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## Stability for Walls in Ferromagnetic Nanowire

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**Summary.** We study the stability of travelling wall profiles for a one dimensional model of ferromagnetic nanowire submitted to an exterior magnetic field. We prove that these profiles are asymptotically stable modulo a translation-rotation for small applied magnetic fields.

### 1 Model for ferromagnetic nanowires

Ferromagnetic materials are characterized by a spontaneous magnetization described by the magnetic moment  $u$  which is a unitary vector field linking the magnetic induction  $B$  with the magnetic field  $H$  by the relation  $B = H + u$ . The variations of  $u$  are described by the Landau-Lifschitz Equation

$$\frac{\partial u}{\partial t} = -u \wedge H_e - u \wedge (u \wedge H_e) \quad (1)$$

where the effective field is given by  $H_e = \Delta u + h_d(u) + H_a$ , and the demagnetizing field  $h_d(u)$  is deduced from  $u$  solving the magnetostatic equations:

$$\operatorname{div} B = \operatorname{div} (H + u) = 0 \text{ and } \operatorname{curl} H = 0$$

where  $H_a$  is an applied magnetic field.

For more details on the ferromagnetism model, see [2], [9], [14] and [18]. For existence results about Landau Lifschitz equations see [3], [4], [10], [17]. For numerical studies see [8], [12] and [13]. For asymptotic studies see [1], [5], [7], [15] and [16].

In this paper we consider an asymptotic one dimensional model of ferromagnetic nanowire submitted to an applied field along the axis of the wire. We denote by  $(e_1, e_2, e_3)$  the canonical basis of  $\mathbb{R}^3$ . The ferromagnetic nanowire is assimilated to the axis  $\mathbb{R}e_1$ . The demagnetizing energy is approximated by the formula  $h_d(u) = -u_2e_2 - u_3e_3$  where  $u = (u_1, u_2, u_3)$  (this approximation

of the demagnetizing energy for a ferromagnetic wire is obtained using a BKW method by D. Sanchez, taking the limit when the diameter of the wire tends to zero in [16]). We assume in addition that an exterior magnetic field  $\delta e_1$  is applied along the wire axis.

To sum up we study the following system

$$\begin{cases} \frac{\partial u}{\partial t} = -u \wedge h_\delta(u) - u \wedge (u \wedge h_\delta(u)) \\ \text{with } h_\delta(u) = \frac{\partial^2 u}{\partial x^2} - u_2 e_2 - u_3 e_3 + \delta e_1 \end{cases} \quad (2)$$

For  $\delta = 0$ , that is without applied field, we observe in physical experiments the formation of a wall breaking down the domain in two parts: one in which the magnetization is almost equal to  $e_1$  and another in which the magnetization is almost equal to  $-e_1$ . Such a distribution is described in our one dimensional model by the following profile  $M_0$ :

$$M_0 = \begin{pmatrix} \text{th } x \\ 0 \\ \frac{1}{\text{ch } x} \end{pmatrix}. \quad (3)$$

This profile is a steady state solution of Equation (2) with  $\delta = 0$ . We prove in [6] the stability of the profile  $M_0$  for Equation (2) without applied field (when  $\delta = 0$ ).

When we apply a magnetic field in the direction  $+e_1$  (that is with  $\delta > 0$ ) since Landau-Lifschitz Equation tends to align the magnetic moment with the effective field, we observe a translation of the wall in the direction  $-e_1$ . Furthermore, we observe a rotation of the magnetic moment around the wire axis. This phenomenon is described by the solution of (2)

$$U_\delta(t, x) = R_{\delta t}(M_0(x + \delta t)) \quad (4)$$

where  $R_\theta$  is the rotation of angle  $\theta$  around the axis  $\mathbb{R}e_1$ :

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

We study in this paper the stability of  $U_\delta$ , we prove that for a small  $\delta$ ,  $U_\delta$  is stable for the  $H^2$  norm and asymptotically stable for the  $H^1$  norm, modulo a translation in the variable  $x$  and a rotation around  $\mathbb{R}e_1$ . This result is claimed in the following theorem:

