguing as above, but now with the second equation of (5.73), we obtain

$$|v(r)-\tilde{v}(r)| \le K+m\int_0^r t^{1-N}\int_0^t s^{N-1}q(s)|u(s)-\tilde{u}(s)|\,ds\,dt. \tag{5.76}$$

Define

$$X(r)=K+m\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)|v(s)-\tilde{v}(s)|\,ds\,dt.$$
$$Y(r)=K+m\int_0^r t^{1-N}\int_0^t s^{N-1}q(s)|u(s)-\tilde{u}(s)|\,ds\,dt.$$

It is clear that X and Y are non-decreasing functions with $X(0) = Y(0) = K$. By a simple calculation together with (5.75) and (5.76) we obtain

$$\begin{aligned} (r^{N-1}X')'(r) &= mr^{N-1}p(r)|v(r) - \tilde{v}(r)| \le mr^{N-1}p(r)Y(r) \\ (r^{N-1}Y')'(r) &= mr^{N-1}q(r)|u(r) - \tilde{u}(r)| \le mr^{N-1}q(r)X(r). \end{aligned} \tag{5.77}$$

Since Y is non-decreasing, we have

$$X(r) \le K + mY(r)A(r) \le K + \frac{m}{N-2}Y(r)\int_0^r tp(t)\,dt \le K + mC_pY(r) \tag{5.78}$$

where $C_p = (1/(N-2))\int_0^\infty tp(t)\,dt$. Using (5.78) in the second inequality of (5.77) we find

$$(r^{N-1}Y')'(r) \le mr^{N-1}q(r)(K + mC_pY(r)).$$

Integrating twice this inequality from 0 to r, we obtain

$$Y(r) \le K(1 + mC_q) + \frac{m^2}{N-2}C_p\int_0^r tq(t)Y(t)\,dt,$$

where $C_q = (1/(N-2))\int_0^\infty tq(t)\,dt$. From Gronwall's inequality, we deduce

$$Y(r) \le K(1 + mC_q)e^{\frac{m^2}{N-2}C_p\int_0^r tq(t)\,dt} \le K(1 + mC_q)e^{m^2C_pC_q}$$

and similarly for X. The conclusion follows now from the above inequality, (5.75) and (5.76). $\square$

Proof of Theorem ts1 completed. Since the radial solutions of (5.68) are solutions of the ordinary differential equations system (5.73) it follows that the radial solutions of (5.68) with $u(0) = a > 0$, $v(0) = b > 0$ satisfy

$$u(r) = a + \int_0^r t^{1-N}\int_0^t s^{N-1}p(s)\,g(v(s))\,ds\,dt, \qquad r \ge 0. \tag{5.79}$$

$$v(r) = b + \int_0^r t^{1-N}\int_0^t s^{N-1}q(s)\,f(u(s))\,ds\,dt, \qquad r \ge 0. \tag{5.80}$$

Define $v_0(r) = b$ for all $r \ge 0$. Let $(u_k)_{k\ge1}$ and $(v_k)_{k\ge1}$ be two sequences of functions given by

$$u_k(r) = a + \int_0^r t^{1-N}\int_0^t s^{N-1}p(s)\,g(v_{k-1}(s))\,ds\,dt, \qquad r \ge 0.$$

$$v_k(r) = b + \int_0^r t^{1-N}\int_0^t s^{N-1}q(s)\,f(u_k(s))\,ds\,dt, \qquad r \ge 0.$$

Since $v_1(r) \ge b$, we find $u_2(r) \ge u_1(r)$ for all $r \ge 0$. This implies $v_2(r) \ge v_1(r)$ which further produces $u_3(r) \ge u_2(r)$ for all $r \ge 0$. Proceeding at the same manner we conclude that

$$u_k(r) \le u_{k+1}(r) \quad \text{and} \quad v_k(r) \le v_{k+1}(r), \qquad \forall r \ge 0 \text{ and } k \ge 1.$$

We now prove that the non-decreasing sequences $(u_k(r))_{k\geq 1}$ and $(v_k(r))_{k\geq 1}$ are bounded from above on bounded sets. Indeed, we have

$$u_k(r) \leq u_{k+1}(r) \leq a + g(v_k(r))A(r), \qquad \forall r \geq 0 \tag{5.81}$$

and

$$v_k(r) \leq b + f\left(u_k(r)\right) B(r), \qquad \forall r \geq 0. \tag{5.82}$$

Let $R > 0$ be arbitrary. By (5.81) and (5.82) we find

$$u_k(R) \leq a + g\left(b + f\left(u_k(R)\right) B(R)\right) A(R), \qquad \forall k \geq 1$$

or, equivalently,

$$1 \leq \frac{a}{u_k(R)} + \frac{g\left(b + f\left(u_k(R)\right) B(R)\right)}{u_k(R)} A(R), \qquad \forall k \geq 1. \tag{5.83}$$

By the monotonicity of $(u_k(R))_{k\geq 1}$, there exists $\lim_{k\to\infty} u_k(R) := L(R)$. We claim that $L(R)$ is finite. Assume the contrary. Then, by taking $k \to \infty$ in (5.83) and using (5.71) we obtain a contradiction. Since $u_k'(r)$, $v_k'(r) \geq 0$ we get that the map $(0,\infty) \ni R \to L(R)$ is non-decreasing on $(0,\infty)$ and

$$u_k(r) \leq u_k(R) \leq L(R), \qquad \forall r \in [0, R],\ \forall k \geq 1. \tag{5.84}$$

$$v_k(r) \leq b + f\left(L(R)\right) B(R), \qquad \forall r \in [0, R],\ \forall k \geq 1. \tag{5.85}$$

It follows that there exists $\lim_{R\to\infty} L(R) = \overline{L} \in (0,\infty]$ and the sequences $(u_k(r))_{k\geq 1}$, $(v_k(r))_{k\geq 1}$ are bounded above on bounded sets. Therefore, we can define $u(r) := \lim_{k\to\infty} u_k(r)$ and $v(r) := \lim_{k\to\infty} v_k(r)$ for all $r \geq 0$. By standard elliptic regularity theory we obtain that (u,v) is a positive entire solution of (5.68) with $u(0) = a$ and $v(0) = b$.

We now assume that, in addition, condition (5.70) is fulfilled. According to Lemma 5.2 we have that $\lim_{r\to\infty} A(r) = \overline{A} < \infty$ and $\lim_{r\to\infty} B(r) = \overline{B} < \infty$. Passing to the limit as $k \to \infty$ in (5.83) we find

$$1 \leq \frac{a}{L(R)} + \frac{g\left(b + f\left(L(R)\right) B(R)\right)}{L(R)} A(R) \leq \frac{a}{L(R)} + \frac{g(b + f\left(L(R)\right) \overline{B})}{L(R)} \overline{A}.$$

Letting $R \to \infty$ and using (5.71) we deduce $\overline{L} < \infty$. Thus, taking into account (5.84) and (5.85), we obtain

$$u_k(r) \leq \overline{L} \quad \text{and} \quad v_k(r) \leq b + f(\overline{L})\overline{B}, \qquad \forall r \geq 0,\ \forall k \geq 1.$$

So, we have found upper bounds for $(u_k(r))_{k\geq 1}$ and $(v_k(r))_{k\geq 1}$ which are independent of r. Thus, the solution (u,v) is bounded from above. This shows that any solution of (5.79) and (5.80) will be bounded from above provided (5.70) holds. Thus, we can apply Lemma 5.3 to achieve the second assertion of *ii*).

Let us now drop the condition (5.70) and assume that (5.69) is fulfilled. In this case, Lemma 5.2 tells us that $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = \infty$. Let (u, v) be an entire positive radial solution of (5.68). Using (5.79) and (5.80) we obtain

$$u(r) \geq a + g(b)\,A(r), \qquad \forall r \geq 0.$$

$$v(r) \geq b + f(a)\,B(r), \qquad \forall r \geq 0.$$

Taking $r \to \infty$ we get that (u, v) is an entire large solution. This concludes the proof of Theorem 5.1. □

If f and g satisfy the stronger regularity $f, g \in C^1[0,\infty)$, then we drop the assumption (5.71) and require, in turn,

$(\mathbf{H_1})$ $\quad f(0) = g(0) = 0,\ \liminf_{u\to\infty} \frac{f(u)}{g(u)} =: \sigma > 0$

and the Keller-Osserman condition

$(\mathbf{H_2})$ $\quad \displaystyle\int_1^\infty \frac{dt}{\sqrt{G(t)}} < \infty$, where $G(t) = \displaystyle\int_0^t g(s)\,ds$.

Observe that assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$ imply that f satisfies condition $(\mathbf{H_2})$, too.

Set $\eta = \min\{p, q\}$.

Our main result in this case is

Theorem 5.4. *Let $f, g \in C^1[0,\infty)$ satisfy $(\mathbf{H_1})$ and $(\mathbf{H_2})$. Assume that (5.70) holds, η is not identically zero at infinity and $\nu := \max\{p(0), q(0)\} > 0$.*

Then any entire radial solution (u, v) of (5.68) with $(u(0), v(0)) \in F(\mathcal{G})$ is large.

Proof. Under the assumptions of Theorem 5.4 we prove the following auxiliary results.

Lemma 5.5. *$\mathcal{G} \neq \emptyset$.*

Proof. Cf. Cîrstea and Rădulescu [26], the problem

$$\Delta\psi = (p + q)(x)(f + g)(\psi) \quad \text{in } \mathbb{R}^N,$$

has a positive radial entire large solution. Since ψ is radial, we have

$$\psi(r) = \psi(0) + \int_0^r t^{1-N} \int_0^t s^{N-1}(p + q)(s)(f + g)(\psi(s))\,ds\,dt, \quad \forall r \geq 0.$$

We claim that $(0, \psi(0)] \times (0, \psi(0)] \subseteq \mathcal{G}$. To prove this, fix $0 < a, b \leq \psi(0)$ and let $v_0(r) \equiv b$ for all $r \geq 0$. Define the sequences $(u_k)_{k\geq1}$ and $(v_k)_{k\geq1}$ by

$$u_k(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g(v_{k-1}(s))\,ds\,dt, \quad \forall r \in [0,\infty), \quad \forall k \geq 1, \tag{5.86}$$

$$v_k(r) = b + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(u_k(s))\,ds\,dt, \quad \forall r \in [0,\infty), \quad \forall k \geq 1. \tag{5.87}$$

We first see that $v_0 \le v_1$ which produces $u_1 \le u_2$. Consequently, $v_1 \le v_2$ which further yields $u_2 \le u_3$. With the same arguments, we obtain that (u_k) and (v_k) are non-decreasing sequences. Since $\psi'(r) \ge 0$ and $b = v_0 \le \psi(0) \le \psi(r)$ for all $r \ge 0$ we find

$$\begin{aligned} u_1(r) &\le a + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g(\psi(s))\, ds\, dt \\ &\le \psi(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} (p+q)(s)(f+g)(\psi(s))\, ds\, dt = \psi(r). \end{aligned}$$

Thus $u_1 \le \psi$. It follows that

$$\begin{aligned} v_1(r) &\le b + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(\psi(s))\, ds\, dt \\ &\le \psi(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} (p+q)(s)(f+g)(\psi(s))\, ds\, dt = \psi(r). \end{aligned}$$

Similar arguments show that

$$u_k(r) \le \psi(r) \quad \text{and} \quad v_k(r) \le \psi(r) \quad \forall r \in [0, \infty), \quad \forall k \ge 1.$$

Thus, (u_k) and (v_k) converge and $(u, v) = \lim_{k\to\infty}(u_k, v_k)$ is an entire radial solution of (5.68) such that $(u(0), v(0)) = (a, b)$. This completes the proof. □

An easy consequence of the above result is

Corollary 5.6. *If* $(a, b) \in \mathcal{G}$, *then* $(0, a] \times (0, b] \subseteq \mathcal{G}$.

Proof. Indeed, the process used before can be repeated by taking

$$u_k(r) = a_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g(v_{k-1}(s))\, ds\, dt, \quad \forall r \in [0, \infty), \quad \forall k \ge 1,$$

$$v_k(r) = b_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(u_k(s))\, ds\, dt, \quad \forall r \in [0, \infty), \quad \forall k \ge 1,$$

where $0 < a_0 \le a$, $0 < b_0 \le b$ and $v_0(r) \equiv b_0$ for all $r \ge 0$.

Letting (U, V) be the entire radial solution of (5.68) with central values (a, b) we obtain as in Lemma 5.5,

$$u_k(r) \le u_{k+1}(r) \le U(r), \quad \forall r \in [0, \infty), \quad \forall k \ge 1,$$

$$v_k(r) \le v_{k+1}(r) \le V(r), \quad \forall r \in [0, \infty), \quad \forall k \ge 1.$$

Set $(u, v) = \lim_{k\to\infty}(u_k, v_k)$. We see that $u \le U$, $v \le V$ on $[0, \infty)$ and (u, v) is an entire radial solution of (5.68) with central values (a_0, b_0). This shows that $(a_0, b_0) \in \mathcal{G}$, so that our assertion is proved. □

Lemma 5.7. *$\mathcal{G}$ is bounded.*

Proof. Set $0 < \lambda < \min\{\sigma, 1\}$ and let $\delta = \delta(\lambda)$ be large enough so that

$$f(t) \geq \lambda g(t), \qquad \forall t \geq \delta. \tag{5.88}$$

Since η is radially symmetric and not identically zero at infinity, we can assume $\eta > 0$ on $\partial B(0, R)$ for some $R > 0$. Let ζ be a positive large solution ζ of the problem

$$\Delta\zeta = \lambda\eta(x)g\left(\frac{\zeta}{2}\right) \qquad \text{in } B(0, R).$$

Arguing by contradiction, we assume that $\mathcal{G}$ is not bounded. Then, there exists $(a, b) \in \mathcal{G}$ such that $a + b > \max\{2\delta, \zeta(0)\}$. Let (u, v) be the entire radial solution of (5.68) such that $(u(0), v(0)) = (a, b)$. Since $u(x) + v(x) \geq a + b > 2\delta$ for all $x \in \mathbb{R}^N$, by (5.88), we find

$$f(u(x)) \geq f\left(\frac{u(x)+v(x)}{2}\right) \geq \lambda g\left(\frac{u(x)+v(x)}{2}\right) \qquad \text{if } u(x) \geq v(x)$$

and

$$g(v(x)) \geq g\left(\frac{u(x)+v(x)}{2}\right) \geq \lambda g\left(\frac{u(x)+v(x)}{2}\right) \qquad \text{if } v(x) \geq u(x).$$

It follows that

$$\Delta(u+v) = p(x)g(v) + q(x)f(u) \geq \eta(x)(g(v) + f(u)) \geq \lambda\eta(x)g\left(\frac{u+v}{2}\right) \qquad \text{in } \mathbb{R}^N.$$

On the other hand, $\zeta(x) \to \infty$ as $|x| \to R$ and $u, v \in C^2(\overline{B(0,R)})$. Thus, by the maximum principle, we conclude that $u + v \leq \zeta$ in $B(0, R)$. But this is impossible since $u(0) + v(0) = a + b > \zeta(0)$. ☐

Lemma 5.8. *$F(\mathcal{G}) \subset \mathcal{G}$.*

Proof. Let $(a, b) \in F(\mathcal{G})$. We claim that $(a - 1/n_0, b - 1/n_0) \in \mathcal{G}$ provided $n_0 \geq 1$ is large enough so that $\min\{a, b\} > 1/n_0$. Indeed, if this is not true, by Corollary 5.6

$$D := \left[a - \frac{1}{n_0}, \infty\right) \times \left[b - \frac{1}{n_0}, \infty\right) \subseteq (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \mathcal{G}.$$

So, we can find a small ball B centered in (a, b) such that $B \subset\subset D$, i.e., $B \cap \mathcal{G} = \emptyset$. But this will contradict the choice of (a, b). Consequently, there exists (u_{n_0}, v_{n_0}) an entire radial solution of (5.68) such that $(u_{n_0}(0), v_{n_0}(0)) = (a - 1/n_0, b - 1/n_0)$. Thus, for any $n \geq n_0$, we can define

$$u_n(r) = a - \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g(v_n(s))\, ds\, dt, \qquad r \geq 0,$$

$$v_n(r) = b - \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(u_n(s))\, ds\, dt, \qquad r \geq 0.$$

Using Corollary 5.6 once more, we conclude that $(u_n)_{n \geq n_0}$ and $(v_n)_{n \geq n_0}$ are non-decreasing sequences. We now prove that (u_n) and (v_n) converge on $\mathbb{R}^N$. To this aim, let $x_0 \in \mathbb{R}^N$ be arbitrary.

But η is not identically zero at infinity so that, for some $R_0 > 0$, we have $\eta > 0$ on $\partial B(0, R_0)$ and $x_0 \in B(0, R_0)$.

Since $\sigma = \liminf_{u\to\infty} \frac{f(u)}{g(u)} > 0$, we find $\tau \in (0, 1)$ such that

$$f(t) \geq \tau g(t), \qquad \forall t \geq \frac{a+b}{2} - \frac{1}{n_0}.$$

Therefore, on the set where $u_n \geq v_n$, we have

$$f(u_n) \geq f\left(\frac{u_n + v_n}{2}\right) \geq \tau g\left(\frac{u_n + v_n}{2}\right).$$

Similarly, on the set where $u_n \leq v_n$, we have

$$g(v_n) \geq g\left(\frac{u_n + v_n}{2}\right) \geq \tau g\left(\frac{u_n + v_n}{2}\right).$$

It follows that, for any $x \in \mathbb{R}^N$,

$$\Delta(u_n + v_n) = p(x)g(v_n) + q(x)f(u_n) \geq \eta(x)[g(v_n) + f(u_n)] \geq \tau\eta(x)g\left(\frac{u_n + v_n}{2}\right).$$

On the other hand, there exists a positive large solution of

$$\Delta\zeta = \tau\eta(x)g\left(\frac{\zeta}{2}\right) \qquad \text{in } B(0, R_0).$$

The maximum principle yields $u_n+v_n \leq \zeta$ in $B(0, R_0)$. So, it makes sense to define $(u(x_0), v(x_0)) = \lim_{n\to\infty}(u_n(x_0), v_n(x_0))$. Since x_0 is arbitrary, the functions u, v exist on $\mathbb{R}^N$. Hence (u, v) is an entire radial solution of (5.68) with central values (a, b), i.e., $(a, b) \in \mathcal{G}$. □

For $(c, d) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \mathcal{G}$, define

$$R_{c,d} = \sup\{r > 0 \,|\, \text{there exists a radial solution of (5.68) in } B(0, r) \text{ so that } (u(0), v(0)) = (c, d)\}. \tag{5.89}$$

Lemma 5.9. *If, in addition, $\nu = \max\{p(0), q(0)\} > 0$, then $0 < R_{c,d} < \infty$ where $R_{c,d}$ is defined by (5.89).*

Proof. Since $\nu > 0$ and $p, q \in C[0, \infty)$, there exists $\epsilon > 0$ such that $(p + q)(r) > 0$ for all $0 \leq r < \epsilon$. Let $0 < R < \epsilon$ be arbitrary. There exists a positive radial large solution of the problem

$$\Delta\psi_R = (p + q)(x)(f + g)(\psi_R) \qquad \text{in } B(0, R).$$

Moreover, for any $0 \leq r < R$,

$$\psi_R(r) = \psi_R(0) + \int_0^r t^{1-N} \int_0^t s^{N-1}(p + q)(s)(f + g)(\psi_R(s))\, ds\, dt.$$

It is clear that $\psi_R'(r) \geq 0$. Thus, we find

$$\psi_R'(r) = r^{1-N} \int_0^r s^{N-1}(p+q)(s)(f+g)(\psi_R(s))\,ds \leq C(f+g)(\psi_R(r))$$

where $C > 0$ is a positive constant such that $\int_0^\epsilon (p+q)(s)\,ds \leq C$.

Since $f+g$ satisfies $(\mathbf{A_1})$ and $(\mathbf{A_2})$, we may invoke Remark 1 in Section 2 to conclude that

$$\int_1^\infty \frac{dt}{(f+g)(t)} < \infty.$$

Therefore, we obtain

$$-\frac{d}{dr}\int_{\psi_R(r)}^\infty \frac{ds}{(f+g)(s)} = \frac{\psi_R'(r)}{(f+g)(\psi_R(r))} \leq C \qquad \text{for any } 0 < r < R.$$

Integrating from 0 to R and recalling that $\psi_R(r) \to \infty$ as $r \nearrow R$, we obtain

$$\int_{\psi_R(0)}^\infty \frac{ds}{(f+g)(s)} \leq CR.$$

Letting $R \searrow 0$ we conclude that

$$\lim_{R \searrow 0} \int_{\psi_R(0)}^\infty \frac{ds}{(f+g)(s)} = 0.$$

This implies that $\psi_R(0) \to \infty$ as $R \searrow 0$. So, there exists $0 < \tilde{R} < \epsilon$ such that $0 < c, d \leq \psi_{\tilde{R}}(0)$. Set

$$u_k(r) = c + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g(v_{k-1}(s))\,ds\,dt, \qquad \forall r \in [0,\infty),\ \forall k \geq 1, \tag{5.90}$$

$$v_k(r) = d + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(u_k(s))\,ds\,dt, \qquad \forall r \in [0,\infty),\ \forall k \geq 1, \tag{5.91}$$

where $v_0(r) = d$ for all $r \in [0,\infty)$. As in Lemma 5.5, we find that (u_k) resp., (v_k) are non-decreasing and

$$u_k(r) \leq \psi_{\tilde{R}}(r) \quad \text{and} \qquad v_k(r) \leq \psi_{\tilde{R}}(r), \qquad \forall r \in [0, \tilde{R}),\ \forall k \geq 1.$$

Thus, for any $r \in [0, \tilde{R})$, there exists $(u(r), v(r)) = \lim_{k\to\infty}(u_k(r), v_k(r))$ which is, moreover, a radial solution of (5.68) in $B(0, \tilde{R})$ such that $(u(0), v(0)) = (c, d)$. This shows that $R_{c,d} \geq \tilde{R} > 0$. By the definition of $R_{c,d}$ we also derive

$$\lim_{r \nearrow R_{c,d}} u(r) = \infty \quad \text{and} \quad \lim_{r \nearrow R_{c,d}} v(r) = \infty. \tag{5.92}$$

On the other hand, since $(c,d) \notin \mathcal{G}$, we conclude that $R_{c,d}$ is finite. □

Proof of Theorem 5.4 completed. Let $(a,b) \in F(\mathcal{G})$ be arbitrary. By Lemma 5.8, $(a,b) \in \mathcal{G}$ so that we can define (U,V) an entire radial solution of (5.68) with $(U(0), V(0)) = (a,b)$. Obviously, for any $n \geq 1$, $(a + 1/n, b + 1/n) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \mathcal{G}$. By Lemma 5.9, $R_{a+1/n,b+1/n}$ (in short, R_n) defined by (5.89) is a positive number. Let (U_n, V_n) be the radial solution of (5.68) in $B(0, R_n)$ with the central values $(a + 1/n, b + 1/n)$. Thus,

$$U_n(r) = a + \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g(V_n(s))\, ds\, dt, \qquad \forall r \in [0, R_n), \tag{5.93}$$

$$V_n(r) = b + \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(U_n(s))\, ds\, dt, \qquad \forall r \in [0, R_n). \tag{5.94}$$

In view of (5.92) we have

$$\lim_{r \nearrow R_n} U_n(r) = \infty \quad \text{and} \quad \lim_{r \nearrow R_n} V_n(r) = \infty, \quad \forall n \geq 1.$$

We claim that $(R_n)_{n \geq 1}$ is a non-decreasing sequence. Indeed, if (u_k), (v_k) denote the sequences of functions defined by (5.90) and (5.91) with $c = a + 1/(n + 1)$ and $d = b + 1/(n + 1)$, then

$$u_k(r) \leq u_{k+1}(r) \leq U_n(r), \qquad v_k(r) \leq v_{k+1}(r) \leq V_n(r), \qquad \forall r \in [0, R_n),\ \forall k \geq 1. \tag{5.95}$$

This implies that $(u_k(r))_{k \geq 1}$ and $(v_k(r))_{k \geq 1}$ converge for any $r \in [0, R_n)$. Moreover, $(U_{n+1}, V_{n+1}) = \lim_{k \to \infty}(u_k, v_k)$ is a radial solution of (5.68) in $B(0, R_n)$ with central values $(a + 1/(n + 1), b + 1/(n + 1))$. By the definition of R_{n+1}, it follows that $R_{n+1} \geq R_n$ for any $n \geq 1$.

