

# Cooling process for inelastic Boltzmann equations for hard spheres, Part I: The Cauchy problem

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## Abstract

We develop the Cauchy theory of the spatially homogeneous inelastic Boltzmann equation for hard spheres, for a general form of collision rate which includes in particular variable restitution coefficients depending on the kinetic energy and the relative velocity as well as the sticky particles model. We prove (local in time) non-concentration estimates in Orlicz spaces, from which we deduce weak stability and existence theorem. Strong stability together with uniqueness and instantaneous appearance of exponential moments are proved under additional smoothness assumption on the initial datum, for a restricted class of collision rates. Concerning the long-time behaviour, we give conditions for the cooling process to occur or not in finite time.

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# 1 Introduction and main results

In this paper we address the Cauchy problem for the spatially homogeneous Boltzmann equation modelling the dynamic of a homogeneous system of inelastic hard spheres which interact only through binary collisions. More precisely, describing the gas by the probability density  $f(t, v) \geq 0$  of particles with velocity  $v \in \mathbb{R}^N$  ( $N \geq 2$ ) at time  $t \geq 0$ , we study the existence, uniqueness and the qualitative behaviour of solutions to the Boltzmann equation for inelastic collision

$$(1.1) \quad \frac{\partial f}{\partial t} = Q(f, f) \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^N,$$

$$(1.2) \quad f(0, \cdot) = f_{\text{in}} \quad \text{in} \quad \mathbb{R}^N.$$

The use of Boltzmann inelastic hard spheres-like models to describe dilute, rapid flows of granular media started with the seminal physics paper [22], and a huge physics literature has developed in the last twenty years. The study of granular systems in such regime is motivated by their unexpected physical behavior (with the phenomena of *collapse* –or “*cooling effect*”– at the kinetic level and *clustering* at the hydrodynamical level), their use to derive hydrodynamical equations for granular fluids, and their applications.

From the mathematical viewpoint, works on the Cauchy problem for Boltzmann models have been first restricted to the so-called *inelastic Maxwell molecules model*, where, in a similar way to the Maxwell model in the elastic framework, the collision rate is independent on the relative velocity. Existence, uniqueness of solutions

and description of the asymptotic cooling has been obtained in [7] for the *inelastic Maxwell molecules model* with constant normal restitution coefficients as well as with some cases of normal restitution coefficients depending on the kinetic energy of the solution. More precise properties of the solutions, such as their convergence to self-similarity, have also been investigated and we refer to our companion paper [27] for more details and more references on this issue. At least in the spatially homogeneous setting, the *inelastic Maxwell molecules model* seems well understood now. The *Maxwell molecules model* is important because of its analytic simplifications (with regards to the hard sphere model) allowing to use powerful Fourier transform tools as introduced by Bobylev (see for instance [5]) for the *elastic Maxwell molecules Boltzmann equation*. Another simplification which has lead to interesting results is the restriction to one-dimensional models (in space and velocity), where, on the contrary to the elastic case, the collision operator has a non-trivial outcome. These models have been considered in [3, 36, 4] for some cases of normal restitution coefficients possibly depending on the relative velocity.

It is possible to modify the collision operator of the *inelastic Maxwell molecules model* by a multiplication by a function of the kinetic energy in order to restore its dimensional homogeneity (see [7]) and thus the rate of cooling. Physically the derivation of this model amounts to replace the collision rate by a mean value independent on the relative velocity, starting from the *inelastic hard spheres model*, and the resulting approximation is named *pseudo-Maxwell molecules* in [7]. However, fine properties of the distribution (such as the behavior of the overpopulated tails or the self-similar solutions) are broken or modified by that approximation with respect to the *inelastic hard spheres model*. The recent papers [18, 8] have studied the case of inelastic hard spheres with constant normal restitution coefficients in any dimension and in various regimes: in particular in a thermal bath, *i.e.*, when a heat source term is added to the equation, and in the self-similar variables of the free regime. Existence and smoothness of solutions to the Cauchy problem and to the associated stationary problem are obtained in [18] for the thermal bath regime, while precise estimates on the tails of the stationary solutions (assuming their existence) for various regimes (including the two ones above-mentioned) are exhibited in [8].

In the present work, we shall construct solutions to the freely cooling Boltzmann equation for *inelastic hard spheres* in any dimension  $N \geq 2$  and for a general framework of distributions of inelasticity (defined by a measure on the set of all possible post-collisional velocities), which covers in particular *variable normal restitution coefficients* possibly depending on the relative velocity and the kinetic energy of the solution. It includes the cases of *visco-elastic hard spheres model* (see [9]) as well as the case of *sticky particles model*. Our framework enables to consider interesting physical features, such as elasticity increasing when the relative velocity or the temperature decrease (*“normal” granular media*) or the opposite phenomenon (*“anomalous” granular media*). We refer to [7, 36, 17, 9] and the references therein for a physical motivation. Let us emphasize that these solutions are new even in the case of a constant normal restitution coefficient as considered in [18, 8]. We also

discuss the uniqueness of solutions, the instantaneous appearance of exponential moments and various conditions on the collisions rate for the collapse to occur or not in finite time. A second part of this work [27] will be concerned with the existence of self-similar solutions and the tail behavior of the distribution. In a third part [28], we shall prove the uniqueness and the asymptotic stability of these self-similar solutions for a small inelasticity.

From the viewpoint of *mathematical tools*, our main new contributions can be summarized as follows:

(i) A generalization of the propagation of the  $L^p$ -norm of the solution for the elastic Boltzmann equation based on Young's inequality as introduced in [12] (see also [30, 16] where similar ideas are used for a different model), into a result of propagation of Orlicz norms for inelastic (and elastic) Boltzmann equations. This *a priori* estimate is used in order to prove the existence of solutions to the inelastic Boltzmann equation with energy dependent inelasticity. Let us emphasize that it also gives an alternative proof of existence of solution for the elastic Boltzmann equation with initial datum having only finite mass and kinetic energy (but possibly infinite entropy).

(ii) A generalization of the DiBlasio uniqueness Theorem for the elastic hard spheres Boltzmann equation (see [10, 20, 39, 31]) and for the inelastic hard spheres Boltzmann equation with constant normal restitution coefficients (see [18, 17]), to the inelastic hard spheres Boltzmann equation with energy dependent normal restitution coefficients (see also [13] where similar tools are developed).

For points (i) and (ii), one of the main ideas of the proof is an appropriate use of the change of variables  $v_* \rightarrow v'$  (for fixed  $(v, \sigma)$ ) and  $v \rightarrow v'$  (for fixed  $(v_*, \sigma)$ ) in the spirit of the proof of the so-called "cancelation lemma" introduced in [37] (see also [1]).

(iii) An improvement of the result of propagation of exponential moments for the elastic Boltzmann equation [6] and for the inelastic Boltzmann equation [8], into a result of instantaneous appearance of exponential moments. This is obtained by combining estimates from [8] together with a simple o.d.e. argument introduced in the context of the Boltzmann equation in [40].

Before we explain our results and methods in details, let us introduce the problem.

## 1.1 A general framework for the collision operator

We denote by  $B$  the rate of occurrence of collisions of two particles with *pre-collisional velocities*  $\{v, v_*\}$  which gives rise to post-collisional velocities  $\{v', v'_*\}$ . The collision may be schematically written

$$(1.3) \quad \{v\} + \{v_*\} \xrightarrow{B} \{v'\} + \{v'_*\} \quad \text{with} \quad \begin{cases} v' + v'_* = v + v_* \\ |v'|^2 + |v'_*|^2 \leq |v|^2 + |v_*|^2. \end{cases}$$

More precisely, for any fixed *pre-collisional velocities*  $v, v_* \in \mathbb{R}^N$ , we introduce a parametrization by  $z \in D := \{w \in \mathbb{R}^N; |w| \leq 1\}$  of all possible resulting *post-collisional velocities*  $\{v', v'_*\}$  in (1.3) in the following way:

$$(1.4) \quad \begin{cases} v' = (v + v_*)/2 + z |v_* - v|/2 \\ v'_* = (v + v_*)/2 - z |v_* - v|/2. \end{cases}$$

The *collision rate*  $B$  takes the form

$$(1.5) \quad B = |u| b, \quad b = \alpha(\mathcal{E}) \beta(\mathcal{E}, u; dz)$$

where  $u = v - v_*$  is the *relative velocity*,  $\alpha$  is an intensity coefficient,  $\beta$  is the normalized cross-section (it is a probability measure on  $D$  for any fixed  $\mathcal{E}, u$ ), and  $\mathcal{E}$  is the *kinetic energy* of the distribution  $f$ , defined by

$$\mathcal{E} := \mathcal{E}(f) = \int_{\mathbb{R}^N} f |v|^2 dv.$$

The expression (1.5) reflects the fact that we are dealing with *hard spheres* which undergo contact interactions. The term  $|u| \alpha(\mathcal{E})$  corresponds to the rate of collisions of two particles with *pre-collisional velocities*  $v, v_* \in \mathbb{R}^N$ , while the term  $\beta$  corresponds to the conditional distributional probability to obtain the two *post-collisional velocities*  $\{v', v'_*\}$ . The non-negative real  $|z|$  is the *restitution coefficient* which measures the loss of energy in the collision, since

$$(1.6) \quad |v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -\frac{1}{2}(1 - |z|^2) |v_* - v|^2 \leq 0.$$

In the above formula,  $|z| = 1$  corresponds to an elastic collision while  $z = 0$  corresponds to a completely inelastic collision (or *sticky collision*).

The bilinear collision operator  $Q(f, f)$  models the interaction of particles by means of inelastic binary collisions (preserving mass and total momentum but dissipating kinetic energy). More precisely, we define the collision operator by its action on test functions (which is related to the evolution of the *observables* of the probability density). Taking  $\varphi = \varphi(v)$  to be some well-suited regular function, we introduce the following weak formulation of the collision operator (valid under the symmetry assumption (1.11) below on  $\beta$ )

$$(1.7) \quad \langle Q(f, f), \varphi \rangle := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_* f \int_D (\varphi'_* + \varphi' - \varphi - \varphi_*) B(\mathcal{E}, u; dz) dv dv_*.$$

Here and below we use the shorthand notations  $\psi := \psi(v)$ ,  $\psi_* := \psi(v_*)$ ,  $\psi' := \psi(v')$  and  $\psi'_* := \psi(v'_*)$  for any function  $\psi$  on  $\mathbb{R}^N$ .

A first simple consequence of the definition of the operator (1.7) and of the parametrization (1.4) is that mass and momentum are conserved

$$\frac{d}{dt} \int_{\mathbb{R}^N} f \begin{pmatrix} 1 \\ v \end{pmatrix} dv = 0,$$

a fact that we easily derive (at least formally), multiplying the equation (1.1) by  $\varphi = 1$  or  $\varphi = v$  and integrating in the velocity variable (using (1.7)). In the same way, multiplying equation (1.1) by  $\varphi = |v|^2$ , integrating and using (1.6) and (1.7), we obtain that the kinetic energy is dissipated

$$(1.8) \quad \frac{d}{dt} \mathcal{E}(t) = -D(f) \leq 0,$$

where we define the energy dissipation functional  $D$  and the energy dissipation rate  $\Delta$ , which measures the (averaged) inelasticity of collisions, by

$$\begin{aligned} D(f) &:= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f f_* |u|^3 \Delta(\mathcal{E}, u) dv dv_*, \\ \Delta(\mathcal{E}, u) &:= \frac{1}{4} \int_D (1 - |z|^2) b(\mathcal{E}, u; dz) \geq 0. \end{aligned}$$

Finally, we introduce the *cooling time*, associated to the process of cooling (possibly in finite time) of granular gases:

$$(1.9) \quad T_c := \inf \left\{ T \geq 0, \mathcal{E}(t) = 0 \forall t > T \right\} = \sup \left\{ S \geq 0, \mathcal{E}(t) > 0 \forall t < S \right\}.$$

This cooling effect (or collapse) is one of the main motivations for the physical and mathematical study of granular media.

The Boltzmann equation (1.1) is complemented with an initial condition (1.2) where the initial datum is supposed to satisfy the moment conditions

$$(1.10) \quad 0 \leq f_{\text{in}} \in L^1_q(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} f_{\text{in}} dv = 1, \quad \int_{\mathbb{R}^N} f_{\text{in}} v dv = 0$$

for some  $q \geq 2$ . Notice that we can assume without loss of generality the two last moment conditions in (1.10), since we may always reduce to that case by a scaling and translation argument. Here we denote, for any integer  $q \in \mathbb{N}$ , the Banach space

$$L^1_q = \left\{ f : \mathbb{R}^N \longrightarrow \mathbb{R} \text{ measurable; } \|f\|_{L^1_q} := \int_{\mathbb{R}^N} |f(v)| (1 + |v|^q) dv < \infty \right\}.$$

We also define the weighted Sobolev spaces  $W_q^{k,1}$  ( $q \in \mathbb{R}$  and  $k \in \mathbb{N}$ ) by the norm

$$\|f\|_{W_q^{k,1}} = \sum_{|s| \leq k} \|\partial^s f (1 + |v|^q)\|_{L^1}.$$

We introduce the space of normalized probability measures on  $\mathbb{R}^N$ , denoted by  $M^1(\mathbb{R}^N)$ , and the space  $BV_q(\mathbb{R}^N)$  ( $q \in \mathbb{R}$ ) of (weighted) Bounded Variation functions, defined as the set of the weak limits in  $\mathcal{D}'(\mathbb{R}^N)$  of sequences of smooth functions which are bounded in  $W_q^{1,1}(\mathbb{R}^N)$ . Throughout the paper we denote by “C” various constants which do not depend on the collision rate  $B$ .

## 1.2 Mathematical assumptions on the collision rate

Let us state the basic assumptions on the collision rate  $B$ :

- The probability measure  $\beta$  satisfies the symmetry property

$$(1.11) \quad \beta(\mathcal{E}, u; dz) = \beta(\mathcal{E}, -u; -dz).$$

- For any  $\varphi \in C_c(\mathbb{R}^N)$  the functions

$$(1.12) \quad (v, v_*, \mathcal{E}) \mapsto \int_D \varphi(v') \beta(\mathcal{E}, u; dz) \quad \text{and} \quad \mathcal{E} \mapsto \alpha(\mathcal{E})$$

are continuous on  $\mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)$  and  $(0, \infty)$  respectively.

- The probability measure  $\beta$  satisfies the following angular spreading property: for any  $\mathcal{E} > 0$ , there is a function  $j_{\mathcal{E}}(\varepsilon) \geq 0$  such that

$$(1.13) \quad \forall \varepsilon > 0, u \in \mathbb{R}^N \quad \int_{\{\hat{u} \cdot z \in [-1, 1] \setminus [-1+\varepsilon, 1-\varepsilon]\}} \beta(\mathcal{E}, u; dz) \leq j_{\mathcal{E}}(\varepsilon)$$

and  $j_{\mathcal{E}}(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  uniformly according to  $\mathcal{E}$  when it is restricted to a compact set of  $(0, +\infty)$ .

We will sometimes restrict our analysis to a kind of *generalized (energy dependent) visco-elastic model* assuming that the cross-section  $b$  reduces to an absolutely continuous measure according to the Hausdorff measure on the sphere

$$(1.14) \quad \mathcal{C}_{u,e} = \frac{1-e}{2} \hat{u} + \frac{1+e}{2} \mathbb{S}^{N-1}.$$

More precisely, we assume that

$$(1.15) \quad b(\mathcal{E}, u; dz) = \delta_{\{z=(1-e)\hat{u}/2+(1+e)\sigma/2\}} \tilde{b}(\mathcal{E}, |u|, \hat{u} \cdot \sigma) d\sigma$$

where  $d\sigma$  is the uniform measure on the unit sphere,  $\tilde{b}$  is a non-negative measurable function and  $e : (0, \infty) \times \mathbb{R}^N \times [-1, 1] \rightarrow [0, 1]$ ,  $e = e(\mathcal{E}, |u|, \hat{u} \cdot \sigma)$  is a continuous function. For a vector  $x \in \mathbb{R}^N \setminus \{0\}$ , we define  $\hat{x} = x/|x|$  and  $\mathbb{S}^{N-1}$  stands for the unit sphere of  $\mathbb{R}^N$ . Roughly speaking, the *generalized energy dependent visco-elastic model* corresponds then to the case where for any direction  $\hat{z} \in \mathbb{S}^{N-1}$ , the post-collisional velocities  $(v', v'_*)$  such that  $(v' - v'_*)/|v' - v'_*| = \hat{z}$  are uniquely (or deterministically) defined by the pre-collisional velocities  $(v, v_*)$ .

For the uniqueness of the energy coupled models, we shall need the following additional assumption:

**H1.** The cross-section  $b$  satisfies (1.15) with  $\tilde{b}$  bounded,  $e = e(\mathcal{E})$  and the following locally Lipschitz conditions holds: for any compact subset  $K \subset (0, \infty)$  there exists a constant  $L_K \in (0, \infty)$  such that for any  $\mathcal{E}, \mathcal{E}' \in K$

$$(1.16) \quad \sup_{u \in \mathbb{R}^N} \|\tilde{b}(\mathcal{E}', u, \cdot) - \tilde{b}(\mathcal{E}, u, \cdot)\|_{L^1(\mathbb{S}^{N-1})} \leq L_K |\mathcal{E}' - \mathcal{E}|$$

and

$$(1.17) \quad |e(\mathcal{E}') - e(\mathcal{E})| \leq L_K |\mathcal{E}' - \mathcal{E}|.$$

In the study of the cooling process, we always assume:

**H2.** The energy dissipation rate  $\Delta(\mathcal{E}, u)$  in (1.9) is continuous on  $(0, +\infty) \times \mathbb{R}^N$  and satisfies

$$(1.18) \quad \Delta(\mathcal{E}, u) > 0 \quad \forall u \in \mathbb{R}^N, \mathcal{E} > 0.$$

We will also need one of the two following additional assumptions:

**H3.** For any  $\mathcal{E}_0, \mathcal{E}_\infty \in (0, \infty)$  (with  $\mathcal{E}_0 \geq \mathcal{E}_\infty$ ) there exists  $\psi$  such that

$$(1.19) \quad \Delta(\mathcal{E}, u) \geq \psi(|u|) \quad \forall \mathcal{E} \in (\mathcal{E}_\infty, \mathcal{E}_0), \forall u \in \mathbb{R}^N,$$

with  $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$  and such that for any  $R > 0$  there exists  $\psi_R > 0$  with

$$(1.20) \quad \psi(|u|) \geq \psi_R |u|^{-1} \quad \forall u \in \mathbb{R}^N, |u| > R/2.$$

This assumption is quite natural. In particular, it holds for a “*normal*” granular media.