**Theorem 1.** *There exists  $\delta_0 > 0$  such that for all  $\delta$  with  $|\delta| < \delta_0$  then for  $\varepsilon > 0$  there exists  $\eta > 0$  such that if  $\|u(t = 0, x) - U_\delta(t = 0, x)\|_{H^2} < \eta$  then the solution  $u$  of Equation (2) with initial data  $u(t = 0, x)$  satisfies:*

$$\forall t > 0, \|u(t, x) - U_\delta(t, x)\|_{H^2} < \varepsilon.$$

In addition there exists  $\sigma_\infty$  and  $\theta_\infty$  such that

$$\|u(t, x) - R_{\theta_\infty}(U_\delta(t, x + \sigma_\infty))\|_{H^1} \longrightarrow 0 \text{ when } t \longrightarrow +\infty.$$

This result is a generalization of the stability result concerning the static walls when  $\delta = 0$  in [6]. It looks like the theorems of stability concerning the travelling waves solutions for semilinear equations like Ginzburg Landau Equation (see Kapitula [11]). Here we have three new difficulties. The first one is that the magnetic moment takes its values in the sphere and not in a linear space. In order to work with maps with values in a linear space we will use a mobile frame adapted to Landau-Lifschitz equation and we will describe in Section 2 the magnetic moment in this mobile frame. The second difficulty is that we have here a two dimensional invariance family for Equation (2) whereas Ginzburg Landau-Equation is only invariant by translation. This is the reason why we must use in the perturbations description the translations and the rotations (see Section 3). The last difficulty is that Landau-Lifschitz Equation is quasilinear, and then we have to couple variational estimates and semi-group estimates to control the perturbations of our profiles. Section 4 is devoted to these estimates.

## 2 Landau-Lifschitz Equation in the mobile frame

### 2.1 First reduction of the problem

For  $u$  a solution of Landau-Lifschitz Equation (2) we define  $v$  by  $v(t, x) = R_{-\delta t}(u(t, x - \delta t))$  (that is  $u(t, x) = R_{\delta t}(v(t, x + \delta t))$ ). A straightforward calculation gives that  $u$  satisfies (2) if and only if  $v$  satisfies

$$\begin{cases} \frac{\partial v}{\partial t} = -v \wedge h(v) - v \wedge (v \wedge h(v)) - \delta \left( \frac{\partial v}{\partial x} + v_1 v - e_1 \right) \\ h(v) = \frac{\partial^2 v}{\partial x^2} - v_2 e_2 - v_3 e_3 \end{cases} \quad (5)$$

In addition  $U_\delta$  is stable for (2) if and only if  $M_0$  is stable for (5), that is we are led to study the stability of a static profile, which is more convenient.

### 2.2 Mobile frame

Let us introduce the mobile frame  $(M_0(x), M_1(x), M_2)$ , where

$$M_1(x) = \begin{pmatrix} 1 \\ \operatorname{ch} x \\ 0 \\ -\operatorname{th} x \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Let  $v : \mathbb{R}_t^+ \times \mathbb{R}_x \longrightarrow S^2 \subset \mathbb{R}^3$  be a little perturbation of  $M_0$ . We can decompose  $v$  in the mobile frame writing

$$v(t, x) = r_1(t, x)M_1(x) + r_2(t, x)M_2 + \sqrt{1 - r_1^2 - r_2^2}M_0(x).$$

Now we can obtain a new version of Landau-Lifschitz Equation:  $v$  satisfies (5) if and only if  $r = (r_1, r_2)$  satisfies

$$\frac{\partial r}{\partial t} = (\mathcal{L} + \delta l)r + G(r)\left(\frac{\partial^2 r}{\partial x^2}\right) + H\left(x, r, \frac{\partial r}{\partial x}\right) \quad (6)$$

where

- the linear operator  $\mathcal{L}$  is given by  $\mathcal{L} = JL$  with  $J = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$  and

$$L = -\frac{\partial^2}{\partial x^2} + 2\text{th}^2 x - 1$$

- the linear perturbation due to the presence of the applied magnetic field  $\delta e_1$  is given by  $\delta l$  with  $l = \frac{\partial}{\partial x} + \text{th} x$ ,
- the higher degree non linear part is  $G(r)\left(\frac{\partial^2 r}{\partial x^2}\right)$ , where  $G(r)$  is a matrix depending on  $r$  with  $G(0) = 0$ ,
- the last non linear term  $H\left(x, r, \frac{\partial r}{\partial x}\right)$  is at least quadratic in the variable  $\left(r, \frac{\partial r}{\partial x}\right)$ .

In addition the stability of the profile  $M_0$  for Equation (5) is equivalent to the stability of the zero solution for Equation (6).