Set $R := \lim_{n \to \infty} R_n$ and let $0 \leq r < R$ be arbitrary. Then, there exists $n_1 = n_1(r)$ such that $r < R_n$ for all $n \geq n_1$. From (5.95) we see that $U_{n+1} \leq U_n$ (resp., $V_{n+1} \leq V_n$) on $[0, R_n)$ for all $n \geq 1$. So, there exists $\lim_{n \to \infty}(U_n(r), V_n(r))$ which, by (5.93) and (5.94), is a radial solution of (5.68) in $B(0, R)$ with central values (a, b). Consequently,

$$\lim_{n \to \infty} U_n(r) = U(r) \quad \text{and} \quad \lim_{n \to \infty} V_n(r) = V(r) \quad \text{for any } r \in [0, R). \tag{5.96}$$

Since $U_n'(r) \geq 0$, from (5.94) we find

$$V_n(r) \leq b + \frac{1}{n} + f(U_n(r)) \int_0^\infty t^{1-N} \int_0^t s^{N-1} q(s)\, ds\, dt.$$

This yields

$$V_n(r) \leq C_1 U_n(r) + C_2 f(U_n(r)) \tag{5.97}$$

where C_1 is an upper bound of $(V(0) + 1/n)/(U(0) + 1/n)$ and

$$C_2 = \int_0^\infty t^{1-N} \int_0^t s^{N-1} q(s)\, ds\, dt \leq \frac{1}{N - 2} \int_0^\infty s q(s)\, ds < \infty.$$

Define $h(t) = g(C_1 t + C_2 f(t))$ for $t \geq 0$. It is easy to check that h satisfies $(\mathbf{A_1})$ and $(\mathbf{A_2})$. Define

$$\Gamma(s) = \int_s^\infty \frac{dt}{h(t)}, \qquad \text{for all } s > 0.$$

But U_n verifies

$$\Delta U_n = p(x)g(V_n)$$

which combined with (5.97) implies

$$\Delta U_n \le p(x)h(U_n).$$

A simple calculation shows that

$$\begin{aligned}\Delta\Gamma(U_n) &= \Gamma'(U_n)\Delta U_n + \Gamma''(U_n)|\nabla U_n|^2 = \frac{-1}{h(U_n)}\Delta U_n + \frac{h'(U_n)}{[h(U_n)]^2}|\nabla U_n|^2 \\ &\ge \frac{-1}{h(U_n)}p(r)h(U_n) = -p(r)\end{aligned}$$

which we rewrite as

$$\left(r^{N-1}\frac{d}{dr}\Gamma(U_n)\right)' \ge -r^{N-1}p(r) \qquad \text{for any } 0 < r < R_n.$$

Fix $0 < r < R$. Then $r < R_n$ for all $n \ge n_1$ provided n_1 is large enough. Integrating the above inequality over $[0, r]$, we get

$$\frac{d}{dr}\Gamma(U_n) \ge -r^{1-N}\int_0^r s^{N-1}p(s)\,ds.$$

Integrating this new inequality over $[r, R_n]$ we obtain

$$-\Gamma(U_n(r)) \ge -\int_r^{R_n} t^{1-N}\int_0^t s^{N-1}p(s)\,ds\,dt, \qquad \forall n \ge n_1,$$

since $U_n(r) \to \infty$ as $r \nearrow R_n$ implies $\Gamma(U_n(r)) \to 0$ as $r \nearrow R_n$. Therefore,

$$\Gamma(U_n(r)) \le \int_r^{R_n} t^{1-N}\int_0^t s^{N-1}p(s)\,ds\,dt, \qquad \forall n \ge n_1.$$

Letting $n \to \infty$ and using (5.96) we find

$$\Gamma(U(r)) \le \int_r^{R} t^{1-N}\int_0^t s^{N-1}p(s)\,ds\,dt,$$

or, equivalently

$$U(r) \ge \Gamma^{-1}\left(\int_r^{R} t^{1-N}\int_0^t s^{N-1}p(s)\,ds\,dt\right).$$

Passing to the limit as $r \nearrow R$ and using the fact that $\lim_{s\searrow 0}\Gamma^{-1}(s) = \infty$ we deduce

$$\lim_{r\nearrow R} U(r) \ge \lim_{r\nearrow R}\Gamma^{-1}\left(\int_r^{R} t^{1-N}\int_0^t s^{N-1}p(s)\,ds\,dt\right) = \infty.$$

But (U, V) is an entire solution so that we conclude $R = \infty$ and $\lim_{r\to\infty} U(r) = \infty$. Since (5.70) holds and $V'(r) \geq 0$ we find

$$\begin{aligned} U(r) &\leq a + g(V(r)) \int_0^\infty t^{1-N} \int_0^t s^{N-1} p(s)\, ds\, dt \\ &\leq a + g(V(r)) \frac{1}{N-2} \int_0^\infty tp(t)\, dt, \qquad \forall r \geq 0. \end{aligned}$$

We deduce $\lim_{r\to\infty} V(r) = \infty$, otherwise we obtain that $\lim_{r\to\infty} U(r)$ is finite, a contradiction. Consequently, (U, V) is an entire large solution of (5.68). This concludes our proof. □

6 Bifurcation problems for singular Lane-Emden-Fowler equations

In this section we study the bifurcation problem

$$\begin{cases} -\Delta u = \lambda f(u) + a(x)g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P_\lambda)$$

where $\lambda \in \mathbb{R}$ is a parameter and $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$. The main feature of this boundary value problem is the presence of the "smooth" nonlinearity f combined with the "singular" nonlinearity g. More exactly, we assume that $0 < f \in C^{0,\beta}[0,\infty)$ and $0 \leq g \in C^{0,\beta}(0,\infty)$ $(0 < \beta < 1)$ fulfill the hypotheses

$(f1)$ f is nondecreasing on $(0,\infty)$ while $f(s)/s$ is nonincreasing for $s > 0$;

$(g1)$ g is nonincreasing on $(0,\infty)$ with $\lim_{s\searrow 0} g(s) = +\infty$;

$(g2)$ there exists $C_0, \eta_0 > 0$ and $\alpha \in (0,1)$ so that $g(s) \leq C_0 s^{-\alpha}$, $\forall s \in (0, \eta_0)$.

The assumption $(g2)$ implies the following Keller-Osserman-type growth condition around the origin

$$\int_0^1 \left(\int_0^t g(s) ds \right)^{-1/2} dt < +\infty. \qquad (6.98)$$

As proved by Bénilan, Brezis and Crandall in [11], condition (6.98) is equivalent to the *property of compact support*, that is, for any $h \in L^1(\mathbb{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbb{R}^N)$ and

$$-\Delta u + g(u) = h \qquad \text{a.e. in } \mathbb{R}^N.$$

In many papers (see, e.g., Dalmasso [36], Kusano and Swanson [64]) the potential $a(x)$ is assumed to depend "almost" radially on x, in the sense that $C_1\, p(|x|) \leq a(x) \leq C_2\, p(|x|)$, where C_1, C_2 are positive constants and $p(|x|)$ is a positive function satisfying some integrability condition.

We do not impose any growth assumption on a, but we suppose throughout this paper that the variable potential $a(x)$ satisfies $a \in C^{0,\beta}(\overline{\Omega})$ and $a > 0$ in Ω.

If $\lambda = 0$ this equation is called the Lane-Emden-Fowler equation and arises in the boundary-layer theory of viscous fluids (see Wong [92]). Problems of this type, as well as the associated evolution equations, describe naturally certain physical phenomena. For example, super-diffusivity equations of this type have been proposed by de Gennes [37] as a model for long range Van der Waals interactions in thin films spreading on solid surfaces.

Our purpose is to study the effect of the asymptotically linear perturbation $f(u)$ in (P_λ), as well as to describe the set of values of the positive parameter λ such that problem (P_λ) admits a solution. In this case, we also prove a uniqueness result. Due to the singular character of (P_λ), we can not expect to find solutions in $C^2(\overline{\Omega})$. However, under the above assumptions we will show that (P_λ) has solutions in the class

$$\mathcal{E} := \{\, u \in C^2(\Omega) \cap C^{1,1-\alpha}(\overline{\Omega});\ \Delta u \in L^1(\Omega)\}.$$

We first observe that, in view of the assumption $(f1)$, there exists

$$m := \lim_{s\to\infty} \frac{f(s)}{s} \in [0,\infty).$$

This number plays a crucial role in our analysis. More precisely, the existence of the solutions to (P_λ) will be separately discussed for $m > 0$ and $m = 0$. Let $a_* = \min_{x\in\overline{\Omega}} a(x)$.

Theorems 6.1–6.4 have been established by Cîrstea, Ghergu, and Rădulescu [23].

Theorem 6.1. *Assume* $(f1)$, $(g1)$, $(g2)$ *and* $m = 0$. *If* $a_* > 0$ *(resp.,* $a_* = 0$*), then* (P_λ) *has a unique solution* $u_\lambda \in \mathcal{E}$ *for all* $\lambda \in \mathbb{R}$ *(resp.,* $\lambda \geq 0$*) with the properties:*

(i) u_λ *is strictly increasing with respect to* λ.

(ii) *there exist two positive constant* $c_1, c_2 > 0$ *depending on* λ *such that* $c_1 d(x) \leq u_\lambda \leq c_2 d(x)$ *in* Ω.

The bifurcation diagram in the "sublinear" case $m = 0$ is depicted in Figure 1.

Proof. We first recall some auxiliary results that we need in the proof.

Lemma 6.2. (Shi and Yao [86]). *Let* $F : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ *be a Hölder continuous function with exponent* $\beta \in (0,1)$ *on each compact subset of* $\overline{\Omega} \times (0,\infty)$ *which satisfies*

(F1) $\limsup_{s\to+\infty} \left(s^{-1} \max_{x\in\overline{\Omega}} F(x,s)\right) < \lambda_1$;

(F2) *for each* $t > 0$, *there exists a constant* $D(t) > 0$ *such that*

$$F(x,r) - F(x,s) \geq -D(t)(r-s), \quad \text{for } x \in \overline{\Omega} \ \text{ and } \ r \geq s \geq t;$$

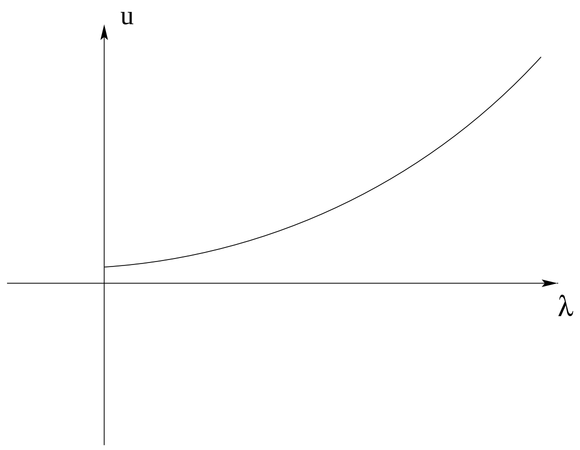

Figure 1: *The "sublinear" case* $m = 0$.

(F3) *there exists* $\eta_0 > 0$ *and an open subset* $\Omega_0 \subset \Omega$ *such that*

$$\min_{x\in\overline{\Omega}} F(x,s) \geq 0 \quad \textit{for } s \in (0,\eta_0),$$

and

$$\lim_{s\searrow 0} \frac{F(x,s)}{s} = +\infty \quad \textit{uniformly for } x \in \Omega_0.$$

Then for any nonnegative function $\phi_0 \in C^{2,\beta}(\partial\Omega)$*, the problem*

$$\begin{cases} -\Delta u = F(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega\,, \\ u = \phi_0 & \text{on } \partial\Omega, \end{cases}$$

has at least one positive solution $u \in C^{2,\beta}(G) \cap C(\overline{\Omega})$*, for any compact set* $G \subset \Omega \cup \{x \in \partial\Omega;\ \phi_0(x) > 0\}$.

Lemma 6.3. (Shi and Yao [86]). *Let* $F : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ *be a continuous function such that the mapping* $(0,\infty) \ni s \longmapsto \dfrac{F(x,s)}{s}$ *is strictly decreasing at each* $x \in \Omega$*. Assume that there exists* v, $w \in C^2(\Omega) \cap C(\overline{\Omega})$ *such that*

(a) $\Delta w + F(x,w) \leq 0 \leq \Delta v + F(x,v)$ *in* Ω*;*

(b) $v, w > 0$ *in* Ω *and* $v \leq w$ *on* $\partial\Omega$*;*

(c) $\Delta v \in L^1(\Omega)$.

Then $v \leq w$ *in* Ω.

Now, we are ready to give the proof of Theorem 6.1. This will be divided into four steps.

STEP 1. **Existence of solutions to problem** (P_λ)**.**

For any $\lambda \in \mathbb{R}$, define the function

$$\Phi_\lambda(x,s) = \lambda f(s) + a(x)g(s), \quad (x,s) \in \overline{\Omega} \times (0,\infty). \tag{6.99}$$

Taking into account the assumptions of Theorem 6.1, it follows that Φ_λ verifies the hypotheses of Lemma 6.2 for $\lambda \in \mathbb{R}$ if $a_* > 0$ and $\lambda \geq 0$ if $a_* = 0$. Hence, for λ in the above range, (P_λ) has at least one solution $u_\lambda \in C^{2,\beta}(\Omega) \cap C(\overline{\Omega})$.

STEP 2. **Uniqueness of solution.**

Fix $\lambda \in \mathbb{R}$ (resp., $\lambda \geq 0$) if $a_* > 0$ (resp., $a_* = 0$). Let u_λ be a solution of (P_λ). Denote $\lambda^- = \min\{0,\lambda\}$ and $\lambda^+ = \max\{0,\lambda\}$. We claim that $\Delta u_\lambda \in L^1(\Omega)$. Since $a \in C^{0,\beta}(\overline{\Omega})$, by [55, Theorem 6.14], there exists a unique nonnegative solution $\zeta \in C^{2,\beta}(\overline{\Omega})$ of

$$\begin{cases} -\Delta\zeta = a(x) & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega\,. \end{cases}$$

By the weak maximum principle (see e.g., [55, Theorem 2.2]), $\zeta > 0$ in Ω. Moreover, we are going to prove that

(a) $z(x) := c\zeta(x)$ is a sub-solution of (P_λ), for $c > 0$ small enough;

(b) $z(x) \geq c_1 d(x)$ in $\overline{\Omega}$, for some positive constant $c_1 > 0$;

(c) $u_\lambda \geq z$ in $\overline{\Omega}$.

Therefore, by (b) and (c), $u_\lambda \geq c_1 d(x)$ in $\overline{\Omega}$. Using $(g2)$, we obtain $g(u_\lambda) \leq Cd^{-\alpha}(x)$ in Ω, where $C > 0$ is a constant. So, $g(u_\lambda) \in L^1(\Omega)$. This implies

$$\Delta u_\lambda \in L^1(\Omega).$$

Proof of (a). Using $(f1)$ and $(g1)$, we have

$$\begin{aligned} \Delta z(x) + \Phi_\lambda(x,z) &= -ca(x) + \lambda f(c\zeta) + a(x)g(c\zeta) \\ &\geq -ca(x) + \lambda^- f(c\|\zeta\|_\infty) + a(x)g(c\|\zeta\|_\infty) \\ &\geq ca(x)\left[\frac{g(c\|\zeta\|_\infty)}{2c} - 1\right] + f(c\|\zeta\|_\infty)\left[a_*\frac{g(c\|\zeta\|_\infty)}{2f(c\|\zeta\|_\infty)} + \lambda^-\right] \end{aligned}$$

for each $x \in \Omega$. Since $\lambda < 0$ corresponds to $a_* > 0$, using $\lim_{t\searrow 0} g(t) = +\infty$ and $\lim_{t\to 0} f(t) \in (0,\infty)$, we can find $c > 0$ small such that

$$\Delta z + \Phi_\lambda(x,z) \geq 0, \quad \forall x \in \Omega.$$

This concludes (a).

Proof of (b). Since $\zeta \in C^{2,\beta}(\overline{\Omega})$, $\zeta > 0$ in Ω and $\zeta = 0$ on $\partial\Omega$, by Lemma 3.4 in Gilbarg and Trudinger [55], we have

$$\frac{\partial \zeta}{\partial \nu}(y) < 0, \quad \forall y \in \partial\Omega.$$

Therefore, there exists a positive constant c_0 such that

$$\frac{\partial \zeta}{\partial \nu}(y) := \lim_{x\in\Omega, x\to y} \frac{\zeta(y) - \zeta(x)}{|x-y|} \leq -c_0, \quad \forall y \in \partial\Omega.$$

So, for each $y \in \Omega$, there exists $r_y > 0$ such that

$$\frac{\zeta(x)}{|x-y|} \geq \frac{c_0}{2}, \quad \forall x \in B_{r_y}(y) \cap \Omega. \tag{6.100}$$

Using the compactness of $\partial\Omega$, we can find a finite number k of balls $B_{r_{y_i}}(y_i)$ such that $\partial\Omega \subset \cup_{i=1}^k B_{r_{y_i}}(y_i)$. Moreover, we can assume that for small $d_0 > 0$,

$$\{x \in \Omega : \ d(x) < d_0\} \subset \cup_{i=1}^k B_{r_{y_i}}(y_i).$$

Therefore, by (6.100) we obtain

$$\zeta(x) \geq \frac{c_0}{2} d(x), \quad \forall x \in \Omega \text{ with } d(x) < d_0.$$

This fact, combined with $\zeta > 0$ in Ω, shows that for some constant $\tilde{c} > 0$

$$\zeta(x) \geq \tilde{c} d(x), \quad \forall x \in \Omega.$$

Thus, (b) follows by the definition of z.

Proof of (c). We distinguish two cases:

Case 1. $\lambda \geq 0$. We see that Φ_λ verifies the hypotheses in Lemma 6.3. Since

$$\begin{aligned} \Delta u_\lambda + \Phi_\lambda(x, u_\lambda) \leq 0 \leq \Delta z + \Phi_\lambda(x, z) \quad &\text{in } \Omega, \\ u_\lambda,\, z > 0 \quad &\text{in } \Omega, \\ u_\lambda = z \quad &\text{on } \partial\Omega, \\ \Delta z \in L^1(\Omega), & \end{aligned}$$

by Lemma 6.3 it follows that $u_\lambda \geq z$ in $\overline{\Omega}$.

Now, if u_1 and u_2 are two solutions of (P_λ), we can use Lemma 6.3 in order to deduce that $u_1 = u_2$.

Case 2. $\lambda < 0$ (corresponding to $a_* > 0$). Let $\varepsilon > 0$ be fixed. We prove that

$$z \leq u_\lambda + \varepsilon(1 + |x|^2)^\tau \quad \text{in } \Omega, \tag{6.101}$$

where $\tau < 0$ is chosen such that $\tau|x|^2 + 1 > 0$, $\forall x \in \Omega$. This is always possible since $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is bounded.

We argue by contradiction. Suppose that there exists $x_0 \in \Omega$ such that $u_\lambda(x_0)+\varepsilon(1+|x_0|)^\tau < z(x_0)$. Then $\min_{x\in\overline{\Omega}}\{u_\lambda(x) + \varepsilon(1 + |x|^2)^\tau - z(x)\} < 0$ is achieved at some point $x_1 \in \Omega$. Since $\Phi_\lambda(x, z)$ is nonincreasing in z, we have

$$\begin{aligned} 0 &\geq -\Delta[u_\lambda(x) - z(x) + \varepsilon(1 + |x|^2)^\tau]|_{x=x_1} \\ &= \Phi_\lambda(x_1, u_\lambda(x_1)) - \Phi_\lambda(x_1, z(x_1)) - \varepsilon\Delta[(1 + |x|^2)^\tau]|_{x=x_1} \\ &\geq -\varepsilon\Delta[(1 + |x|^2)^\tau]|_{x=x_1} = -2\varepsilon\tau(1 + |x_1|^2)^{\tau-2}[(N + 2\tau - 2)|x_1|^2 + N] \\ &\geq -4\varepsilon\tau(1 + |x_1|^2)^{\tau-2}(\tau|x_1|^2 + 1) > 0. \end{aligned}$$

This contradiction proves (6.101). Passing to the limit $\varepsilon \to 0$, we obtain (c).

In a similar way we can prove that (P_λ) has a unique solution.

Step 3. **Dependence on λ.**

We fix $\lambda_1 < \lambda_2$, where λ_1, $\lambda_2 \in \mathbb{R}$ if $a_* > 0$ resp., λ_1, $\lambda_2 \in [0, \infty)$ if $a_* = 0$. Let u_{λ_1}, u_{λ_2} be the corresponding solutions of (P_{λ_1}) and (P_{λ_2}) respectively.

If $\lambda_1 \geq 0$, then Φ_{λ_1} verifies the hypotheses in Lemma 6.3. Furthermore, we have

$$\begin{aligned} &\Delta u_{\lambda_2} + \Phi_{\lambda_1}(x, u_{\lambda_2}) \leq 0 \leq \Delta u_{\lambda_1} + \Phi_{\lambda_1}(x, u_{\lambda_1}) \quad \text{in } \Omega, \\ &u_{\lambda_1}, u_{\lambda_2} > 0 \quad \text{in } \Omega, \\ &u_{\lambda_1} = u_{\lambda_2} \quad \text{on } \partial\Omega, \\ &\Delta u_{\lambda_1} \in L^1(\Omega). \end{aligned}$$

Again by Lemma 6.3, we conclude that $u_{\lambda_1} \leq u_{\lambda_2}$ in $\overline{\Omega}$. Moreover, by the maximum principle, $u_{\lambda_1} < u_{\lambda_2}$ in Ω.

Let $\lambda_2 \leq 0$; we show that $u_{\lambda_1} \leq u_{\lambda_2}$ in $\overline{\Omega}$. Indeed, supposing the contrary, there exists $x_0 \in \Omega$ such that $u_{\lambda_1}(x_0) > u_{\lambda_2}(x_0)$. We conclude now that $\max\limits_{x\in\overline{\Omega}}\{u_{\lambda_1}(x) - u_{\lambda_2}(x)\} > 0$ is achieved at some point in Ω. At that point, say $\bar{x}$, we have

$$0 \leq -\Delta(u_{\lambda_1} - u_{\lambda_2})(\bar{x}) = \Phi_{\lambda_1}(\bar{x}, u_{\lambda_1}(\bar{x})) - \Phi_{\lambda_2}(\bar{x}, u_{\lambda_2}(\bar{x})) < 0,$$

which is a contradiction. It follows that $u_{\lambda_1} \leq u_{\lambda_2}$ in $\overline{\Omega}$, and by maximum principle we have $u_{\lambda_1} < u_{\lambda_2}$ in Ω.

If $\lambda_1 < 0 < \lambda_2$, then $u_{\lambda_1} < u_0 < u_{\lambda_2}$ in Ω. This finishes the proof of Step 3.

Step 4. **Regularity of the solution.**

Fix $\lambda \in \mathbb{R}$ and let $u_\lambda \in C^2(\Omega) \cap C(\overline{\Omega})$ be the unique solution of (P_λ). An important result in our approach is the following estimate

$$c_1 d(x) \leq u_\lambda(x) \leq c_2 d(x), \qquad \text{for all } x \in \Omega, \tag{6.102}$$

where c_1, c_2 are positive constants. The first inequality in (6.102) was established in Step 2. For the second one, we apply an idea found in Gui and Lin [57].

Using the smoothness of $\partial\Omega$, we can find $\delta \in (0,1)$ such that for all $x_0 \in \Omega_\delta := \{x \in \Omega\,;\, d(x) \leq \delta\}$, there exists $y \in \mathbb{R}^N \setminus \overline{\Omega}$ with $d(y, \partial\Omega) = \delta$ and $d(x_0) = |x_0 - y| - \delta$.