**H4.** The cross-section  $b$  satisfies (1.15) with  $e = e(\mathcal{E}, |u|)$  and there exists  $b_0, b_1 \in (0, \infty)$  such that  $b_0 \leq \tilde{b} \leq b_1$  a.e. and  $x \mapsto \tilde{b}(\mathcal{E}, |u|, x)$  is nondecreasing and convex on  $(-1, 1)$  for any fixed  $\mathcal{E} \in (0, \infty)$  and  $u \in \mathbb{R}^N$ .

Notice that under assumption (1.15) with  $\tilde{b} = \tilde{b}(\hat{u} \cdot \sigma)$  and  $e = e(\mathcal{E}, u)$  the energy dissipation rate just writes

$$(1.21) \quad \Delta(\mathcal{E}, u) = C_N (1 - e^2),$$

where  $C_N$  is a constant depending on the dimension.

Let us emphasize that the classical Boltzmann collision operator for inelastic hard spheres with a constant normal restitution coefficient  $e \in [0, 1]$ , as studied in [7] and [18], is included as a particular case of our model, and satisfies all the assumptions above. But the formalism described from (1.3) to (1.13) is much more general than this case. In particular, we may also consider:

1. Uniformly inelastic collision processes such that

$$(1.22) \quad \exists z_0 \in [0, 1) \quad \text{s.t.} \quad \text{supp } B(\mathcal{E}, u, \cdot) \subset D(0, z_0) \quad \forall u \in \mathbb{R}^N, \forall \mathcal{E} > 0,$$

which includes the *sticky particles model* when  $z_0 = 0$ .

2. The physically important case (1.14,1.15) of collisions defined by a normal restitution coefficient  $e$  and the cross-section  $\tilde{b}$  which possibly depend on  $\mathcal{E}$ ,  $|u|$  and  $\hat{u} \cdot \sigma$ . In particular it covers the kind of models studied in [7] (where  $e$  depends on  $\mathcal{E}$ , and  $\tilde{b}$  is independent on  $\mathcal{E}$  and  $|u|$ ). It includes also the important case of the *visco-elastic hard spheres model* where  $\tilde{b} = \tilde{b}(\hat{u} \cdot \sigma)$  and the normal restitution coefficient depends (smoothly) on the normal component of the relative velocity, that is  $|u| |\hat{u} - \sigma|/2$  in our notation (see [9]).

3. This formalism also covers multidimensional versions of the kind of models proposed in [36], which corresponds to the case where  $b$  is the product of a measure depending on  $|u|$ ,  $|z|$  and a measure of  $\hat{u} \cdot \hat{z}$  absolutely continuous according to the Hausdorff measure. One easily checks that our assumptions (1.5,1.11,1.12,1.13) on the collision rate are quite natural for this kind of models as well. Note that our measure framework for  $B$  can also models situations where, in the opposite to the *generalized visco-elastic case*, there is some stochasticity or uncertainty on the degree of inelasticity of the collisions, for instance due to some experimental noise, or due to the fact that particles in the gas are a mixture of different inelasticity behaviors, which are therefore handled statistically.

The fact that  $b$  is a finite measure on  $D$  allows to define the splitting  $Q = Q^+ - Q^-$  where  $Q^+$  and  $Q^-$  are defined in weak form by

$$(1.23) \quad \langle Q^+(g, f), \varphi \rangle := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g_* f \int_D \varphi' |u| b(\mathcal{E}, u; dz) dv dv_*$$

and

$$(1.24) \quad \langle Q^-(g, f), \varphi \rangle := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g_* f \int_D \varphi |u| b(\mathcal{E}, u; dz) dv dv_*,$$

where  $v'$  is defined by (1.4). A straightforward computation shows that it is possible to give a very simple strong form of  $Q^-$  as follows

$$(1.25) \quad Q^-(g, f) = L(g) f,$$

where  $L$  is the convolution operator

$$(1.26) \quad L(g)(v) := \alpha(\mathcal{E}) \int_{\mathbb{R}^N} g(v_*) |v - v_*| dv_*.$$

Under assumption (1.15), the expression of  $Q^+$  reduces to

$$(1.27) \quad \langle Q^+(g, f), \varphi \rangle := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g_* f |u| \int_{\mathbb{S}^{N-1}} \varphi' \tilde{b}(\mathcal{E}, |u|, \hat{u} \cdot \sigma) d\sigma dv dv_*,$$

where  $v'$  is defined by the formula (deduced from (1.4) and (1.14))

$$(1.28) \quad v' = v - \frac{1+e}{4} [u - |u|\sigma], \quad v'_* = v_* + \frac{1+e}{4} [u - |u|\sigma].$$

### 1.3 Statement of the main results

Let us now define the notion of solutions we deal with in this paper.

**Definition 1.1** Consider an initial datum  $f_{\text{in}}$  satisfying (1.10) with  $q = 2$ . A non-negative function  $f$  on  $[0, T] \times \mathbb{R}^N$  is said to be a solution to the Boltzmann equation (1.1)-(1.2) if

$$(1.29) \quad f \in C([0, T]; L^1_2(\mathbb{R}^N)),$$

and if (1.1)-(1.2) holds in the sense of distributions, that is,

$$(1.30) \quad \int_0^T \left\{ \int_{\mathbb{R}^N} f \frac{\partial \phi}{\partial t} dv \langle Q(f, f), \phi \rangle \right\} dt = \int_{\mathbb{R}^N} f_{\text{in}} \phi(0, \cdot) dv$$

for any  $\phi \in C_c^1([0, T] \times \mathbb{R}^N)$ .

It is worth mentioning that (1.29) ensures that the collision term  $Q(f, f)$  is well defined as a function of  $L^1(\mathbb{R}^N)$ . Indeed, on the one hand, we deduce from  $f \in C([0, T]; L^1_2(\mathbb{R}^N))$  that  $\mathcal{E}(t) \in K_1$  on  $[0, T]$  and thus  $\alpha(\mathcal{E}(t)) \in K_2$  on  $[0, T]$  for some compact sets  $K_i \subset (0, \infty)$ . On the other hand, from the dual form (1.23) it is immediate that  $Q^\pm$  is bounded from  $L^1_1 \times L^1_1$  into  $L^1$ , with bound  $\alpha(\mathcal{E})$  (see also [18, 27] for some *strong forms* of the  $Q^+(f, f)$  term). It turns out that a solution  $f$ , defined as above, is also a solution of (1.1)-(1.2) in the mild sense:

$$f(t, \cdot) = f_{\text{in}} + \int_0^t Q(f(s, \cdot), f(s, \cdot)) ds \quad \text{a.e. in } \mathbb{R}^N.$$

Another straightforward consequence is that if  $f \in L^\infty([0, T], L^1_q)$  then  $f$  satisfies the following *chain rule*

$$(1.31) \quad \frac{d}{dt} \int_{\mathbb{R}^N} \Xi(f) \phi dv = \langle Q(f, f), \Xi'(f) \phi \rangle \quad \text{in } \mathcal{D}'([0, T]),$$

for any  $\Xi \in C^1(\mathbb{R}) \cap W^{1, \infty}(\mathbb{R})$ ,  $\phi \in L^\infty_{1-q}(\mathbb{R}^N)$ , in the sense of distribution on  $[0, T]$ .

Let us state the main results of this paper. First, we give a Cauchy Theorem valid when the collision rate  $B$  is independent on the kinetic energy.

**Theorem 1.2** Assume that  $B$  satisfies the assumptions (1.5)-(1.11)-(1.12)-(1.13) with  $b = b(u; dz)$ : the cross-section does not depend on the kinetic energy. Take an initial datum  $f_{\text{in}}$  satisfying (1.10) with  $q = 3$ . Then

- (i) For all  $T > 0$ , there exists a unique solution  $f \in C([0, T]; L^1_2) \cap L^\infty(0, T; L^1_3)$  to the Boltzmann equation (1.1)-(1.2). This solution conserves mass and momentum,

$$(1.32) \quad \int_{\mathbb{R}^N} f(t, v) dv = 1, \quad \int_{\mathbb{R}^N} f(t, v) v dv = 0 \quad \forall t \in [0, T],$$

and has a positive and decreasing kinetic energy

$$(1.33) \quad 0 < \mathcal{E}(t_2) \leq \mathcal{E}(t_1) \leq \mathcal{E}_{\text{in}} = \mathcal{E}(0) \quad \forall t_i \in [0, T], \quad t_1 \leq t_2.$$

In particular, the life time of the solution (as introduced in (1.9)) is  $T_c = +\infty$ .

(ii) Moreover, assuming **H2-H3** or **H2-H4** (with  $e$  and  $\tilde{b}$  independent on the kinetic energy), there holds

$$(1.34) \quad \mathcal{E}(t) \rightarrow 0 \text{ and } f(t, \cdot) \rightharpoonup \delta_{v=0} \text{ in } M^1(\mathbb{R}^N)\text{-weak* when } t \rightarrow T_c.$$

In other words, the cooling process does not occur in finite time, but asymptotically in large time.

**Remarks 1.3** Let us discuss the assumptions and conclusions of this theorem.

1. Under assumption **H4** and when the collision rate is independent on the kinetic energy, one can prove in fact that there exists a unique solution  $f \in C([0, \infty); L^1)$  satisfying (1.32) and (1.33) for any initial condition  $f_{\text{in}}$  satisfying (1.10) with  $q = 2$ . The proof is quite more technical and we refer to [31] where the result is presented for the true elastic collision Boltzmann equation; nevertheless the proof may be readily adapted to the inelastic collisional framework.

2. The existence and uniqueness part of Theorem 1.2 (point (i)) extends to a collision rate  $B = B(u; dz) \geq 0$  which satisfies the sole assumptions

$$\left\{ \begin{array}{l} B(-u; -dz) = B(u; dz), \\ \int_D B dz \leq C_0 (1 + |v| + |v_*|) \\ (v, v_*) \mapsto \int_D \varphi(v') B(u; dz) \in C(\mathbb{R}^N \times \mathbb{R}^N) \quad \forall \varphi \in C_c(\mathbb{R}^N) \end{array} \right.$$

for some constant  $C_0 \in \mathbb{R}_+$ . This corresponds to the so-called cut-off hard potentials (or variable hard spheres) model in the context of inelastic gases.

3. For a uniformly dissipative collision model, i.e., such that

$$\Delta(u) \geq \Delta_0 \in (0, \infty),$$

a fact which holds under assumption (1.22) or under assumption **H4** with a normal restitution coefficient  $e$  satisfying  $e(|u|) \leq e_0 \in [0, 1)$  for any  $u \in \mathbb{R}^N$ , we may prove the additional a priori bound

$$\int_0^{+\infty} \|f(t, \cdot)\|_{L^1_3} dt \leq C(\|f_{\text{in}}\|_{L^1_2}, \Delta_0).$$

As a consequence, one can easily adapt the proof of existence and uniqueness in Theorem 1.2 and then one can easily establish that the existence part of Theorem 1.2 holds for any initial datum  $f_{\text{in}}$  satisfying (1.10) with  $q = 2$ .

4. The existence and uniqueness part of Theorem 1.2 (point (i)) immediately extends to a time dependent collision rate  $B = |u| \gamma(t) b(t, u; dz)$  where  $b(t, u; \cdot)$  is a probability measure for any  $u \in \mathbb{R}^N$ ,  $t \in [0, T]$  such that  $b(t, u; dz) = b(t, -u; -dz)$ , and  $\gamma(t)$  is a non-negative function in  $L^\infty(0, T)$ .

5. Finally let us emphasize that Theorem 1.2 applies to the important (non-coupled) model of visco-elastic hard spheres. Indeed the collision rate of this model satisfies assumptions (1.5,1.11,1.12,1.13) as well as **H2** and **H3**, with  $\tilde{b}$  and  $e$  independent of  $\mathcal{E}$ . We refer to the work in preparation [29] which shall be devoted to the detailed study of this particular model.

Now, let us turn to the case where the collision rate depends on the kinetic energy of the solution.

**Theorem 1.4** *Assume now that  $B$  satisfies the assumptions (1.5)-(1.11)-(1.12)-(1.13) and that the cross-section  $b = b(\mathcal{E}, u; dz)$  indeed depends on the kinetic energy  $\mathcal{E}$ . Take an initial datum  $f_{\text{in}}$  satisfying (1.10) with  $q = 3$ .*

- (i) *There exists at least one maximal solution  $f \in C([0, T]; L^1_2) \cap L^\infty(0, T; L^1_3)$ ,  $\forall T \in (0, T_c)$ , for some  $T_c \in (0, +\infty]$ , which satisfies the conservation laws (1.32) and the decay of the kinetic energy (1.33).*
- (ii) *If the collision rate satisfies the additional assumption **H1**, and the initial datum satisfies the additional assumption  $f_{\text{in}} \in BV_4 \cap L^1_5$ , then this solution is unique among the class of functions  $C([0, T], L^1_2) \cap L^\infty(0, T; L^1_3)$ , for any  $T \in (0, T_c)$ .*
- (iii) *The asymptotic convergence (1.34) holds under the additional assumptions **H2-H3** or **H2-H4**.*
- (iv) *If one of following assumptions a. or b. is satisfied, then  $T_c = +\infty$ :*
  - a.  *$\alpha$  is bounded near  $\mathcal{E} = 0$  and  $j_\mathcal{E}$  converges to 0 as  $\varepsilon \rightarrow 0$  uniformly near  $\mathcal{E} = 0$ ;*
  - b.  *$B$  satisfies **H4**,  $\Delta$  is bounded by an increasing function  $\Delta_0$  which only depends on the energy, and  $f_{\text{in}} e^{a_\eta |v|^\eta} \in L^1$  with  $\eta \in (1, 2]$ ,  $a_\eta > 0$ .*
- (v) *If  $\Delta(\mathcal{E}, u) \geq \Delta_0 \mathcal{E}^\delta$  with  $\Delta_0 > 0$  and  $\delta < -1/2$ , then  $T_c < +\infty$ .*

**Remark 1.5** *Under the assumptions of point (ii) on the initial datum, by using a bootstrap a posteriori argument as introduced in [31], one can prove that there exists a unique solution  $f \in C([0, \infty); L^1)$  satisfying (1.32) and (1.33) for any initial condition  $f_{\text{in}}$  satisfying (1.10) with  $q > 4$  and  $f_{\text{in}} \in BV_4$ .*

## 1.4 Plan of the paper

We gather in Section 2 some new integrability estimates on the collision operator which can be of independent interest. We prove convolution-like estimates in Orlicz spaces for the *gain term*. We give then estimates on the *global operator* in Orlicz space, which show essentially that even if the bilinear collision operator is not bounded, its evolution semi-group is bounded in any Orlicz space (with bound

depending on time). In Section 3 we start looking at *solutions* of the Boltzmann equation. We prove Povzner lemma and several moments estimates in  $L^1$ , from which we deduce the existence and uniqueness part of Theorem 1.2. In Section 4, we extend the existence result to collision rates depending on the kinetic energy of the solution by proving a weak stability result on the basis of (local in time) non-concentration estimates obtained by the study of Section 2, to obtain the existence part of Theorem 1.4. The uniqueness part of Theorem 1.4 is obtained by proving a strong stability result valid for smooth solution. In Section 5 we study the cooling process and prove the remaining parts of Theorem 1.2 and Theorem 1.4.

## 2 Estimates in Orlicz spaces

In this section we gather some new functional estimates on the collision operator in Orlicz spaces, that will be used in the sequel to obtain (local in time) non-concentration estimates. Let us introduce the following decomposition  $b = b_\varepsilon^t + b_\varepsilon^r$  of the cross-section  $b$  for  $\varepsilon \in (0, 1)$ :

$$(2.1) \quad \begin{cases} b_\varepsilon^t(\mathcal{E}, u; dz) = b(\mathcal{E}, u; dz) \mathbf{1}_{\{-1+\varepsilon \leq \hat{u} \cdot z \leq 1-\varepsilon\}} \\ b_\varepsilon^r(\mathcal{E}, u; dz) = b(\mathcal{E}, u; dz) - b_\varepsilon^t(\mathcal{E}, u; dz) \end{cases}$$

where  $\mathbf{1}_{\{-1+\varepsilon \leq \hat{u} \cdot z \leq 1-\varepsilon\}}$  denotes the usual indicator function of the set  $\{-1 + \varepsilon \leq \hat{u} \cdot z \leq 1 - \varepsilon\}$ . When no confusion is possible the subscript  $\varepsilon$  shall be omitted.

In the sequel,  $\Lambda$  denotes a function  $C^2$  strictly increasing, convex satisfying the assumptions (A.1), (A.2) and (A.3) (see the appendix). This function defines the Orlicz space  $L^\Lambda(\mathbb{R}^N)$ , which is a Banach space (see the definition in the appendix).

### 2.1 Convolution-like estimates on the gain term

In this subsection we shall prove convolution-like estimates in Orlicz spaces. These estimates extend existing results in Lebesgue spaces: see [20, 21, 33, 12] in the elastic case and [18] in the inelastic case. The proof relies only upon elementary tools, essentially Young's inequality, in the spirit of [12]. Moreover it has several advantages: its simplicity, the fact that it handles only the dual form of  $Q^+$  and the fact that it is naturally well-suited to deal with Orlicz spaces, since it is based on Young's inequality.