### 3 A new system of coordinates

We remark that  $L$  is a self adjoint operator on  $L^2(\mathbb{R})$ , with domain  $H^2(\mathbb{R})$ . Furthermore,  $L$  is positive since we can write  $L = l^* \circ l$  with  $l = \frac{\partial}{\partial x} + \text{th} x$ , and  $\text{Ker } L$  is the one dimensional space generated by  $\frac{1}{\text{ch } x}$ .

The matrix  $J$  being invertible,  $\text{Ker } \mathcal{L}$  is the two dimensional space generated by  $v_1$  and  $v_2$  with

$$v_1(x) = \begin{pmatrix} 0 \\ 1 \\ \text{ch } x \end{pmatrix}, \quad v_2(x) = \begin{pmatrix} 1 \\ \text{ch } x \\ 0 \end{pmatrix}$$

We introduce  $\mathcal{E} = (\text{Ker } \mathcal{L})^\perp$ . We denote by  $Q$  the orthogonal projection onto  $\mathcal{E}$  for the  $L^2(\mathbb{R})$  scalar product.

Landau-Lifschitz equation (5) is invariant by translation in the variable  $x$  and by rotation around the axis  $e_1$ . Therefore for  $\Lambda = (\theta, \sigma)$  fixed in  $\mathbb{R}^2$ ,  $M_\Lambda$  defined by  $M_\Lambda(x) = R_\theta(M_0(x-\sigma))$  is a solution of Equation (5). We introduce  $R_\Lambda(x)$  the coordinates of  $M_\Lambda(x)$  in the mobile frame  $(M_1(x), M_2(x))$ :

$$R_\Lambda(x) = \begin{pmatrix} M_\Lambda(x) \cdot M_1(x) \\ M_\Lambda(x) \cdot M_2(x) \end{pmatrix}$$

The map  $\Psi$  given by

$$\begin{aligned} \Psi : \mathbb{R}^2 \times \mathcal{E} &\longrightarrow H^2(\mathbb{R}) \\ (\Lambda, W) &\longmapsto r(x) = R_\Lambda(x) + W(x) \end{aligned}$$

is a diffeomorphism in a neighborhood of zero. Thus we can write the solution  $r$  of Equation (6) on the form :

$$r(t, x) = R_{\Lambda(t)}(x) + W(t, x)$$

where for all  $t$ ,  $W(t) \in \mathcal{E}$  and where  $\Lambda : \mathbb{R}_t^+ \mapsto \mathbb{R}^2$ .

We will re-write Equation (5) in the coordinates  $(\Lambda, W)$ . Taking the scalar product of (5) with  $v_1$  and  $v_2$  we obtain the equation satisfied by  $\Lambda$ , and using  $Q$  the orthogonal projection onto  $\mathcal{E}$ , we deduce the equation satisfied by  $W$ . After this calculation we obtain that  $r$  is solution of Equation (5) if and only if  $(\Lambda, W)$  satisfies the following system

$$\begin{cases} \frac{\partial W}{\partial t} = (\mathcal{L} + \delta l + \mathcal{K}_\Lambda)W + \mathcal{R}_1(x, \Lambda, W)\left(\frac{\partial^2 W}{\partial x^2}\right) + \mathcal{R}_2(x, \Lambda, W, \frac{\partial W}{\partial x}) \\ \frac{d\Lambda}{dt} = \mathcal{M}(W, \frac{\partial W}{\partial x}, \Lambda) \end{cases} \quad (7)$$

where

- $\mathcal{K}_\Lambda : H^2(\mathbb{R}) \longrightarrow \mathcal{E}$  is a linear map satisfying

$$\exists K_1, \forall \Lambda \in \mathbb{R}^2, \forall W \in \mathcal{E}, \|\mathcal{K}_\Lambda W\|_{L^2(\mathbb{R})} \leq K_1 |\Lambda| \|W\|_{H^2(\mathbb{R})} \quad (8)$$

- the non linear terms take their values in  $\mathcal{E}$  and satisfy that there exists a constant  $K_2$  such that for  $|\Lambda| \leq 1$  and for all  $W \in \mathcal{E}$

$$\|\mathcal{R}_1(\cdot, \Lambda, W)\left(\frac{\partial^2 W}{\partial x^2}\right)\|_{L^2(\mathbb{R})} \leq K_2 \|W\|_{H^1(\mathbb{R})} \|W\|_{H^2(\mathbb{R})} \quad (9)$$

$$\|\mathcal{R}_2(\cdot, \Lambda, W, \frac{\partial W}{\partial x})\|_{H^1(\mathbb{R})} \leq K_2 \|W\|_{H^1(\mathbb{R})}^2$$