Let $K > 1$ be such that $\operatorname{diam}(\Omega) < (K-1)\delta$ and let w be the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta w = \lambda^+ f(w) + g(w) & \text{in } B_K(0) \setminus \overline{B_1}(0), \\ w > 0 & \text{in } B_K(0) \setminus \overline{B_1}(0), \\ w = 0 & \text{on } \partial(B_K(0) \setminus \overline{B_1}(0)), \end{cases} \tag{6.103}$$

where $B_r(0)$ is the open ball in $\mathbb{R}^N$ of radius r and centered at the origin. By uniqueness, w is radially symmetric. Hence $w(x) = \tilde{w}(|x|)$ and

$$\begin{cases} \tilde{w}'' + \dfrac{N-1}{r}\tilde{w}' + \lambda^+ f(\tilde{w}) + g(\tilde{w}) = 0 & \text{for } r \in (1,K), \\ \tilde{w} > 0 & \text{in } (1,K), \\ \tilde{w}(1) = \tilde{w}(K) = 0. \end{cases} \tag{6.104}$$

Integrating in (6.104) we have

$$\begin{aligned} \tilde{w}'(t) &= \tilde{w}'(a)a^{N-1}t^{1-N} - t^{1-N}\int_a^t r^{N-1}\left[\lambda^+ f(\tilde{w}(r)) + g(\tilde{w}(r))\right] dr, \\ &= \tilde{w}'(b)b^{N-1}t^{1-N} + t^{1-N}\int_t^b r^{N-1}\left[\lambda^+ f(\tilde{w}(r)) + g(\tilde{w}(r))\right] dr, \end{aligned}$$

where $1 < a < t < b < K$. Since $g(\tilde{w}) \in L^1(1,K)$, we deduce that both $\tilde{w}'(1)$ and $\tilde{w}'(K)$ are finite, so $\tilde{w} \in C^2(1,K) \cap C^1[1,K]$. Furthermore,

$$w(x) \leq C\min\{K - |x|, |x| - 1\}, \qquad \text{for any} \quad x \in B_K(0) \setminus B_1(0). \tag{6.105}$$

Let us fix $x_0 \in \Omega_\delta$. Then we can find $y_0 \in \mathbb{R}^N \setminus \overline{\Omega}$ with $d(y_0, \partial\Omega) = \delta$ and $d(x_0) = |x_0 - y| - \delta$. Thus, $\Omega \subset B_{K\delta}(y_0) \setminus B_\delta(y_0)$. Define $v(x) = cw\left(\dfrac{x - y_0}{\delta}\right)$, $x \in \overline{\Omega}$. We show that v is a super-solution of (P_λ), provided that c is large enough. Indeed, if $c > \max\{1, \delta^2\|a\|_\infty\}$, then for all $x \in \Omega$ we have

$$\begin{aligned} \Delta v + \lambda f(v) + a(x)g(v) \leq \frac{c}{\delta^2}&\left(\tilde{w}''(r) + \frac{N-1}{r}\tilde{w}'(r)\right) \\ &+ \lambda^+ f(c\tilde{w}(r)) + a(x)g(c\tilde{w}(r)), \end{aligned}$$

where $r = \dfrac{|x - y_0|}{\delta} \in (1,K)$. Using the assumption $(f1)$ we get $f(c\tilde{w}) \leq cf(\tilde{w})$ in $(1,K)$. The above relations lead us to

$$\begin{aligned} \Delta v + \lambda f(v) + a(x)g(v) &\leq \frac{c}{\delta^2}\left(\tilde{w}'' + \frac{N-1}{r}\tilde{w}'\right) + \lambda^+ c f(\tilde{w}) + \|a\|_\infty g(\tilde{w}) \\ &\leq \frac{c}{\delta^2}\left(\tilde{w}'' + \frac{N-1}{r}\tilde{w}' + \lambda^+ f(\tilde{w}) + g(\tilde{w})\right) \\ &= 0. \end{aligned}$$

Since $\Delta u_\lambda \in L^1(\Omega)$, with a similar proof as in Step 2 we get $u_\lambda \le v$ in Ω. This combined with (6.105) yields

$$u_\lambda(x_0) \le v(x_0) \le C \min\{K - \frac{|x_0 - y_0|}{\delta}, \frac{|x_0 - y_0|}{\delta} - 1\} \le \frac{C}{\delta} d(x_0).$$

Hence $u_\lambda \le \frac{C}{\delta} d(x)$ in Ω_δ and the last inequality in (6.102) follows.

Let G be the Green's function associated with the Laplace operator in Ω. Then, for all $x \in \Omega$ we have

$$u_\lambda(x) = -\int_\Omega G(x,y)\,[\lambda f(u_\lambda(y)) + a(y)g(u_\lambda(y))]\,dy,$$

and

$$\nabla u_\lambda(x) = -\int_\Omega G_x(x,y)\,[\lambda f(u_\lambda(y)) + a(y)g(u_\lambda(y))]\,dy.$$

If $x_1, x_2 \in \Omega$, using $(g2)$ we obtain

$$\begin{aligned}|\nabla u_\lambda(x_1) - \nabla u_\lambda(x_2)| \le |\lambda| \int_\Omega |G_x(x_1,y) - G_x(x_2,y)| \cdot f(u_\lambda(y))dy \\ + \tilde{c} \int_\Omega |G_x(x_1,y) - G_x(x_2,y)| \cdot u_\lambda^{-\alpha}(y)dy.\end{aligned}$$

Now, taking into account that $u_\lambda \in C(\overline{\Omega})$, by the standard regularity theory (see Gilbarg and Trudinger [55]) we get

$$\int_\Omega |G_x(x_1,y) - G_x(x_2,y)| \cdot f(u_\lambda(y)) \le \tilde{c}_1 |x_1 - x_2|.$$

On the other hand, with the same proof as in [57, Theorem 1], we deduce

$$\int_\Omega |G_x(x_1,y) - G_x(x_2,y)| \cdot u_\lambda^{-\alpha}(y) \le \tilde{c}_2 |x_1 - x_2|^{1-\alpha}.$$

The above inequalities imply $u_\lambda \in C^2(\Omega) \cap C^{1,1-\alpha}(\overline{\Omega})$. The proof of Theorem 6.1 is now complete. □

Next, consider the case $m > 0$. The results in this case are different from those presented in Theorem 6.1. A careful examination of (P_λ) reveals the fact that the singular term $g(u)$ is not significant. Actually, the conclusions are close to those established in Mironescu and Rădulescu [78, Theorem A], where an elliptic problem associated to an asymptotically linear function is studied.

Let λ_1 be the first Dirichlet eigenvalue of $(-\Delta)$ in Ω and $\lambda^* = \dfrac{\lambda_1}{m}$. Our result in this case is the following.

Theorem 6.4. *Assume $(f1)$, $(g1)$, $(g2)$ and $m > 0$. Then the following hold.*

(i) *If $\lambda \ge \lambda^*$, then (P_λ) has no solutions in $\mathcal{E}$.*

(ii) *If* $a_* > 0$ *(resp.* $a_* = 0$*) then* (P_λ) *has a unique solution* $u_\lambda \in \mathcal{E}$ *for all* $-\infty < \lambda < \lambda^*$ *(resp.* $0 < \lambda < \lambda^*$*) with the properties:*

(ii1) u_λ *is strictly increasing with respect to* λ*;*

(ii2) *there exists two positive constants* $c_1, c_2 > 0$ *depending on* λ *such that* $c_1 d(x) \le u_\lambda \le c_2 d(x)$ *in* Ω*;*

(ii3) $\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$*, uniformly on compact subsets of* Ω*.*

The bifurcation diagram in the "linear" case $m > 0$ is depicted in Figure 2.

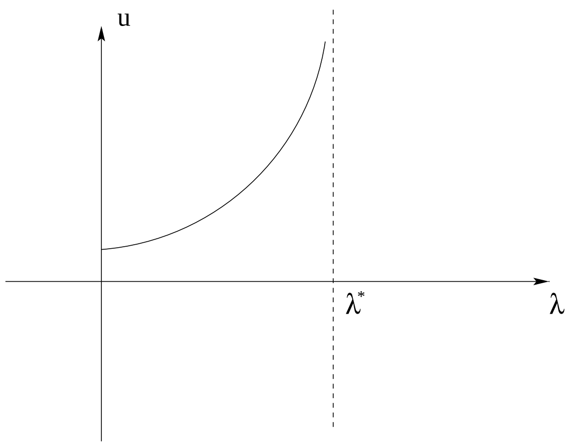

Figure 2: *The "linear" case* $m > 0$*.*

Proof. (i) Let ϕ_1 be the first eigenfunction of the Laplace operator in Ω with Dirichlet boundary condition. Arguing by contradiction, let us suppose that there exists $\lambda \ge \lambda^*$ such that (P_λ) has a solution $u_\lambda \in \mathcal{E}$.

Multiplying by ϕ_1 in (P_λ) and then integrating over Ω we get

$$-\int_\Omega \phi_1 \, \Delta u_\lambda = \lambda \int_\Omega f(u_\lambda)\phi_1 + \int_\Omega a(x) g(u_\lambda)\phi_1 \tag{6.106}$$

Since $\lambda \ge \dfrac{\lambda_1}{m}$, in view of the assumption $(f1)$ we get $\lambda f(u_\lambda) \ge \lambda_1 u_\lambda$ in Ω. Using this fact in (6.106) we obtain

$$-\int_\Omega \phi_1 \, \Delta u_\lambda > \lambda_1 \int_\Omega u_\lambda \phi_1.$$

The regularity of u_λ yields $-\int_\Omega u_\lambda \Delta\phi_1 > \lambda_1 \int_\Omega u_\lambda \phi_1$. This is clearly a contradiction since $-\Delta\phi_1 = \lambda_1\phi_1$ in Ω. Hence (P_λ) has no solutions in $\mathcal{E}$ for any $\lambda \ge \lambda^*$.

(ii) From now on, the proof of the existence, uniqueness and regularity of solution is the same as in Theorem 6.1.

(ii3) In what follows we shall apply some ideas developed in Mironescu and Rădulescu [78]. Due to the special character of our problem, we will be able to prove that, in certain cases, L^2–boundedness implies H_0^1–boundedness!

Let $u_\lambda \in \mathcal{E}$ be the unique solution of (P_λ) for $0 < \lambda < \lambda^*$. We prove that $\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$, uniformly on compact subsets of Ω. Suppose the contrary. Since $(u_\lambda)_{0<\lambda<\lambda^*}$ is a sequence of nonnegative super-harmonic functions in Ω, by Theorem 4.1.9 in Hörmander [61], there exists a subsequence of $(u_\lambda)_{\lambda<\lambda^*}$ (still denoted by $(u_\lambda)_{\lambda<\lambda^*}$) which is convergent in $L^1_{\mathrm{loc}}(\Omega)$.

We first prove that $(u_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. We argue by contradiction. Suppose that $(u_\lambda)_{\lambda<\lambda^*}$ is not bounded in $L^2(\Omega)$. Thus, passing eventually at a subsequence we have $u_\lambda = M(\lambda)w_\lambda$, where

$$M(\lambda) = ||u_\lambda||_{L^2(\Omega)} \to \infty \text{ as } \lambda \nearrow \lambda^* \text{ and } w_\lambda \in L^2(\Omega), \ \ \|w_\lambda\|_{L^2(\Omega)} = 1. \tag{6.107}$$

Using $(f1)$, $(g2)$ and the monotonicity assumption on g, we deduce the existence of A, B, C, $D > 0$ $(A > m)$ such that

$$f(t) \le At + B, \quad g(t) \le Ct^{-\alpha} + D, \qquad \text{for all } t > 0. \tag{6.108}$$

This implies

$$\frac{1}{M(\lambda)}\left(\lambda f(u_\lambda) + a(x)g(u_\lambda)\right) \to 0 \quad \text{ in } L^1_{\mathrm{loc}}(\Omega) \text{ as } \lambda \nearrow \lambda^*$$

that is,

$$-\Delta w_\lambda \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\Omega) \text{ as } \lambda \nearrow \lambda^*. \tag{6.109}$$

By Green's first identity, we have

$$\int_\Omega \nabla w_\lambda \cdot \nabla\phi\, dx = -\int_\Omega \phi\, \Delta w_\lambda\, dx = -\int_{\mathrm{Supp}\,\phi} \phi\, \Delta w_\lambda\, dx \quad \forall \phi \in C_0^\infty(\Omega). \tag{6.110}$$

Using (6.109) we derive that

$$\begin{aligned} \left|\int_{\mathrm{Supp}\,\phi} \phi\, \Delta w_\lambda\, dx\right| &\le \int_{\mathrm{Supp}\,\phi} |\phi||\Delta w_\lambda|\, dx \\ &\le \|\phi\|_{L^\infty} \int_{\mathrm{Supp}\,\phi} |\Delta w_\lambda|\, dx \to 0 \quad \text{as } \lambda \nearrow \lambda^*. \end{aligned} \tag{6.111}$$

Combining (6.110) and (6.111), we arrive at

$$\int_\Omega \nabla w_\lambda \cdot \nabla \phi\, dx \to 0 \ \text{ as } \lambda \nearrow \lambda^*, \quad \forall \phi \in C_0^\infty(\Omega). \tag{6.112}$$

By definition, the sequence $(w_\lambda)_{0<\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$.

We claim that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$. Indeed, using (6.108) and Hölder's inequality, we have

$$\begin{aligned}
\int_\Omega |\nabla w_\lambda|^2 &= -\int_\Omega w_\lambda \Delta w_\lambda = \frac{-1}{M(\lambda)}\int_\Omega w_\lambda \Delta u_\lambda \\
&= \frac{1}{M(\lambda)}\int_\Omega [\lambda w_\lambda f(u_\lambda) + a(x)g(u_\lambda)w_\lambda] \\
&\le \frac{\lambda}{M(\lambda)}\int_\Omega w_\lambda(Au_\lambda + B) + \frac{||a||_\infty}{M(\lambda)}\int_\Omega w_\lambda(Cu_\lambda^{-\alpha} + D) \\
&= \lambda A\int_\Omega w_\lambda^2 + \frac{||a||_\infty C}{M(\lambda)^{1+\alpha}}\int_\Omega w_\lambda^{1-\alpha} + \frac{\lambda B + \|a\|_\infty D}{M(\lambda)}\int_\Omega w_\lambda \\
&\le \lambda^* A + \frac{||a||_\infty C}{M(\lambda)^{1+\alpha}}|\Omega|^{(1+\alpha)/2} + \frac{\lambda B + \|a\|_\infty D}{M(\lambda)}|\Omega|^{1/2}.
\end{aligned}$$

From the above estimates, it is easy to see that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$, so the claim is proved. Then, there exists $w \in H_0^1(\Omega)$ such that (up to a subsequence)

$$w_\lambda \rightharpoonup w \quad \text{weakly in} \quad H_0^1(\Omega) \ \text{ as } \lambda \nearrow \lambda^* \tag{6.113}$$

and, because $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$,

$$w_\lambda \to w \quad \text{strongly in} \quad L^2(\Omega) \ \text{ as } \lambda \nearrow \lambda^*. \tag{6.114}$$

On the one hand, by (6.107) and (6.114), we derive that $\|w\|_{L^2(\Omega)} = 1$. Furthermore, using (6.112) and (6.113), we infer that

$$\int_\Omega \nabla w \cdot \nabla \phi \, dx = 0, \quad \forall \phi \in C_0^\infty(\Omega).$$

Since $w \in H_0^1(\Omega)$, using the above relation and the definition of $H_0^1(\Omega)$, we get $w = 0$. This contradiction shows that $(u_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. As above for w_λ, we can derive that u_λ is bounded in $H_0^1(\Omega)$. So, there exists $u^* \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\begin{cases}
u_\lambda \rightharpoonup u^* \ \text{ weakly in } H_0^1(\Omega) \ \text{ as } \lambda \nearrow \lambda^*, \\
u_\lambda \to u^* \ \text{ strongly in } L^2(\Omega) \ \text{ as } \lambda \nearrow \lambda^*, \\
u_\lambda \to u^* \ \text{ a.e. in } \Omega \text{ as } \lambda \nearrow \lambda^*.
\end{cases} \tag{6.115}$$

Now we can proceed to get a contradiction. Multiplying by ϕ_1 in (P_λ) and integrating over Ω we have

$$-\int_\Omega \varphi_1 \Delta u_\lambda = \lambda \int_\Omega f(u_\lambda)\varphi_1 + \int_\Omega a(x)g(u_\lambda)\varphi_1, \quad \text{for all } \ 0 < \lambda < \lambda^*. \tag{6.116}$$

On the other hand, by $(f1)$ it follows that $f(u_\lambda) \ge mu_\lambda$ in Ω, for all $0 < \lambda < \lambda^*$. Combining this with (6.116) we obtain

$$\lambda_1 \int_\Omega u_\lambda \varphi_1 \ge \lambda m \int_\Omega u_\lambda \varphi_1 + \int_\Omega a(x)g(u_\lambda)\varphi_1, \quad \text{for all } \ 0 < \lambda < \lambda^*. \tag{6.117}$$

Notice that by $(g1)$, (6.115) and the monotonicity of u_λ with respect to λ we can apply the Lebesgue convergence theorem to find

$$\int_\Omega a(x)g(u_\lambda)\varphi_1\,dx \to \int_\Omega a(x)g(u^*)\varphi_1\,dx \quad \text{as } \lambda \nearrow \lambda_1.$$

Passing to the limit in (6.117) as $\lambda \nearrow \lambda^*$, and using (6.115), we get

$$\lambda_1 \int_\Omega u^*\varphi_1 \geq \lambda_1 \int_\Omega u^*\varphi_1 + \int_\Omega a(x)g(u^*)\varphi_1. \tag{6.118}$$

Hence $\int_\Omega a(x)g(u^*)\varphi_1 = 0$, which is a contradiction. This fact shows that $\lim_{\lambda\nearrow\lambda^*} u_\lambda = +\infty$, uniformly on compact subsets of Ω. This ends the proof. □

7 Sublinear singular elliptic problems with two bifurcation parameters

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$. In this section we study the existence or the nonexistence of solutions to the following boundary value problem

$$\begin{cases} -\Delta u + K(x)g(u) = \lambda f(x,u) + \mu h(x) & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$P_{\lambda,\mu}$}$$

Here $K, h \in C^{0,\gamma}(\overline{\Omega})$, with $h > 0$ on Ω and λ, μ are positive real numbers. We suppose that $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is positive on $\overline{\Omega} \times (0,\infty)$. We also assume that f is nondecreasing with respect to the second variable and is sublinear, that is,

$(f1)$ the mapping $(0,\infty) \ni s \longmapsto \dfrac{f(x,s)}{s}$ is nonincreasing for all $x \in \overline{\Omega}$;

$(f2)$ $\lim_{s\downarrow 0} \dfrac{f(x,s)}{s} = +\infty$ and $\lim_{s\to\infty} \dfrac{f(x,s)}{s} = 0$, uniformly for $x \in \overline{\Omega}$.

We assume that $g \in C^{0,\gamma}(0,\infty)$ is a nonnegative and nonincreasing function satisfying

$(g1)$ $\lim_{s\downarrow 0} g(s) = +\infty$;

$(g2)$ there exists $C, \delta_0 > 0$ and $\alpha \in (0,1)$ such that $g(s) \leq Cs^{-\alpha}$ for all $s \in (0,\delta_0)$.

Our framework includes the Emden–Fowler equation that corresponds to $g(s) = s^{-\gamma}$, $\gamma > 0$ (see Wong [92]).

Denote $\mathcal{E} = \{\, u \in C^2(\Omega) \cap C(\overline{\Omega});\ g(u) \in L^1(\Omega)\}$.

We show in this section that $(P_{\lambda,\mu})$ has at least one solution in $\mathcal{E}$ for λ, μ belonging to a certain range. We also prove that in some cases $(P_{\lambda,\mu})$ has no solutions in $\mathcal{E}$, provided that λ and μ are sufficiently small.

Remark 5. *(i)* If $u \in \mathcal{E}$, $v \in C^2(\Omega) \cap C(\overline{\Omega})$ and $0 < u < v$ in Ω, then $v \in \mathcal{E}$.
(ii) Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of $(P_{\lambda,\mu})$. Then $u \in \mathcal{E}$ if and only if $\Delta u \in L^1(\Omega)$.

A fundamental role will be played in our analysis by the numbers

$$K^* = \max_{x \in \overline{\Omega}} K(x), \; K_* = \min_{x \in \overline{\Omega}} K(x).$$

Our main results (see Ghergu and Rădulescu [47]) are the following.

Theorem 7.1. *Assume that* $K_* > 0$ *and* f *satisfies* $(f1) - (f2)$.
If $\int_0^1 g(s)ds = +\infty$, *then* $(P_{\lambda,\mu})$ *has no solution in* $\mathcal{E}$ *for any* $\lambda, \mu > 0$.

Theorem 7.2. *Assume that* $K_* > 0$, f *satisfies* $(f1) - (f2)$ *and* g *satisfies* $(g1) - (g2)$. *Then there exists* $\lambda_*, \mu_* > 0$ *such that*

$(P_{\lambda,\mu})$ *has at least one solution in* $\mathcal{E}$ *if* $\lambda > \lambda_*$ *or* $\mu > \mu_*$.

$(P_{\lambda,\mu})$ *has no solution in* $\mathcal{E}$ *if* $\lambda < \lambda_*$ *and* $\mu < \mu_*$.

Moreover, if $\lambda > \lambda_*$ *or* $\mu > \mu_*$, *then* $(P_{\lambda,\mu})$ *has a maximal solution in* $\mathcal{E}$ *which is increasing with respect to* λ *and* μ.

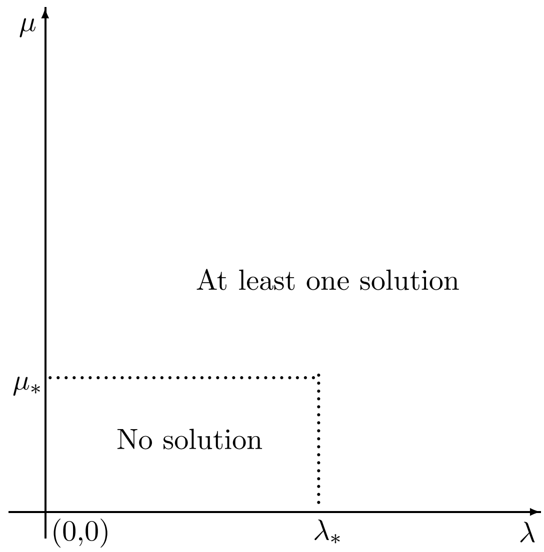

Figure 3: The dependence on λ and μ in Theorem 7.2

Theorem 7.3. *Assume that* $K^* \leq 0$, f *satisfies* $(f1) - (f2)$ *and* g *satisfies* $(g1) - (g2)$. *Then* $(P_{\lambda,\mu})$ *has a unique solution* $u_{\lambda,\mu} \in \mathcal{E}$ *for any* λ, $\mu > 0$. *Moreover,* $u_{\lambda,\mu}$ *is increasing with respect to* λ *and* μ.

Theorems 7.2 and 7.3 also show the role played by the sublinear term f and the sign of $K(x)$. Indeed, if f becomes linear then the situation changes radically. First, by the results established by Crandall, Rabinowitz, and Tartar [35], the problem

$$\begin{cases} -\Delta u - u^{-\alpha} = -u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution, for any $\alpha > 0$. Next, as showed in Chen [19], the problem

$$\begin{cases} -\Delta u + u^{-\alpha} = u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has no solution, provided $0 < \alpha < 1$ and $\lambda_1 \geq 1$ (that is, if Ω is "small"), where λ_1 denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$.

Theorem 7.4. *Assume that $K^* > 0 > K_*$, f satisfies $(f1) - (f2)$ and g verifies $(g1) - (g2)$. Then there exists λ_*, $\mu_* > 0$ such that $(P_{\lambda,\mu})$ has at least one solution $u_{\lambda,\mu} \in \mathcal{E}$ if $\lambda > \lambda_*$ or $\mu > \mu_*$. Moreover, for $\lambda > \lambda_*$ or $\mu > \mu_*$, $u_{\lambda,\mu}$ is increasing with respect to λ and μ.*

Before giving the proofs, we state some auxiliary results.

Let ϕ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{7.119}$$

Lemma 7.5. (Lazer and McKenna [68]). $\int_\Omega \phi_1^{-s} dx < +\infty$ *if and only if* $s < 1$.

Next, we observe that the hypotheses of Lemmas 6.2 and 6.3 are fulfilled for

$$\Phi_{\lambda,\mu}(x,s) = \lambda f(x,s) + \mu h(x), \tag{7.120}$$

$$\Psi_{\lambda,\mu}(x,s) = \lambda f(x,s) - K(x)g(s) + \mu h(x), \quad \text{provided } K^* \leq 0. \tag{7.121}$$

Lemma 7.6. *Let f satisfying $(f1) - (f2)$ and g satisfying $(g1) - (g2)$. Then there exists $\overline{\lambda} > 0$ such that the problem*

$$\begin{cases} -\Delta v + g(v) = \lambda f(x,v) + \mu h(x) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.122}$$

has at least one solution $v_{\lambda,\mu} \in \mathcal{E}$ for all $\lambda > \overline{\lambda}$ and for any $\mu > 0$.

Proof. Let λ, $\mu > 0$. According to Lemmas 6.2 and 6.3, the boundary value problem

$$\begin{cases} -\Delta U = \lambda f(x, U) + \mu h(x) & \text{in } \Omega, \\ U > 0 & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega \end{cases} \tag{7.123}$$

has a unique solution $U_{\lambda,\mu} \in C^{2,\gamma}(\Omega) \cap C(\overline{\Omega})$. Then $\overline{v}_{\lambda,\mu} = U_{\lambda,\mu}$ is a super-solution of (7.122). The main point is to find a sub-solution of (7.122). For this purpose, let $H : [0, \infty) \to [0, \infty)$ be such that

$$\begin{cases} H''(t) = g(H(t)), & \text{for all } t > 0, \\ H'(0) = H(0) = 0. \end{cases} \tag{7.124}$$

Obviously, $H \in C^2(0, \infty) \cap C^1[0, \infty)$ exists by our assumption $(g2)$. From (7.124) it follows that H'' is nonincreasing, while H and H' are nondecreasing on $(0, \infty)$. Using this fact and applying the mean value theorem, we deduce that for all $t > 0$ there exists ξ_t^1, $\xi_t^2 \in (0, t)$ such that

$$\frac{H(t)}{t} = \frac{H(t) - H(0)}{t - 0} = H'(\xi_t^1) \leq H'(t);$$

$$\frac{H'(t)}{t} = \frac{H'(t) - H'(0)}{t - 0} = H''(\xi_t^2) \geq H''(t).$$

The above inequalities imply

$$H(t) \leq tH'(t) \leq 2H(t), \qquad \text{for all } t > 0.$$

Hence

$$1 \leq \frac{tH'(t)}{H(t)} \leq 2, \qquad \text{for all } t > 0. \tag{7.125}$$

On the other hand, by $(g2)$ and (7.124), there exists $\eta > 0$ such that

$$\begin{cases} H(t) \leq \delta_0, & \text{for all } t \in (0, \eta), \\ H''(t) \leq CH^{-\alpha}(t), & \text{for all } t \in (0, \eta), \end{cases} \tag{7.126}$$

which yields

$$H(t) \leq c\, t^{2/(\alpha+1)}, \quad \text{for all } t \in (0, \eta), \tag{7.127}$$

where $c > 0$ is a constant.