As shown by the formula for the differential of the Orlicz norm in the appendix, the crucial quantity to estimate is

$$\int_{\mathbb{R}^N} Q^+(f, f) \Lambda' \left( \frac{f}{\|f\|_{L^\Lambda}} \right) dv.$$

Most of the difficulty is related to the fact that the bilinear operator  $Q^+$  is not bounded because of the term  $|v - v_*|$  in the collision rate. Nevertheless it is possible to prove a compactness-like estimate with respect to this algebraic weight. When

combined with the damping effect of the loss term this estimate shall show that the evolution semi-group of the global collision operator is bounded in any Orlicz space.

Let us state the result

**Theorem 2.1** *Assume that  $B$  satisfies (1.5)-(1.11)-(1.12)-(1.13). For any function  $f \in L^1_1 \cap L^\Lambda$ , for any  $\varepsilon \in (0, 1)$ , there is an explicit constant  $C_\varepsilon^+(\varepsilon)$  such that*

$$(2.2) \quad \int_{\mathbb{R}^N} Q^+(f, f) \Lambda' \left( \frac{f}{\|f\|_{L^\Lambda}} \right) dv \leq \alpha(\mathcal{E}) \left[ C_\varepsilon^+(\varepsilon) N^{\Lambda^*} \left( \Lambda' \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) \right) \|f\|_{L^1_1} \|f\|_{L^\Lambda} \right. \\ \left. + (2 + 2^{N+2}) j_\varepsilon(\varepsilon) \|f\|_{L^1_1} \int_{\mathbb{R}^N} f \Lambda' \left( \frac{f}{\|f\|_{L^\Lambda}} \right) |v| dv \right].$$

**Remark 2.2** *Let us comment on the conclusions of this theorem.*

1. *We establish estimates for the quadratic Boltzmann collision operator but similar bilinear estimates could be proved under additional assumption on  $b$ , namely that either no frontal collision occurs, i.e.,  $b(\mathcal{E}, u; dz)$  should vanish for  $\hat{u}$  close to  $z$ , or no grazing collision occurs, i.e.,  $b(\mathcal{E}, ; dz)$  should vanish for  $\hat{u}$  close to  $-z$ . For more details on these bilinear estimates and the corresponding assumptions, we refer to [33] where they are proved in Lebesgue spaces in the elastic framework.*

2. *Let us emphasize that for  $z \sim 0$  (close to sticky collisions), the jacobian of the pre-postcollisional change of variable  $(v, v_*) \rightarrow (v', v'_*)$  (both velocities at the same time) is blowing up. However in our method, we only use the changes of variable  $v \rightarrow v'$  and  $v_* \rightarrow v'_*$ , keeping the other velocity unchanged, and the jacobians of these changes of variable remain uniformly bounded as  $z \rightarrow 0$ . This explains why our bounds includes the sticky particules model, and are uniform as  $z \rightarrow 0$ .*

3. *When  $\Lambda(t) = t^p/p$ , estimate (2.3) just writes*

$$(2.3) \quad \int_{\mathbb{R}^N} Q^+(f, f) f^{p-1} dv \leq \tilde{C}_\varepsilon^+(\varepsilon) \|f\|_{L^1_1} \|f\|_{L^p}^p + \tilde{j}_\varepsilon(\varepsilon) \|f\|_{L^1_1} \|f\|_{L^\Lambda} |v|^{1/p} \|f\|_{L^p}^p,$$

*for any  $\varepsilon \in (0, 1)$  and for some explicit constants  $\tilde{C}_\varepsilon^+(\varepsilon), \tilde{j}_\varepsilon(\varepsilon) \in (0, \infty)$  with  $\tilde{j}_\varepsilon(\varepsilon) \rightarrow 0$  when  $\varepsilon \rightarrow 0$ . Although the quantities involved in these previously mentioned papers are slightly different, one can see that estimate (2.3) (or the  $L^p$  version of Theorem 2.6) generalizes [12, Proposition 2.5] to the inelastic Boltzmann operator and that it improves [18, Lemma 4.1] because of the better control of the norm  $\|f |v|^{1/p}\|_{L^p}$ .*

Let us start with an elementary geometrical lemma that we shall need several times in the sequel, in order to justify the change of variables  $v_* \rightarrow v'$  (keeping  $v, z$  fixed) and  $v \rightarrow v'$  (keeping  $v_*, z$  fixed). This lemma is close to the spirit of the proof of these changes of variables in the proof of the so-called ‘‘cancellation lemma’’ in [37, 1].

**Lemma 2.3** For any  $z \in D$  and  $\gamma \in (-1, 1)$  we define the map

$$(2.4) \quad \Phi_z : \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad u \mapsto w = \Phi_z(u) := u + |u|z,$$

its Jacobian function  $J_z := \det(D\Phi_z)$  and the cone  $\Omega_\gamma := \{u \in \mathbb{R}^N \setminus \{0\}, \hat{u} \cdot \hat{z} > \gamma\}$ . Then  $\Phi_z$  is a  $C^\infty$ -diffeomorphism from  $\Omega_\gamma$  onto  $\Omega_\delta$  with

$$\delta = \frac{\gamma + |z|}{(1 + 2\gamma|z| + |z|^2)^{1/2}}$$

and there exists  $C_\gamma \in (0, \infty)$  such that

$$(2.5) \quad C_\gamma^{-1} \leq J_z \leq C_\gamma \quad \text{on} \quad \Omega_\gamma$$

uniformly with respect to the parameter  $z \in D$ .

*Proof of Lemma 2.3.* We may assume  $z \neq 0$  since otherwise the conclusion is clear. Let start proving that  $\Phi_z$  is one-to-one on  $\Omega_{-1} = \mathbb{R}^N \setminus (\mathbb{R}_- z)$ . For any  $x \in \mathbb{R}^N$  we introduce the decomposition  $x = x_1 \hat{z} + x_2 := (x_1, x_2)$  such that  $x_1 \in \mathbb{R}$ ,  $x_2 \in \mathbb{R}^N$ ,  $x_2 \cdot \hat{z} = 0$ . The expression (2.4) then writes equivalently

$$w_1 = u_1 + (u_1^2 + |u_2|^2)^{1/2} |z|, \quad w_2 = u_2.$$

For any  $u, u' \in \Omega_{-1}$  the relation  $\Phi_z(u) = \Phi_z(u') =: w$  implies immediately  $u_2 = u'_2 = w_2$  and we conclude observing that for any  $z \in D$  and  $w_2 \in \mathbb{R}^N$  the map

$$\varphi_{w_2, |z|} : u_1 \mapsto w_1 := u_1 + (u_1^2 + |w_2|^2)^{1/2} |z|$$

is strictly increasing from  $\mathbb{R}$  onto  $\mathbb{R}$  if  $|z| < 1$ , from  $\mathbb{R}$  onto  $\mathbb{R}_+$  if  $|z| = 1$  and  $w_2 \neq 0$ , and from  $\mathbb{R}_+$  onto  $\mathbb{R}_+$  if  $|z| = 1$  and  $w_2 = 0$ . That proves that  $\Phi$  is one-to-one. Moreover, any point  $\hat{u} = (u_1, u_2) \in \mathbb{S}^{N-1}$  such that  $\hat{u}_1 = \gamma$  is mapped to the point  $w = (\gamma + |z|, u_2)$  with square norm  $|w|^2 = 1 + 2\gamma|z| + |z|^2$ . We conclude that  $\Phi(\Omega_\gamma) = \Omega_\delta$  thanks to the homogeneity property  $\Phi_z(ru) = r\Phi_z(u)$  for any  $r > 0$  and  $u \in \mathbb{R}^N$ . We next compute  $D\Phi_z(u) = Id + \hat{u} \otimes z$  and thus  $J_z(u) = 1 + \hat{u} \cdot z$  from which (2.5) easily follows. Finally, the fact that  $\Phi_z$  is a  $C^\infty$ -diffeomorphism is a direct consequence of the local inversion Theorem.  $\square$

*Proof of Theorem 2.1.* Let us denote

$$\varphi(f) = \Lambda' \left( \frac{f}{\|f\|_{L^\Lambda}} \right).$$

Using the decomposition  $b = b^t + b^r$ , we control separately the two terms  $I^t$  and  $I^r$  in the decomposition

$$\begin{aligned} \int_{\mathbb{R}^N} Q^+(f, f) \varphi(f) dv &= \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f f_* \varphi(f') |u| b^t(\mathcal{E}, u; dz) dv dv_* \\ &+ \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f f_* \varphi(f') |u| b^r(\mathcal{E}, u; dz) dv dv_* =: I^t + I^r. \end{aligned}$$

Using the bound

$$|u| = |v - v_*| \leq |v| + |v_*|$$

we have

$$\begin{aligned} I^t &\leq \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} (f|v|) f_* \varphi(f') b^t(\mathcal{E}, u; dz) dv dv_* \\ &\quad + \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f(f_*|v_*|) \varphi(f') |u| b^t(\mathcal{E}, u; dz) dv dv_* =: I_1^t + I_2^t. \end{aligned}$$

For the term  $I_1^t$ , by applying the Young's inequality (A.4)

$$f_* \varphi(f') = \|f\|_{L^\Lambda} \left( \frac{f_*}{\|f\|_{L^\Lambda}} \right) \varphi(f') \leq \|f\|_{L^\Lambda} \Lambda \left( \frac{f_*}{\|f\|_{L^\Lambda}} \right) + \|f\|_{L^\Lambda} \Lambda^*(\varphi(f')),$$

we get

$$\begin{aligned} I_1^t &\leq \|f\|_{L^\Lambda} \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f|v| \Lambda \left( \frac{f_*}{\|f\|_{L^\Lambda}} \right) b^t(\mathcal{E}, u; dz) dv dv_* \\ &\quad + \|f\|_{L^\Lambda} \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f|v| \Lambda^*(\varphi(f')) b^t(\mathcal{E}, u; dz) dv dv_* =: I_{1,1}^t + I_{1,2}^t. \end{aligned}$$

On the one hand, using

$$\forall x \in \mathbb{R}_+, \quad \Lambda(x) \leq x \Lambda'(x),$$

which is a trivial consequence of the fact that  $\Lambda(0) = 0$  and  $\Lambda'$  is increasing, we have

$$I_{1,1}^t \leq \alpha(\mathcal{E}) \|f\|_{L_1^1} \int_{\mathbb{R}^N} f \varphi(f) dv.$$

Hölder's inequality in Orlicz spaces (A.5) recalled in the appendix then yields

$$(2.6) \quad I_{1,1}^t \leq \alpha(\mathcal{E}) N^{\Lambda^*} \left( \Lambda' \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) \right) \|f\|_{L_1^1} \|f\|_{L^\Lambda}.$$

On the other hand, using that  $\Lambda^*(y) = y(\Lambda')^{-1}(y) - \Lambda((\Lambda')^{-1}(y))$ , we get

$$I_{1,2}^t \leq \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f|v| \varphi(f') f' b^t(\mathcal{E}, u; dz) dv dv_*.$$

We make the change of variables  $v_* \rightarrow v'$  (while the other integration variables are kept fixed) or more precisely  $\Psi : (v, v_*, z) \rightarrow (v, \psi_{v,z}(v_*), z)$  with  $\psi_{v,z}(v_*) = v' = v + 2^{-1} \Phi_z(v_* - v)$ . Thanks to the truncation (2.1) on  $b_\varepsilon^t$  and Lemma 2.3, the application  $\Psi$  is a  $C^\infty$ -diffeomorphism from  $\{(v, v_*, z) \in \mathbb{R}^{2N} \times D, \hat{u} \cdot z \neq 1\}$  onto its image and its jacobian  $J_\Psi = 2^{-N} (1 - \hat{u} \cdot z)$  satisfies  $|J_\Psi^{-1}| \leq 2^N \varepsilon^{-1}$  on  $\{(v, v_*, z) \in \mathbb{R}^{2N} \times D, \hat{u} \cdot z \leq 1 - \varepsilon\}$ . We then get

$$\begin{aligned} I_{1,2}^t &\leq \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f|v| f' \varphi(f') J_\Psi^{-1} b^t(\mathcal{E}, v - \psi_{v,z}^{-1}(v'); dz) dv dv' \\ &\leq \alpha(\mathcal{E}) 2^N \varepsilon^{-1} \|f\|_{L_1^1} \int_{\mathbb{R}^N} f \varphi(f) dv. \end{aligned}$$

As previously, Hölder's inequality (A.5) then yields

$$(2.7) \quad I_{1,2}^t \leq \alpha(\mathcal{E}) 2^N \varepsilon^{-1} \|f\|_{L_1^1} N^{\Lambda^*} \left( \Lambda' \left( \frac{\|f\|}{\|f\|_{L^\Lambda}} \right) \right) \|f\|_{L^\Lambda}.$$

Next, the term  $I_2^t$  is exactly similar to  $I_1^t$ , except that one has to use the change of variable  $v \rightarrow v' = v_* + 2^{-1} \Phi_z(v - v_*)$  instead of  $v_* \rightarrow v'$ . Therefore, gathering (2.6), (2.7) and the same estimate for  $I_2^t$ , we obtain

$$(2.8) \quad I^t \leq 2\alpha(\mathcal{E}) (1 + 2^N \varepsilon^{-1}) \|f\|_{L_1^1} \left[ N^{\Lambda^*} \left( \Lambda' \left( \frac{\|f\|}{\|f\|_{L^\Lambda}} \right) \right) \right] \|f\|_{L^\Lambda}.$$

Finally, for the term  $I^r$ , we can split it as

$$\begin{aligned} I^r &\leq \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f f_* \varphi(f') \mathbf{1}_{\{\hat{u} \cdot z \leq 0\}} |u| b^r(\mathcal{E}, u; dz) dv dv_* \\ &\quad + \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f f_* \varphi(f') \mathbf{1}_{\{\hat{u} \cdot z \geq 0\}} |u| b^r(\mathcal{E}, u; dz) dv dv_* =: I_1^r + I_2^r. \end{aligned}$$

For  $I_1^r$ , we use Young's inequality (A.4) on  $x = f_*$  and  $y = \varphi(f')$  to obtain

$$\begin{aligned} I_1^r &\leq \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f f_* \varphi(f') \mathbf{1}_{\{\hat{u} \cdot z \leq 0\}} |u| b^r(\mathcal{E}, u; dz) dv dv_* \\ &\quad + \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f f' \varphi(f') \mathbf{1}_{\{\hat{u} \cdot z \leq 0\}} |u| b^r(\mathcal{E}, u; dz) dv dv_*. \end{aligned}$$

In the second integral we make again the change of variable defined by  $\Psi$  for which there holds  $|J_\Psi^{-1}| \leq 2^N$  on the domain of integration because of the truncation  $\hat{u} \cdot z \leq 0$ . We also observe thanks to a direct computation starting from (1.3) that under the truncation  $\hat{u} \cdot z \leq 0$  there holds

$$|v - v_*| \leq 2|v' - v| \leq 2(1 + |v'|)(1 + |v|).$$

Hence we obtain

$$\begin{aligned} I_1^r &\leq (1 + 2^{N+1}) \left( \sup_{u \in \mathbb{R}^N} \int_D b^r(\mathcal{E}, u; dz) \right) \|f\|_{L_1^1} \int_{\mathbb{R}^N} f \varphi(f) (1 + |v|) dv \\ &\leq (1 + 2^{N+1}) \alpha(\mathcal{E}) j_\mathcal{E}(\varepsilon) \|f\|_{L_1^1} \int_{\mathbb{R}^N} f \varphi(f) (1 + |v|) dv. \end{aligned}$$

The term  $I_2^r$  is treated similarly using Young's inequality (this time on  $x = f$  and  $y = \varphi(f')$ ) and the change of variable  $v \rightarrow v'$  instead of  $v_* \rightarrow v'$ . It satisfies therefore the same estimate. Thus we obtain the estimate

$$(2.9) \quad I^r \leq (2 + 2^{N+2}) \alpha(\mathcal{E}) j_\mathcal{E}(\varepsilon) \|f\|_{L_1^1} \int_{\mathbb{R}^N} f \varphi(f) (1 + |v|) dv.$$

Defining

$$(2.10) \quad C_\mathcal{E}^+(\varepsilon) = 2(1 + 2^N \varepsilon^{-1}) + (2 + 2^{N+2}) j_\mathcal{E}(\varepsilon),$$

we conclude the proof gathering (2.8) and (2.9).  $\square$

## 2.2 Minoration of the loss term

In this subsection we recall a well-known result about the minoration of the loss term  $Q^-$ . Let us recall first the following classical estimate.