- $\mathcal{M} : H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$  satisfies

$$\exists K_3, \forall \Lambda \text{ such that } |\Lambda| \leq 1, \forall W \in \mathcal{E}, \left| \mathcal{M}(W, \frac{\partial W}{\partial x}, \Lambda) \right| \leq K_3 \|W\|_{H^1(\mathbb{R})} \quad (10)$$

Theorem 1 is equivalent to the following Proposition:

**Proposition 1.** *There exists  $\delta_0 > 0$  such that for  $\delta$  with  $|\delta| < \delta_0$ , we have the following stability result for Equation (7): for  $\varepsilon > 0$  there exists  $\eta > 0$  such that if  $|A_0| < \eta$  and if  $\|W_0\|_{H^2} < \eta$  then the solution  $(A, W)$  of (7) with initial value  $(A_0, W_0)$  satisfies*

1. for all  $t > 0$ ,  $\|W(t)\|_{H^2} \leq \varepsilon$  and  $|A| \leq \varepsilon$ ,
2.  $\|W(t)\|_{H^1}$  tends to zero when  $t$  tends to  $+\infty$ ,
3. there exists  $\Lambda_\infty \in \mathbb{R}^2$  such that  $\Lambda(t)$  tends to  $\Lambda_\infty$  when  $t$  tends to  $+\infty$ .

The last section is devoted to the proof of Proposition 1.

## 4 Estimates for the perturbations

### 4.1 Linear semi group estimates

On  $\mathcal{E}$  we have  $Re(\text{sp } \mathcal{L}) \subset ]-\infty, -1]$ . In particular this fact implies that the  $H^2$  norm is equivalent on  $\mathcal{E}$  to the norm  $\|\mathcal{L}u\|_{L^2}$ . Furthermore it implies good decreasing properties for the semigroup generated by  $\mathcal{L}$ . We first prove that this decreasing property is preserved for the linear part of the Equation on  $W$  in (7) for a little applied field, and if we assume that  $A$  remains little.

The operator  $l$  is an order one operator dominated on  $\mathcal{E}$  by  $\mathcal{L}$ , thus there exists  $\delta_0 > 0$  such that if  $|\delta| < \delta_0$ ,  $Re(\text{sp } \mathcal{L} + \delta l) \subset ]-\infty, -1/2[$ .

Let us fix  $\delta$  such that  $|\delta| < \delta_0$ . With Estimate (10), if  $A$  remains small,  $\mathcal{K}_A$  is a little perturbation of  $\mathcal{L} + \delta l$ . This implies that for  $A$  little, the semigroup generated by  $\mathcal{L} + \delta l + Q\mathcal{K}_A$  has the same good decreasing properties than  $\mathcal{L}$ , that is there exists  $\nu_0 > 0$  such that if  $|A(t)|$  remains less than  $\nu_0$  for all  $t$ , then there exists  $K_4$  and  $\beta > 0$  such that

$$\begin{aligned} \|S_A(t)W_0\|_{H^1} &\leq K_4 e^{-\beta t} \|W_0\|_{H^1} \\ &\leq K_4 \frac{e^{-\beta t}}{\sqrt{t}} \|W_0\|_{L^2}. \end{aligned} \quad (11)$$

We can then use the Duhamel formula to solve the equation on  $W$  in (7):

$$W(t) = S_A(t)W_0 + \int_0^t S_A(t-s)\mathcal{R}_1(s)ds + \int_0^t S_A(t-s)\mathcal{R}_2(s)ds$$

and then using the estimates (9) and (11) we obtain that if  $|A(t)|$  remains less than  $\nu_0$  then there exists  $K_5$  such that

$$\begin{aligned} \|W(t)\|_{H^1} &\leq K_5 e^{-\beta t} \|W_0\|_{H^1} + \int_0^t K_5 \frac{e^{-\beta(t-s)}}{\sqrt{t-s}} \|W(s)\|_{H^1} \|W(s)\|_{H^2} \\ &\quad + \int_0^t K_5 e^{-\beta(t-s)} \|W(s)\|_{H^1}^2 \end{aligned} \quad (12)$$