Now we look for a sub-solution of the form $\underline{v}_{\lambda,\mu} = MH(\phi_1)$, for some constant $M > 0$. We have

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) = \lambda_1 MH'(\phi_1)\phi_1 + g(MH(\phi_1)) - Mg(H(\phi_1))|\nabla\phi_1|^2 \quad \text{in } \Omega. \tag{7.128}$$

Take $M \geq 1$. The monotonicity of g leads to

$$g(MH(\phi_1)) \leq g(H(\phi_1)) \quad \text{in } \Omega,$$

and, by (7.128),

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leq \lambda_1 MH'(\phi_1)\phi_1 + g(H(\phi_1))\left(1 - M|\nabla\phi_1|^2\right) \quad \text{in } \Omega. \tag{7.129}$$

We claim that

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leq 2\lambda_1 MH'(\phi_1)\phi_1 \quad \text{in } \Omega. \tag{7.130}$$

Indeed, by Hopf's maximum principle, there exists $\delta > 0$ and $\omega \subset\subset \Omega$ such that

$$|\nabla\phi_1| \geq \delta \quad \text{in } \Omega \setminus \omega,$$

$$\phi_1 \geq \delta \quad \text{in } \omega.$$

On $\Omega \setminus \omega$ we choose $M \geq M_1 = \max\{1, \delta^{-2}\}$. Then, by (7.129) we obtain

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leq \lambda_1 MH'(\phi_1)\phi_1 \quad \text{in } \Omega \setminus \omega. \tag{7.131}$$

Fix $M \geq \max\left\{M_1, \frac{g(H(\delta))}{\lambda_1 H'(\delta)\delta}\right\}$. Then

$$g(H(\phi_1)) \leq g(H(\delta)) \leq \lambda_1 MH'(\delta)\delta \leq \lambda_1 MH'(\phi_1)\phi_1 \quad \text{in } \omega.$$

From (7.129) we deduce

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leq 2\lambda_1 MH'(\phi_1)\phi_1 \quad \text{in } \omega. \tag{7.132}$$

Hence our claim (7.130) follows from (7.131) and (7.132).

Since $\phi_1 > 0$ in Ω, from (7.125) we have

$$1 \leq \frac{H'(\phi_1)\phi_1}{H(\phi_1)} \leq 2 \qquad \text{in } \Omega. \tag{7.133}$$

Thus, (7.130) and (7.133) yield

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leq 4\lambda_1 MH(\phi_1) = 4\lambda_1 \underline{v}_{\lambda,\mu} \quad \text{in } \Omega. \tag{7.134}$$

Take $\overline{\lambda} = 4\lambda_1 c^{-1}|\underline{v}_{\lambda,\mu}|_\infty$, where $c = \inf_{x\in\overline{\Omega}} f(x, |\underline{v}_{\lambda,\mu}|_\infty) > 0$. If $\lambda > \overline{\lambda}$, the assumption $(f1)$ produces

$$\lambda\,\frac{f(x,\underline{v}_{\lambda,\mu})}{\underline{v}_{\lambda,\mu}} \geq \overline{\lambda}\,\frac{f(x,|\underline{v}_{\lambda,\mu}|_\infty)}{|\underline{v}_{\lambda,\mu}|_\infty} \geq 4\lambda_1, \quad \text{for all } x \in \Omega.$$

This combined with (7.134) gives

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leq \lambda\, f(x,\underline{v}_{\lambda,\mu}) \quad \text{in } \Omega.$$

Hence $\underline{v}_{\lambda,\mu}$ is a sub-solution of (7.122), for all $\lambda > \overline{\lambda}$ and $\mu > 0$.

We now prove that $\underline{v}_{\lambda,\mu} \in \mathcal{E}$, that is $g(\underline{v}_{\lambda,\mu}) \in L^1(\Omega)$. Denote $\Omega_0 = \{x \in \Omega;\ \phi_1(x) < \eta\}$. By (7.126) and (7.127) it follows that

$$g(\underline{v}_{\lambda,\mu}) = g(MH(\phi_1)) \le g(H(\phi_1)) \le CH^{-\alpha}(\phi_1) \le C_0\, \phi_1^{-2\alpha/(1+\alpha)} \qquad \text{in } \Omega_0,$$

$$g(\underline{v}_{\lambda,\mu}) \le g(MH(\eta)) \qquad \text{in } \Omega \setminus \Omega_0.$$

These estimates combined with Lemma 7.5 yield $g(\underline{v}_{\lambda,\mu}) \in L^1(\Omega)$ and so $\Delta \underline{v}_{\lambda,\mu} \in L^1(\Omega)$. Hence

$$\Delta \overline{v}_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, \overline{v}_{\lambda,\mu}) \le 0 \le \Delta \underline{v}_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, \underline{v}_{\lambda,\mu}) \quad \text{in } \Omega,$$

$$\underline{v}_{\lambda,\mu},\ \overline{v}_{\lambda,\mu} > 0 \quad \text{in } \Omega,$$

$$\underline{v}_{\lambda,\mu} = \overline{v}_{\lambda,\mu} \quad \text{on } \partial\Omega,$$

$$\Delta \underline{v}_{\lambda,\mu} \in L^1(\Omega).$$

By Lemma 6.3, it follows that $\underline{v}_{\lambda,\mu} \le \overline{v}_{\lambda,\mu}$ on $\overline{\Omega}$. Now, standard elliptic arguments guarantee the existence of a solution $v_{\lambda,\mu} \in C^2(\Omega) \cap C(\overline{\Omega})$ for (7.122) such that $\underline{v}_{\lambda,\mu} \le v_{\lambda,\mu} \le \overline{v}_{\lambda,\mu}$ in $\overline{\Omega}$. Since $\underline{v}_{\lambda,\mu} \in \mathcal{E}$, by Remark 5 we deduce that $v_{\lambda,\mu} \in \mathcal{E}$. Hence, for all $\lambda > \overline{\lambda}$ and $\mu > 0$, problem (7.122) has at least a solution in $\mathcal{E}$. The proof of Lemma 7.6 is now complete. □

We shall often refer in what follows to the following approaching problem of $(P_{\lambda,\mu})$:

$$\begin{cases} -\Delta u + K(x)g(u) = \lambda f(x,u) + \mu h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \dfrac{1}{k} & \text{on } \partial\Omega. \end{cases} \qquad (P^k_{\lambda,\mu})$$

where k is a positive integer. We observe that any solution of $(P_{\lambda,\mu})$ is a sub-solution of $(P^k_{\lambda,\mu})$.

Proof of Theorem 7.1. Suppose to the contrary that there exists λ and μ such that $(P_{\lambda,\mu})$ has a solution $u_{\lambda,\mu} \in \mathcal{E}$ and let $U_{\lambda,\mu}$ be the solution of (7.123). Since

$$\Delta U_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, U_{\lambda,\mu}) \le 0 \le \Delta u_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, u_{\lambda,\mu}) \quad \text{in } \Omega,$$

by Lemma 6.3 we get $u_{\lambda,\mu} \le U_{\lambda,\mu}$ in $\overline{\Omega}$.

Consider the perturbed problem

$$\begin{cases} -\Delta u + K_* g(u+\varepsilon) = \lambda f(x,u) + \mu h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (7.135)$$

Since $K_* > 0$, it follows that $u_{\lambda,\mu}$ and $U_{\lambda,\mu}$ are sub and super-solution for (7.135), respectively. So, by elliptic regularity, there exists $u_\varepsilon \in C^{2,\gamma}(\overline{\Omega})$ a solution of (7.135) such that

$$u_{\lambda,\mu} \le u_\varepsilon \le U_{\lambda,\mu} \quad \text{in } \Omega. \tag{7.136}$$

Integrating in (7.135) we deduce

$$-\int_\Omega \Delta u_\varepsilon dx + K_* \int_\Omega g(u_\varepsilon + \varepsilon)dx = \int_\Omega [\lambda f(x, u_\varepsilon) + \mu h(x)]dx.$$

Hence

$$-\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n} ds + K_* \int_\Omega g(u_\varepsilon + \varepsilon)dx \le M, \tag{7.137}$$

where $M > 0$ is a constant. Since $\dfrac{\partial u_\varepsilon}{\partial n} \le 0$ on $\partial\Omega$, relation (7.137) yields $K_* \displaystyle\int_\Omega g(u_\varepsilon+\varepsilon)dx \le M$, and so $K_* \displaystyle\int_\Omega g(U_{\lambda,\mu} + \varepsilon)dx \le M$. Thus, for any compact subset $\omega \subset\subset \Omega$ we have

$$K_* \int_\omega g(U_{\lambda,\mu} + \varepsilon)dx \le M.$$

Letting $\varepsilon \to 0$, the above relation leads to $K_* \displaystyle\int_\omega g(U_{\lambda,\mu})dx \le M$. Therefore

$$K_* \int_\Omega g(U_{\lambda,\mu})dx \le M. \tag{7.138}$$

Choose $\delta > 0$ sufficiently small and define $\Omega_\delta := \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) \le \delta\}$. Taking into account the regularity of domain, there exists $k > 0$ such that

$$U_{\lambda,\mu} \le k \operatorname{dist}(x, \partial\Omega) \quad \text{for all } x \in \Omega_\delta.$$

Then

$$\int_\Omega g(U_{\lambda,\mu})dx \ge \int_{\Omega_\delta} g(U_{\lambda,\mu})dx \ge \int_{\Omega_\delta} g\left(k \operatorname{dist}(x, \partial\Omega)\right) dx = +\infty,$$

which contradicts (7.138). It follows that the problem $(P_{\lambda,\mu})$ has no solutions in $\mathcal{E}$ and the proof of Theorem 7.1 is now complete. □

Using the same method as in Zhang [93, Theorem 2], we can prove that $(P_{\lambda,\mu})$ has no solution in $C^2(\Omega) \cap C^1(\overline{\Omega})$, as it was pointed out in Choi, Lazer, and McKenna [21, Remark 2].

Proof of Theorem 7.2. We split the proof into several steps.

Step I. Existence of the solutions of $(P_{\lambda,\mu})$ for λ large. By Lemma 7.6, there exists $\overline{\lambda}$ such that for all $\lambda > \overline{\lambda}$ and $\mu > 0$ the problem

$$\begin{cases} -\Delta v + K^* g(v) = \lambda f(x, v) + \mu h(x) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least one solution $v_{\lambda,\mu} \in \mathcal{E}$. Then $v_k = v_{\lambda,\mu} + \dfrac{1}{k}$ is a sub-solution of $(P^k_{\lambda,\mu})$ for all positive integers $k \geq 1$.

From Lemma 6.2, let $w \in C^{2,\gamma}(\overline{\Omega})$ be the solution of

$$\begin{cases} -\Delta w = \lambda f(x,w) + \mu h(x) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 1 & \text{on } \partial\Omega. \end{cases}$$

It follows that w is a super-solution of $(P^k_{\lambda,\mu})$ for all $k \geq 1$ and

$$\Delta w + \Phi_{\lambda,\mu}(x,w) \leq 0 \leq \Delta v_1 + \Phi_{\lambda,\mu}(x,v_1) \quad \text{in } \Omega,$$

$$w,\ v_1 > 0 \quad \text{in } \Omega,$$

$$w = v_1 \quad \text{on } \partial\Omega,$$

$$\Delta v_1 \in L^1(\Omega).$$

Therefore, by Lemma 6.3, $1 \leq v_1 \leq w$ in $\overline{\Omega}$. Standard elliptic arguments imply that there exists a solution $u^1_{\lambda,\mu} \in C^{2,\gamma}(\overline{\Omega})$ of $(P^1_{\lambda,\mu})$ such that $v_1 \leq u^1_{\lambda,\mu} \leq w$ in $\overline{\Omega}$. Now, taking $u^1_{\lambda,\mu}$ and v_2 as a pair of super and sub-solutions for $(P^2_{\lambda,\mu})$, we obtain a solution $u^2_{\lambda,\mu} \in C^{2,\gamma}(\overline{\Omega})$ of $(P^2_{\lambda,\mu})$ such that $v_2 \leq u^2_{\lambda,\mu} \leq u^1_{\lambda,\mu}$ in $\overline{\Omega}$. In this manner we find a sequence $\{u^n_{\lambda,\mu}\}$ such that

$$v_n \leq u^n_{\lambda,\mu} \leq u^{n-1}_{\lambda,\mu} \leq w \quad \text{in } \overline{\Omega}. \tag{7.139}$$

Define $u_{\lambda,\mu}(x) = \lim_{n\to\infty} u^n_{\lambda,\mu}(x)$ for all $x \in \overline{\Omega}$. Standard bootstrap arguments imply that $u_{\lambda,\mu}$ is a solution of $(P_{\lambda,\mu})$. From (7.139) we have $v_{\lambda,\mu} \leq u_{\lambda,\mu} \leq w$ in $\overline{\Omega}$. Since $v_{\lambda,\mu} \in \mathcal{E}$, by Remark 5 it follows that $u_{\lambda,\mu} \in \mathcal{E}$. Consequently, problem $(P_{\lambda,\mu})$ has at least a solution in $\mathcal{E}$ for all $\lambda > \overline{\lambda}$ and $\mu > 0$.

Step II. Existence of the solutions of $(P_{\lambda,\mu})$ for μ large. Let us first notice that g verifies the hypotheses of Theorem 2 in Díaz, Morel, and Oswald [39]. We also remark that the assumption $(g2)$ and Lemma 7.5 is essential to find a sub-solution in the proof of Theorem 2 in Díaz, Morel, and Oswald [39].

According to this result, there exists $\overline{\mu} > 0$ such that the problem

$$\begin{cases} -\Delta v + K^* g(v) = \mu h(x) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least a solution $v_\mu \in \mathcal{E}$ provided that $\mu > \overline{\mu}$. Fix $\lambda > 0$ and denote $v_k = v_\mu + \dfrac{1}{k}$, $k \geq 1$. Hence v_k is a sub-solution of $(P^k_{\lambda,\mu})$, for all $k \geq 1$. Similarly to the previous step we obtain a solution $u_{\lambda,\mu} \in \mathcal{E}$ for all $\lambda > 0$ and $\mu > \overline{\mu}$.

Step III. Nonexistence for λ,μ small. Let $\lambda,\ \mu>0$. Since $K_*>0$, the assumption $(g1)$ implies $\lim_{s\downarrow 0}\Psi_{\lambda,\mu}(x,s)=-\infty$, uniformly for $x\in\overline{\Omega}$. So, there exists $c>0$ such that

$$\Psi_{\lambda,\mu}(x,s)<0 \quad \text{for all } (x,s)\in\overline{\Omega}\times(0,c). \tag{7.140}$$

Let $s\geq c$. From $(f1)$ we deduce

$$\frac{\Psi_{\lambda,\mu}(x,s)}{s}\leq\lambda\,\frac{f(x,s)}{s}+\mu\,\frac{h(x)}{s}\leq\lambda\,\frac{f(x,c)}{c}+\mu\,\frac{|h|_\infty}{s},$$

for all $x\in\overline{\Omega}$. Fix $\mu<\dfrac{c\lambda_1}{2|h|_\infty}$ and let $M=\sup_{x\in\overline{\Omega}}\dfrac{f(x,c)}{c}>0$. From the above inequality we have

$$\frac{\Psi_{\lambda,\mu}(x,s)}{s}\leq\lambda M+\frac{\lambda_1}{2},\quad \text{for all } (x,s)\in\overline{\Omega}\times[c,+\infty). \tag{7.141}$$

Thus, (7.140) and (7.141) yield

$$\Psi_{\lambda,\mu}(x,s)\leq a(\lambda)s+\frac{\lambda_1}{2}s,\quad \text{for all } (x,s)\in\overline{\Omega}\times(0,+\infty). \tag{7.142}$$

Moreover, $a(\lambda)\to 0$ as $\lambda\to 0$. If $(P_{\lambda,\mu})$ has a solution $u_{\lambda,\mu}$, then

$$\begin{aligned}\lambda_1\int_\Omega u^2_{\lambda,\mu}(x)dx \quad &\leq\int_\Omega|\nabla u_{\lambda,\mu}|^2dx=-\int_\Omega u_{\lambda,\mu}(x)\Delta u_{\lambda,\mu}(x)dx\\ &\leq\int_\Omega u_{\lambda,\mu}(x)\Psi(x,u_{\lambda,\mu}(x))dx.\end{aligned}$$

Using (7.142), we get

$$\lambda_1\int_\Omega u^2_{\lambda,\mu}(x)dx\leq\left[a(\lambda)+\frac{\lambda_1}{2}\right]\int_\Omega u^2_{\lambda,\mu}(x)dx.$$

Since $a(\lambda)\to 0$ as $\lambda\to 0$, the above relation leads to a contradiction for $\lambda,\mu>0$ sufficiently small.

Step IV. Existence of a maximal solution of $(P_{\lambda,\mu})$. We show that if $(P_{\lambda,\mu})$ has a solution $u_{\lambda,\mu}\in\mathcal{E}$, then it has a maximal solution. Let $\lambda,\mu>0$ be such that $(P_{\lambda,\mu})$ has a solution $u_{\lambda,\mu}\in\mathcal{E}$. If $U_{\lambda,\mu}$ is the solution of (7.123), by Lemma 6.3 we have $u_{\lambda,\mu}\leq U_{\lambda,\mu}$ in $\overline{\Omega}$. For any $j\geq 1$, denote

$$\Omega_j=\left\{x\in\Omega;\ \mathrm{dist}(x,\partial\Omega)>\frac{1}{j}\right\}.$$

Let $U_0=U_{\lambda,\mu}$ and U_j be the solution of

$$\begin{cases}-\Delta\zeta+K(x)g(U_{j-1})=\lambda f(x,U_{j-1})+\mu h(x) & \text{in } \Omega_j,\\ \zeta=U_{j-1} & \text{in } \Omega\setminus\Omega_j.\end{cases}$$

Using the fact that $\Psi_{\lambda,\mu}$ is nondecreasing with respect to the second variable, we get

$$u_{\lambda,\mu}\leq U_j\leq U_{j-1}\leq U_0 \quad \text{in } \overline{\Omega}.$$

If $\overline{u}_{\lambda,\mu}(x) = \lim_{j\to\infty} U_j(x)$ for all $x \in \overline{\Omega}$, by standard elliptic arguments (see Gilbarg and Trudinger [55]) it follows that $\overline{u}_{\lambda,\mu}$ is a solution of $(P_{\lambda,\mu})$. Since $u_{\lambda,\mu} \le \overline{u}_{\lambda,\mu}$ in $\overline{\Omega}$, by Remark 5 we have $\overline{u}_{\lambda,\mu} \in \mathcal{E}$. Moreover, $\overline{u}_{\lambda,\mu}$ is a maximal solution of $(P_{\lambda,\mu})$.

Step V. Dependence on λ and μ. We first show the dependence on λ of the maximal solution $\overline{u}_{\lambda,\mu} \in \mathcal{E}$ of $(P_{\lambda,\mu})$. For this purpose, fix $\mu > 0$ and define

$$A := \{\lambda > 0;\ (P_{\lambda,\mu}) \text{ has at least a solution } u_{\lambda,\mu} \in \mathcal{E}\}.$$

Let $\lambda_* = \inf A$. From the previous steps we have $A \ne \emptyset$ and $\lambda_* > 0$. Let $\lambda_1 \in A$ and $\overline{u}_{\lambda_1,\mu}$ be the maximal solution of $(P_{\lambda_1,\mu})$. We prove that $(\lambda_1, +\infty) \subset A$. If $\lambda_2 > \lambda_1$ then $\overline{u}_{\lambda_1,\mu}$ is a sub-solution of $(P_{\lambda_2,\mu})$. On the other hand,

$$\Delta U_{\lambda_2,\mu} + \Phi_{\lambda_2,\mu}(x, U_{\lambda_2,\mu}) \le 0 \le \Delta \overline{u}_{\lambda_1,\mu} + \Phi_{\lambda_2,\mu}(x, \overline{u}_{\lambda_1,\mu}) \quad \text{in } \Omega,$$

$$U_{\lambda_2,\mu},\ \overline{u}_{\lambda_1,\mu} > 0 \quad \text{in } \Omega,$$

$$U_{\lambda_2,\mu} \ge \overline{u}_{\lambda_1,\mu} \quad \text{on } \partial\Omega,$$

$$\Delta \overline{u}_{\lambda_1,\mu} \in L^1(\Omega).$$

By Lemma 6.3, $\overline{u}_{\lambda_1,\mu} \le U_{\lambda_2,\mu}$ in $\overline{\Omega}$. In the same way as in Step IV we find a solution $u_{\lambda_2,\mu} \in \mathcal{E}$ of $(P_{\lambda_2,\mu})$ such that

$$\overline{u}_{\lambda_1,\mu} \le u_{\lambda_2,\mu} \le U_{\lambda_2,\mu} \quad \text{in } \overline{\Omega}.$$

Hence $\lambda_2 \in A$ and so $(\lambda_*, +\infty) \subset A$. If $\overline{u}_{\lambda_2,\mu} \in \mathcal{E}$ is the maximal solution of $(P_{\lambda_2,\mu})$, the above relation implies $\overline{u}_{\lambda_1,\mu} \le \overline{u}_{\lambda_2,\mu}$ in $\overline{\Omega}$. By the maximum principle, it follows that $\overline{u}_{\lambda_1,\mu} < \overline{u}_{\lambda_2,\mu}$ in Ω. So, $\overline{u}_{\lambda,\mu}$ is increasing with respect to λ.

To prove the dependence on μ, we fix $\lambda > 0$ and define

$$B := \{\mu > 0;\ (P_{\lambda,\mu}) \text{ has at least one solution } u_{\lambda,\mu} \in \mathcal{E}\}.$$

Let $\mu_* = \inf B$. The conclusion follows in the same manner as above. The proof of Theorem 7.2 is now complete. □

Proof of Theorem 7.3. Let $\lambda, \mu > 0$. We recall that the function $\Psi_{\lambda,\mu}$ defined in (7.121) verifies the hypotheses of Lemma 6.2, since $K^* \le 0$. So, there exists $u_{\lambda,\mu} \in C^{2,\gamma}(\Omega) \cap C(\overline{\Omega})$ a solution of $(P_{\lambda,\mu})$. If $U_{\lambda,\mu}$ is the solution of (7.123), then

$$\Delta u_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, u_{\lambda,\mu}) \le 0 \le \Delta U_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, U_{\lambda,\mu}) \quad \text{in } \Omega,$$

$$u_{\lambda,\mu},\ U_{\lambda,\mu} > 0 \quad \text{in } \Omega,$$

$$u_{\lambda,\mu} = U_{\lambda,\mu} = 0 \quad \text{on } \partial\Omega.$$

Since $\Delta U_{\lambda,\mu} \in L^1(\Omega)$, by Lemma 6.3 we get $u_{\lambda,\mu} \ge U_{\lambda,\mu}$ in $\overline{\Omega}$.

We claim that there exists $c > 0$ such that

$$U_{\lambda,\mu} \geq c\phi_1 \quad \text{in } \Omega. \tag{7.143}$$

Indeed, if not, there exists $\{x_n\} \subset \Omega$ and $\varepsilon_n \to 0$ such that

$$(U_{\lambda,\mu} - \varepsilon_n \phi_1)(x_n) < 0. \tag{7.144}$$

Moreover, we can choose the sequence $\{x_n\}$ with the additional property

$$\nabla (U_{\lambda,\mu} - \varepsilon_n \phi_1)(x_n) = 0. \tag{7.145}$$

Passing eventually at a subsequence, we can assume that $x_n \to x_0 \in \overline{\Omega}$. From (7.144) it follows that $U_{\lambda,\mu}(x_0) \leq 0$ which implies $U_{\lambda,\mu}(x_0) = 0$, that is $x_0 \in \partial\Omega$. Furthermore, from (7.145) we have $\nabla U_{\lambda,\mu}(x_0) = 0$. This is a contradiction since $\dfrac{\partial U_{\lambda,\mu}}{\partial n}(x_0) < 0$, by Hopf's strong maximum principle. Our claim follows and so

$$u_{\lambda,\mu} \geq U_{\lambda,\mu} \geq c\phi_1 \quad \text{in } \Omega. \tag{7.146}$$

Then, $g(u_{\lambda,\mu}) \leq g(U_{\lambda,\mu}) \leq g(c\phi_1)$ in Ω. From the assumption $(g2)$ and Lemma 2.2 (using the same method as in the proof of Lemma 7.6) it follows that $g(c\phi_1) \in L^1(\Omega)$. Hence $u_{\lambda,\mu} \in \mathcal{E}$.