**Lemma 2.4** *For any non-negative measurable function  $f$  such that*

$$(2.11) \quad f \in L_1^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} f \, dv = 1, \quad \int_{\mathbb{R}^N} f v \, dv = 0,$$

we have

$$\forall v \in \mathbb{R}^N, \quad \int_{\mathbb{R}^N} f_* |v - v_*| \, dv_* \geq |v|.$$

*Proof of Lemma 2.4.* Use Jensen's inequality

$$\int_{\mathbb{R}^N} \varphi(g_*) \, d\mu_* \geq \varphi \left( \int_{\mathbb{R}^N} g_* \, d\mu_* \right)$$

with the probability measure  $d\mu_* = f_* \, dv_*$ , the measurable function  $v_* \mapsto g_* = v - v_*$  and the convex function  $\varphi(s) = |s|$ .  $\square$

Then the proof of the following proposition is straightforward:

**Proposition 2.5** *Assume that  $B$  satisfies (1.5). For a non-negative function  $f$  satisfying (2.11), we have*

$$(2.12) \quad \int_{\mathbb{R}^N} Q^-(f, f) \Lambda' \left( \frac{f}{\|f\|_{L^\Lambda}} \right) \, dv \geq \alpha(\mathcal{E}) \int_{\mathbb{R}^N} f \Lambda' \left( \frac{f}{\|f\|_{L^\Lambda}} \right) |v| \, dv.$$

## 2.3 Estimate on the global collision operator and a priori estimate on the solutions

Combining Theorem 2.1 and Proposition 2.5 we get

**Theorem 2.6** *Assume that  $B$  satisfies (1.5)-(1.11)-(1.12)-(1.13). Let us consider a non-negative function  $f$  satisfying (2.11). Then there is an explicit constant  $C_\mathcal{E}$  depending on the collision rate through the functions  $\alpha$  and  $j_\mathcal{E}$  such that*

$$\int_{\mathbb{R}^N} Q(f, f) \Lambda' \left( \frac{f}{\|f\|_{L^\Lambda}} \right) \, dv \leq C_\mathcal{E} \left[ N^{\Lambda^*} \left( \Lambda' \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) \right) \right] \|f\|_{L_1^1} \|f\|_{L^\Lambda}.$$

More precisely,  $C_\mathcal{E} = \alpha(\mathcal{E}) C_\mathcal{E}^+(\varepsilon_0)$ , with  $\varepsilon_0$  such that  $j_\mathcal{E}(\varepsilon_0) \leq (2 + 2^{N+2})^{-1} \|f\|_{L_1^1}^{-1}$  and where  $C_\mathcal{E}^+$  is defined in (2.10).

*Proof of Theorem 2.6.* One just has to combine (2.2) and (2.12) and pick a  $\varepsilon_0$  small enough such that

$$(2 + 2^{N+2}) \|f\|_{L_1^1} j_\mathcal{E}(\varepsilon_0) \leq 1.$$

$\square$

**Corollary 2.7** *Assume that  $B$  satisfies (1.5)-(1.11)-(1.12)-(1.13) and let us consider a solution  $f \in C([0, T]; L_2^1)$  to the Boltzmann equation (1.1)-(1.2) associated to an initial datum  $f_{\text{in}} \in L_2^1$  and to the collision rate  $B$ . Assume moreover that (1.32) holds and there exists a compact set  $K \subset (0, +\infty)$  such that*

$$\forall t \in [0, T], \quad \mathcal{E}(t) \in K.$$

*Then, there exists a  $C^2$ , strictly increasing and convex function  $\Lambda$  satisfying the assumptions (A.1), (A.2) and (A.3) (which only depends on  $f_{\text{in}}$ ) and a constant  $C_T$  (which depends on  $K, T$  and  $B$ ) such that*

$$\sup_{[0, T]} \|f(t, \cdot)\|_{L^\Lambda} \leq C_T.$$

**Remark 2.8** *Let us emphasize that these non-concentration bounds are valid for the sticky particles model (in this case they provide an exponentially growing bound in  $L^\Lambda$  for all times). As a particular case we deduce some explicit bounds on the entropy when it is finite initially. Moreover, since our bounds are uniform as  $b \rightarrow \delta_{z=0}$ , we also deduce a proof of the sticky particles limit (for a cross-section being a diffuse measure converging to a Dirac mass at  $z = 0$ ) by the Dunford-Pettis Lemma. This shows moreover that this limit is not singular.*

*Proof of Corollary 2.7.* Since  $f_{\text{in}} \in L^1(\mathbb{R}^N)$ , as recalled in the appendix, a refined version of the De la Vallée-Poussin Theorem [25, Proposition I.1.1] (see also [23, 24]) guarantees that there exists a function  $\Lambda$  satisfying the properties listed in the statement of Corollary 2.7 and such that

$$\int_{\mathbb{R}^N} \Lambda(|f_{\text{in}}|) dv < +\infty.$$

Then the  $L^\Lambda$  norm of  $f$  satisfies

$$\frac{d}{dt} \|f_t\|_{L^\Lambda} = \left[ N^{\Lambda^*} \left( \Lambda' \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) \right) \right]^{-1} \int_{\mathbb{R}^N} Q(f, f) \Lambda' \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) dv$$

thanks to Theorem A.2, and thus using Theorem 2.6, we get

$$\forall t \in [0, T], \quad \frac{d}{dt} \|f_t\|_{L^\Lambda} \leq C_{\mathcal{E}(t)} \|f_t\|_{L_1^1} \|f_t\|_{L^\Lambda}.$$

Thanks to the assumptions on  $B$ , the constant  $C_{\mathcal{E}(t)}$  provided by Theorem 2.6 is uniform when the kinetic energy belongs to a compact set. Thus we deduce

$$(2.13) \quad \forall t \in [0, T], \quad \frac{d}{dt} \|f_t\|_{L^\Lambda} \leq C_K \|f_t\|_{L_1^1} \|f_t\|_{L^\Lambda}.$$

for some explicit constant  $C_K > 0$  depending on  $K$  and the collision rate. We conclude thanks to a Gronwall argument.  $\square$

### 3 Proof of the Cauchy theorem for non-coupled collision rate

In this section we fix  $T_* > 0$  and we assume that the collision rate  $B$  satisfies

$$(3.1) \quad B = B(t, u; dz) = |u| \gamma(t) b(t, u; dz),$$

where  $b$  is a probability measure on  $D$  for any  $t \in [0, T_*]$  and  $u \in \mathbb{R}^N$  satisfying

$$(3.2) \quad \forall t \in [0, T_*], \forall u \in \mathbb{R}^N, \quad b(t, u; dz) = b(t, -u; -dz)$$

and where  $\gamma$  satisfies

$$(3.3) \quad 0 \leq \gamma(t) \leq \gamma_* \quad \text{on} \quad (0, T_*).$$

#### 3.1 Propagation of moments

In this subsection we establish several moments estimates which are well known for the Boltzmann equation with elastic collision, see [6, 31, 26] and the references therein, as well as the recent works [18, 8] for the inelastic case. Let us emphasize that these moment estimates are uniform with respect to the normal restitution coefficient  $e$  or more generally to the support of  $b(t, u; \cdot)$  in  $D$ .

First we give a result of propagation of moments valid for general collision rates using a rough version of the Povzner inequality.

**Proposition 3.1** *Assume that  $B$  satisfies (3.1)–(3.3). For any  $0 \leq f_{\text{in}} \in L^1_q(\mathbb{R}^N)$  with  $q > 2$  and  $T > 0$ , there exists  $C_T$  such that any solution  $f$  to the inelastic Boltzmann equation (1.1)–(1.2) on  $[0, T]$  satisfies, at least formally,*

$$\sup_{[0, T]} \|f(t, \cdot)\|_{L^1_q} \leq C_T.$$

*Proof of Proposition 3.1.* We write the proof for the third moment, the general moment estimate being similar. For any function  $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}_+$  such that  $\Psi(v) := \psi(|v|^2)$  for some function  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , the evolution of the associated moment is given by

$$\frac{d}{dt} \int_{\mathbb{R}^N} f \Psi dv = \int_{\mathbb{R}^N \times \mathbb{R}^N} f f_* K_\Psi dv dv_*,$$

where

$$K_\Psi := \frac{1}{2} \int_D (\Psi' + \Psi'_* - \Psi - \Psi_*) B(t, u; dz).$$

For  $\psi(z) = z^s$ ,  $s > 1$ , the function  $\psi$  is super-additive, that is  $\psi(x) + \psi(y) \leq \psi(x+y)$ , and it is an increasing function. As a consequence,

$$\begin{aligned} \Psi' + \Psi'_* - \Psi - \Psi_* &\leq \psi(|v'|^2) + \psi(|v'_*|^2) - \psi(|v'|^2 + |v'_*|^2) \\ &\quad + \psi(|v|^2 + |v_*|^2) - \psi(|v|^2) - \psi(|v_*|^2) \\ &\leq \psi(|v|^2 + |v_*|^2) - \psi(|v|^2) - \psi(|v_*|^2), \end{aligned}$$

which implies

$$K_\Psi \leq \frac{\gamma(t)}{2} |v - v_*| \left[ \psi(|v|^2 + |v_*|^2) - \psi(|v|^2) - \psi(|v_*|^2) \right].$$

Making the choice  $\psi(x) = x^{3/2}$  and using the inequality

$$(3.4) \quad \begin{aligned} (x^{1/2} + y^{1/2}) [(x + y)^{3/2} - x^{3/2} - y^{3/2}] &\leq C (x^{1/2} + y^{1/2}) (x^{1/2}y + xy^{1/2}) \\ &\leq C (2xy + x^{1/2}y^{3/2} + x^{3/2}y^{1/2}) \end{aligned}$$

for any  $x, y > 0$ , we get

$$(3.5) \quad \frac{d}{dt} \int_{\mathbb{R}^N} f |v|^3 dv \leq C \gamma(t) \int_{\mathbb{R}^N \times \mathbb{R}^N} f f_* (|v|^2 |v_*|^2 + |v| |v_*|^3) dv dv_*,$$

and we conclude thanks to a Gronwall argument.  $\square$

Finally we give a much more precise result on the evolution of moments in the case when assumption **H4** is made. On the one hand, we state uniform in time propagation of algebraic moments (as introduced in [34, 2, 15]) and exponential moments (for which the first results were obtained in [6]). On the other hand, we prove appearance of some exponential moments (while appearance of algebraic moments was initiated in [11, 39, 40]) using carefully estimates developed in [8]. These estimates may be seen as *a priori* bounds, but in fact, by the bootstrap argument introduced in [31], they can be obtained *a posteriori* for any solution given by the existence part of Theorem 1.2 and Theorem 1.4.

**Proposition 3.2** *We make the assumption **H4** on  $B$ . A solution  $f$  to the inelastic Boltzman equation (1.1)-(1.2) on  $[0, T_c)$  satisfies the additional moment properties:*

(i) *For any  $s > 2$ , there exists  $C_s > 0$  such that*

$$(3.6) \quad \sup_{t \in [0, T_c)} \|f(t, \cdot)\|_{L_s^1} \leq \max \{ \|f_{\text{in}}\|_{L_s^1}, C_s \}.$$

(ii) *If  $f_{\text{in}} e^{r|v|^\eta} \in L^1(\mathbb{R}^N)$  for  $r > 0$  and  $\eta \in (0, 2]$ , there exists  $C_1, r' > 0$ , such that*

$$(3.7) \quad \sup_{t \in [0, T_c)} \int_{\mathbb{R}^N} f(t, v) e^{r'|v|^\eta} dv \leq C_1.$$

(iii) *For any  $\eta \in (0, 1/2)$  and  $\tau \in (0, T_c)$  there exists  $a_\eta, C_\eta \in (0, \infty)$  such that*

$$(3.8) \quad \sup_{t \in [\tau, T_c)} \int_{\mathbb{R}^N} f(t, v) e^{a_\eta |v|^\eta} dv \leq C_\eta.$$

*Let us emphasize that none of these constants depends on the inelasticity coefficient  $e$  (so that the estimates are uniform with respect to the inelasticity of the Boltzmann operator) and that the constant  $C_s, a_\eta, C_\eta$  may depend on  $f_{\text{in}}$  only through its kinetic energy  $\mathcal{E}_{\text{in}}$ .*

**Remark 3.3** *The proof of (i) is very classical for the elastic Boltzmann equation [34, 2, 15] and it has been extended to the inelastic operator in [18]. Estimate (ii) has been proved in [6] for the elastic Boltzmann equation and it has been generalized in [8] to the (stationary) inelastic Boltzmann equation. We refer to [6, 31, 26, 38] for development around the Povzner inequalities. Since (ii) is a straightforward consequence of the Povzner inequality proved in [8], we just have to prove (iii). Nevertheless, since the proof of (iii) requires some tools and notations introduced in [18, 8] we begin (step 1 and step 2) by briefly presenting the proof of (ii). Let us emphasize again that (iii) is new even for the elastic equation. In the elastic framework, an extension of (iii) to hard potentials with cutoff has been used recently in the proof of the exponential return to equilibrium with explicit rate for initial data with finite mass and energy, see [32].*

*Proof of Proposition 3.2.* Let us define

$$m_p := \int_{\mathbb{R}^N} f |v|^{2p} dv.$$

*Step 1. Differential inequalities on the moments.* Taking  $\psi(x) = x^{p/2}$  and  $B$  of the above form, there holds

$$(3.9) \quad \frac{d}{dt} m_p = \int_{\mathbb{R}^N} Q(f, f) |v|^{2p} dv = \alpha(\mathcal{E}) \int_{\mathbb{R}^N \times \mathbb{R}^N} f f_* |v - v_*| K_p(v, v_*) dv dv_*,$$

where

$$(3.10) \quad K_p(v, v_*) := \frac{1}{2} \int_{\mathbb{S}^{N-1}} (|v'|^{2p} + |v_*'|^{2p} - |v|^{2p} - |v_*|^{2p}) \frac{\tilde{b}(\mathcal{E}, |u|, \sigma \cdot \hat{u})}{\alpha(\mathcal{E})} d\sigma.$$

From [8, Lemma 1, Corollary 3], there holds

$$(3.11) \quad K_p(v, v_*) \leq \gamma_p (|v|^2 + |v_*|^2)^p - |v|^{2p} - |v_*|^{2p}$$

where  $(\gamma_p)_{p=3/2, 2, \dots}$  is a decreasing sequence of real numbers such that

$$(3.12) \quad 0 < \gamma_p < \min \left\{ 1, \frac{4}{p+1} \right\}$$

(notice that the assumptions [8, (2.11)-(2.12)-(2.13)] are satisfied under our assumptions on the collision rate). Let us emphasize that the estimate (3.11) does not depend on the inelasticity coefficient  $e(\mathcal{E}, |u|)$ . Then, from [8, Lemma 2 and Lemma 3], we have

$$(3.13) \quad \frac{1}{\alpha(\mathcal{E})} \int_{\mathbb{R}^N} Q(f, f) |v|^{2p} dv \leq \gamma_p S_p - (1 - \gamma_p) m_{p+1/2}$$

with

$$S_p := \sum_{k=1}^{k_p} \binom{p}{k} (m_{k+1/2} m_{p-k} + m_k m_{p-k+1/2}),$$

where  $k_p := [(p+1)/2]$  is the integer part of  $(p+1)/2$  and  $\binom{p}{k}$  stands for the binomial coefficient. Gathering (3.9) and (3.13), we get

$$(3.14) \quad \frac{d}{dt} m_p \leq \alpha(\mathcal{E}) (\gamma_p S_p - (1 - \gamma_p) m_{p+1/2}) \quad \forall p = 3/2, 2, \dots$$

By Hölder's inequality and the conservation of mass,

$$m_p^{1+\frac{1}{2p}} \leq m_{p+1/2}$$

and, by [8, Lemma 4], for any  $a \geq 1$ , there exists  $A > 0$  such that

$$S_p \leq A \Gamma(ap + a/2 + 1) Z_p$$

with

$$Z_p := \max_{k=1, \dots, k_p} \{z_{k+1/2} z_{p-k}, z_k z_{p-k+1/2}\}, \quad z_p := \frac{m_p}{\Gamma(ap + 1/2)}.$$

We may then rewrite (3.14) as

$$(3.15) \quad \frac{dz_p}{dt} \leq \alpha(\mathcal{E}) \left( A \gamma_p \frac{\Gamma(ap + a/2 + 1)}{\Gamma(ap + 1/2)} Z_p - (1 - \gamma_p) \Gamma(ap + 1/2)^{1/2p} z_p^{1+1/2p} \right)$$

for any  $p = 3/2, 2, \dots$ . On the one hand, from (3.12), there exists  $A'$  such that

$$(3.16) \quad A \gamma_p \frac{\Gamma(ap + a/2 + 1)}{\Gamma(ap + 1/2)} \leq A' p^{a/2-1/2} \quad \forall p = 3/2, 2, \dots$$

On the other hand, thanks to Stirling's formula  $n! \sim n^n e^{-n} \sqrt{2\pi n}$  when  $n \rightarrow \infty$  and the estimate (3.12), there exists  $A'' > 0$  such that

$$(3.17) \quad (1 - \gamma_p) \Gamma(ap + 1/2)^{1/2p} \geq A'' p^{a/2} \quad \forall p = 3/2, 2, \dots$$

Gathering (3.15), (3.16) and (3.17), we obtain the differential inequality

$$(3.18) \quad \frac{dz_p}{dt} \leq \alpha(\mathcal{E}) (A' p^{a/2-1/2} Z_p - A'' p^{a/2} z_p^{1+1/2p})$$

for any  $p = 3/2, 2, \dots$

*Step 2. Proof of (3.7).* On the one hand, we remark, by an induction argument, that taking  $p_0 := \max\{3/2, (2A'/A'')^2\}$ , the sequence of functions  $z_p := x^p$  is a sequence of supersolutions of (3.18) for any  $x > 0$  and for  $p \geq p_0$ . On the other hand, choosing  $x_0$  large enough, which may depend on  $p_0$ , we have from (i) that the sequence of functions  $z_p := x^p$  is a sequence of supersolutions of (3.18) for any  $x \geq x_0$  and for  $p \in \{3/2, \dots, p_0\}$ . As a consequence, since  $z_p$  for  $p = 0, 1/2, 1$  are bounded by  $\|f_{\text{in}}\|_{L^1_2}$ , we have proved that there exists  $x_0$  such that the set

$$(3.19) \quad \mathcal{C}_x := \left\{ z = (z_p); \quad z_p \leq x^p \quad \forall p \in \frac{1}{2} \mathbb{N} \right\}$$

is invariant under the flow generated by the Boltzmann equation for any  $x \geq x_0$ : if  $f(t_1) \in \mathcal{C}_x$  then  $f(t_2) \in \mathcal{C}_x$  for any  $t_2 \geq t_1$ . We set  $a := 2/\eta \geq 1$ . Noticing that

$$(3.20) \quad \int_{\mathbb{R}^N} f(v) e^{r|v|^\eta} dv = \sum_{k=0}^{\infty} \frac{r^k}{k!} m_{k\eta/2}$$

we get, from the assumption made on  $f_{\text{in}}$ , that

$$m_{k/a}(0) \leq C_0 \frac{k!}{r^k} \quad \forall k \in \mathbb{N}.$$

Since we may assume  $r \in (0, 1]$ , the function  $y \mapsto C_0 \Gamma(y+1)r^{-y}$  is increasing, and we deduce by Hölder's inequality that for any  $p$

$$m_p(0) \leq C_0 \frac{\ell_p!}{r^{\ell_p}} \leq C_0 \frac{\Gamma(ap+2)}{r^{ap+2}} \quad \text{with} \quad \ell_p := [ap] + 1.$$

From the definition of  $z_p$  we deduce

$$(3.21) \quad z_p(0) \leq C_0 \frac{ap(ap+1)}{r^{ap+2}} \leq x_1^p$$

for any  $p$  and for some constant  $x_1 \in (0, \infty)$ . Choosing  $x := \max\{x_0, x_1\}$  we get from (3.19) and (3.21) that  $z_p(t) \leq x^p \forall t \in [0, T_c)$  for any  $p$ . Therefore, we have

$$m_p(t) \leq \Gamma(ap+1/2) x^p \quad \forall p = 3/2, 2, \dots, \quad \forall t \in [0, T_c).$$

The function  $y \mapsto \Gamma(y+1/2) x^y$  being increasing, we deduce from Hölder's inequality that for any  $k \in \mathbb{N}^*$  that  $m_{k/a}(t) \leq \Gamma(ap+1/2) x^p \leq \Gamma(k+a/2+1/2) x^{k/a+1/2}$  with  $p := [2k/a]/2 + 1/2$ . For  $r' < 2x^{-1/a}(1+a)^{-1}$  we conclude

$$\begin{aligned} \forall t \in [0, T_c) \quad \int_{\mathbb{R}^N} f(t, v) e^{r'|v|^\eta} dv &\leq \sum_{k=0}^{\infty} \frac{\Gamma(k+a/2+1/2)}{k!} x^{k/a+1/2} (r')^k \\ &\leq C \sum_{k=0}^{\infty} \left( \left( \frac{a+1}{2} \right) x^{1/a} r' \right)^k < +\infty \end{aligned}$$

from which (3.7) follows.