## 4.2 Variational estimates

We see that Estimate (12) is not sufficient to conclude since we have the  $H^2$  norm of  $W$  in the right hand side of this estimate. In order to dominate this  $H^2$  norm, we multiply the equation on  $W$  in (7) by  $J^2 \mathcal{L}^2 W$  and we obtain that there exists a constant  $K_6$ :

$$\frac{d}{dt} \|LW\|_{L^2}^2 + \|L^{\frac{3}{2}}W\|_{L^2}^2 (1 - K_6 \|LW\|_{L^2}) \leq 0$$

From this estimate we deduce that if  $\|LW\|_{L^2} < \frac{1}{K_6}$ , then  $1 - K_6 \|LW\|_{L^2}$  is positive, thus  $\frac{d}{dt} \|LW\|_{L^2}^2$  is negative and  $\|LW\|_{L^2}$  remains less than  $\frac{1}{K_6}$ . So if  $\|LW_0\|_{L^2} < \frac{1}{K_6}$ , then for all  $t$   $\|LW(t)\|_{L^2} \leq \|LW_0\|_{L^2}$ . This property gives a bound for the  $H^2$  norm of  $W$  since the  $H^2$  norm is equivalent on  $\mathcal{E}$  to  $\|LW\|_{L^2}$ , and reducing the  $H^2$  norm of  $W_0$ , we obtain the first part of the conclusion 1 in Proposition 1.

## 4.3 Conclusion

Let us assume that  $\|LW_0\|_{L^2(\mathbb{R})} \leq \frac{1}{K_6}$ . Then for all  $t$ ,  $\|W(t)\|_{H^2(\mathbb{R})} \leq C_1 \|LW(t)\|_{L^2} \leq \|LW_0\|_{L^2} \leq C_2 \|W_0\|_{H^2(\mathbb{R})}$ , where  $C_1$  and  $C_2$  are constants.

Multiplying (12) by  $(1+t)^2$ , defining  $G(t) = \max_{[0,T]} (1+s)^2 \|W(s)\|_{H^1}$ , we obtain that there exists a constant  $K_7$  such that if  $|\Lambda(t)|$  remains less than  $\nu_0$  we have:

$$G(t) \leq K_7 G(0) + K_7 G(t) \|W_0\|_{H^2} + K_7 (G(t))^2$$

If we suppose in addition that  $\|W_0\|_{H^2} \leq \frac{1}{2K_7}$  we obtain that

$$0 \leq K_7 G(0) - \frac{1}{2} G(t) + K_7 (G(t))^2 := P(G(t)) \quad (13)$$

The polynomial map  $P(\xi) = K_7 \xi^2 - \frac{1}{2} \xi + K_7 G(0)$  has for  $G(0)$  small enough two positive roots. We denote by  $\xi(G(0))$  the smallest one. For  $G(0)$  little enough we have  $G(0) \leq \xi(G(0)) \leq 2K_7 G(0)$  (we can a priori assume that  $K_7 \geq 1$  for example). Estimate (13) implies that for all  $t$ ,  $G(t) \leq \xi(G(0))$  that is

$$\forall t > 0, \|W(t)\|_{H^1(\mathbb{R})} \leq \frac{\xi(G(0))}{1+t^2} \leq \frac{2K_7 G(0)}{1+t^2}. \quad (14)$$

This implies that  $\|W(t)\|_{H^1(\mathbb{R})}$  tends to zero when  $t$  tends to  $+\infty$ . It remains to prove that  $\Lambda$  remains less than  $\nu_0$  and admits a limit when  $t$  tends to  $+\infty$ .

Plugging Estimate (14) in the equation on  $\Lambda$  in (7) and using (10), we obtain that  $\frac{d\Lambda}{dt}$  is integrable on  $\mathbb{R}^+$ , that is  $\Lambda$  admits a limit when  $t$  tends to  $+\infty$ . Furthermore, by integration we have

$$\forall t, |\Lambda(t)| \leq |\Lambda(0)| + \int_0^t K_3 \frac{2K_7 G(0)}{1+s^2} ds \leq |\Lambda(0)| + \pi K_3 K_7 G(0)$$

Reducing  $|\Lambda_0|$  and  $G(0) = \|W_0\|_{H^1(\mathbb{R})}$  we obtain that for all  $t$ ,  $|\Lambda(t)|$  remains less than  $\nu_0$ , which justifies all our estimates a posteriori.

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