Let us now assume that $u^1_{\lambda,\mu}$, $u^2_{\lambda,\mu} \in \mathcal{E}$ are two solutions of $(P_{\lambda,\mu})$. In order to prove the uniqueness, it is enough to show that $u^1_{\lambda,\mu} \geq u^2_{\lambda,\mu}$ in $\overline{\Omega}$. This follows by Lemma 6.3.

Let us show now the dependence on λ of the solution of $(P_{\lambda,\mu})$. For this purpose, let $0 < \lambda_1 < \lambda_2$ and $u_{\lambda_1,\mu}$, $u_{\lambda_2,\mu}$ be the unique solutions of $(P_{\lambda_1,\mu})$ and $(P_{\lambda_2,\mu})$ respectively, with $\mu > 0$ fixed. Since $u_{\lambda_1,\mu}$, $u_{\lambda_2,\mu} \in \mathcal{E}$ and

$$\Delta u_{\lambda_2,\mu} + \Phi_{\lambda_2,\mu}(x, u_{\lambda_2,\mu}) \leq 0 \leq \Delta u_{\lambda_1,\mu} + \Phi_{\lambda_2,\mu}(x, u_{\lambda_1,\mu}) \quad \text{in } \Omega,$$

in virtue of Lemma 6.3 we find $u_{\lambda_1,\mu} \leq u_{\lambda_2,\mu}$ in $\overline{\Omega}$. So, by the maximum principle, $u_{\lambda_1,\mu} < u_{\lambda_2,\mu}$ in Ω.

The dependence on μ follows similarly. The proof of Theorem 7.3 is now complete. □

Proof of Theorem 7.4. **Step I.** EXISTENCE. Using the fact that $K^* > 0$, from Theorem 7.2 it follows that there exists $\lambda_*, \mu_* > 0$ such that the problem

$$\begin{cases} -\Delta v + K^* g(v) = \lambda f(x,v) + \mu h(x) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

has a maximal solution $v_{\lambda,\mu} \in \mathcal{E}$, provided $\lambda > \lambda_*$ or $\mu > \mu_*$. Moreover, $v_{\lambda,\mu}$ is increasing with respect to λ and μ. Then $v_k = v_{\lambda,\mu} + \dfrac{1}{k}$ is a sub-solution of $(P^k_{\lambda,\mu})$, for all $k \geq 1$. On the other hand, by Lemma 6.2, the boundary value problem

$$\begin{cases} -\Delta w + K_* g(w) = \lambda f(x,w) + \mu h(x) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = \dfrac{1}{k} & \text{on } \partial\Omega. \end{cases}$$

has a solution $w_k \in C^{2,\gamma}(\overline{\Omega})$. Obviously, w_k is a super-solution of $(P^k_{\lambda,\mu})$.

Since $K^* > 0 > K_*$, we have

$$\Delta w_k + \Phi_{\lambda,\mu}(x, w_k) \le 0 \le \Delta v_k + \Phi_{\lambda,\mu}(x, v_k) \quad \text{in } \Omega,$$

and

$$w_k,\ v_k > 0 \quad \text{in } \Omega,$$

$$w_k = v_k \quad \text{on } \partial\Omega,$$

$$\Delta v_k \in L^1(\Omega).$$

From Lemma 6.3 it follows that $v_k \le w_k$ in $\overline{\Omega}$. By standard super and sub-solution argument, there exists a minimal solution $u^1_{\lambda,\mu} \in C^{2,\gamma}(\overline{\Omega})$ of $(P^1_{\lambda,\mu})$ such that $v_1 \le u^1_{\lambda,\mu} \le w_1$ in $\overline{\Omega}$. Now, taking $u^1_{\lambda,\mu}$ and v_2 as a pair of super and sub-solutions for $(P^2_{\lambda,\mu})$, we deduce that there exists a minimal solution $u^2_{\lambda,\mu} \in C^{2,\gamma}(\overline{\Omega})$ of $(P^2_{\lambda,\mu})$ such that $v_2 \le u^2_{\lambda,\mu} \le u^1_{\lambda,\mu}$ in $\overline{\Omega}$. Arguing in the same manner, we obtain a sequence $\{u^k_{\lambda,\mu}\}$ such that

$$v_k \le u^k_{\lambda,\mu} \le u^{k-1}_{\lambda,\mu} \le w_1 \quad \text{in } \overline{\Omega}. \tag{7.147}$$

Define $u_{\lambda,\mu}(x) = \lim_{k\to\infty} u^k_{\lambda,\mu}(x)$ for all $x \in \overline{\Omega}$. With a similar argument to that used in the proof of Theorem 7.2, we find that $u_{\lambda,\mu} \in \mathcal{E}$ is a solution of $(P_{\lambda,\mu})$. Hence, problem $(P_{\lambda,\mu})$ has at least a solution in $\mathcal{E}$, provided that $\lambda > \lambda_*$ or $\mu > \mu_*$.

Step II. Dependence on λ and μ. As above, it is enough to justify only the dependence on λ. Fix $\lambda_* < \lambda_1 < \lambda_2$, $\mu > 0$ and let $u_{\lambda_1,\mu}, u_{\lambda_2,\mu} \in \mathcal{E}$ be the solutions of $(P_{\lambda_1,\mu})$ and $(P_{\lambda_2,\mu})$ respectively that we have obtained in Step I. It follows that $u^k_{\lambda_2,\mu}$ is a super-solution of $(P^k_{\lambda_1,\mu})$. So, Lemma 6.3 combined with the fact that $v_{\lambda,\mu}$ is increasing with respect to $\lambda > \lambda_*$ yield

$$u^k_{\lambda_2,\mu} \ge v_{\lambda_2,\mu} + \frac{1}{k} \ge v_{\lambda_1,\mu} + \frac{1}{k} \quad \text{in } \overline{\Omega}.$$

Thus, $u^k_{\lambda_2,\mu} \ge u^k_{\lambda_1,\mu}$ in $\overline{\Omega}$ since $u^k_{\lambda_1,\mu}$ is the minimal solution of $(P^k_{\lambda_1,\mu})$ which satisfies $u^k_{\lambda_1,\mu} \ge v_{\lambda_1,\mu} + 1/k$ in $\overline{\Omega}$. It follows that $u_{\lambda_2,\mu} \ge u_{\lambda_1,\mu}$ in $\overline{\Omega}$. By the maximum principle we deduce that $u_{\lambda_2,\mu} > u_{\lambda_1,\mu}$ in Ω. This concludes the proof. $\square$

8 Bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with a convection term

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with a smooth boundary. In this section we are concerned with singular elliptic problems of the following type

$$\begin{cases} -\Delta u = g(u) + \lambda |\nabla u|^p + \mu f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.148}$$

where $0 < p \leq 2$ and $\lambda, \mu \geq 0$. As remarked by Choquet-Bruhat and Leray [22] and by Kazdan and Warner [62], the requirement that the nonlinearity grows at most quadratically in $|\nabla u|$ is natural in order to apply the maximum principle.

Throughout this section we suppose that $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is nondecreasing with respect to the second variable and is positive on $\overline{\Omega} \times (0,\infty)$. We assume that $g : (0,\infty) \to (0,\infty)$ is a Hölder continuous function which is nonincreasing and $\lim_{s \searrow 0} g(s) = +\infty$.

Many papers have been devoted to the case $\lambda = 0$, where the problem (8.148) becomes

$$\begin{cases} -\Delta u = g(u) + \mu f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.149}$$

If $\mu = 0$, then (8.149) has a unique solution (see Crandall, Rabinowitz, and Tartar [35], Lazer and McKenna [68]). When $\mu > 0$, the study of (8.149) emphasizes the role played by the nonlinear term $f(x,u)$. For instance, if one of the following assumptions are fulfilled

$(f1)$ there exists $c > 0$ such that $f(x,s) \geq cs$ for all $(x,s) \in \overline{\Omega} \times [0,\infty)$;

$(f2)$ the mapping $(0,\infty) \ni s \longmapsto \frac{f(x,s)}{s}$ is nondecreasing for all $x \in \overline{\Omega}$,

then problem (8.149) has solutions only if $\mu > 0$ is small enough (see Coclite and Palmieri [34]). In turn, when f satisfies the following assumptions

$(f3)$ the mapping $(0,\infty) \ni s \longmapsto \frac{f(x,s)}{s}$ is nonincreasing for all $x \in \overline{\Omega}$;

$(f4)$ $\lim_{s\to\infty} \frac{f(x,s)}{s} = 0$, uniformly for $x \in \overline{\Omega}$,

then problem (8.149) has at least one solutions for all $\mu > 0$ (see Coclite and Palmieri [34], Shi and Yao [86] and the references therein). The same assumptions will be used in the study of (8.148).

By the monotonicity of g, there exists

$$a = \lim_{s\to\infty} g(s) \in [0,\infty).$$

The main results in this section have been obtained by Ghergu and Rădulescu [53, 54].

We are first concerned with the case $\lambda = 1$ and $1 < p \leq 2$. In the statement of the following result we do not need assumptions $(f1) - (f4)$; we just require that f is a Hölder continuous function which is nondecreasing with respect to the second variable and is positive on $\overline{\Omega} \times (0, \infty)$.

Theorem 8.1. *Assume $\lambda = 1$ and $1 < p \leq 2$.*
(i) *If $p = 2$ and $a \geq \lambda_1$, then (8.148) has no solutions;*
(ii) *If $p = 2$ and $a < \lambda_1$ or $1 < p < 2$, then there exists $\mu^* > 0$ such that (8.148) has at least one classical solution for $\mu < \mu^*$ and no solutions exist if $\mu > \mu^*$.*

If $\lambda = 1$ and $0 < p \leq 1$ the study of existence is close related to the asymptotic behaviour of the nonlinear term $f(x, u)$. In this case we prove

Theorem 8.2. *Assume $\lambda = 1$ and $0 < p \leq 1$.*
(i) *If f satisfies $(f1)$ or $(f2)$, then there exists $\mu^* > 0$ such that (8.148) has at least one classical solution for $\mu < \mu^*$ and no solutions exist if $\mu > \mu^*$;*
(ii) *If $0 < p < 1$ and f satisfies $(f3) - (f4)$, then (8.148) has at least one solution for all $\mu \geq 0$.*

Next we are concerned with the case $\mu = 1$. Our result is the following

Theorem 8.3. *Assume $\mu = 1$ and f satisfies assumptions $(f3)$ and $(f4)$. Then the following properties hold true.*
(i) *If $0 < p < 1$, then (8.148) has at least one classical solution for all $\lambda \geq 0$;*
(ii) *If $1 \leq p \leq 2$, then there exists $\lambda^* \in (0, \infty]$ such that (8.148) has at least one classical solution for $\lambda < \lambda^*$ and no solution exists if $\lambda > \lambda^*$. Moreover, if $1 < p \leq 2$, then λ^* is finite.*

Related to the above result we raise the following **open problem:** if $p = 1$ and $\mu = 1$, is λ^* a finite number?

Theorem 8.3 shows the importance of the convection term $\lambda|\nabla u|^p$ in (8.148). Indeed, according to Theorem 7.3 and for any $\mu > 0$, the boundary value problem

$$\begin{cases} -\Delta u = u^{-\alpha} + \lambda|\nabla u|^p + \mu u^\beta & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{8.150}$$

has a unique solution, provided $\lambda = 0$, α, $\beta \in (0, 1)$. The above theorem shows that if λ is not necessarily 0, then the following situations may occur : (i) problem (8.150) has solutions if $p \in (0, 1)$ and for all $\lambda \geq 0$; (ii) if $p \in (1, 2)$ then there exists $\lambda^* > 0$ such that problem (8.150) has a solution for any $\lambda < \lambda^*$ and no solution exists if $\lambda > \lambda^*$.

To see the dependence between λ and μ in (8.148), we consider the special case $f \equiv 1$ and $p = 2$. In this case we can say more about the problem (8.148). More precisely we have

Theorem 8.4. *Assume that $p = 2$ and $f \equiv 1$.*
(i) *The problem (8.148) has solution if and only if $\lambda(a+\mu) < \lambda_1$;*
(ii) *Assume $\mu > 0$ is fixed, g is decreasing and let $\lambda^* = \dfrac{\lambda_1}{a+\mu}$. Then (8.148) has a unique solution u_λ for all $\lambda < \lambda^*$ and the sequence $(u_\lambda)_{\lambda<\lambda^*}$ is increasing with respect to λ.*
Moreover, if $\limsup_{s\searrow 0} s^\alpha g(s) < +\infty$, for some $\alpha \in (0,1)$, then the sequence of solutions $(u_\lambda)_{0<\lambda<\lambda^*}$ has the following properties

(ii1) *For all $0 < \lambda < \lambda^*$ there exist two positive constants c_1, c_2 depending on λ such that $c_1 \operatorname{dist}(x,\partial\Omega) \le u_\lambda \le c_2 \operatorname{dist}(x,\partial\Omega)$ in Ω;*

(ii2) *$u_\lambda \in C^{1,1-\alpha}(\overline{\Omega}) \cap C^2(\Omega)$;*

(ii3) *$u_\lambda \longrightarrow +\infty$ as $\lambda \nearrow \lambda^*$, uniformly on compact subsets of Ω.*

As regards the uniqueness of the solutions to problem (8.148), we may say that this does not seem to be a feature easy to achieve. Only when $f(x,u)$ is constant in u we can use classical methods in order to prove the uniqueness. It is worth pointing out here that the uniqueness of the solution is a delicate issue even for the simpler problem (8.149). We have already observed that if f fulfills $(f3)-(f4)$ and g satisfies the same growth condition as in Theorem 8.4, then this solution is unique, provided that problem (8.149) has a solution. On the other hand, if f satisfies $(f2)$, the uniqueness generally does not occur. In that sense we refer the interested reader to Haitao [58]. In the case $f(x,u) = u^q$, $g(u) = u^{-\gamma}$, $0 < \gamma < \frac{1}{N}$ and $1 < q < \frac{N+2}{N-2}$, we learn from [58] that problem (8.149) has at least two classical solutions provided μ belongs to a certain range.

Our approach relies on finding of appropriate sub- and super-solutions of (8.148). This will allows us to enlarge the study of bifurcation to a class of problems more generally to that studied in Zhang and Yu [95]. However, neither the method used in [95], nor our method gives a precise answer if λ^* is finite or not in the case $p = 1$ and $\mu = 1$.

We start with some auxiliary results.

Let φ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of $(-\Delta)$ in $H_0^1(\Omega)$. As it is well known $\lambda_1 > 0$, $\varphi_1 \in C^2(\overline{\Omega})$ and

$$C_1 \operatorname{dist}(x,\partial\Omega) \le \varphi_1 \le C_2 \operatorname{dist}(x,\partial\Omega) \quad \text{in } \Omega, \tag{8.151}$$

for some positive constants $C_1, C_2 > 0$. From the characterization of λ_1 and φ_1 we state the following elementary result. For the convenience of the reader we shall give a complete proof.

Lemma 8.5. *Let $F : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ be a continuous function such that $F(x,s) \ge \lambda_1 s + b$ for some $b > 0$ and for all $(x,s) \in \overline{\Omega} \times (0,\infty)$. Then the problem*

$$\begin{cases} -\Delta u = F(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.152}$$

has no solutions.

Proof. By contradiction, suppose that (8.152) admits a solution. This will provide a super-solution of the problem

$$\begin{cases} -\Delta u = \lambda_1 u + b & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.153}$$

Since 0 is a sub-solution, by the sub and super-solution method and classical regularity theory it follows that (8.152) has a solution $u \in C^2(\overline{\Omega})$. Multiplying by φ_1 in (8.153) and then integrating over Ω, we get

$$-\int_\Omega \varphi_1 \Delta u = \lambda_1 \int_\Omega \varphi_1 u + b \int_\Omega \varphi_1,$$

that is $\lambda_1 \int_\Omega \varphi_1 u = \lambda_1 \int_\Omega \varphi_1 u + b \int_\Omega \varphi_1$, which implies $\int_\Omega \varphi_1 = 0$. This is clearly a contradiction since $\varphi_1 > 0$ in Ω. Hence (8.152) has no solutions. □

According to Lemma 6.2, there exists $\zeta \in C^2(\overline{\Omega})$ a solution of the problem

$$\begin{cases} -\Delta \zeta = g(\zeta) & \text{in } \Omega, \\ \zeta > 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.154}$$

Clearly ζ is a sub-solution of (8.148) for all $\lambda \geq 0$. It is worth pointing out here that the sub-super solution method still works for the problem (8.148). With the same proof as in Zhang and Yu [95, Lemmma 2.8] that goes back to the pioneering work of Amann [3] we state the following result.

Lemma 8.6. *Let $\lambda, \mu \geq 0$. If (8.148) has a super-solution $\overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\zeta \leq \overline{u}$ in Ω, then (8.148) has at least a solution.*

Lemma 8.7. (Alaa and Pierre [1]). *If $p > 1$, then there exists a real number $\bar{\sigma} > 0$ such that the problem*

$$\begin{cases} -\Delta u = |\nabla u|^p + \sigma & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.155}$$

has no solutions for $\sigma > \bar{\sigma}$.

Lemma 8.8. *Let $F : \overline{\Omega} \times (0, \infty) \to [0, \infty)$ and $G : (0, \infty) \to (0, \infty)$ be two Hölder continuous functions that verify*

(*A1*) $F(x, s) > 0$, *for all* $(x, s) \in \overline{\Omega} \times (0, \infty)$;

(*A2*) *The mapping* $[0, \infty) \ni s \longmapsto F(x, s)$ *is nondecreasing for all* $x \in \overline{\Omega}$;

(*A3*) *G is nonincreasing and* $\lim_{s \searrow 0} G(s) = +\infty$.

Assume that $\tau > 0$ is a positive real number. Then the following holds.
(i) *If $\tau \lim_{s\to\infty} G(s) \geq \lambda_1$, then the problem*

$$\begin{cases} -\Delta u = G(u) + \tau|\nabla u|^2 + \mu F(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.156}$$

has no solutions.
(ii) *If $\tau \lim_{s\to\infty} G(s) < \lambda_1$, then there exists $\bar{\mu} > 0$ such that the problem (8.156) has at least one solution for all $0 \leq \mu < \bar{\mu}$.*

Proof. (i) With the change of variable $v = e^{\tau u} - 1$, the problem (8.156) takes the form

$$\begin{cases} -\Delta v = \Psi_\mu(x,u) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.157}$$

where

$$\Psi_\mu(x,s) = \tau(s+1)G\left(\frac{1}{\tau}\ln(s+1)\right) + \mu\tau(s+1)F\left(x, \frac{1}{\tau}\ln(s+1)\right),$$

for all $(x,s) \in \overline{\Omega} \times (0,\infty)$.
Taking into account the fact that G is nonincreasing and $\tau \lim_{s\to\infty} G(s) \geq \lambda_1$, we get

$$\Psi_\mu(x,s) \geq \lambda_1(s+1) \quad \text{in } \overline{\Omega} \times (0,\infty), \text{ for all } \mu \geq 0.$$

By Lemma 8.5 we conclude that (8.157) has no solutions. Hence (8.156) has no solutions.
(ii) Since

$$\lim_{s\to+\infty} \frac{\tau(s+1)G\left(\frac{1}{\tau}\ln(s+1)\right) + 1}{s} < \lambda_1$$

and

$$\lim_{s\searrow 0} \frac{\tau(s+1)G\left(\frac{1}{\tau}\ln(s+1)\right) + 1}{s} = +\infty,$$

we deduce that the mapping $(0,\infty) \ni s \longmapsto \tau(s+1)G\left(\frac{1}{\tau}\ln(s+1)\right) + 1$ fulfills the hypotheses in Lemma 6.2. According to this one, there exists $\overline{v} \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of the problem

$$\begin{cases} -\Delta v = \tau(v+1)G\left(\frac{1}{\tau}\ln(v+1)\right) + 1 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

Define

$$\bar{\mu} := \frac{1}{\tau(\|\overline{v}\|_\infty + 1)} \cdot \frac{1}{\max\limits_{x\in\overline{\Omega}} F\left(x, \frac{1}{\tau}\ln(\|\overline{v}\|_\infty + 1)\right)}.$$

It follows that $\overline{v}$ is a super-solution of (8.157) for all $0 \le \mu < \bar{\mu}$.
Next we provide a sub-solution $\underline{v}$ of (8.157) such that $\underline{v} \le \overline{v}$ in Ω. To this aim, we apply Lemma 6.2 to get that there exists $\underline{v} \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of the problem

$$\begin{cases} -\Delta v = \tau G\left(\frac{1}{\tau}\ln(v+1)\right) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly, $\underline{v}$ is a sub-solution of (8.157) for all $0 \le \mu < \bar{\mu}$. Let us prove now that $\underline{v} \le \overline{v}$ in Ω. Assuming the contrary, it follows that $\max_{x\in\overline{\Omega}}\{\underline{v} - \overline{v}\} > 0$ is achieved in Ω. At that point, say x_0, we have

$$\begin{aligned} 0 &\le -\Delta(\underline{v} - \overline{v})(x_0) \\ &\le \tau\left[G\left(\frac{1}{\tau}\ln(\underline{v}(x_0)+1)\right) - G\left(\frac{1}{\tau}\ln(\overline{v}(x_0)+1)\right)\right] - 1 < 0, \end{aligned}$$

which is a contradiction. Thus, $\underline{v} \le \overline{v}$ in Ω. We have proved that $(\underline{v}, \overline{v})$ is an ordered pair of sub-super solutions of (8.157) provided $0 \le \mu < \bar{\mu}$. It follows that (8.156) has at least one classical solution for all $0 \le \mu < \bar{\mu}$ and the proof of Lemma 8.8 is now complete. □

Proof of Theorem 8.1. According to Lemma 8.8(i) we deduce that (8.148) has no solutions if $p = 2$ and $a \ge \lambda_1$. Furthermore, if $p = 2$ and $a < \lambda_1$, in view of Lemma 8.8(ii), we deduce that (8.148) has at least one classical solution if μ is small enough. Assume now $1 < p < 2$ and let us fix $C > 0$ such that

$$aC^{p/2} + C^{p-1} < \lambda_1. \tag{8.158}$$

Define

$$\psi : [0,\infty) \to [0,\infty), \quad \psi(s) = \frac{s^p}{s^2 + C}.$$

A careful examination reveals the fact that ψ attains its maximum at $\bar{s} = \left(\frac{Cp}{2-p}\right)^{2-p}$. Hence

$$\psi(s) \le \psi(\bar{s}) = \frac{p^{p/2}(2-p)^{(2-p)/2}}{2C^{1-p/2}}, \quad \text{for all } s \ge 0.$$

By the classical Young's inequality we deduce

$$p^{p/2}(2-p)^{(2-p)/2} \le 2,$$

which yields $\psi(s) \le C^{p/2-1}$, for all $s \ge 0$. Thus, we have proved

$$s^p \le C^{p/2}s^2 + C^{p/2-1}, \quad \text{for all } s \ge 0. \tag{8.159}$$

Consider the problem

$$\begin{cases} -\Delta u = g(u) + C^{p/2-1} + C^{p/2}|\nabla u|^2 + \mu f(x,u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.160}$$

By virtue of (8.159), any solution of (8.160) is a super-solution of (8.148).

Using now (8.158) we get

$$\lim_{s\to\infty} C^{p/2}(g(u) + C^{p/2-1}) < \lambda_1.$$

The above relation enables us to apply Lemma 8.8(ii) with $G(s) = g(s) + C^{p/2-1}$ and $\tau = C^{p/2}$. It follows that there exists $\bar{\mu} > 0$ such that (8.160) has at least a solution u. With a similar argument to that used in the proof of Lemma 8.8, we obtain $\zeta \le u$ in Ω, where ζ is defined in (8.154). By Lemma 8.6 we get that (8.148) has at least one solution if $0 \le \mu < \bar{\mu}$.

We have proved that (8.148) has at least one classical solution for both cases $p = 2$ and $a < \lambda_1$ or $1 < p < 2$, provided μ is nonnegative small enough. Define next

$$A = \{\mu \ge 0; \text{ problem (8.148) has at least one solution}\}.$$

The above arguments implies that A is nonempty. Let $\mu^* = \sup A$. We first show that $[0, \mu^*) \subseteq A$. For this purpose, let $\mu_1 \in A$ and $0 \le \mu_2 < \mu_1$. If u_{μ_1} is a solution of (8.148) with $\mu = \mu_1$, then u_{μ_1} is a super-solution of (8.148) with $\mu = \mu_2$. It is easy to prove that $\zeta \le u_{\mu_1}$ in Ω and by virtue of Lemma 8.6 we conclude that the problem (8.148) with $\mu = \mu_2$ has at least one solution.