*Step 3. Proof of (3.8).* Let us fix  $\tau \in (0, T_c)$ . We claim that there exists  $x$  large enough and some increasing sequence of times  $(t_p)_{p \geq p_0}$  which are bounded by  $\tau$  such that for any  $p$

$$(3.22) \quad \forall t \in [t_p, T_c) \quad z_p(t) \leq x^p.$$

We already know by classical arguments (see [31, 38]) that for  $p_0$  (defined at the beginning of Step 2) there exists  $x_1$ , larger than  $x_0$  defined in (3.19), such that (3.22) holds for any  $p \leq p_0$  and  $t_p = \tau/2$ . We then argue by induction, assuming that for  $p \geq p_0$  there holds:

$$(3.23) \quad z_k \leq x^k \quad \text{on} \quad [t_{p-1/2}, T_c) \quad \forall k \leq p-1/2$$

$$(3.24) \quad z_p \geq x^p \quad \text{on} \quad [t_{p-1/2}, t_p),$$

for some  $x \geq x_1$  to be defined. If (3.24) does not hold, there is nothing to prove thanks to Step 2. Gathering (3.23), (3.24) with (3.18) we get from the definition of  $p_0$  and the fact that  $\mathcal{E}(t) \in [\mathcal{E}(\tau), \mathcal{E}(0)]$  so that  $\alpha(\mathcal{E}) \geq \alpha_0 > 0$

$$(3.25) \quad \frac{dz_p}{dt} \leq -\alpha_0 \frac{A''}{2} p^{a/2} z_p^{1+1/2p} \quad \text{on} \quad (t_{p-1/2}, t_p).$$

Integrating this differential inequality we obtain

$$-z_p^{-\frac{1}{2p}}(t_p) \leq z_p^{-\frac{1}{2p}}(t_{p-1/2}) - z_p^{-\frac{1}{2p}}(t_p) \leq -\frac{1}{2p} \frac{A'' \alpha_0}{2} p^{a/2} (t_p - t_{p-1/2}).$$

Defining  $(t_p)$  in the following way:

$$t_0 := \frac{\tau}{2}, \quad t_p := t_{p-1/2} + \frac{\tau}{2} \frac{p^{1-a/2}}{s_a}, \quad s_a := \sum_{p=0}^{\infty} p^{1-a/2}$$

and defining  $x_2 := (8 s_a)^2 / (A'' \alpha_0 \tau)^2$  we have then proved  $z_p(t_p) \leq x_2^p$  and therefore  $z_p(t) \leq x^p$  for any  $t \geq (t_p, T_c)$  with  $x = \max\{x_1, x_2\}$  thanks to Step 2. Setting  $a := 2/\eta > 4$  ( $\eta < 1/2$ ) we have

$$(3.26) \quad \sum_{k=0}^{\infty} t_{1+k/2} \leq \tau$$

and we conclude as in the end of Step 2.  $\square$

### 3.2 Stability estimate in $L^1_2$ and proof of the uniqueness part of Theorem 1.2

**Proposition 3.4** *Assume that  $B$  satisfies (3.1)–(3.3). For any two solutions  $f$  and  $g$  of the inelastic Boltzmann equation (1.1)–(1.2) on  $[0, T]$  ( $T \leq T_*$ ) we have*

$$(3.27) \quad \frac{d}{dt} \int_{\mathbb{R}^N} |f-g| (1+|v|^2) dv \leq C \gamma_* \int_{\mathbb{R}^N} (f+g) (1+|v|^3) dv \int_{\mathbb{R}^N} |f-g| (1+|v|^2) dv.$$

We deduce that there is  $C_T > 0$  depending on  $B$  and  $\sup_{t \in [0, T]} \|f+g\|_{L^1_3}$  such that

$$\forall t \in [0, T], \quad \|f_t - g_t\|_{L^1_2} \leq \|f_{\text{in}} - g_{\text{in}}\|_{L^1_2} e^{C_T t}.$$

*In particular, there exists at most one solution to the Cauchy problem for the inelastic Boltzmann equation in  $C([0, T]; L^1_2) \cap L^1(0, T; L^1_3)$ .*

*Proof of Proposition 3.4.* We multiply the equation satisfied by  $(f-g)$  by  $\phi(t, y) = \text{sgn}(f(t, y) - g(t, y)) k$ , where  $k = (1 + |v|^2)$ . Using the chain rule (1.31), we get for

all  $t \geq 0$

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^N} |f - g| k dv &= \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} [(f - g)g_* + f(f_* - g_*)] \\
&\quad (\phi' + \phi'_* - \phi - \phi_*) B(t, u; dz) dv_* dv \\
&= \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} (f - g)(f_* + g_*) \\
&\quad (\phi' + \phi'_* - \phi - \phi_*) B(t, u; dz) dv_* dv \\
&\leq \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} |f - g|(f_* + g_*) \\
&\quad (k' + k'_* - k + k_*) B(t, u; dz) dv_* dv,
\end{aligned}$$

where we have just use the symmetry hypothesis (3.1), (3.2) on  $B$  and a change of variable  $(v, v_*) \rightarrow (v_*, v)$ . Then, thanks to the bounds (3.1), (3.3) we deduce

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^N} |f - g| k dv &\leq \gamma_* \int_{\mathbb{R}^N \times \mathbb{R}^N} |u| |f - g| (f_* + g_*) k_* dv_* dv \\
&\leq \gamma_* \int_{\mathbb{R}^N} |f - g| k dv \int_{\mathbb{R}^N} (f_* + g_*) k_*^{3/2} dv_*
\end{aligned}$$

which yields the differential inequality (3.27). The end of the proof is straightforward by a Gronwall argument.  $\square$

The uniqueness in  $C([0, T]; L_2^1) \cap L^1(0, T; L_3^1)$  as stated in Theorem 1.2 is given by Proposition 3.4.

### 3.3 Sketch of the proof of the existence part of Theorem 1.2

As for the existence part, we briefly sketch the proof. We follow a method introduced in [31] and developed in [17]. We split the proof into three steps.

*Step 1.* Let us first consider an initial datum  $f_{\text{in}}$  satisfying (1.10) with  $q = 4$  and let us define the truncated collision rates  $B_n = B \mathbf{1}_{|u| \leq n}$ . The associated collision operators  $Q_n$  are bounded in any  $L_q^1$ ,  $q \geq 1$ , and are Lipschitz in  $L_2^1$  on any bounded subset of  $L_2^1$ . Therefore following a classical argument from Arkeryd, see [2], we can use the Banach fixed point Theorem and obtain the existence of a solution  $0 \leq f_n \in C([0, T]; L_2^1) \cap L^\infty(0, T; L_4^1)$  for any  $T > 0$ , to the associated Boltzmann equation (1.1)-(1.2), which satisfies (1.32)-(1.33).

*Step 2.* From Proposition 3.1, for any  $T > 0$ , there exists  $C_T$  such that

$$\sup_{[0, T]} \|f_n\|_{L_4^1} \leq C_T.$$

Moreover, coming back to the proof of Proposition 3.4 (see also the first step in the proof of [17, Theorem 2.6]), we may establish the differential inequality

$$\frac{d}{dt} \|f_n - f_m\|_{L_2^1} \leq C_1 \|f_n + f_m\|_{L_3^1} \|f_n - f_m\|_{L_2^1} + \frac{C_2}{n} \|f_n + f_m\|_{L_4^1}^2$$

for any integers  $m \geq n$ . Gathering these two informations we easily deduce that  $(f_n)$  is a Cauchy sequence in  $C([0, T]; L_2^1)$  for any  $T > 0$ . Denoting by  $f \in C([0, T]; L_2^1) \cap L^\infty(0, T; L_4^1)$  its limit, we obtain that  $f$  is a solution to the Boltzmann equation (1.1)-(1.2) associated to the collision rate  $B$  and the initial datum  $f_{\text{in}}$  by passing to the limit in the weak formulation (1.30) of the Boltzmann equation written for  $f_n$ .

*Step 3.* When the initial datum  $f_{\text{in}}$  satisfies (1.10) with  $q = 3$  we introduce the sequence of initial data  $f_{\text{in}, \ell} := f_{\text{in}} \mathbf{1}_{\{|v| \leq \ell\}}$ . Since  $f_{\text{in}, \ell} \in L_4^1$ , the preceding step give the existence of a sequence of solutions  $f_\ell \in C([0, T]; L_2^1) \cap L^\infty(0, T; L_3^1)$  for any  $T > 0$  to the Boltzmann equation (1.1)-(1.2) associated to the initial datum  $f_{\text{in}, \ell}$ . From Proposition 3.1, for any  $T > 0$ , there exists  $C_T$  such that

$$\sup_{[0, T]} \|f_\ell\|_{L_3^1} \leq C_T.$$

Thanks to (3.27) we establish that  $(f_\ell)$  is a Cauchy sequence in  $C([0, T]; L_2^1)$  and we conclude as before.  $\square$

**Remark 3.5** *Note here that an alternative path to the proof of existence could have been the use of the result of propagation of Orlicz norm which shows that the solution is uniformly bounded for  $t \in [0, T]$  in a certain Orlicz space. Together with the propagation of moments and Dunford-Pettis Lemma, it would yield the existence of a solution by classical approximation arguments and weak stability results as presented below. More generally the propagation of Orlicz norm by the collision operator can be seen as a new tool (as well as a clarification) for the theory of solutions to the spatially homogeneous Boltzmann equation with no entropy bound, as in the inelastic case, or in the elastic case when the initial datum has infinite entropy, see also [2, 31] where other strategies of proof are presented.*

## 4 Proof of the Cauchy theorem for coupled collision rate

### 4.1 Weak stability and proof of the existence part of Theorem 1.4

**Proposition 4.1** *Consider a sequence  $B_n = B_n(t, u; dz)$  of collision rates satisfying the structure conditions (3.1)-(3.2) and the uniform bound*

$$0 \leq \gamma_n(t) \leq \gamma_T \quad \forall t \in [0, T], \quad \forall n \in \mathbb{N}^*,$$

*and let us denote by  $f_n \in C([0, T]; L_2^1) \cap L^\infty(0, T; L_3^1)$  the solution associated to  $B_n$  thanks to the existence result of the preceding section (existence and uniqueness part of Theorem 1.2 and Remark 1.3 4th point). Assume furthermore that  $(f_n)$  belongs*

to a weak compact set of  $L^1((0, T) \times \mathbb{R}^N)$  and that there exists a collision rate  $B$  satisfying (3.1)-(3.2)-(3.3) and such that for any  $\psi \in C_c(\mathbb{R}^N)$

$$\gamma_n \rightarrow \gamma \quad \text{and} \quad \int_D \psi(v') b_n(t, u; dz) \rightarrow \int_D \psi(v') b(t, u; dz) \quad \text{a.e.}$$

Then there exists a function  $f \in C([0, T]; L^1_2) \cap L^\infty(0, T; L^1_3)$  and a subsequence  $f_{n_k}$  such that

$$f_{n_k} \rightharpoonup f \quad \text{weakly in} \quad L^1((0, T) \times \mathbb{R}^N),$$

and  $f$  is a solution to the Boltzmann equation (1.1)-(1.2) associated to  $B$ .

Such a stability/compactness result is very classical and we refer to [2, 14] for its proof.

*Proof of the existence part of Theorem 1.4.* We assume without restriction that there exists a decreasing function  $\alpha_0$  such that  $\alpha \leq \alpha_0$  on  $[0, \mathcal{E}_{\text{in}}]$ . We proceed in three steps.

*Step 1.* We start with some *a priori* bounds. We set  $Y_3 := \|f\|_{L^1_3}$ . From the Povzner inequality (3.5) (with  $\gamma(t) = \alpha(\mathcal{E}(t))$ ) and the dissipation of energy equation (1.8), we have

$$(4.1) \quad \frac{d}{dt} Y_3 \leq C_1 \alpha_0(\mathcal{E}) Y_3, \quad Y_3(0) Y_3(f_{\text{in}})$$

and

$$(4.2) \quad \frac{d}{dt} \mathcal{E} \geq -C_1 \alpha_0(\mathcal{E}) Y_3, \quad \mathcal{E}(0) = \mathcal{E}_{\text{in}},$$

for some constant  $C_1$  (which depends on  $\mathcal{E}_{\text{in}}$ ). There exists  $T_*$  such that any solution  $(Y_3, \mathcal{E})$  to the above differential inequalities system is defined on  $[0, T_*]$  and satisfies

$$(4.3) \quad \sup_{[0, T_*]} Y_3(t) \leq 2 Y_3(f_{\text{in}}), \quad \inf_{[0, T_*]} \mathcal{E}(t) \geq \mathcal{E}_{\text{in}}/2.$$

More precisely, we choose  $T_*$  such that

$$C_1 \alpha_0(\mathcal{E}_{\text{in}}/2) T_* \leq Y_3(f_{\text{in}}) \quad \text{and} \quad C_1 \alpha_0(\mathcal{E}_{\text{in}}/2) 2 Y_3(f_{\text{in}}) T_* \leq \mathcal{E}_{\text{in}}/2,$$

in such a way that if  $(Y_3, \mathcal{E})$  satisfies  $Y_3 \leq 2 Y_3(f_{\text{in}})$  and (4.2) on  $(0, T_*)$  or if  $(Y_3, \mathcal{E})$  satisfies  $\mathcal{E} \geq \mathcal{E}_{\text{in}}/2$  and (4.1) on  $(0, T_*)$  then (4.3) holds. We introduce

$$X := \left\{ \mathcal{E} \in C([0, T_*]), \mathcal{E}_{\text{in}}/2 \leq \mathcal{E}(t) \leq \mathcal{E}_{\text{in}} \text{ on } (0, T_*) \right\}.$$

*Step 2.* Let us consider a function  $\mathcal{E}_1 \in X$  and define  $B_2(t, u; dz) := B(\mathcal{E}_1(t), u; dz)$ . From assumptions (1.5)-(1.11)-(1.12)-(1.13) we may write

$$B_2(t, u; dz) = |u| \gamma_2(t) b_2(t, u; dz)$$

where  $b_2$  is a probability measure and  $\gamma_2(t)$  satisfies

$$\gamma_2(t) = \alpha(\mathcal{E}_1(t)) \leq \alpha_0(\mathcal{E}_{\text{in}}/2) < +\infty \quad \forall t \in [0, T_*].$$

Thanks to Theorem 1.2 there exists a unique solution  $f_2 \in C([0, T_*]; L_2^1) \cap L^\infty(0, T_*; L_3^1)$  to the Boltzmann equation (1.1)-(1.2) associated to the collision rate  $B_2$  and we set  $\mathcal{E}_2 := \mathcal{E}(f_2)$ . In such a way we have defined a map  $\Phi : X \rightarrow X$ ,  $\Phi(\mathcal{E}_1) = \mathcal{E}_2$ .

In order to apply the Schauder fixed point Theorem, we aim to prove that  $\Phi$  is continuous and compact from  $X$  to  $X$ . Consider  $(\mathcal{E}_1^n)$  a sequence of  $X$  which uniformly converges to  $\mathcal{E}_1$ . Since  $(\mathcal{E}_1^n)$  belongs to the compact set  $[\mathcal{E}_{\text{in}}/2, \mathcal{E}_{\text{in}}]$  for any  $n$  and any  $t \in [0, T_*]$ , we deduce by applying Corollary 2.7 to the sequence  $(f_2^n)$  associated to  $B_2^n(t, u; dz)B(\mathcal{E}_1^n(t), u; dz)$  that

$$(4.4) \quad \forall n \geq 0, \quad \sup_{[0, T_*]} \int_{\mathbb{R}^N} \Lambda(f_2^n(t, v)) dv \leq C_2,$$

for a superlinear function  $\Lambda$  and a constant  $C_2 > 0$ . Moreover, from Proposition 3.1 we have

$$(4.5) \quad \forall n \geq 0, \quad \sup_{[0, T_*]} \int_{\mathbb{R}^N} f_2^n(t, v) |v|^3 dv \leq C_3$$

for some constant  $C_3 > 0$ .

On the one hand, gathering (4.4), (4.5) and using the Dunford-Pettis Lemma, we obtain that  $(f_2^n)$  belongs to a weak compact set of  $L^1((0, T_*) \times \mathbb{R}^3)$ . Proposition 4.1 then implies that there exists  $f_2 \in C([0, T_*]; L_2^1) \cap L^\infty(0, T_*; L_3^1)$  such that, up to a subsequence,  $f_2^n \rightharpoonup f_2$  weakly in  $L^1(0, T; L_2^1)$  and  $f_2$  is a solution to the Boltzmann equation associated to  $B_2(t, u; dz) = B(\mathcal{E}_1(t), u; dz)$ . Since this limit is unique by the previous study, the whole sequence  $(f_2^n)$  converges weakly to  $f_2$ , and in particular

$$(4.6) \quad \mathcal{E}_2^n \rightharpoonup \mathcal{E}_2 \quad \text{weakly in } L^1(0, T)$$

where  $\mathcal{E}_2$  is the kinetic energy of  $f_2$ .

On the other hand, there holds

$$\frac{d}{dt} \mathcal{E}_2^n = - \int_{\mathbb{R}^N \times \mathbb{R}^N} f_2^n f_{2*}^n |u|^3 \Delta(\mathcal{E}_1^n, u) dv dv_* =: -D_2^n.$$

Since  $\Delta(\mathcal{E}_1^n, u) \leq \alpha(\mathcal{E}_1^n)/4 \leq \alpha_0(\mathcal{E}_{\text{in}}/2)/4$ , we deduce from (3.1) that  $D_2^n$  is bounded in  $L^\infty(0, T)$  which in turn implies

$$(4.7) \quad \|\mathcal{E}_2^n\|_{W^{1, \infty}(0, T)} \leq C_4.$$

From Ascoli's Theorem we infer that the sequence  $(\mathcal{E}_2^n)$  belongs to a compact set of  $C([0, T])$ . Since the cluster points for the uniform norm are included in the set of cluster points for the  $L^1(0, T)$  weak topology, it then follows from (4.6) that  $\Phi(\mathcal{E}_1^n) = \mathcal{E}(f_2^n)$  converges to  $\Phi(\mathcal{E}_1) = \mathcal{E}(f_2)$  for the uniform norm on  $C([0, T])$ , which ends the proof of the continuity of  $\Phi$ . Of course, the *a priori* bound (4.7) and Ascoli's Theorem also imply that  $\Phi$  is a compact map on  $X$ . We may thus use the Schauder fixed point Theorem to conclude to the existence of at least one  $\bar{\mathcal{E}} \in X$  such that

$\Phi(\bar{\mathcal{E}}) = \bar{\mathcal{E}}$ . Then, the solution  $\bar{f} \in C([0, T_*]; L_2^1) \cap L^\infty(0, T_*; L_3^1)$  to the Boltzmann equation associated to  $\bar{B}(t, u; dz) := B(\bar{\mathcal{E}}(t), u; dz)$  satisfies

$$\int_{\mathbb{R}^N} \bar{f}(t, v) |v|^2 dv = \Phi(\bar{\mathcal{E}})(t) = \bar{\mathcal{E}}(t)$$

and therefore  $\bar{f}$  is a solution to the Boltzmann equation associated to  $B$  in  $C([0, T_*]; L_2^1) \cap L^\infty(0, T_*; L_3^1)$ .

*Step 3.* We then consider the class of solution  $f : (0, T_1) \rightarrow L_3^1$  such that  $f \in C([0, T]; L_2^1) \cap L^\infty(0, T; L_3^1)$  for any  $T \in (0, T_1)$ ,  $\mathcal{E}$  is decreasing,  $f$  is mass conserving. By Zorn's Lemma, there exists a maximal interval  $[0, T_c)$  such that

$$(T_c < \infty \text{ and } \mathcal{E}(t) \rightarrow 0 \text{ when } t \rightarrow T_c) \quad \text{or} \quad T_c = +\infty.$$

In order to end the proof, the only thing one has to remark is that if  $T_c < +\infty$  and  $\lim_{t \nearrow T_c} \mathcal{E}(t) = \mathcal{E}_c > 0$ , then  $\lim_{t \nearrow T_c} Y_3(t) < \infty$  (by (4.1)) so that  $f \in C([0, T_c]; L_2^1) \cap L^\infty(0, T_c; L_3^1)$  and we may extend the solution  $f$  to a larger time interval.  $\square$

## 4.2 Strong stability and uniqueness part of Theorem 1.4

In this subsection we give a quantitative stability result in strong sense, under the additional assumption of some smoothness on the initial datum and the collision rate. Let us first prove a simple result of propagation of the total variation of the distribution.

**Proposition 4.2** *Let  $B$  be a collision rate satisfying assumptions (3.1)-(3.2)-(3.3) and  $0 \leq f_{\text{in}} \in BV_4 \cap L_5^1$  an initial datum. Then there exists  $C_{T_*}$ , depending on  $\gamma_*$  and  $\|f_{\text{in}}\|_{L_5^1}$ , such that any solution  $f \in C([0, T_*], L_2^1) \cap L^\infty(0, T_*, L_3^1)$  to the Boltzmann equation constructed in the previous step satisfies*

$$\forall t \in [0, T_*], \quad \|f_t\|_{BV_4} \leq \|f_{\text{in}}\|_{BV_4} e^{C_{T_*} t}.$$

*Proof of Proposition 4.2.* The proof is based on the same kind of Povzner inequality as above. Let us first prove the estimate by *a priori* approach, for the sake of clearness. We have the following formula for the differential of  $Q$ :

$$\nabla_v Q(f, f) = Q(\nabla_v f, f) + Q(f, \nabla_v f).$$

This property is proved in the elastic case in [38] but it is strictly related to the invariance property of the collision operator

$$\tau_h Q(f, f) = Q(\tau_h f, \tau_h f)$$

where the translation operator  $\tau_h$  is defined by

$$\forall v \in \mathbb{R}^N, \quad \tau_h f(v) = f(v - h).$$

It is easily seen that it remains true in the inelastic case under our assumptions. The propagation of the  $L_5^1$  norm has already been established. Then we estimate the time derivative of the  $L_4^1$  norm of the gradient along the flow:

$$\begin{aligned}
\frac{d}{dt} \|\nabla_v f_t\|_{L_4^1} &= \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f(\nabla_v f_*) \left[ (1 + |v'|^4) \operatorname{sgn}(\nabla_v f)' + (1 + |v_*'|^4) \operatorname{sgn}(\nabla_v f)'_* \right. \\
&\quad \left. - (1 + |v|^4) \operatorname{sgn}(\nabla_v f) - (1 + |v_*|^4) \operatorname{sgn}(\nabla_v f)_* \right] B dv dv_* \\
&\leq \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f |\nabla_v f_*| \left[ (1 + |v'|^4) + (1 + |v_*'|^4) - (1 + |v|^4) \right. \\
&\quad \left. - (1 + |v_*|^4) \right] B dv dv_* + 4\gamma_* \|f_t(1 + |v|^5)\|_{L^1} \|\nabla_v f(1 + |v|)\|_{L^1} \\
&\leq C \|f_t\|_{L_5^1} \|\nabla_v f\|_{L_4^1}
\end{aligned}$$

using a Povzner inequality as in (3.4). This shows the *a priori* propagation of the  $BV_4$  norm by a Gronwall argument.

Now let us explain how to obtain the same estimate by a *posteriori* approach. First concerning the *a posteriori* propagation of the  $L_5^1$  norm, it is similar to the method in [31] and does not lead to any difficulty. Concerning the propagation of  $BV_4$  norm, we look at some “discretized derivative”. Let us denote  $k = \operatorname{sgn}(\tau_h f - f)(1 + |v|^4)$ . We can compute by the chain rule the following time derivative (using the invariance property of the collision operator)

$$\begin{aligned}
\frac{d}{dt} \|\tau_h f_t - f_t\|_{L_4^1} &= \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} (\tau_h f \tau_h f_* - f f_*) [k' - k] B dv dv_* \\
&= \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} (\tau_h f - f) f_* [k' + k'_* - k - k_*] B dv dv_* \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} (\tau_h f - f)(\tau_h f_* - f_*) [k' + k'_* - k - k_*] B dv dv_* \\
&\leq \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} |\tau_h f - f| f_* \left[ |v'|^4 + |v_*'|^4 - |v|^4 + |v_*|^4 \right] B dv dv_* \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} |\tau_h f - f| |\tau_h f_* - f_*| \\
&\quad \quad \quad \left[ |v'|^4 + |v_*'|^4 + |v|^4 + |v_*|^4 \right] B dv dv_*.
\end{aligned}$$

Then using the same rough Povzner inequality as in the proof of Proposition 3.1, we have

$$\left[ |v'|^4 + |v_*'|^4 + |v|^4 + |v_*|^4 \right] |v - v_*| \leq C \left[ (1 + |v|^4)(1 + |v_*|^5) + (1 + |v_*|^4)(1 + |v|^5) \right].$$

Hence we deduce that

$$\frac{d}{dt} \|\tau_h f_t - f_t\|_{L_4^1} \leq C \gamma_* \|\tau_h f_t - f_t\|_{L_4^1} \left[ \|f\|_{L_5^1} + \|\tau_h f_t - f_t\|_{L_5^1} \right]$$

and for  $|h| \leq 1$ , we deduce

$$\frac{d}{dt} \|\tau_h f_t - f_t\|_{L^1_4} \leq C \gamma_* \|\tau_h f_t - f_t\|_{L^1_4} \|f\|_{L^1_5}.$$

By a Gronwall argument it shows for any  $|h| \leq 1$  that

$$\forall t \in [0, T_*], \quad \|\tau_h f_t - f_t\|_{L^1_4} \leq \|\tau_h f_{\text{in}} - f_{\text{in}}\|_{L^1_4} e^{C_{T_*} t}$$

for a constant  $C_{T_*}$  depending on  $\gamma_*$  and  $\sup_{t \in [0, T_*]} \|f_t\|_{L^1_5}$ . By dividing by  $h$  and letting  $h$  goes to 0, we conclude that

$$\forall t \in [0, T_*], \quad \|\nabla_v f_t\|_{M^1_4} \leq \|\nabla_v f_{\text{in}}\|_{M^1_4} e^{C_{T_*} t}$$

which ends the proof.  $\square$

Assume now that the collision rate satisfies (1.5)-(1.11)-(1.12)-(1.13) plus the additional assumption **H1**. Let us take  $f_{\text{in}} \in BV_4 \cap L^1_5$  and let us consider two solutions  $f, g \in C([0, T_c]; L^1_2) \cap L^\infty(0, T; L^1_3)$  constructed by the previous steps. For these two solutions the function  $e(\mathcal{E})$  is locally Lipschitz, so is the function  $\Delta(\mathcal{E})$  and the differential equation (1.8) satisfied by  $\mathcal{E}(f_t)$  on  $[0, T_*]$  implies that it is bounded from below on this interval. Thus thanks to the continuity of  $\alpha$ , the assumptions of Proposition 4.2 are satisfied, and thus the  $BV_4$  norm is bounded on any time interval  $[0, T_*] \subset [0, T_c)$ .

**Proposition 4.3** *Let  $B$  be a collision rate satisfying (1.5)-(1.11)-(1.12)-(1.13) plus the additional assumption **H1**. Let  $f, g \in C([0, T_*]; L^1_2) \cap L^\infty(0, T_*; L^1_3)$  be two solutions with mass 1 and momentum 0, with initial data  $f_{\text{in}}$  and  $g_{\text{in}}$ , and such that  $\mathcal{E}(f(t, \cdot)), \mathcal{E}(g(t, \cdot)) \in K$  on  $[0, T_*]$  with  $K$  a compact of  $(0, +\infty)$  and*

$$\forall t \in [0, T_*], \quad \|f(t, \cdot)\|_{BV_4}, \|g(t, \cdot)\|_{BV_4} \leq C_{T_*}.$$

*Then there is a constant  $C'_{T_*}$  depending on  $B, K$  and  $C_{T_*}$  such that*

$$\forall t \in [0, T_*], \quad \|f(t, \cdot) - g(t, \cdot)\|_{L^1_2} \leq \|f_{\text{in}} - g_{\text{in}}\|_{L^1_2} e^{C'_{T_*} t}.$$

We need the following geometrical lemma which is a more accurate version of Lemma 2.3 when the collision process is of the generalized visco-elastic type (1.14,1.15).

**Lemma 4.4** *For any  $e \in (0, 1]$  and  $\sigma \in \mathbb{S}^{N-1}$  we define*

$$(4.8) \quad \phi_e^* = \phi_{e, v^*, \sigma}^* : \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad v_* \mapsto v' = v + \frac{1+e}{4} \Phi_\sigma(v_* - v)$$

$$(4.9) \quad \phi_e = \phi_{e, v_*, \sigma} : \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad v \mapsto v' = v_* + \frac{3-e}{4} \Phi_{r_e \sigma}(v - v_*), \quad r_e = \frac{1+e}{3-e},$$

*(where  $\Phi_z$  was defined in Lemma 2.3) and the Jacobian functions  $J_e^* = \det(D\phi_{e, v^*, \sigma}^*)$ ,  $J_e = \det(D\phi_{e, v_*, \sigma})$ .*

Then for any  $\gamma \in (-1, 1)$ ,  $\phi_e^*$  defines a  $C^\infty$ -diffeomorphism from  $v + \Omega_\gamma$  onto  $v + \Omega_{\omega^*(\gamma)}$  with  $\omega^*(\gamma) = ((1 + \gamma)/2)^{1/2}$  and  $\phi_e$  defines a  $C^\infty$ -diffeomorphism from  $v_* + \Omega_\gamma$  onto  $v_* + \Omega_{\omega_e(\gamma)}$  with  $\omega_e(\gamma) = (\gamma + r_e)/(1 + 2\gamma r_e + r_e^2)^{1/2}$ . Moreover, there exists  $C_\gamma \in (0, \infty)$  such that

$$(4.10) \quad C_\gamma^{-1} |v - v_*| \leq |\phi_e(v) - v_*| \leq 2 |v - v_*|,$$

$$(4.11) \quad |\phi_e^{-1}(v') - \phi_{e'}^{-1}(v')| \leq C_\gamma |e' - e| |v' - v_*|,$$

$$(4.12) \quad |J_e| \leq C_\gamma, \quad |J_e^{-1}| \leq C_\gamma, \quad |J_e^{-1} - J_{e'}^{-1}| \leq C_\gamma |e' - e|$$

on  $v_* + \Omega_\gamma$  uniformly with respect to the parameters  $e, e' \in [0, 1]$ ,  $\sigma \in \mathbb{S}^{N-1}$ ,  $v_* \in \mathbb{R}^N$ . The same estimates hold for  $\phi_e^*$ .

Finally, for any  $e, e' \in [0, 1]$ ,  $\sigma \in \mathbb{S}^{N-1}$ ,  $v_* \in \mathbb{R}^N$  and  $t \in [0, 1]$  there holds

$$(4.13) \quad t \phi_e^{-1} + (1 - t) \phi_{e'}^{-1} = \phi_{e''}^{-1}$$

for some  $e''$  into the segment with extremal points  $e$  and  $e'$ . The same result holds for  $\phi_e^*$ .

*Proof of Lemma 4.4.* We only establish the result for the function  $\phi_e$ , since the proof for  $\phi_e^*$  is similar (and even simpler). First, (4.12) and the fact that  $\phi_e$  defines a  $C^\infty$ -diffeomorphism from  $v_* + \Omega_\gamma$  onto  $v_* + \Omega_{\omega_e(\gamma)}$  come straightforwardly from Lemma 2.3 and its proof.

Second, for any  $z \in D$ ,  $|\Phi_z(u)| = |u + |u|z| \leq 2|u|$  and

$$|\Phi_z(u)|^2 \geq |u|^2 + 2|u|z \cdot u + |u|^2|z|^2 \geq |u|^2(1 - \gamma^2)$$

for any  $u \in \mathbb{R}^N$ ,  $\hat{u} \cdot \hat{z} \geq \gamma$ . That proves (4.10).

Third, using the notation of Lemma 2.3 we write  $\Phi_{r\sigma}^{-1}(w) = (\varphi_{w_2, r}^{-1}(w_1), w_2)$  for any  $w = (w_1, w_2)$ ,  $w_1 \in \mathbb{R}$ ,  $w_2 \in \mathbb{R}^N$ ,  $w_2 \cdot \sigma = 0$ . The map  $(u_1, r) \mapsto \varphi_{w_2, r}(u_1)$  is smooth and has positive partial derivatives on  $\mathbb{R} \times [0, 1]$  if  $w_2 \neq 0$  and on  $(0, \infty) \times [0, 1]$  if  $w_2 = 0$ . On the one hand, we deduce that  $(w_1, r) \mapsto \varphi_{w_2, r}^{-1}(w_1)$  is smooth and increasing in both variables and that the same holds for

$$(w_1, e) \mapsto \frac{4}{3 - e} \varphi_{w_2, r_e}^{-1}(w_1).$$

The intermediate values Theorem then implies that for any  $e \leq e' \in [0, 1]$ ,  $t \in [0, 1]$  there holds

$$t \frac{4}{3 - e} \varphi_{w_2, r_e}^{-1}(w_1) + (1 - t) \frac{4}{3 - e'} \varphi_{w_2, r_{e'}}^{-1}(w_1) = \frac{4}{3 - e''} \varphi_{w_2, r_{e''}}^{-1}(w_1)$$

for some  $e'' \in [e, e']$  from which (4.13) follows.