Thus we have proved $[0, \mu^*) \subseteq A$. Next we show $\mu^* < +\infty$.
Since $\lim_{s\searrow 0} g(s) = +\infty$, we can choose $s_0 > 0$ such that $g(s) > \bar{\sigma}$ for all $s \le s_0$. Let

$$\mu_0 = \frac{\bar{\sigma}}{\min_{x\in\overline{\Omega}} f(x, s_0)}.$$

Using the monotonicity of f with respect to the second argument, the above relations yields

$$g(s) + \mu f(x,s) \ge \bar{\sigma}, \quad \text{for all } (x,s) \in \overline{\Omega} \times (0,\infty) \text{ and } \mu > \mu_0.$$

If (8.148) has a solution for $\mu > \mu_0$, this would be a super-solution of the problem

$$\begin{cases} -\Delta u = |\nabla u|^p + \bar{\sigma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.161}$$

Since 0 is a sub-solution, we deduce that (8.161) has at least one solution. According to Lemma 8.7, this is a contradiction. Hence $\mu^* \le \mu_0 < +\infty$. This concludes the proof of Theorem 8.1. □

Proof of Theorem 8.2 (i) We fix $p \in (0,1]$ and define

$$q = q(p) = \begin{cases} p+1 & \text{if } 0 < p < 1, \\ 3/2 & \text{if } p = 1. \end{cases}$$

Consider the problem

$$\begin{cases} -\Delta u = g(u) + 1 + |\nabla u|^q + \mu f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.162}$$

Since $s^p \leq s^q + 1$, for all $s \geq 0$, we deduce that any solution of (8.162) is a super-solution of (8.148). Furthermore, taking into account the fact that $1 < q < 2$, we can apply Theorem 8.1(ii) in order to get that (8.162) has at least one solution if μ is small enough. Thus, by Lemma 8.6 we deduce that (8.148) has at least one classical solution. Following the method used in the proof of Theorem 8.1, we set

$$A = \{\mu \geq 0;\ \text{ problem (8.148) has at least one solution}\}$$

and let $\mu^* = \sup A$. With the same arguments we prove that $[0, \mu^*) \subseteq A$. It remains only to show that $\mu^* < +\infty$.

Let us assume first that f satisfies $(f1)$. Since $\lim_{s\searrow 0} g(s) = +\infty$, we can choose $\mu_0 > \frac{2\lambda_1}{c}$ such that $\frac{1}{2}\mu_0 cs + g(s) \geq 1$ for all $s > 0$. Then

$$g(s) + \mu f(x,s) \geq \lambda_1 s + 1, \qquad \text{for all } \ (x,s) \in \overline{\Omega} \times (0,\infty) \ \text{ and } \ \mu \geq \mu_0.$$

By virtue of Lemma 8.5 we obtain that (8.148) has no classical solutions if $\mu \geq \mu_0$, so μ^* is finite.

Assume now that f satisfies $(f2)$. Since $\lim_{s\searrow 0} g(s) = +\infty$, there exists $s_0 > 0$ such that

$$g(s) \geq \lambda_1(s+1) \qquad \text{for all } \ 0 < s < s_0. \tag{8.163}$$

On the other hand, the assumption $(f2)$ and the fact that Ω is bounded implies that the mapping

$$(0,\infty) \ni s \longmapsto \frac{\min_{x\in\overline{\Omega}} f(x,s)}{s+1}$$

is nondecreasing, so we can choose $\tilde{\mu} > 0$ with the property

$$\tilde{\mu} \cdot \frac{\min_{x\in\overline{\Omega}} f(x,s)}{s+1} \geq \lambda_1 \qquad \text{for all } \ s \geq s_0. \tag{8.164}$$

Now (8.163) combined with (8.164) yields

$$g(s) + \mu f(x,s) \geq \lambda_1(s+1), \qquad \text{for all } \ (x,s) \in \overline{\Omega} \times (0,\infty) \ \text{ and } \ \mu \geq \tilde{\mu}.$$

Using Lemma 8.5, we deduce that (8.148) has no solutions if $\mu > \tilde{\mu}$, that is, μ^* is finite. The first part in Theorem 8.2 is therefore established.

(ii) The strategy is to find a super-solution $\overline{u}_\mu \in C^2(\Omega) \cap C(\overline{\Omega})$ of (8.148) such that $\zeta \le \overline{u}_\mu$ in Ω. To this aim, let $h \in C^2(0,\eta] \cap C[0,\eta]$ be such that

$$\begin{cases} h''(t) = -g(h(t)), & \text{for all } 0 < t < \eta, \\ h(0) = 0, \\ h > 0 \quad \text{in } (0,\eta]. \end{cases} \tag{8.165}$$

The existence of h follows by classical arguments of ODE. Since h is concave, there exists $h'(0+) \in (0,+\infty]$. By taking $\eta > 0$ small enough, we can assume that $h' > 0$ in $(0,\eta]$, so h is increasing on $[0,\eta]$.

Lemma 8.9. (i) $h \in C^1[0,\eta]$ *if and only if* $\int_0^1 g(s)ds < +\infty$;
(ii) *If* $0 < p \le 2$, *then there exist* $c_1, c_2 > 0$ *such that*

$$(h')^p(t) \le c_1 g(h(t)) + c_2, \quad \textit{for all } \ 0 < t < \eta.$$

Proof. (i) Multiplying by h' in (8.165) and then integrating on $[t,\eta]$, $0 < t < \eta$, we get

$$(h')^2(t) - (h')^2(\eta) = 2\int_t^\eta g(h(s))h'(s)ds = 2\int_{h(t)}^{h(\eta)} g(\tau)d\tau. \tag{8.166}$$

This gives

$$(h')^2(t) = 2G(h(t)) + (h')^2(\eta) \quad \text{ for all } \ 0 < t < \eta, \tag{8.167}$$

where $G(t) = \int_t^{h(\eta)} g(s)ds$. From (8.167) we deduce that $h'(0+)$ is finite if and only if $G(0+)$ is finite, so (i) follows.
(ii) Let $p \in (0,2]$. Taking into account the fact that g is nonincreasing, the inequality (8.167) leads to

$$(h')^2(t) \le 2h(\eta)g(h(t)) + (h')^2(\eta), \quad \text{ for all } 0 < t < \eta. \tag{8.168}$$

Since $s^p \le s^2 + 1$, for all $s \ge 0$, from (8.168) we have

$$(h')^p(t) \le c_1 g(h(t)) + c_2, \quad \text{ for all } 0 < t < \eta \tag{8.169}$$

where $c_1 = 2h(\eta)$ and $c_2 = (h')^2(\eta) + 1$. This completes the proof of our Lemma. □

Proof of Theorem 8.2 completed. Let $p \in (0,1)$ and $\mu \ge 0$ be fixed. We also fix $c > 0$ such that $c\|\varphi_1\|_\infty < \eta$. By Hopf's maximum principle, there exist $\delta > 0$ small enough and $\theta_1 > 0$ such that

$$|\nabla \varphi_1| > \theta_1 \quad \text{ in } \ \Omega_\delta, \tag{8.170}$$

where $\Omega_\delta := \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) \le \delta\}$.
Moreover, since $\lim_{s\searrow 0} g(h(s)) = +\infty$, we can pick δ with the property

$$(c\theta_1)^2 g(h(c\varphi_1)) - 3\mu f(x, h(c\varphi_1)) > 0 \quad \text{in } \Omega_\delta. \tag{8.171}$$

Let $\theta_2 := \inf\limits_{\Omega\setminus\Omega_\delta} \varphi_1 > 0$. We choose $M > 1$ with

$$M(c\theta_1)^2 > 3, \tag{8.172}$$

$$Mc\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty) > 3g(h(c\theta_2)). \tag{8.173}$$

Since $p < 1$, we also may assume

$$(Mc)^{1-p}\lambda_1 (h')^{1-p}(c\|\varphi_1\|_\infty) \ge 3\|\nabla\varphi_1\|_\infty^p. \tag{8.174}$$

On the other hand, by Lemma 8.9(ii) we can choose $M > 1$ such that

$$3(h'(c\varphi_1))^p \le M^{1-p}(c\theta_1)^{2-p} g(h(c\varphi_1)) \quad \text{in } \Omega_\delta. \tag{8.175}$$

The assumption $(f4)$ yields

$$\lim_{s\to\infty} \frac{3\mu f(x, sh(c\|\varphi_1\|_\infty))}{sh(c\|\varphi_1\|_\infty)} = 0.$$

So we can choose $M > 1$ large enough such that

$$\frac{3\mu f(x, Mh(c\|\varphi_1\|_\infty))}{Mh(c\|\varphi_1\|_\infty)} < \frac{c\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty)}{h(c\|\varphi_1\|_\infty)},$$

uniformly in Ω. This leads us to

$$3\mu f(x, Mh(c\|\varphi_1\|_\infty)) < Mc\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty), \quad \text{for all } x \in \Omega. \tag{8.176}$$

For M satisfying (8.172)-(8.176), we prove that $\overline{u}_\mu = Mh(c\varphi_1)$ is a super-solution of (8.148). We have

$$-\Delta\overline{u}_\lambda = Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1 h'(c\varphi_1) \quad \text{in } \Omega. \tag{8.177}$$

First we prove that

$$Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 \ge g(\overline{u}_\mu) + |\nabla\overline{u}_\mu|^p + \mu f(x, \overline{u}_\mu) \quad \text{in } \Omega_\delta. \tag{8.178}$$

From (8.170) and (8.172) we get

$$\frac{1}{3}Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 \ge g(h(c\varphi_1)) \ge g(Mh(c\varphi_1)) = g(\overline{u}_\mu) \quad \text{in } \Omega_\delta. \tag{8.179}$$

By (8.170) and (8.175) we also have

$$\frac{1}{3}Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 \ge (Mc)^p (h')^p(c\varphi_1))|\nabla\varphi_1|^p = |\nabla\overline{u}_\mu|^p \quad \text{in } \Omega_\delta. \tag{8.180}$$

The assumption $(f3)$ and (8.171) produce

$$\frac{1}{3}Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 \geq \mu M f(x, h(c\varphi_1)) \geq \mu f(x, Mh(c\varphi_1)) \quad \text{in } \Omega_\delta. \tag{8.181}$$

Now, by (8.179), (8.180) and (8.181) we conclude that (8.178) is fulfilled.

Next we prove

$$Mc\lambda_1\varphi_1 h'(c\varphi_1) \geq g(\overline{u}_\mu) + |\nabla\overline{u}_\mu|^p + \mu f(x, \overline{u}_\mu) \quad \text{in } \Omega\setminus\Omega_\delta. \tag{8.182}$$

From (8.173) we obtain

$$\frac{1}{3}Mc\lambda_1\varphi_1 h'(c\varphi_1) \geq g(h(c\varphi_1)) \geq g(Mh(c\varphi_1)) = g(\overline{u}_\mu) \quad \text{in } \Omega\setminus\Omega_\delta. \tag{8.183}$$

From (8.174) we get

$$\frac{1}{3}Mc\lambda_1\varphi_1 h'(c\varphi_1) \geq (Mc)^p (h')^p(c\varphi_1)|\nabla\varphi_1|^p = |\nabla\overline{u}_\mu|^p \quad \text{in } \Omega\setminus\Omega_\delta. \tag{8.184}$$

By (8.176) we deduce

$$\frac{1}{3}Mc\lambda_1\varphi_1 h'(c\varphi_1) \geq \mu f(x, Mh(c\varphi_1)) = \mu f(x, \overline{u}_\mu) \quad \text{in } \Omega\setminus\Omega_\delta. \tag{8.185}$$

Obviously, (8.182) follows now by (8.183), (8.184) and (8.185).
Combining (8.177) with (8.178) and (8.182) we find that $\overline{u}_\mu$ is a super-solution of (8.148). Moreover, $\zeta \leq \overline{u}_\mu$ in Ω. Applying Lemma 8.6, we deduce that (8.148) has at least one solution for all $\mu \geq 0$. This finishes the proof of Theorem 8.2. □

Proof of Theorem 8.3 The proof case relies on the same arguments used in the proof of Theorem 8.2. In fact, the main point is to find a super-solution $\overline{u}_\lambda \in C^2(\Omega) \cap (\overline{\Omega})$ of (8.148), while ζ defined in (8.154) is a sub-solution. Since g is nonincreasing, the inequality $\zeta \leq \overline{u}_\lambda$ in Ω can be proved easily and the existence of solutions to (8.148) follows by Lemma 8.6.

Define c, δ and θ_1, θ_2 as in the proof of Theorem 8.2. Let M satisfying (8.172) and (8.173). Since $g(h(s)) \to +\infty$ as $s \searrow 0$, we can choose $\delta > 0$ such that

$$(c\theta_1)^2 g(h(c\varphi_1)) - 3f(x, h(c\varphi_1)) > 0 \quad \text{in } \Omega_\delta. \tag{8.186}$$

The assumption $(f4)$ produces

$$\lim_{s\to\infty}\frac{f(x, sh(c\|\varphi_1\|_\infty))}{sh(c\|\varphi_1\|_\infty)} = 0, \qquad \text{uniformly for } x\in\overline{\Omega}.$$

Thus, we can take $M > 3$ large enough, such that

$$\frac{f(x, Mh(c\|\varphi_1\|_\infty))}{Mh(c\|\varphi_1\|_\infty)} < \frac{c\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty)}{3h(c\|\varphi_1\|_\infty)}.$$

The above relation yields

$$3f(x, Mh(c\|\varphi_1\|_\infty)) < Mc\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty), \quad \text{for all} \quad x \in \overline{\Omega}. \tag{8.187}$$

Using Lemma 8.9(ii) we can take $\lambda > 0$ small enough such that the following inequalities hold

$$3\lambda M^{p-1}(h')^p(c\varphi_1) \le g(h(c\varphi_1))(c\theta_1)^{2-p} \quad \text{in } \Omega_\delta \tag{8.188}$$

$$\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty) > 3\lambda(Mc)^{p-1}(h')^p(c\theta_2)\|\nabla\varphi_1\|_\infty^p. \tag{8.189}$$

For M and λ satisfying (8.172)-(8.173) and (8.186)-(8.189), we claim that $\overline{u}_\lambda = Mh(c\varphi_1)$ is a super-solution of (8.148). First we have

$$-\Delta\overline{u}_\lambda = Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1 h'(c\varphi_1) \quad \text{in } \Omega. \tag{8.190}$$

Arguing as in the proof of Theorem 8.2, from (8.170), (8.172), (8.186), (8.188) and the assumption $(f3)$ we obtain

$$Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 \ge g(\overline{u}_\lambda) + \lambda|\nabla\overline{u}_\lambda|^p + f(x, \overline{u}_\lambda) \quad \text{in } \Omega_\delta. \tag{8.191}$$

On the other hand, (8.173), (8.187) and (8.189) gives

$$Mc\lambda_1\varphi_1 h'(c\varphi_1) \ge g(\overline{u}_\lambda) + \lambda|\nabla\overline{u}_\lambda|^p + f(x, \overline{u}_\lambda) \quad \text{in } \Omega \setminus \Omega_\delta. \tag{8.192}$$

Using now (8.190) and (8.191)-(8.192) we find that $\overline{u}_\lambda$ is a super-solution of (8.148) so our claim follows.
As we have already argued at the beginning of this case, we easily get that $\zeta \le \overline{u}_\lambda$ in Ω and by Lemma 8.6 we deduce that problem (8.148) has at least one solution if $\lambda > 0$ is sufficiently small.

Set

$$A = \{\ \lambda \ge 0;\ \text{problem (8.148) has at least one classical solution}\}.$$

From the above arguments, A is nonempty. Let $\lambda^* = \sup A$. First we claim that if $\lambda \in A$, then $[0, \lambda) \subseteq A$. For this purpose, let $\lambda_1 \in A$ and $0 \le \lambda_2 < \lambda_1$. If u_{λ_1} is a solution of (8.148) with $\lambda = \lambda_1$, then u_{λ_1} is a super-solution for (8.148) with $\lambda = \lambda_2$ while ζ defined in (8.154) is a sub-solution. Using Lemma 8.6 once more, we have that (8.148) with $\lambda = \lambda_2$ has at least one classical solution. This proves the claim. Since $\lambda \in A$ was arbitrary chosen, we conclude that $[0, \lambda^*) \subset A$.

Let us assume now $p \in (1, 2]$. We prove that $\lambda^* < +\infty$. Set

$$m := \inf_{(x,s)\in\overline{\Omega}\times(0,\infty)} \Big(g(s) + f(x, s)\Big).$$

Since $\lim_{s\searrow 0} g(s) = +\infty$ and the mapping $(0,\infty) \ni s \longmapsto \min_{x\in\overline{\Omega}} f(x,s)$ is positive and nondecreasing, we deduce that m is a positive real number. Let $\lambda > 0$ be such that (8.148) has a solution u_λ. If $v = \lambda^{1/(p-1)} u_\lambda$, then v verifies

$$\begin{cases} -\Delta v \geq |\nabla v|^p + \lambda^{1/(p-1)} m & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.193}$$

It follows that v is a super-solution of (8.155) for $\sigma = \lambda^{1/(p-1)} m$. Since 0 is a sub-solution, we obtain that (8.155) has at least one classical solution for σ defined above. According to Lemma 8.7, we have $\sigma \leq \bar{\sigma}$, and so $\lambda \leq \left(\frac{\bar{\sigma}}{m}\right)^{p-1}$. This means that λ^* is finite.

Assume now $p \in (0,1)$ and let us prove that $\lambda^* = +\infty$. Recall that ζ defined in (8.154) is a sub-solution. To get a super-solution, we proceed in the same manner. Fix $\lambda > 0$. Since $p < 1$ we can find $M > 1$ large enough such that (8.172)-(8.173) and (8.187)-(8.189) hold. From now on, we follow the same steps as above. The proof of Theorem 8.3 is now complete. □

We remark that if $\int_0^1 g(s)ds < \infty$, then the above method can be applied in order to extend the study of (8.148) to the case $\mu = 1$ and $p > 2$. Indeed, by Lemma 8.9(i) it follows $h \in C^1[0,\eta]$. Using this fact, we can choose $c_1, c_2 > 0$ large enough such that the conclusion of Lemma 8.9(ii) holds. Repeating the above arguments we prove that if $p > 2$ then there exists a real number $\lambda^* > 0$ such that (8.148) has at least one solution if $\lambda < \lambda^*$ and no solutions exist if $\lambda > \lambda^*$.

Proof of Theorem 8.4. (i) If $\lambda = 0$, the existence of the solution follows by using Lemma 6.2. Next we assume that $\lambda > 0$ and let us fix $\mu \geq 0$. With the change of variable $v = e^{\lambda u} - 1$, the problem (8.148) becomes

$$\begin{cases} -\Delta v = \Phi_\lambda(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.194}$$

where

$$\Phi_\lambda(s) = \lambda(s+1) g\left(\frac{1}{\lambda}\ln(s+1)\right) + \lambda\mu(s+1),$$

for all $s \in (0,\infty)$. Obviously Φ_λ is not monotone but we still have that the mapping $(0,\infty) \ni s \mapsto \frac{\Phi_\lambda(s)}{s}$ is decreasing for all $\lambda > 0$ and

$$\lim_{s\to+\infty} \frac{\Phi_\lambda(s)}{s} = \lambda(a+\mu) \quad \text{and} \quad \lim_{s\searrow 0} \frac{\Phi_\lambda(s)}{s} = +\infty,$$

uniformly for $\lambda > 0$.

We first remark that Φ_λ satisfies the hypotheses in Lemma 6.2 provided $\lambda(a+\mu) < \lambda_1$. Hence (8.194) has at least one solution.

On the other hand, since $g \geq a$ on $(0, \infty)$, we get

$$\Phi_\lambda(s) \geq \lambda(a+\mu)(s+1), \qquad \text{for all } \ \lambda, s \in (0,\infty). \tag{8.195}$$

Using now Lemma 8.5 we deduce that (8.194) has no solutions if $\lambda(a+\mu) \geq \lambda_1$. The proof of the first part in Theorem 8.4 is therefore complete.

(ii) We split the proof into several steps.

STEP 1. **Existence of solutions.** This follows directly from (i).

STEP 2. **Uniqueness of the solution.**
Fix $\lambda \geq 0$. Let u_1 and u_2 be two classical solutions of (8.148) with $\lambda < \lambda^*$. We show that $u_1 \leq u_2$ in Ω. Supposing the contrary, we deduce that $\max_{\overline{\Omega}}\{u_1 - u_2\} > 0$ is achieved in a point $x_0 \in \Omega$. This yields $\nabla(u_1 - u_2)(x_0) = 0$ and

$$0 \leq -\Delta(u_1 - u_2)(x_0) = g(u_1(x_0)) - g(u_2(x_0)) < 0,$$

a contradiction. We conclude that $u_1 \leq u_2$ in Ω; similarly $u_2 \leq u_1$. Therefore $u_1 = u_2$ in Ω and the uniqueness is proved.

STEP 3. **Dependence on λ.** Fix $0 \leq \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_1}, u_{λ_2} be the unique solutions of (8.148) with $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively. If $\{x \in \Omega; u_{\lambda_1} > u_{\lambda_2}\}$ is nonempty, then $\max_{\overline{\Omega}}\{u_{\lambda_1} - u_{\lambda_2}\} > 0$ is achieved in Ω. At that point, say $\bar{x}$, we have $\nabla(u_{\lambda_1} - u_{\lambda_2})(\bar{x}) = 0$ and

$$0 \leq -\Delta(u_{\lambda_1} - u_{\lambda_2})(\bar{x}) = g(u_{\lambda_1}(\bar{x})) - g(u_{\lambda_2}(\bar{x})) + (\lambda_1 - \lambda_2)|\nabla u_{\lambda_1}|^p(\bar{x}) < 0,$$

which is a contradiction.
Hence $u_{\lambda_1} \leq u_{\lambda_2}$ in $\overline{\Omega}$. The maximum principle also gives $u_{\lambda_1} < u_{\lambda_2}$ in Ω.

STEP 4. **Regularity.** We fix $0 < \lambda < \lambda^*$, $\mu > 0$ and assume that $\limsup_{s \searrow 0} s^\alpha g(s) < +\infty$. This means that $g(s) \leq cs^{-\alpha}$ in a small positive neighborhood of the origin. To prove the regularity, we will use again the change of variable $v = e^{\lambda u} - 1$. Thus, if u_λ is the unique solution of (8.148), then $v_\lambda = e^{\lambda u_\lambda} - 1$ is the unique solution of (8.194). Since $\lim_{s \searrow 0} \dfrac{e^{\lambda s} - 1}{s} = \lambda$, we conclude that (ii1) and (ii2) in Theorem 8.4 are established if we prove

(a) $\tilde{c}_1 \operatorname{dist}(x, \partial\Omega) \leq v_\lambda(x) \leq \tilde{c}_2 \operatorname{dist}(x, \partial\Omega)$ in Ω, for some positive constants $\tilde{c}_1, \tilde{c}_2 > 0$.

(b) $v_\lambda \in C^{1,1-\alpha}(\overline{\Omega})$.

Proof of (a). By the monotonicity of g and the fact that $g(s) \leq cs^{-\alpha}$ near the origin, we deduce the existence of $A, B, C > 0$ such that

$$\Phi_\lambda(s) \leq As + Bs^{-\alpha} + C, \qquad \text{for all } 0 < \lambda < \lambda^* \text{ and } \ s > 0. \tag{8.196}$$

Let us fix $m > 0$ such that $m\lambda_1\|\varphi_1\|_\infty < \lambda\mu$. Combining this with (8.195) we deduce

$$-\Delta(v_\lambda - m\varphi_1) = \Phi_\lambda(v_\lambda) - m\lambda_1\varphi_1 \geq \lambda\mu - m\lambda_1\varphi_1 \geq 0 \tag{8.197}$$

in Ω. Since $v_\lambda - m\varphi_1 = 0$ on $\partial\Omega$, we conclude

$$v_\lambda \geq m\varphi_1 \quad \text{in } \Omega. \tag{8.198}$$

Now, (8.198) and (8.151) imply $v_\lambda \geq \tilde{c}_1 \operatorname{dist}(x, \partial\Omega)$ in Ω, for some positive constant $\tilde{c}_1 > 0$. The first inequality in the statement of (a) is therefore established. For the second one, we apply an idea found in Gui and Lin [57]. Using (8.198) and the estimate (8.196), by virtue of Lemma 7.5 we deduce $\Phi_\lambda(v_\lambda) \in L^1(\Omega)$, that is, $\Delta v_\lambda \in L^1(\Omega)$.

Using the smoothness of $\partial\Omega$, we can find $\delta \in (0,1)$ such that for all $x_0 \in \Omega_\delta := \{x \in \Omega\,;\, \operatorname{dist}(x, \partial\Omega) \leq \delta\}$, there exists $y \in \mathbb{R}^N \setminus \overline{\Omega}$ with $\operatorname{dist}(y, \partial\Omega) = \delta$ and $\operatorname{dist}(x_0, \partial\Omega) = |x_0 - y| - \delta$.