On the other hand,  $r \mapsto \Phi_{r\sigma}^{-1}(\hat{w})$  is smooth for any  $\hat{w} \in \mathbb{S}^{N-1} \setminus \{-\sigma\}$  and therefore there exists  $C_\gamma$  such that  $|\Phi_{r'\sigma}^{-1}(\hat{w}) - \Phi_{r\sigma}^{-1}(\hat{w})| \leq C_\gamma |r' - r|$  uniformly for any  $\hat{w} \in \mathbb{S}^{N-1}$ ,  $\hat{w} \cdot \sigma \geq \gamma$ . Thanks to the homogeneity property  $\Phi_z^{-1}(\lambda w) = \lambda \Phi_z^{-1}(w)$  we deduce

$$|\Phi_{r'\sigma}^{-1}(w) - \Phi_{r\sigma}^{-1}(w)| = |w| |\Phi_{r'\sigma}^{-1}(\hat{w}) - \Phi_{r\sigma}^{-1}(\hat{w})| \leq C_\gamma |r' - r| |w|,$$

from which (4.11) follows.  $\square$

*Proof of Proposition 4.3.* Let us denote  $Q_f$  (resp.  $Q_g$ ) the collision operator with collision rate associated with  $\mathcal{E} = \mathcal{E}(f)$  (resp.  $\mathcal{E} = \mathcal{E}(g)$ ),  $D := f - g$ ,  $S := f + g$  and  $k := (1 + |v|^2) \operatorname{sgn}(D)$ . The evolution equation on  $D$  writes

$$\frac{\partial}{\partial t} D = \frac{1}{2} [Q_f(D, S) + Q_f(S, D)] + [Q_f(g, g) - Q_g(g, g)]$$

and thus the time derivative of the  $L_2^1$  norm of  $D$  is

$$\begin{aligned} \frac{d}{dt} \|D\|_{L_2^1} &= \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} SD_* \left[ k(v'_{e(f)}) + k(v'_{*,e(f)}) - k - k_* \right] |u| \tilde{b}_{\mathcal{E}(f)} dv dv_* d\sigma \\ &+ \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} gg_* \left[ k(v'_{e(g)}) - k \right] |u| \left[ \tilde{b}_{\mathcal{E}(f)} - \tilde{b}_{\mathcal{E}(g)} \right] dv dv_* d\sigma \\ &+ \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} gg_* \left[ k(v'_{e(f)}) - k(v'_{e(g)}) \right] |u| \tilde{b}_{\mathcal{E}(f)}^+ dv dv_* d\sigma \\ &+ \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} gg_* \left[ k(v'_{e(f)}) - k(v'_{e(g)}) \right] |u| \tilde{b}_{\mathcal{E}(f)}^- dv dv_* d\sigma \\ &=: I_1 + I_2 + I_3 + I_4, \end{aligned}$$

the subscripts recalling that the post-collisional velocities  $v'_{e(f)}$ ,  $v'_{*,e(f)}$  and  $v'_{e(g)}$  defined by (1.28) depend on the choice of the normal restitution coefficient  $e$  and thus on the kinetic energies  $\mathcal{E}(f)$  and  $\mathcal{E}(g)$ . Here we have set  $\tilde{b}_{\mathcal{E}}(x) = \tilde{b}(\mathcal{E}, x)$  and  $\tilde{b}_{\mathcal{E}}^\pm(x) = \tilde{b}(\mathcal{E}, x) \mathbf{1}_{\pm x \geq 0}$  and for the sake of brevity we just write  $e(h)$  instead of  $e(\mathcal{E}(h))$  for any function  $h \in L_2^1$ .

The first term is easily dealt with by the same arguments as in the non-coupled case:

$$I_1 \leq \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} S |D_*| (1 + |v|^2) |u| \tilde{b}_{\mathcal{E}(f)} dv dv_* d\sigma \leq \alpha(\mathcal{E}(f)) \|S\|_{L_3^1} \|f - g\|_{L_1^1}.$$

Using  $|u| (|k| + |k(v'_{e(g)})|) \leq 2(1 + |v|^2)^{3/2} (1 + |v_*|^2)^{3/2}$ , the second term  $I_2$  is controlled by

$$I_2 \leq 2 \|\tilde{b}_{\mathcal{E}(f)} - \tilde{b}_{\mathcal{E}(g)}\|_{L^1(\mathbb{S}^{N-1})} \|g\|_{L_3^1}^2.$$

Using now the locally Lipschitz assumption (1.16) and the fact that  $\mathcal{E}(f), \mathcal{E}(g) \in K$  we get for some constant  $C_K$  depending on  $\tilde{b}$  and  $K$ :

$$I_2 \leq C_K |\mathcal{E}(f) - \mathcal{E}(g)| \|g\|_{L_3^1}^2 \leq C_K \|f - g\|_{L_2^1} \|g\|_{L_3^1}^2.$$

As for the third term  $I_3$ , we use twice the change of variable  $v \mapsto v' = \phi_e(v)$  with  $v_*, \sigma$  fixed and  $e = e(f)$  or  $e = e(g)$ . We get

$$\begin{aligned} I_3 &= \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \int_{\mathcal{O}_{e(f)}} g_* k' G(\phi_{e(f)}^{-1}) J_{e(f)}^{-1} \tilde{b}_{\mathcal{E}(f)}^+ dv' dv_* d\sigma \\ &- \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \int_{\mathcal{O}_{e(g)}} g_* k' G(\phi_{e(g)}^{-1}) J_{e(g)}^{-1} \tilde{b}_{\mathcal{E}(f)}^+ dv' dv_* d\sigma, \end{aligned}$$

where we have introduced the notations  $G(w) := |v_* - w| g(w)$  for any  $w \in \mathbb{R}^N$  and  $\mathcal{O}_e = v_* + \Omega_{\omega_e(0)}$ . Without restriction we may assume  $e(f) \leq e(g)$  and therefore  $\mathcal{O}_{e(g)} \subset \mathcal{O}_{e(f)}$  since  $e \mapsto \omega_e(0)$  is an increasing function. We then split  $I_3$  as

$$\begin{aligned} I_3 &= \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \int_{\mathcal{O}_{e(f)} \setminus \mathcal{O}_{e(g)}} g_* k' G(\phi_{e(f)}^{-1}) J_{e(f)}^{-1} \tilde{b}_{\mathcal{E}(f)}^+ dv' dv_* d\sigma \\ &+ \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \int_{\mathcal{O}_{e(g)}} g_* k' \left[ J_{e(f)}^{-1} - J_{e(g)}^{-1} \right] G(\phi_{e(g)}^{-1}) \tilde{b}_{\mathcal{E}(f)}^+ dv' dv_* d\sigma \\ &+ \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \int_{\mathcal{O}_{e(g)}} g_* k' \left[ G(\phi_{e(f)}^{-1}) - G(\phi_{e(g)}^{-1}) \right] J_{e(f)}^{-1} \tilde{b}_{\mathcal{E}(f)}^+ dv' dv_* d\sigma \\ &= I_{3,1} + I_{3,2} + I_{3,3}. \end{aligned}$$

For the first term  $I_{3,1}$  we use the backward change of variables  $v' \mapsto v = \phi_{e(f)}^{-1}(v')$  and we get

$$I_{3,1} = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \int_{\mathbb{R}^N} g_* k(v'_{e(f)}) G \tilde{b}_{\mathcal{E}(f)} \mathbf{1}_{0 \leq \hat{u} \cdot \sigma \leq \eta} dv dv_* d\sigma$$

with  $\eta := \omega_{e(f)}^{-1} \circ \omega_{e(g)}(0)$ . By inspection, the functions  $(e, \gamma) \in [0, 1] \times (-1, 1] \mapsto \omega_e(\gamma), \omega_e^{-1}(\gamma) \in (-1, 1]$  are smooth with respect to both variables. From this smoothness and the fact that  $\omega_e^{-1} \circ \omega_e(0) = 0$  we deduce  $|\omega_{e'}^{-1} \circ \omega_e(0)| \leq C |e - e'|$  for any  $e, e' \in [0, 1]$  and for some constant  $C \in (0, \infty)$ . As a consequence, thanks to the Lipschitz assumption (1.17), we obtain

$$\begin{aligned} I_{3,1} &\leq \|\tilde{b}\|_{L^\infty} \int_{\mathbb{R}^N \times \mathbb{R}^N} g(1 + |v|)^3 g_*(1 + |v_*|)^3 \left\{ \int_{\mathbb{S}^{N-1}} \mathbf{1}_{-C(e(g)-e(f)) \leq \hat{u} \cdot \sigma \leq 0} d\sigma \right\} dv dv_* \\ &\leq C \|g\|_{L_3^1}^2 |e(f) - e(g)| \leq C \|g\|_{L_3^1}^2 \|f - g\|_{L_2^1}. \end{aligned}$$

For the term  $I_{3,2}$ , using the estimate (4.12) and the Lipschitz assumption (1.17), we get

$$\left| J_{e(f)}^{-1} - J_{e(g)}^{-1} \right| \leq C |e(f) - e(g)| \leq C_K \|f - g\|_{L_2^1}.$$

Then doing the backward change of variable  $v' \mapsto v = \phi_{e(g)}^{-1}(v')$  and observing that  $J_{e(f)}$  is bounded on  $\{u, \hat{u} \cdot \sigma \geq 0\}$  thanks to (4.10), we get

$$I_{3,2} \leq C_K \|f - g\|_{L_2^1} \|g\|_{L_3^1}^2.$$

We now aim to prove that for any functions  $f, g$  which energies  $\mathcal{E}_f$  and  $\mathcal{E}_g$  belonging to a compact  $K \subset (0, \infty)$  there exists a constant  $C_K$  such that the following functional inequality holds

$$(4.14) \quad I_{3,3} \leq C_K \|f - g\|_{L_2^1} \|g\|_{L_4^1} \|g\|_{BV_4}.$$

Let us first assume that  $f$  and  $g$  are smooth functions, say  $f, g \in \mathcal{D}(\mathbb{R}^N)$ . From (4.11) and (4.13) we have

$$\begin{aligned} & \left| G(\phi_{e(f)}^{-1})(v') - G(\phi_{e(g)}^{-1})(v') \right| \leq \\ & \leq |\phi_{e(f)}^{-1}(v') - \phi_{e(g)}^{-1}(v')| \int_0^1 \left| \nabla_w G((1-t)\phi_{e(f)}^{-1}(v') + t\phi_{e(g)}^{-1}(v')) \right| dt \\ & \leq C |e(f) - e(g)| |v' - v| \int_0^1 \left| \nabla_w G(\phi_{e_t}^{-1}(v')) \right| dt \end{aligned}$$

with  $e_t \in [e(f), e(g)]$ . Since then  $\mathcal{O}_{e(g)} \subset \mathcal{O}_{e_t}$  for any  $t \in [0, 1]$ , we deduce

$$I_{3,3} \leq C |e(f) - e(g)| \int_0^1 \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \int_{\mathcal{O}_{e_t}} g_* |k'| |v' - v| \left| \nabla_w G(\phi_{e_t}^{-1}(v')) \right| dv' dv d\sigma dt.$$

Using finally the backward change of variable  $v' \mapsto v = \phi_{e_t}^{-1}(v')$  and the uniform bound (4.12) on the Jacobian  $J_{e_t}$  on  $v_* + \Omega_0$  we get

$$I_{3,3} \leq C |e(f) - e(g)| \|g\|_{L^1_4} \|g\|_{BV_4}.$$

Therefore we obtain (4.14) for smooth functions. When  $f, g \in BV_4$  we argue by density, introducing two sequences of smooth functions  $(f_n)$  and  $(g_n)$  which converge respectively to  $f$  and  $g$  in  $L^1$  and are bounded in  $BV_4$ , we pass to the limit  $n \rightarrow \infty$  in the functional inequality (4.14) written for the functions  $f_n$  and  $g_n$ . We then easily conclude that (4.14) also holds for  $f$  and  $g$ .

The term  $I_4$  can be dealt with similarly to the term  $I_3$ . Collecting all the estimates we thus get

$$\frac{d}{dt} \|f_t - g_t\|_{L^1_2} \leq C'_{T_*} \|f_t - g_t\|_{L^1_2}$$

where  $C'_{T_*}$  depends on  $K, \tilde{b}$  and on some uniform bounds on  $\|f\|_{L^1_3}$  and  $\|g\|_{BV_4}$ . This concludes the proof by a Gronwall argument.  $\square$

The uniqueness part of Theorem 1.4 follows straightforwardly from Proposition 4.3 and the discussion made just before its statement.

## 5 Study of the cooling process

In this section we prove the cooling asymptotic as stated in point (ii) of Theorem 1.2 and points (iii), (iv), (v) of Theorem 1.4. We first prove the collapse of the distribution function in the sense of weak \* convergence to the Dirac mass in the set of measures.

**Proposition 5.1** *Let  $T_c \in (0, +\infty]$  be the time of life of the solution. Under the sole additional assumption **H2**, there holds*

$$(5.1) \quad f(t, \cdot) \xrightarrow[t \rightarrow T_c]{} \delta_{v=0} \text{ weakly* in } M^1(\mathbb{R}^N).$$

*Proof of Proposition 5.1.* We split the proof in two steps.

*Step 1.* Assume first that  $\mathcal{E} \rightarrow 0$  when  $t \rightarrow T_c$ . This is always the case when  $T_c < +\infty$  (since the convergence to 0 of the kinetic energy follows from the existence proof in this case) and it will be established under additional assumptions on  $B$  when  $T_c = +\infty$  but it probably holds true under the sole assumption **H2** in this case as well. For any  $0 \leq \varphi \in \mathcal{D}(\mathbb{R}^N \setminus \{0\})$ , there exists  $r > 0$  such that  $\varphi = 0$  on  $D(0, r)$  and then, there exists  $C_\varphi = C_\varphi(r, \|\varphi\|_\infty)$  such that  $|\varphi(v)| \leq C_\varphi |v|^2$ . As a consequence,

$$\int_{\mathbb{R}^N} f \varphi dv \leq C_\varphi \mathcal{E}(t) \rightarrow 0,$$

from which we deduce that any weak  $*$  limit  $\bar{\mu}$  of  $f$  in  $M^1$  satisfies  $\text{supp } \bar{\mu} \subset \{0\}$ . Therefore, (5.1) follows using the conservations (1.32) and the energy bound (1.33).

*Step 2.* Assume next that  $\mathcal{E} \rightarrow \mathcal{E}_\infty > 0$  (and thus also  $T_c = +\infty$ ). Then for a fixed time  $T > 0$  and for any non-negative sequence  $(t_n)$  increasing and going to  $+\infty$ , there exists a subsequence  $(t_{n_k})$  and a measure  $\bar{\mu} \in L^\infty(0, T; M_2^1)$  such that the sequence  $f_k(t, v) := f(t_{n_k} + t, v)$  satisfies

$$(5.2) \quad f_k \rightharpoonup \bar{\mu} \text{ weakly } * \text{ in } L^\infty(0, T; M^1).$$

Moreover, for any  $\varphi \in C_c(\mathbb{R}^N)$ , there holds

$$\frac{d}{dt} \int_{\mathbb{R}^N} f_k \varphi dv = \langle Q(f_k, f_k), \varphi \rangle \quad \text{on } (0, T),$$

with  $\langle Q(f_k, f_k), \varphi \rangle$  bounded in  $L^\infty(0, T)$ . From Ascoli's Theorem, we get

$$\int_{\mathbb{R}^N} f_k \varphi dv \rightarrow \int_{\mathbb{R}^N} \varphi d\bar{\mu}(v) \quad \text{uniformly on } [0, T].$$

As a consequence, for any given function  $\chi_\varepsilon \in C_c(\mathbb{R}^3 \times \mathbb{R}^3)$  such that  $0 \leq \chi_\varepsilon \leq 1$  and  $\chi_\varepsilon(v, v_*) = 1$  for every  $(v, v_*)$  such that  $|v| \leq \varepsilon^{-1}$  and  $|v_*| \leq \varepsilon^{-1}$  we may pass to the limit (using the continuity of  $\Delta = \Delta(\mathcal{E}, u)$  which is uniform on the compact set determined by  $[\mathcal{E}_\infty, \mathcal{E}_0]$  and the support of  $\chi_\varepsilon$ )

$$(5.3) \quad \int_0^T D_\varepsilon(f_k) dt \xrightarrow{k \rightarrow +\infty} \int_0^T \int_{\mathbb{R}^N \times \mathbb{R}^N} |u|^3 \Delta(\mathcal{E}_\infty, u) \chi_\varepsilon(v, v_*) d\bar{\mu} d\bar{\mu}_* dt,$$

where we have defined for any measure (or function)  $\lambda$ :

$$D_\varepsilon(\lambda) := \int_{\mathbb{R}^N \times \mathbb{R}^N} |u|^3 \Delta(\mathcal{E}, u) \chi_\varepsilon(v, v_*) d\lambda(v) d\lambda(v_*).$$

From the dissipation of energy (1.8) and the estimate from below (1.19), there holds

$$\frac{d}{dt} \mathcal{E}(t) \leq -D(f) \quad \text{with } D(f) := \int_{\mathbb{R}^N \times \mathbb{R}^N} |u|^3 \Delta(\mathcal{E}, u) f f_* dv dv_*,$$

which in turn implies that  $t \mapsto D(f(t, \cdot)) \in L^1(0, \infty)$ , and then

$$(5.4) \quad \int_0^T D_\varepsilon(f_k) dt \leq \int_0^T D(f_k) dt = \int_{t_{n_k}}^{t_{n_k}+T} D(f) dt \xrightarrow[k \rightarrow \infty]{} 0.$$

Gathering (5.3) and (5.4), and letting  $\varepsilon$  goes to 0, we deduce that

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} |u|^3 \Delta(\mathcal{E}_\infty, u) d\bar{\mu} d\bar{\mu}_* = 0 \quad \text{on } (0, T).$$

The positivity (1.18) of  $\Delta(\mathcal{E}_\infty, u)$  then implies that  $\bar{\mu} = \bar{c} \delta_{v=\bar{w}}$  for some measurable functions  $\bar{w} : (0, T) \rightarrow \mathbb{R}^N$  and  $\bar{c} : (0, T) \rightarrow \mathbb{R}_+$ . Moreover, from the conservation of mass and momentum (1.32) and the bound of energy (1.33) we deduce that  $\bar{c} = 1$  and  $\bar{w} = 0$  a.e. It is then classical to deduce (by the uniqueness of the limit and the fact that it is independent on time) that (5.1) holds.  $\square$

To conclude that this weak convergence of the distribution to the Dirac mass as time goes to infinity implies the convergence of the kinetic energy to 0 (*i.e.*, the kinetic energy of the Dirac mass) we have to show that no kinetic energy is escaping at infinity as  $t \rightarrow T_c$ . To this purpose we put stronger assumptions on the collision rate. The first additional assumption **H3** roughly speaking means that the energy dissipation functional is strong enough to forbid it, whereas the second additional assumption **H4** allows to use the uniform propagation of moments of order strictly greater than 2 to forbid it.

**Proposition 5.2** *Let  $T_c \in (0, +\infty]$  be the time of life of the solution. Then if either  $T_c < +\infty$ , or  $T_c = +\infty$  and  $B$  satisfies additional assumptions **H2-H3** or **H2-H4**, we have*

$$(5.5) \quad \mathcal{E}(t) \rightarrow 0 \quad \text{when } t \rightarrow T_c.$$

*Proof of Proposition 5.2.* We split the proof in three steps.

*Step 1.* Assume first  $T_c < +\infty$ . The claim follows from the existence proof.

*Step 2.* Assume now  $T_c = +\infty$  and that  $B$  satisfies assumption **H3**: (1.19)-(1.20). We argue by contradiction: assume that  $\mathcal{E}(t) \not\rightarrow 0$ , that is, there exists  $\mathcal{E}_\infty > 0$  such that  $\mathcal{E}(t) \in (\mathcal{E}_\infty, \mathcal{E}_{\text{in}})$ . Reasoning as in Proposition 5.1, we get, for a fixed time  $T > 0$  and for any sequence  $(t_n)$  increasing and going to infinity, that there exists a subsequence  $(t_{n_k})$  and a measure  $\bar{\mu} \in L^\infty(0, T; M_2^1)$  such that the function  $f_k(t, v) := f(t_{n_k} + t, v)$  satisfies (5.2) and

$$(5.6) \quad \int_0^T D_\varepsilon^0(f_k) dt \rightarrow \int_0^T D_\varepsilon^0(\bar{\mu}) dt,$$

where we have defined for any measure (or function)  $\lambda$ :

$$D_\varepsilon^0(\lambda) := \int_{\mathbb{R}^N \times \mathbb{R}^N} |u|^3 \psi(|u|) \chi_\varepsilon(v, v_*) d\lambda(v) d\lambda(v_*).$$

From the dissipation of energy (1.8) and the estimate from below (1.19), there holds

$$(5.7) \quad \frac{d}{dt} \mathcal{E}(t) \leq -D^0(f) \quad \text{with} \quad D^0(f) := \int_{\mathbb{R}^N \times \mathbb{R}^N} |u|^3 \psi(|u|) f f_* dv dv_*,$$

which in turn implies that  $t \mapsto D^0(f(t, \cdot)) \in L^1(0, \infty)$ , and then

$$(5.8) \quad \int_0^T D_\varepsilon^0(f_k) dt \leq \int_0^T D^0(f_k) dt = \int_{t_{n_k}}^{t_{n_k}+T} D^0(f) dt \xrightarrow[k \rightarrow \infty]{} 0.$$

Gathering (5.6) and (5.8), and letting  $\varepsilon$  goes to 0, we deduce that  $D^0(\bar{\mu}) = 0$  on  $(0, T)$ . The positivity of  $\psi$  implies as in Proposition 5.1 that  $\text{supp } \bar{\mu} \subset \{0\}$  and  $\bar{\mu} \delta_{v=0}$ . As this limit is unique and independent on time we deduce that (5.1) holds.

Now, on the one hand, taking  $R = \sqrt{\mathcal{E}_\infty/2}$  there holds

$$(5.9) \quad \int_{B_R^c} f |v|^2 dv = \int_{\mathbb{R}^N} f |v|^2 dv - \int_{B_R} f |v|^2 dv \geq \mathcal{E}_\infty - R^2 \geq \mathcal{E}_\infty/2$$

for any  $t \geq 0$ . On the other hand, for  $T$  large enough, there holds thanks to (5.1)

$$(5.10) \quad \int_{B_{R/2}} f dv \geq \frac{1}{2} \quad \text{for any } t \geq T.$$

Remarking that on  $B_{R/2} \times B_R^c$  there holds, thanks to (1.20),

$$(5.11) \quad |u|^3 \psi(|u|) \geq \frac{|v_*|^3}{8} \psi\left(\frac{|v_*|}{2}\right) \geq \psi_R \frac{|v_*|^2}{4},$$

we may put together (5.7)-(5.11) and we get thanks to (5.9) and (5.10)

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &\leq - \int_{B_{R/2}} \int_{B_R^c} |v - v_*|^3 \psi(|v - v_*|) f f_* dv dv_* \\ &\leq - \frac{\psi_R}{4} \int_{B_{R/2}} f dv \int_{B_R^c} f_* |v_*|^2 dv_* \leq - \frac{\psi_R}{4} \frac{1}{2} \frac{\mathcal{E}_\infty}{2} \end{aligned}$$

for any  $t \geq T$ . This implies that  $\mathcal{E}$  becomes negative in finite time and we get a contradiction.

*Step 3.* Finally, assume that  $T_c = +\infty$  and  $B$  satisfies assumption **H4**. On the one hand, thanks to (3.6), there holds

$$\sup_{[0, \infty)} \int_{\mathbb{R}^N} f(t, v) |v|^3 dv < \infty.$$

On the other hand, arguing as in Step 2, we obtain (keeping the same notations) that (5.2) and then (from the uniform bound in  $L_3^1$ )

$$\mathcal{E}(f_k) \rightarrow \bar{\mathcal{E}} = \mathcal{E}(\bar{\mu}) \quad \text{and} \quad D(\bar{\mu}) = 0.$$

The dissipation of energy vanishing implies that

$$|u|^3 \bar{\mu} \bar{\mu}_* \equiv 0 \quad \text{or} \quad \Delta(\bar{\mathcal{E}}, u) \text{ is not positive on } (0, T) \times \mathbb{R}^{2N}.$$

In the first case we deduce that  $\bar{\mu} = \delta_{v=0}$  as in Step 2 and then  $\bar{\mathcal{E}} = \mathcal{E}(\delta_{v=0}) = 0$ . In the second case we deduce, from (1.18), that  $\bar{\mathcal{E}}$  is not positive. In both case, there exists  $\tau_k$  such that  $\tau_k \rightarrow \infty$  and  $\mathcal{E}(\tau_k) \rightarrow 0$  and therefore (5.2) holds since  $\mathcal{E}$  is decreasing.  $\square$

Now we turn to some criterions for the cooling process to occur or not in finite time.

**Proposition 5.3** *Assume that  $\alpha$  is bounded near  $\mathcal{E} = 0$ , and  $j_{\mathcal{E}}$  converges to 0 as  $\varepsilon \rightarrow 0$  uniformly near  $\mathcal{E} = 0$ , then  $T_c = +\infty$ .*

*Proof of Proposition 5.3.* It is enough to remark that, thanks to the hypothesis made on  $\alpha$  and  $j_{\mathcal{E}}$ , the *a priori* bound in Orlicz norm that one deduces from (2.13) as in Corollary 2.7 extends to all times:

$$\forall t \geq 0 \quad \|f_t\|_{L^\Lambda} \leq \|f_{\text{in}}\|_{L^\Lambda} \exp\left(C \|f_{\text{in}}\|_{L_2^1} t\right)$$

for some constant  $C$  depending on the collision rate. It shows that the energy cannot vanish in finite time.  $\square$

**Proposition 5.4** *Assume that  $B$  satisfies **H4**, that for some increasing and positive function  $\Delta_0$  there holds  $\Delta(\mathcal{E}, u) \leq \Delta_0(\mathcal{E})$  for any  $u \in \mathbb{R}^N$ ,  $\mathcal{E} \geq 0$ , and that  $f_{\text{in}} e^{r|v|^\eta} \in L^1$  for some  $r > 0$  and  $\eta \in (1, 2]$ , then  $T_c = +\infty$ .*

*Proof of Proposition 5.4.* From the dissipation of energy (1.8), the bound on  $\Delta$  and the decay of the energy (1.33), we have

$$\frac{d\mathcal{E}}{dt} \geq -\Delta_0(\mathcal{E}_{\text{in}}) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f f_* |u|^3 dv dv_* =: -\Delta_0(\mathcal{E}_{\text{in}}) (I_{1,R} + I_{2,R})$$

where

$$\begin{cases} I_{1,R} := \int_{\mathbb{R}^N \times \mathbb{R}^N} |u|^3 \mathbf{1}_{\{|u| \leq R\}} f f_* dv dv_*, \\ I_{2,R} := \int_{\mathbb{R}^N \times \mathbb{R}^N} |u|^3 \mathbf{1}_{\{|u| \geq R\}} f f_* dv dv_*. \end{cases}$$

On the one hand, for any  $R > 0$ , we have using (1.32)

$$I_{1,R} \leq R \int_{\mathbb{R}^N \times \mathbb{R}^N} |u|^2 f f_* dv dv_* = 2R\mathcal{E}.$$

On the other hand, we infer from Proposition 3.2 (since  $B$  satisfies **H4**) that

$$\sup_{t \in [0, T_c)} \int_{\mathbb{R}^N} f(t, v) e^{2r'|v|^\eta} dv \leq C_1$$

for some  $r', C_1 \in (0, \infty)$ . Therefore

$$\begin{aligned} I_{2,R} &\leq \int_{\mathbb{R}^N \times \mathbb{R}^N} (4|v|^3 + 4|v_*|^3) 2 \mathbf{1}_{\{|v| > R/2\}} f f_* dv dv_* \\ &\leq 8 e^{-r' R^\eta} \int_{\mathbb{R}^N} (1 + |v|^3) e^{r'|v|^\eta} f dv \int_{\mathbb{R}^N} (1 + |v_*|^3) f_* dv_* \leq C_2 e^{-r' R^\eta}. \end{aligned}$$

Gathering these three estimates, we deduce

$$\frac{d}{dt} \mathcal{E} \geq -C_3 R \mathcal{E} - C_3 e^{-r' R^\eta},$$

which in turns implies, thanks to a Gronwall argument,

$$\forall R > 0, \quad \inf_{t \in [0, T]} \mathcal{E}(t) \geq \mathcal{E}_{\text{in}} e^{-C_3 R T} - \frac{e^{-r' R^\eta}}{R}.$$

We conclude that  $\mathcal{E}(t) > 0$  for any  $t \in [0, T]$  and any fixed  $T > 0$ , choosing  $R$  large enough (using that  $\eta > 1$ ).  $\square$

**Proposition 5.5** *Assume  $\Delta(\mathcal{E}, u) \geq \Delta_0 \mathcal{E}^\delta$  with  $\Delta_0 > 0$  and  $\delta < -1/2$ , then  $T_c < +\infty$ .*

*Proof of Proposition 5.5.* On the one hand, from the dissipation of energy (1.8) and the bound on  $\Delta$ , we have

$$\frac{d\mathcal{E}}{dt} \leq -\Delta_0 \mathcal{E}^\delta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f f_* |u|^3 dv dv_*.$$

On the other hand, from Jensen's inequality and the conservation of mass and momentum, there holds

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f f_* |u|^3 dv dv_* \geq \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f f_* |u|^2 dv dv_* \right)^{3/2} = (2\mathcal{E})^{3/2}.$$

Gathering these two estimates, we get

$$\frac{d}{dt} \mathcal{E} \leq -\Delta_0 \mathcal{E}^{\delta+3/2}$$

and  $\mathcal{E}$  vanishes in finite time.  $\square$

## Appendix: Some facts about Orlicz spaces

The goal of this appendix is to gather some results about Orlicz spaces in order to make this paper as self-contained as possible. The definition and Hölder's inequality are recalls of results which can be found in [35] for instance. We also state and prove a simple formula for the differential of Orlicz norms, which is most probably not new, but for which we were not able to find a reference.

## Definition

We recall here the definition of Orlicz spaces on  $\mathbb{R}^N$  according to the Lebesgue measure. Let  $\Lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a function  $C^2$  strictly increasing, convex, such that

$$(A.1) \quad \Lambda(0) = \Lambda'(0) = 0,$$

$$(A.2) \quad \forall t \geq 0, \quad \Lambda(2t) \leq c_\Lambda \Lambda(t),$$

for some constant  $c_\Lambda > 0$ , and which is superlinear, in the sense that

$$(A.3) \quad \frac{\Lambda(t)}{t} \xrightarrow{t \rightarrow +\infty} +\infty.$$

We define  $L^\Lambda$  the set of measurable functions  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  such that

$$\int_{\mathbb{R}^N} \Lambda(|f(v)|) dv < +\infty.$$

Then  $L^\Lambda$  is a Banach space for the norm

$$\|f\|_{L^\Lambda} = \inf \left\{ \lambda > 0 \mid \int_{\mathbb{R}^N} \Lambda\left(\frac{|f(v)|}{\lambda}\right) dv \leq 1 \right\}$$

and it is called the *Orlicz space* associated with  $\Lambda$ . The proof of this last point can be found in [35, Chapter III, Theorem 3]. Note that the usual Lebesgue spaces  $L^p$  for  $1 \leq p < +\infty$  are recovered as particular cases of this definition for  $\Lambda(t) = t^p/p$ .

Let us mention that for any  $f \in L^1(\mathbb{R}^N)$ , a refined version of the De la Vallée-Poussin Theorem [25, Proposition I.1.1] (see also [23, 24]) guarantees that there exists a function  $\Lambda$  satisfying all the properties above and such that

$$\int_{\mathbb{R}^N} \Lambda(|f(v)|) dv < +\infty.$$

## Hölder's inequality in Orlicz spaces

Let  $\Lambda$  be a function  $C^2$  strictly increasing, convex satisfying the assumptions (A.1), (A.2) and (A.3), and  $\Lambda^*$  its *complementary Young function*, given (when  $\Lambda$  is  $C^1$ ) by

$$\forall y \geq 0, \quad \Lambda^*(y) = y(\Lambda')^{-1}(y) - \Lambda((\Lambda')^{-1}(y)).$$

It is straightforward to check that  $\Lambda^*$  satisfies the same assumptions as  $\Lambda$ . Recall Young's inequality

$$(A.4) \quad \forall x, y \geq 0, \quad xy \leq \Lambda(x) + \Lambda^*(y).$$

Then one can define the following norm on the Orlicz space  $L^{\Lambda^*}$ :

$$N^{\Lambda^*}(f) = \sup \left\{ \int_{\mathbb{R}^N} |fg| dv; \int_{\mathbb{R}^N} \Lambda(|g|) dv \leq 1 \right\}.$$

One can extract from [35, Chapter III, Section 3.4, Propositions 6 and 9] the following result

**Theorem A.1** (i) We have the following Hölder's inequality for any  $f \in L^\Lambda$ ,  $g \in L^{\Lambda^*}$ :

$$(A.5) \quad \int_{\mathbb{R}^N} |fg| dv \leq \|f\|_{L^\Lambda} N^{\Lambda^*}(g).$$

(ii) There is equality in (A.5) if and only if there is a constant  $0 < k^* < +\infty$  such that

$$(A.6) \quad \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) \left( \frac{k^*|g|}{N^{\Lambda^*}(g)} \right) = \Lambda \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) + \Lambda^* \left( \frac{k^*|g|}{N^{\Lambda^*}(g)} \right)$$

for almost every  $v \in \mathbb{R}^N$ .

## Differential of Orlicz norms

In order to propagate bounds on Orlicz norms along the flow of the Boltzmann equation, we shall need a formula for the time derivative of the Orlicz norm.

**Theorem A.2** Let  $\Lambda$  be a function  $C^2$  strictly increasing, convex satisfying (A.1), (A.2), (A.3), and let  $0 \leq f \in C^1([0, T], L^\Lambda)$  such that  $f(t, \cdot) \not\equiv 0$  for all  $t \in [0, T]$ . Then we have

$$(A.7) \quad \frac{d}{dt} \|f_t\|_{L^\Lambda} = \left[ N^{\Lambda^*} \left( \Lambda' \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) \right) \right]^{-1} \int_{\mathbb{R}^N} \partial_t f \Lambda' \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) dv.$$

*Proof of Theorem A.2.* From [35, Chapter III, Proposition 6]), our assumptions on  $\Lambda$  imply that

$$(A.8) \quad \int_{\mathbb{R}^N} \Lambda \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) dv = 1$$

for all  $0 \neq f \in L^\Lambda$ . By differentiating this quantity along  $t$  we deduce:

$$0 = \int_{\mathbb{R}^N} \partial_t f \Lambda' \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) dv - \frac{1}{\|f_t\|_{L^\Lambda}} \frac{d}{dt} \|f_t\|_{L^\Lambda} \int_{\mathbb{R}^N} f \Lambda' \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) dv.$$

Now using the case of equality in Hölder's inequality (A.5) we have

$$\int_{\mathbb{R}^N} f \Lambda' \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) dv = \|f\|_{L^\Lambda} N^{\Lambda^*} \left( \Lambda' \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right) \right)$$

since the equality (A.6) is trivially satisfied with

$$g = \Lambda' \left( \frac{|f|}{\|f\|_{L^\Lambda}} \right)$$

and  $k^* = N^{\Lambda^*}(g)$ , using that

$$xy = \Lambda(x) + \Lambda^*(y)$$

as soon as  $y = \Lambda'(x)$ . This concludes the proof.  $\square$

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