Let $K > 1$ be such that $\operatorname{diam}(\Omega) < (K-1)\delta$ and let ξ be the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta\xi = \Phi_\lambda(\xi) & \text{in } B_K(0) \setminus B_1(0), \\ \xi > 0 & \text{in } B_K(0) \setminus B_1(0), \\ \xi = 0 & \text{on } \partial(B_K(0) \setminus B_1(0)). \end{cases}$$

where $B_r(0)$ denotes the open ball in $\mathbb{R}^N$ of radius r and centered at the origin. By uniqueness, ξ is radially symmetric. Hence $\xi(x) = \tilde{\xi}(|x|)$ and

$$\begin{cases} \tilde{\xi}'' + \dfrac{N-1}{r}\tilde{\xi}' + \Phi_\lambda(\tilde{\xi}) = 0 & \text{in } (1, K), \\ \tilde{\xi} > 0 & \text{in } (1, K), \\ \tilde{\xi}(1) = \tilde{\xi}(K) = 0. \end{cases} \tag{8.199}$$

Integrating in (8.199) we have

$$\begin{aligned} \tilde{\xi}'(t) &= \tilde{\xi}'(a)a^{N-1}t^{1-N} - t^{1-N}\int_a^t r^{N-1}\Phi_\lambda(\tilde{\xi}(r))dr \\ &= \tilde{\xi}'(b)b^{N-1}t^{1-N} + t^{1-N}\int_t^b r^{N-1}\Phi_\lambda(\tilde{\xi}(r))dr, \end{aligned}$$

where $1 < a < t < b < K$. With the same arguments as above we have $\Phi_\lambda(\tilde{\xi}) \in L^1(1, K)$ which implies that both $\tilde{\xi}(1)$ and $\tilde{\xi}(K)$ are finite. Hence $\tilde{\xi} \in C^2(1, K) \cap C^1[1, K]$. Furthermore,

$$\xi(x) \leq \tilde{C}\min\{K - |x|, |x| - 1\}, \qquad \text{for any} \quad x \in B_K(0) \setminus B_1(0). \tag{8.200}$$

Let us fix $x_0 \in \Omega_\delta$. Then we can find $y_0 \in \mathbb{R}^N \setminus \overline{\Omega}$ with $\operatorname{dist}(y_0, \partial\Omega) = \delta$ and $\operatorname{dist}(x_0, \partial\Omega) = |x_0 - y| - \delta$. Thus, $\Omega \subset B_{K\delta}(y_0) \setminus B_\delta(y_0)$. Define $\overline{v}(x) = \xi\left(\dfrac{x - y_0}{\delta}\right)$, for all $x \in \overline{\Omega}$. We show that $\overline{v}$ is a super-solution of (8.194). Indeed, for all $x \in \Omega$ we have

$$\begin{aligned} \Delta\overline{v} + \Phi_\lambda(\overline{v}) &= \frac{1}{\delta^2}\left(\tilde{\xi}'' + \frac{N-1}{r}\tilde{\xi}'\right) + \Phi_\lambda(\tilde{\xi}) \\ &\leq \frac{1}{\delta^2}\left(\tilde{\xi}'' + \frac{N-1}{r}\tilde{\xi}' + \Phi_\lambda(\tilde{\xi})\right) \\ &= 0, \end{aligned}$$

where $r = \dfrac{|x-y_0|}{\delta}$. We have obtained that

$$\Delta\overline{v} + \Phi_\lambda(\overline{v}) \le 0 \le \Delta v_\lambda + \Phi_\lambda(v_\lambda) \quad \text{in } \Omega,$$

$$\overline{v}, v_\lambda > 0 \quad \text{in } \Omega,\ \overline{v} = v_\lambda \quad \text{on } \partial\Omega$$

$$\Delta v_\lambda \in L^1(\Omega).$$

By Lemma 6.3 we get $v_\lambda \le \overline{v}$ in Ω. Combining this with (8.200) we obtain

$$v_\lambda(x_0) \le \overline{v}(x_0) \le \tilde{C}\min\left\{K - \frac{|x_0-y_0|}{\delta}, \frac{|x_0-y_0|}{\delta} - 1\right\} \le \frac{\tilde{C}}{\delta}\text{dist}(x_0, \partial\Omega).$$

Hence $v_\lambda \le \frac{\tilde{C}}{\delta}\text{dist}(x, \partial\Omega)$ in Ω_δ and the second inequality in the statement of (a) follows.

Proof of (b). Let G be the Green's function associated with the Laplace operator in Ω. Then, for all $x \in \Omega$ we have

$$v_\lambda(x) = -\int_\Omega G(x,y)\Phi_\lambda(v_\lambda(y))dy$$

and

$$\nabla v_\lambda(x) = -\int_\Omega G_x(x,y)\Phi_\lambda(v_\lambda(y))dy.$$

If $x_1, x_2 \in \Omega$, using (8.196) we obtain

$$\begin{aligned} |\nabla v_\lambda(x_1) - \nabla v_\lambda(x_2)| \quad &\le \int_\Omega |G_x(x_1,y) - G_x(x_2,y)| \cdot (Av_\lambda + C)dy \\ &+B\int_\Omega |G_x(x_1,y) - G_x(x_2,y)| \cdot v_\lambda^{-\alpha}(y)dy. \end{aligned}$$

Now, taking into account that $v_\lambda \in C(\overline{\Omega})$, by the standard regularity theory (see Gilbarg and Trudinger [55]) we get

$$\int_\Omega |G_x(x_1,y) - G_x(x_2,y)| \cdot (Av_\lambda + C)dy \le \tilde{c}_1|x_1 - x_2|.$$

On the other hand, with the same proof as in [57, Theorem 1], we deduce

$$\int_\Omega |G_x(x_1,y) - G_x(x_2,y)| \cdot v_\lambda^{-\alpha}(y) \le \tilde{c}_2|x_1 - x_2|^{1-\alpha}.$$

The above inequalities imply $u_\lambda \in C^2(\Omega) \cap C^{1,1-\alpha}(\overline{\Omega})$.

Step 5. **Asymptotic behaviour of the solution.** This follows with the same lines as in the proof of Theorem 6.4. □

We are concerned in what follows with the closely related Dirichlet problem

$$\begin{cases} -\Delta u + K(x)g(u) + |\nabla u|^a = \lambda f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1$_\lambda$}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, $\lambda > 0$, $0 < a \leq 2$ and $K \in C^{0,\gamma}(\overline{\Omega})$, $0 < \gamma < 1$. We assume from now on that $f : \overline{\Omega} \times [0, \infty) \to [0, \infty)$ is a Hölder continuous function which is positive on $\overline{\Omega} \times (0, \infty)$ such that f is nondecreasing with respect to the second variable and is sublinear, in the sense that the mapping

$$(0, \infty) \ni s \longmapsto \frac{f(x,s)}{s} \quad \text{is nonincreasing for all } x \in \overline{\Omega}$$

and

$$\lim_{s \to 0^+} \frac{f(x,s)}{s} = +\infty \quad \text{and} \quad \lim_{s \to \infty} \frac{f(x,s)}{s} = 0, \quad \text{uniformly for } x \in \overline{\Omega}.$$

We also assume that $g \in C^{0,\gamma}(0, \infty)$ is a nonnegative and nonincreasing function satisfying

$$\lim_{s \to 0^+} g(s) = +\infty.$$

Problem $(1)_\lambda$ has been considered in Section 7 in the absence of the gradient term $|\nabla u|^a$ and assuming that the singular term $g(t)$ behaves like $t^{-\alpha}$ around the origin, with $t \in (0, 1)$. In this case it has been shown that the sign of the extremal values of K plays a crucial role. In this sense, we have proved in Section 7 that if $K < 0$ in $\overline{\Omega}$, then problem $(1)_\lambda$ (with $a = 0$) has a unique solution in the class $\mathcal{E} = \{u \in C^2(\Omega) \cap C(\overline{\Omega});\ g(u) \in L^1(\Omega)\}$, for all $\lambda > 0$. On the other hand, if $K > 0$ in $\overline{\Omega}$, then there exists λ^* such that problem $(1)_\lambda$ has solutions in $\mathcal{E}$ if $\lambda > \lambda^*$ and no solution exists if $\lambda < \lambda^*$. The case where f is asymptotically linear, $K \leq 0$, and $a = 0$ has been discussed in Section 6. In this framework, a major role is played by $\lim_{s \to \infty} f(s)/s = m > 0$. More precisely, there exists a solution (which is unique) $u_\lambda \in C^2(\Omega) \cap C^1(\overline{\Omega})$ if and only if $\lambda < \lambda^* := \lambda_1/m$. An additional result asserts that the mapping $(0, \lambda^*) \longmapsto u_\lambda$ is increasing and $\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$ uniformly on compact subsets of Ω.

Due to the singular character of our problem $(1)_\lambda$, we cannot expect to have solutions in $C^2(\overline{\Omega})$. We are seeking in this paper classical solutions of $(1)_\lambda$, that is, solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that verify $(1)_\lambda$. Closely related to our problem is the following one, which has been considered in the first part of this Section:

$$\begin{cases} -\Delta u = g(u) + |\nabla u|^a + \lambda f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.201}$$

where f and g verifies the above assumptions. We recall that we have proved that if $0 < a < 1$ then problem (8.201) has at least one classical solution for all $\lambda \geq 0$. In turn, if $1 < a \leq 2$, then problem (8.201) has no solutions for large values of $\lambda > 0$.

The existence results for our problem $(1)_\lambda$ are quite different to those of (8.201) presented in the first part of this Section. More exactly, we prove in what follows that problem $(1)_\lambda$ has at least one solution only when $\lambda > 0$ is large enough and g satisfies a naturally growth condition around the origin. Thus, we extend the results in Barles, G. Díaz, and J. I. Díaz [10, Theorem 1], corresponding to $K \equiv 0$, $f \equiv f(x)$ and $a \in [0,1)$.

The main difficulty in the treatment of $(1)_\lambda$ is the lack of the usual maximal principle between super and sub-solutions, due to the singular character of the equation. To overcome it, we state an improved comparison principle that fit to our problem $(1)_\lambda$ (see Lemma 8.13 below).

In our first result we assume that $K < 0$ in Ω. Note that K may vanish on $\partial\Omega$ which leads us to a competition on the boundary between the potential $K(x)$ and the singular term $g(u)$. We prove the following result.

Theorem 8.10. *Assume that $K < 0$ in Ω. Then, for all $\lambda > 0$, problem $(1)_\lambda$ has at least one classical solution.*

Next, we assume that $K > 0$ in $\overline{\Omega}$. In this case, the existence of a solution to $(1)_\lambda$ is closely related to the decay rate around its singularity. In this sense, we prove that problem $(1)_\lambda$ has no solution, provided that g has a "strong" singularity at the origin. More precisely, we have

Theorem 8.11. *Assume that $K > 0$ in $\overline{\Omega}$ and $\int_0^1 g(s)ds = +\infty$. Then problem $(1)_\lambda$ has no classical solutions.*

In the following result, assuming that $\int_0^1 g(s)ds < +\infty$, we show that problem $(1)_\lambda$ has at least one solution, provided that $\lambda > 0$ is large enough. More precisely, we prove

Theorem 8.12. *Assume that $K > 0$ in $\overline{\Omega}$ and $\int_0^1 g(s)ds < +\infty$. Then there exists $\lambda^* > 0$ such that problem $(1)_\lambda$ has at least one classical solution if $\lambda > \lambda^*$ and no solution exists if $\lambda < \lambda^*$.*

A very useful auxiliary result in the proofs of the above theorems is the following comparison principle that improves Lemma 6.3. Our proof uses some ideas from Shi and Yao [86], that go back to the pioneering work by Brezis and Kamin [14].

Lemma 8.13. *Let $\Psi : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ be a continuous function such that the mapping $(0,\infty) \ni s \longmapsto \dfrac{\Psi(x,s)}{s}$ is strictly decreasing at each $x \in \Omega$. Assume that there exists v, $w \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

(a) $\quad \Delta w + \Psi(x,w) \le 0 \le \Delta v + \Psi(x,v)$ *in* Ω;

(b) $\quad v, w > 0$ *in* Ω *and* $v \le w$ *on* $\partial\Omega$;

(c) $\quad \Delta v \in L^1(\Omega)$ *or* $\Delta w \in L^1(\Omega)$.

Then $v \le w$ in Ω.

Proof. We argue by contradiction and assume that $v \geq w$ is not true in Ω. Then, we can find $\varepsilon_0, \delta_0 > 0$ and a ball $B \subset\subset \Omega$ such that $v - w \ge \varepsilon_0$ in B and

$$\int_B vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)dx \ge \delta_0. \tag{8.202}$$

The case $\Delta v \in L^1(\Omega)$ was stated in Lemma 6.3. Let us assume now that $\Delta w \in L^1(\Omega)$ and set $M = \max\{1, \|\Delta w\|_{L^1(\Omega)}\}$, $\varepsilon = \min\{1, \varepsilon_0, 2^{-2}\delta_0/M\}$. Consider a nondecreasing function $\theta \in C^1(\mathbb{R})$ such that $\theta(t) = 0$, if $t \le 1/2$, $\theta(t) = 1$, if $t \ge 1$, and $\theta(t) \in (0,1)$ if $t \in (1/2, 1)$. Define

$$\theta_\varepsilon(t) = \theta\left(\frac{t}{\varepsilon}\right), \quad t \in \mathbb{R}.$$

Since $w \ge v$ on $\partial\Omega$, we can find a smooth subdomain $\Omega^* \subset\subset \Omega$ such that

$$B \subset \Omega^* \quad \text{and} \quad v - w < \frac{\varepsilon}{2} \quad \text{in} \quad \Omega \setminus \Omega^*.$$

Using the hypotheses (a) and (b) we deduce

$$\int_{\Omega^*} (w\Delta v - v\Delta w)\theta_\varepsilon(v - w)dx \ge \int_{\Omega^*} vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)\theta_\varepsilon(v - w)dx. \tag{8.203}$$

By (8.202) we have

$$\begin{aligned}\int_{\Omega^*} vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)\theta_\varepsilon(v-w)dx \quad &\ge \int_B vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)\theta_\varepsilon(v-w)dx \\ &= \int_B vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)dx \ge \delta_0.\end{aligned}$$

To raise a contradiction, we need only to prove that the left-hand side in (8.203) is smaller than δ_0. For this purpose, we define

$$\Theta_\varepsilon(t) = \int_0^t s\theta'_\varepsilon(s)ds, \quad t \in \mathbb{R}.$$

It is easy to see that

$$\Theta_\varepsilon(t) = 0, \ \text{ if } t < \frac{\varepsilon}{2} \quad \text{and} \quad 0 \le \Theta_\varepsilon(t) \le 2\varepsilon, \ \text{ for all } t \in \mathbb{R}. \tag{8.204}$$

Now, using the Green theorem, we evaluate the left-hand side of (8.203):

$$\begin{aligned}&\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx \\ = \ &\int_{\partial\Omega^*} w\theta_\varepsilon(v-w)\frac{\partial v}{\partial n}d\sigma - \int_{\Omega^*}(\nabla w \cdot \nabla v)\theta_\varepsilon(v-w)dx \\ &-\int_{\Omega^*} w\theta'_\varepsilon(v-w)\nabla v \cdot \nabla(v-w)dx - \int_{\partial\Omega^*} v\theta_\varepsilon(v-w)\frac{\partial w}{\partial n}d\sigma \\ &+\int_{\Omega^*}(\nabla w \cdot \nabla v)\theta_\varepsilon(v-w)dx + \int_{\Omega^*} v\theta'_\varepsilon(v-w)\nabla w \cdot \nabla(v-w)dx \\ = \ &\int_{\Omega^*}\theta'_\varepsilon(v-w)(v\nabla w - w\nabla v)\cdot\nabla(v-w)dx.\end{aligned}$$

The above relation can also be rewritten as

$$\begin{aligned}\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx = \quad &\int_{\Omega^*} w\theta'_\varepsilon(v-w)\nabla(w-v)\cdot\nabla(v-w)dx \\ &+\int_{\Omega^*}(v-w)\theta'_\varepsilon(v-w)\nabla w \cdot \nabla(v-w)dx.\end{aligned}$$

Since $\int_{\Omega^*} w\theta_\varepsilon'(v-w)\nabla(w-v)\cdot\nabla(v-w)dx \le 0$, the last equality yields

$$\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx \le \int_{\Omega^*}(v-w)\theta_\varepsilon'(v-w)\nabla w\cdot\nabla(v-w)dx,$$

that is,

$$\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx \le \int_{\Omega^*}\nabla w\cdot\nabla(\Theta_\varepsilon(v-w))dx.$$

Again by Green's first formula and by (8.204) we have

$$\begin{aligned}\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx &\le \int_{\partial\Omega^*}\Theta_\varepsilon(v-w)\frac{\partial v}{\partial n}d\sigma - \int_{\Omega^*}\Theta_\varepsilon(v-w)\Delta w dx\\ &\le -\int_{\Omega^*}\Theta_\varepsilon(v-w)\Delta w dx \le 2\varepsilon\int_{\Omega^*}|\Delta w|dx\\ &\le 2\varepsilon M < \frac{\delta_0}{2}.\end{aligned}$$

Thus, we have obtained a contradiction. Hence $v \le w$ in Ω and the proof of Lemma 8.13 is now complete. □

We are now ready to prove our main results.

Proof of Theorem 8.10. Fix $\lambda > 0$. Obviously, $\Psi(x,s) = \lambda f(x,s) - K(x)g(s)$ satisfies the hypotheses in Lemma 6.2 since $K < 0$ in Ω. Hence, there exists a solution $\overline{u}_\lambda$ of the problem

$$\begin{cases} -\Delta u = \lambda f(x,u) - K(x)g(u) & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases}$$

We observe that $\overline{u}_\lambda$ is a super-solution of problem $(1)_\lambda$. To find a sub-solution, let us denote

$$p(x) = \min\{\lambda f(x,1); -K(x)g(1)\},\quad x\in\overline{\Omega}.$$

Using the monotonicity of f and g, we observe that $p(x) \le \lambda f(x,s) - K(x)g(s)$ for all $(x,s) \in \Omega\times(0,\infty)$. We now consider the problem

$$\begin{cases} -\Delta v + |\nabla v|^a = p(x) & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega.\end{cases}\tag{8.205}$$

First, we observe that $v = 0$ is a sub-solution of (8.205) while w defined by

$$\begin{cases} -\Delta w = p(x) & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega,\end{cases}$$

is a super-solution. Since $p > 0$ in Ω we deduce that $w \geq 0$ in Ω. Thus, the problem (8.205) has at least one classical solution v. We claim that v is positive in Ω. Indeed, if v has a minimum in Ω, say at x_0, then $\nabla v(x_0) = 0$ and $\Delta v(x_0) \geq 0$. Therefore

$$0 \geq -\Delta v(x_0) + |\nabla v|^a(x_0) = p(x_0) > 0,$$

which is a contradiction. Hence $\min_{x\in\overline{\Omega}} v = \min_{x\in\partial\Omega} v = 0$, that is, $v > 0$ in Ω. Now $\underline{u}_\lambda = v$ is a sub-solution of $(1)_\lambda$ and we have

$$-\Delta \underline{u}_\lambda = p(x) \leq \lambda f(x, \overline{u}_\lambda) - K(x)g(\overline{u}_\lambda) = -\Delta \overline{u}_\lambda \quad \text{in } \Omega.$$

Since $\underline{u}_\lambda = \overline{u}_\lambda = 0$ on $\partial\Omega$, from the above relation we may conclude that $\underline{u}_\lambda \leq \overline{u}_\lambda$ in Ω and so, there exists at least one classical solution for $(1)_\lambda$. The proof of Theorem 8.10 is now complete. □

Proof of Theorem 8.11. We give a direct proof, without using any change of variable, as in Zhang [94]. Let us assume that there exists $\lambda > 0$ such that the problem $(1)_\lambda$ has a classical solution u_λ. By our hypotheses on f, we deduce by Lemma 6.2 that for all $\lambda > 0$ there exists $U_\lambda \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta U_\lambda = \lambda f(x, U_\lambda) & \text{in } \Omega, \\ U_\lambda > 0 & \text{in } \Omega, \\ U_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.206}$$

Moreover, there exist $c_1, c_2 > 0$ such that

$$c_1 \operatorname{dist}(x, \partial\Omega) \leq U_\lambda(x) \leq c_2 \operatorname{dist}(x, \partial\Omega) \quad \text{for all } x \in \Omega. \tag{8.207}$$

Consider the perturbed problem

$$\begin{cases} -\Delta u + K_* g(u + \varepsilon) = \lambda f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.208}$$

where $K_* = \min_{x\in\overline{\Omega}} K(x) > 0$. It is clear that u_λ and U_λ are respectively sub and super-solution of (8.208). Furthermore, we have

$$\Delta U_\lambda + f(x, U_\lambda) \leq 0 \leq \Delta u_\lambda + f(x, u_\lambda) \quad \text{in } \Omega,$$

$$U_\lambda, u_\lambda > 0 \quad \text{in } \Omega,$$

$$U_\lambda = u_\lambda = 0 \quad \text{on } \partial\Omega,$$

$$\Delta U_\lambda \in L^1(\Omega) \;\; (\text{ since } \; U_\lambda \in C^2(\overline{\Omega})).$$

In view of Lemma 8.13 we get $u_\lambda \le U_\lambda$ in Ω. Thus, a standard bootstrap argument (see Gilbarg and Trudinger [55]) implies that there exists a solution $u_\varepsilon \in C^2(\overline{\Omega})$ of (8.208) such that

$$u_\lambda \le u_\varepsilon \le U_\lambda \quad \text{in } \Omega.$$

Integrating in (8.208) we obtain

$$-\int_\Omega \Delta u_\varepsilon dx + K_* \int_\Omega g(u_\varepsilon + \varepsilon)dx = \lambda \int_\Omega f(x, u_\varepsilon)dx.$$

Hence

$$-\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n} ds + K_* \int_\Omega g(u_\varepsilon + \varepsilon)dx \le M, \tag{8.209}$$

where $M > 0$ is a positive constant. Taking into account the fact that $\dfrac{\partial u_\varepsilon}{\partial n} \le 0$ on $\partial\Omega$, relation (8.209) yields $K_* \displaystyle\int_\Omega g(u_\varepsilon + \varepsilon)dx \le M$. Since $u_\varepsilon \le U_\lambda$ in $\overline{\Omega}$, from the last inequality we can conclude that $\displaystyle\int_\Omega g(U_\lambda + \varepsilon)dx \le C$, for some $C > 0$. Thus, for any compact subset $\omega \subset\subset \Omega$ we have

$$\int_\omega g(U_\lambda + \varepsilon)dx \le C.$$

Letting $\varepsilon \to 0^+$, the above relation produces $\displaystyle\int_\omega g(U_\lambda)dx \le C$. Therefore

$$\int_\Omega g(U_\lambda)dx \le C. \tag{8.210}$$

On the other hand, using (8.207) and the hypothesis $\int_0^1 g(s)ds = +\infty$, it follows

$$\int_\Omega g(U_\lambda)dx \ge \int_\Omega g(c_2 \text{dist}\,(x, \partial\Omega))dx = +\infty,$$

which contradicts (8.210). Hence, $(1)_\lambda$ has no classical solutions and the proof of Theorem 8.11 is now complete. □

Proof of Theorem 8.12. Fix $\lambda > 0$. We first note that U_λ defined in (8.206) is a super-solution of $(1)_\lambda$. We now focuss on finding a sub-solution $\underline{u}_\lambda$ such that $\underline{u}_\lambda \le U_\lambda$ in Ω.

Let $h : [0, \infty) \to [0, \infty)$ be such that

$$\begin{cases} h''(t) = g(h(t)), & \text{for all } t > 0, \\ h > 0, & \text{in } (0, \infty), \\ h(0) = 0. \end{cases} \tag{8.211}$$

Multiplying by h' in (8.211) and then integrating over $[s, t]$ we have

$$(h')^2(t) - (h')^2(s) = 2\int_{h(s)}^{h(t)} g(\tau)d\tau, \quad \text{for all } t > s > 0.$$

Since $\int_0^1 g(\tau)d\tau < \infty$, from the above equality we deduce that we can extend h' in origin by taking $h'(0) = 0$ and so $h \in C^2(0,\infty) \cap C^1[0,\infty)$. Taking into account the fact that h' is increasing and h'' is decreasing on $(0,\infty)$, the mean value theorem implies that

$$\frac{h'(t)}{t} = \frac{h'(t) - h'(0)}{t - 0} \geq h''(t), \quad \text{for all } t > 0.$$

Hence $h'(t) \geq th''(t)$, for all $t > 0$. Integrating in the last inequality we get

$$th'(t) \leq 2h(t), \quad \text{for all } t > 0. \tag{8.212}$$

Let ϕ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

It is well known that $\phi_1 \in C^2(\overline{\Omega})$. Furthermore, by Hopf's maximum principle there exist $\delta > 0$ and $\Omega_0 \subset\subset \Omega$ such that $|\nabla\phi_1| \geq \delta$ in $\Omega\backslash\Omega_0$. Let $M = \max\{1, 2K^*\delta^{-2}\}$, where $K^* = \max_{x\in\overline{\Omega}} K(x)$. Since

$$\lim_{\text{dist}\,(x,\partial\Omega)\to 0^+} \Big\{ -K^* g(h(\phi_1)) + M^a (h')^a(\phi_1)|\nabla\phi_1|^a \Big\} = -\infty,$$

by letting Ω_0 close enough to the boundary of Ω we can assume that

$$-K^* g(h(\phi_1)) + M^a(h')^a(\phi_1)|\nabla\phi_1|^a < 0 \quad \text{in } \Omega\setminus\Omega_0. \tag{8.213}$$

We now are able to show that $\underline{u}_\lambda = Mh(\phi_1)$ is a sub-solution of $(1)_\lambda$ provided $\lambda > 0$ is sufficiently large. Using the monotonicity of g and (8.212) we have

$$\begin{aligned} &-\Delta\underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla\underline{u}_\lambda|^a = \\ &\leq -Mg(h(\phi_1))|\nabla\phi_1|^2 + \lambda_1 Mh'(\phi_1)\phi_1 + K^* g(Mh(\phi_1)) + M^a(h')^a(\phi_1)|\nabla\phi_1|^a \\ &\leq g(h(\phi_1))(K^* - M|\nabla\phi_1|^2) + \lambda_1 Mh'(\phi_1)\phi_1 + M^a(h')^a(\phi_1)|\nabla\phi_1|^a \\ &\leq g(h(\phi_1))(K^* - M|\nabla\phi_1|^2) + 2\lambda_1 Mh(\phi_1) + M^a(h')^a(\phi_1)|\nabla\phi_1|^a. \end{aligned} \tag{8.214}$$

The definition of M and (8.213) yield

$$-\Delta\underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla\underline{u}_\lambda|^a \leq 2\lambda_1 Mh(\phi_1) = 2\lambda_1\underline{u}_\lambda \quad \text{in } \Omega\setminus\Omega_0. \tag{8.215}$$

Let us choose $\lambda > 0$ such that

$$\lambda\frac{\min_{x\in\overline{\Omega}_0} f(x, Mh(\|\phi_1\|_\infty))}{M\|\phi_1\|_\infty} \geq 2\lambda_1. \tag{8.216}$$

Then, by virtue of the assumptions on f and using (8.216), we have

$$\lambda\frac{f(x,\underline{u}_\lambda)}{\underline{u}_\lambda} \geq \lambda\frac{f(x, Mh(\|\phi_1\|_\infty))}{M\|\phi_1\|_\infty} \geq 2\lambda_1 \quad \text{in } \Omega\setminus\Omega_0.$$

The last inequality combined with (8.215) yield

$$-\Delta \underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \leq 2\lambda_1 \underline{u}_\lambda \leq \lambda f(x, \underline{u}_\lambda) \quad \text{in } \Omega \setminus \Omega_0. \tag{8.217}$$

On the other hand, from (8.214) we obtain

$$-\Delta \underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \leq K^* g(h(\phi_1)) + 2\lambda_1 M h(\phi_1) + M^a (h')^a(\phi_1)|\nabla \phi_1|^a \quad \text{in } \Omega_0. \tag{8.218}$$

Since $\phi_1 > 0$ in $\overline{\Omega}_0$ and f is positive on $\overline{\Omega}_0 \times (0, \infty)$, we may choose $\lambda > 0$ such that

$$\lambda \min_{x \in \overline{\Omega}_0} f(x, Mh(\phi_1)) \geq \max_{x \in \overline{\Omega}_0} \Big\{ K^* g(h(\phi_1)) + 2\lambda_1 M h(\phi_1) + M^a (h')^a(\phi_1)|\nabla \phi_1|^a \Big\}. \tag{8.219}$$

From (8.218) and (8.219) we deduce

$$-\Delta \underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \leq \lambda f(x, \underline{u}_\lambda) \quad \text{in } \Omega_0. \tag{8.220}$$

Now, (8.217) together with (8.220) shows that $\underline{u}_\lambda = Mh(\phi_1)$ is a sub-solution of $(1)_\lambda$ provided $\lambda > 0$ satisfy (8.216) and (8.219). With the same arguments as in the proof of Theorem 8.11 and using Lemma 8.13, one can prove that $\underline{u}_\lambda \leq U_\lambda$ in Ω. By a standard bootstrap argument (see Gilbarg and Trudinger [55]) we obtain a classical solution u_λ such that $\underline{u}_\lambda \leq u_\lambda \leq U_\lambda$ in Ω.

We have proved that $(1)_\lambda$ has at least one classical solution when $\lambda > 0$ is large. Set

$$A = \{\lambda > 0;\ \text{problem } (1)_\lambda \text{ has at least one classical solution}\}.$$

From the above arguments we deduce that A is nonempty. Let $\lambda^* = \inf A$. We claim that if $\lambda \in A$, then $(\lambda, +\infty) \subseteq A$. To this aim, let $\lambda_1 \in A$ and $\lambda_2 > \lambda_1$. If u_{λ_1} is a solution of $(1)_{\lambda_1}$, then u_{λ_1} is a sub-solution for $(1)_{\lambda_2}$ while U_{λ_2} defined in (8.206) for $\lambda = \lambda_2$ is a super-solution. Moreover, we have

$$\Delta U_{\lambda_2} + \lambda_2 f(x, U_{\lambda_2}) \leq 0 \leq \Delta u_{\lambda_1} + \lambda_2 f(x, u_{\lambda_1}) \quad \text{in } \Omega,$$

$$U_{\lambda_2}, u_{\lambda_1} > 0 \quad \text{in } \Omega,$$

$$U_{\lambda_2} = u_{\lambda_1} = 0 \quad \text{on } \partial\Omega$$

$$\Delta U_{\lambda_2} \in L^1(\Omega).$$

Again by Lemma 8.13 we get $u_{\lambda_1} \leq U_{\lambda_2}$ in Ω. Therefore, the problem $(1)_{\lambda_2}$ has at least one classical solution. This proves the claim. Since $\lambda \in A$ was arbitrary chosen, we conclude that $(\lambda^*, +\infty) \subset A$.

To end the proof, it suffices to show that $\lambda^* > 0$. In that sense, we will prove that there exists $\lambda > 0$ small enough such that $(1)_\lambda$ has no classical solutions. We first remark that

$$\lim_{s \to 0^+} (f(x, s) - K(x)g(s)) = -\infty \quad \text{uniformly for } x \in \Omega.$$

Hence, there exists $c > 0$ such that

$$f(x,s) - K(x)g(s) < 0, \qquad \text{for all } (x,s) \in \Omega \times (0,c). \tag{8.221}$$

On the other hand, the assumptions on f yield

$$\frac{f(x,s) - K(x)g(s)}{s} \leq \frac{f(x,s)}{s} \leq \frac{f(x,c)}{c} \quad \text{for all } (x,s) \in \Omega \times [c,+\infty). \tag{8.222}$$

Let $m = \max_{x\in\overline{\Omega}} \frac{f(x,c)}{c}$. Combining (8.221) with (8.222) we find

$$f(x,s) - K(x)g(s) < ms, \qquad \text{for all } (x,s) \in \Omega \times (0,+\infty). \tag{8.223}$$

Set $\lambda_0 = \min\{1, \lambda_1/2m\}$. We show that problem $(1)_{\lambda_0}$ has no classical solution. Indeed, if u_0 would be a classical solution of $(1)_{\lambda_0}$, then, according to (8.223), u_0 is a sub-solution of

$$\begin{cases} -\Delta u = \dfrac{\lambda_1}{2} u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.224}$$

Obviously, ϕ_1 is a super-solution of (8.224) and by Lemma 8.13 we get $u_0 \leq \phi_1$ in Ω. Thus, by standard elliptic arguments, problem (8.224) has a solution $u \in C^2(\overline{\Omega})$. Multiplying by ϕ_1 in (8.224) and then integrating over Ω we have

$$-\int_\Omega \phi_1 \Delta u dx = \frac{\lambda_1}{2} \int_\Omega u\phi_1 dx,$$

that is,

$$-\int_\Omega u \Delta \phi_1 dx = \frac{\lambda_1}{2} \int_\Omega u\phi_1 dx.$$

The above equality yields $\int_\Omega u\phi_1 dx = 0$, which is clearly a contradiction, since u and ϕ_1 are positive in Ω. If follows that problem $(1)_{\lambda_0}$ has no classical solutions which means that $\lambda^* > 0$. This completes the proof of Theorem 8.12. □

References

[1] N. E. Alaa and M. Pierre, Weak solutions of some quasilinear elliptic equations with data measures, *SIAM J. Math. Anal.* **24** (1993), 23-35.

[2] S. Alama and G. Tarantello, On the solvability of a semilinear elliptic equation via an associated eigenvalue problem, *Math. Z.*, **221** (1996), 467-493.

[3] H. Amann, Existence and multiplicity theorems for semilinear elliptic boundary value problems, *Math. Z.* **150** (1976), 567-597.

[4] R. Aris, *The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts*, Clarendon Press, Oxford, 1975.

[5] C. Bandle, Asymptotic behaviour of large solutions of quasilinear elliptic problems, *Z. Angew. Math. Phys.* **54** (2003), 731-738.

[6] C. Bandle and M. Essèn, On the solutions of quasilinear elliptic problems with boundary blow-up, in *Partial differential equations of elliptic type* (Cortona, 1992), Sympos. Math. **35**, Cambridge Univ. Press, Cambridge, 1994, p. 93-111.

[7] C. Bandle and E. Giarrusso, Boundary blow-up for semilinear elliptic equations with nonlinear gradient terms, *Advances in Differential Equations* **1** (1996), 133-150.

[8] C. Bandle and M. Marcus, 'Large' solutions of semilinear elliptic equations: Existence, uniqueness, and asymptotic behaviour, *J. Anal. Math.* **58** (1992), 9-24.

[9] C. Bandle and M. Marcus, Dependence of blowup rate of large solutions of semilinear elliptic equations on the curvature of the boundary, *Complex Variables, Theory Appl.* **49** (2004), 555-570.

[10] G. Barles, G. Díaz, and J. I. Díaz, Uniqueness and continuum of foliated solutions for a quasilinear elliptic equation with a non lipschitz nonlinearity, *Comm. Partial Differential Equations* **17** (1992), 1037-1050.

[11] P. Bénilan, H. Brezis, and M. Crandall, A semilinear equation in $L^1(\mathbb{R}^N)$, *Ann. Scuola Norm. Sup. Pisa* **4** (1975), 523-555.

[12] L. Bieberbach, $\Delta u = e^u$ und die automorphen Funktionen, *Math. Ann.* **77** (1916), 173-212.

[13] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, 1987.

[14] H. Brezis and S. Kamin, Sublinear elliptic equations in $\mathbb{R}^N$, *Manuscripta Math.* **74** (1992), 87-106.

[15] H. Brezis and L. Oswald, Remarks on sublinear elliptic equations, *Nonlinear Anal., T.M.A.* **10** (1986), 55-64.

[16] L. Caffarelli, R. Hardt, and L. Simon, Minimal surfaces with isolated singularities, *Manuscripta Math.* **48** (1984), 1-18.

[17] A. Callegari and A. Nachman, Some singular nonlinear equations arising in boundary layer theory, *J. Math. Anal. Appl.* **64** (1978), 96-105.

[18] A. Callegari and A. Nachman, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, *SIAM J. Appl. Math.* **38** (1980), 275-281.

[19] H. Chen, On a singular nonlinear elliptic equation, *Nonlinear Anal., T.M.A.* **29** (1997), 337-345.

[20] M. Chipot, *Elements of Nonlinear Analysis*, Birkhäuser Advanced Texts, Birkhäuser Verlag, 2000.

[21] Y. S. Choi, A. C. Lazer, and P. J. McKenna, Some remarks on a singular elliptic boundary value problem, *Nonlinear Anal., T.M.A.* **3** (1998), 305-314.

[22] Y. Choquet-Bruhat and J. Leray, Sur le problème de Dirichlet quasilinéaire d'ordre 2, *C. R. Acad. Sci. Paris, Ser. A* **274** (1972), 81-85.

[23] F.-C. Cîrstea, M. Ghergu, and V. Rădulescu, Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane-Emden-Fowler type, *J. Math. Pures Appl.* **84** (2005), 493-508.

[24] F.-C. Cîrstea and V. Rădulescu, Blow-up solutions for semilinear elliptic problems, *Nonlinear Analysis, T.M.A.* **48** (2002), 541-554.

[25] F.-C. Cîrstea and V. Rădulescu, Uniqueness of the blow-up boundary solution of logistic equations with absorption, *C. R. Acad. Sci. Paris, Ser. I* **335** (2002), 447-452.

[26] F.-C. Cîrstea and V. Rădulescu, Entire solutions blowing-up at infinity for semilinear elliptic systems, *J. Math. Pures Appliquées* **81** (2002), 827-846.

[27] F.-C. Cîrstea and V. Rădulescu, Existence and uniqueness of blow-up solutions for a class of logistic equations, *Commun. Contemp. Math.* **4** (2002), 559-586.

[28] F.-C. Cîrstea and V. Rădulescu, Asymptotics for the blow-up boundary solution of the logistic equation with absorption, *C. R. Acad. Sci. Paris, Ser. I* **336** (2003), 231-236.

[29] F.-C. Cîrstea and V. Rădulescu, Solutions with boundary blow-up for a class of nonlinear elliptic problems, *Houston J. Math.* **29** (2003), 821-829.

[30] F.-C. Cîrstea and V. Rădulescu, Extremal singular solutions for degenerate logistic-type equations in anisotropic media, *C. R. Acad. Sci. Paris, Ser. I* **339** (2004), 119-124.

[31] F.-C. Cîrstea and V. Rădulescu, Nonlinear problems with boundary blow-up: a Karamata regular variation theory approach", *Asymptotic Analysis*, in press.

[32] F.-C. Cîrstea and V. Rădulescu, Boundary blow-up in nonlinear elliptic equations of Bieberbach–Rademacher type", *Transactions Amer. Math. Soc.*, in press.

[33] D. S. Cohen and H. B. Keller, Some positive problems suggested by nonlinear heat generators, *J. Math. Mech.* **16** (1967), 1361-1376.

[34] M. Coclite and G. Palmieri, On a singular nonlinear Dirichlet problem, *Commun. Partial Diff. Equations* **14** (1989), 1315-1327.

[35] M. G. Crandall, P. H. Rabinowitz, and L. Tartar, On a Dirichlet problem with a singular nonlinearity, *Commun. Partial Diff. Equations* **2** (1977), 193-222.

[36] R. Dalmasso, Solutions d'équations elliptiques semi-linéaires singulières, *Ann. Mat. Pura Appl.* **153** (1989), 191-201.

[37] P. G. de Gennes, Wetting: statics and dynamics, *Review of Modern Physics* **57** (1985), 827x-863.

[38] J. I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries. Vol. I. Elliptic Equations*, Research Notes in Mathematics, vol. 106, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[39] J. I. Díaz, J. M. Morel, and L. Oswald, An elliptic equation with singular nonlinearity, *Comm. Partial Differential Equations* **12** (1987), 1333-1344.

[40] Y. Du and Q. Huang, Blow-up solutions for a class of semilinear elliptic and parabolic equations, *SIAM J. Math. Anal.* **31** (1999), 1-18.

[41] L. Dupaigne, M. Ghergu, and V. Rădulescu, Singular elliptic problems with convection term in anisotropic media, in preparation.

[42] W. Fulks and J. S. Maybee, A singular nonlinear equation, *Osaka J. Math.* **12** (1960), 1-19.

[43] V. Galaktionov and J.-L. Vázquez, The problem of blow-up in nonlinear parabolic equations, *Discrete Contin. Dynam. Systems, Ser. A* **8** (2002), 399-433.

[44] J. García-Melián, R. Letelier-Albornoz, and J. Sabina de Lis, Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up, *Proc. Amer. Math. Soc.* **129** (2001), 3593-3602.

[45] M. Ghergu, C. Niculescu, and V. Rădulescu, Explosive solutions of elliptic equations with absorption and nonlinear gradient term, *Proc. Indian Acad. Sci. (Math. Sci.)* **112** (2002), 441-451.

[46] M. Ghergu and V. Rădulescu, Bifurcation and asymptotics for the Lane-Emden-Fowler equation, *C. R. Acad. Sci. Paris, Ser. I* **337** (2003), 259-264.

[47] M. Ghergu and V. Rădulescu, Sublinear singular elliptic problems with two parameters, *J. Differential Equations* **195** (2003), 520-536.

[48] M. Ghergu and V. Rădulescu, Explosive solutions of semilinear elliptic systems with gradient term, *RACSAM Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **97** (2003), 437-445.

[49] M. Ghergu and V. Rădulescu, Existence and non-existence of entire solutions to the logistic differential equation, *Abstract and Applied Analysis* **17** (2003), 995-1003.

[50] M. Ghergu and V. Rădulescu, Bifurcation and asymptotics for the Lane-Emden-Fowler equation, *C. R. Acad. Sci. Paris, Ser. I* **337** (2003), 259-264.

[51] M. Ghergu and V. Rădulescu, Nonradial blow-up solutions of sublinear elliptic equations with gradient term, *Commun. Pure Appl. Anal.* **3** (2004), 465-474.

[52] M. Ghergu and V. Rădulescu, Bifurcation for a class of singular elliptic problems with quadratic convection term, *C. R. Acad. Sci. Paris, Ser. I* **338** (2004), 831-836.

[53] M. Ghergu and V. Rădulescu, Multiparameter bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with a convection term, *Proc. Royal Soc. Edinburgh Sect. A* **135** (2005), 61-84.

[54] M. Ghergu and V. Rădulescu, On a class of sublinear singular elliptic problems with convection term, *J. Math. Anal. Appl.* **311** (2005), 635-646.

[55] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer Verlag, Berlin, 1983.

[56] S. M. Gomes, On a singular nonlinear elliptic problem, *SIAM J. Math. Anal.* **17** (1986) 1359-1369.

[57] C. Gui and F. H. Lin, Regularity of an elliptic problem with a singular nonlinearity, *Proc. Royal Soc. Edinburgh Sect. A* **123** (1993), 1021-1029.

[58] Y. Haitao, Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem, *J. Differential Equations* **189** (2003), 487-512.

[59] J. Hernández, F. J. Mancebo, and J. M. Vega, On the linearization of some singular nonlinear elliptic problems and applications, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **19** (2002), 777-813.

[60] J. Hernández, F. J. Mancebo, and J. M. Vega, Nonlinear singular elliptic problems: recent results and open problems, *Preprint*, 2005.

[61] L. Hörmander, *The Analysis of Linear Partial Differential Operators I.* Springer Verlag, Berlin, 1983.

[62] J. Kazdan and F. W. Warner, Remarks on some quasilinear elliptic equations, *Comm. Pure Appl. Math.* **28** (1975), 567-597.

[63] J. B. Keller, On solutions of $\Delta u = f(u)$, *Comm. Pure Appl. Math.* **10** (1957), 503-510.

[64] T. Kusano and C. A. Swanson, Entire positive solutions of singular elliptic equations, *Japan J. Math.* **11** (1985), 145-155.

[65] A. V. Lair and A. W. Shaker, Existence of entire large positive solutions of semilinear elliptic systems, *J. Differential Equations* **164** (2000), 380-394.

[66] A. V. Lair and A. W. Wood, Large solutions of semilinear elliptic equations with nonlinear gradient terms, *Internat. J. Math. Math. Sci.* **22** (1999), 869-883.

[67] J. M. Lasry and P.-L. Lions, Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints; the model problem, *Math. Ann.* **283** (1989), 583-630.

[68] A. C. Lazer and P. J. McKenna, On a singular nonlinear elliptic boundary value problem, *Proc. Amer. Math. Soc.* **3** (1991), 720-730.

[69] A. C. Lazer and P. J. McKenna, On a problem of Bieberbach and Rademacher, *Nonlinear Anal., T.M.A.* **21** (1993), 327-335.

[70] A. C. Lazer and P. J. McKenna, Asymptotic behaviour of solutions of boundary blowup problems, *Differential Integral Equations* **7** (1994), 1001-1019.

[71] J. F. Le Gall, A path-valued Markov process and its connections with partial differential equations, in *First European Congress of Mathematics*, Vol. II (Paris, 1992), 185-212, Progr. Math., 120, Birkhäuser Verlag, Basel, 1994.

[72] J. Karamata, Sur un mode de croissance régulière de fonctions. Théorèmes fondamentaux, *Bull. Soc. Math. France* **61** (1933), 55-62.

[73] C. Loewner and L. Nirenberg, Partial differential equations invariant under conformal or projective transformations, in *Contribution to Analysis*, Academic Press, New York, 1974, p. 245-272.

[74] M. Marcus, On solutions with blow-up at the boundary for a class of semilinear elliptic equations, in *Developments in Partial Differential Equations and Applications to Mathematical Physics* (G. Buttazzo *et al.*, Eds.), Plenum Press, New York (1992), 65-77.

[75] M. Marcus and L. Véron, Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **14** (1997), 237-274.

[76] M. Marcus and L. Véron, Existence and uniqueness results for large solutions of general nonlinear elliptic equations, *J. Evol. Equations* **3** (2003), 637-652.

[77] A. Meadows, Stable and singular solutions of the equation $\Delta u = 1/u$, *Indiana Univ. Math. J.* **53** (2004), 1681-1703.

[78] P. Mironescu and V. Rădulescu, The study of a bifurcation problem associated to an asymptotically linear function, *Nonlinear Anal., T.M.A.* **26** (1996), 857-875.

[79] R. Osserman, On the inequality $\Delta u \geq f(u)$, *Pacific J. Math.* **7** (1957), 1641-1647.

[80] M. del Pino, A global estimate for the gradient in a singular elliptic boundary value problem, *Proc. Roy. Soc. Edinburgh Sect. A* **122** (1992), 341-352.

[81] P. Quittner, Blow-up for semilinear parabolic equations with a gradient term, *Math. Meth. Appl. Sci.* **14** (1991), 413-417.

[82] H. Rademacher, Einige besondere Probleme der partiellen Differentialgleichungen, in *Die Differential und Integralgleichungen der Mechanik und Physik I*, 2nd. edition, (P. Frank und R. von Mises, eds.), Rosenberg, New York, 1943, p. 838-845.

[83] A. Ratto, M. Rigoli, and L. Véron, Scalar curvature and conformal deformation of hyperbolic space, *J. Funct. Anal.* **121** (1994), 15-77.

[84] V. Rădulescu, Bifurcation and asymptotics for elliptic problems with singular nonlinearity, in *Studies in Nonlinear Partial Differential Equations: In Honor of Haim Brezis, Fifth European Conference on Elliptic and Parabolic Problems: A special tribute to the work of Haim Brezis*, Gaeta, Italy, May 30–June 3, 2004 (C. Bandle, H. Berestycki, B. Brighi, A. Brillard, M. Chipot, J.-M. Coron, C. Sbordone, I. Shafrir, V. Valente, G. Vergara Caffarelli, Eds.), Birkhäuser Verlag, 2005, pp. 349-362.

[85] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics 508, Springer Verlag, Berlin Heidelberg, 1976.

[86] J. Shi and M. Yao, On a singular nonlinear semilinear elliptic problem, *Proc. Royal Soc. Edinburgh, Sect. A* **128** (1998), 1389-1401.

[87] J. Shi and M. Yao, Positive solutions for elliptic equations with singular nonlinearity, *Electronic Journal of Differential Equations* **4** (2005), 1-11.

[88] C. A. Stuart, Existence and approximation of solutions of nonlinear elliptic equations, *Math. Z.* **147** (1976), 53-63.

[89] C. A. Stuart, Self-trapping of an electromagnetic field and bifurcation from the essential spectrum, *Arch. Rational Mech. Anal.* **113** (1991), 65-96.

[90] C. A. Stuart and H.–S. Zhou, A variational problem related to self-trapping of an electromagnetic field, *Math. Methods Appl. Sci.* **19** (1996), 1397-1407.

[91] L. Véron, *Singularities of Solutions of Second Order Quasilinear Equations*, Pitman Res. Notes Math. Ser., Vol. 353, Longman, Harlow, 1996.

[92] J. S. W. Wong, On the generalized Emden-Fowler equation, *SIAM Rev.* **17** (1975), 339-360.

[93] Z. Zhang, On a Dirichlet problem with a singular nonlinearity, *J. Math. Anal. Appl.* **194** (1995), 103-113.

[94] Z. Zhang, Nonexistence of positive classical solutions of a singular nonlinear Dirichlet problem with a convection term, *Nonlinear Anal., T.M.A.* **8** (1996), 957-961.

[95] Z. Zhang and J. Yu, On a singular nonlinear Dirichlet problem with a convection term, *SIAM J. Math. Anal.* **4** (2000), 916-927.