GEOMETRIC ANALYSIS OF MINIMUM TIME KEPLERIAN ORBIT TRANSFERS

B. Bonnard[1], **J.-B. Caillau**[2], **E. Trélat**[3]

1) Université de Bourgogne, F-21078 Dijon, `Bernard.Bonnard@u-bourgogne.fr`

2) ENSEEIHT-IRIT, F-31071 Toulouse, `caillau@enseeiht.fr`

3) Université Paris-Sud, F-91405 Orsay, `emmanuel.trelat@math.u-psud.fr`

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Abstract

The minimum time control of the Kepler equation is considered. The typical application is the transfer of a satellite from an orbit around the Earth to another one, both orbits being elliptic. We recall the standard model to represent the system. Its Lie algebraic structure is first analyzed, and controllability is established for two different single-input subsystems, the control being oriented by the velocity or by the orthoradial direction. In both cases, a preliminary analysis of singular and regular extremals is also given, using the usual concept of order to classify the contacts. Moreover, the singularity of the multi-input model—which is a particular case of a subriemannian system with drift—is resolved, and the related nilpotent model is given. Finally, second order optimality conditions are recalled, for smooth regular and singular extremals. For both, the algorithms to compute conjugate points are detailed and applied to check numerically the time optimality of orbit transfers.

Extended Abstract

This work is aimed at investigating the control of the Kepler equation

$$\ddot{q} = -\mu \frac{q}{|q|^3} + \frac{F}{m}\cdot \tag{1}$$

Hereabove, q stands for the position in $\mathbf{R}^3$, m is the mass of the body (a satellite, in practice [4]) in the gravitational field (μ is the gravitation constant of the Earth) and F is the thrust of the engine. We take into account the variation of the mass due to fuel consumption,

$$\dot{m} = -\beta |F| \tag{2}$$

where β is a positive constant, and there is a constraint on the thrust:

$$|F| = \sqrt{F^2} \leq F_{\max}. \tag{3}$$

If $c = q \wedge \dot{q}$ is the angular momentum, $L = -\mu q/|q| + \dot{q} \wedge c$ the Laplace integral, and $H = 1/2\ \dot{q}^2 - \mu/|q|$ the energy, we define the elliptic domain

$$\Sigma_e = \{c \neq 0,\ H < 0\}.$$

Then, to each (c, L) is this domain corresponds a unique oriented ellipse describing the free motion of (1). In order to have a geometric representation of the ellipse osculating to a controlled trajectory, it is convenient to choose equinoctial elements as coordinates in the elliptic domain: the so-called Gauss equations arise from such a choice once the moving frame for the thrust is fixed. There are two possibilities: a tangent-normal frame F_t, F_n, F_c where F_t is oriented by $\dot{q}$,

$$F_t = \frac{\dot{q}}{|\dot{q}|}\frac{\partial}{\partial \dot{q}} \quad F_c = \frac{q \wedge \dot{q}}{|q \wedge \dot{q}|}\frac{\partial}{\partial \dot{q}}$$

and $F_n = F_c \wedge F_t$, or a radial-orthoradial frame F_r, F_{or}, F_c with

$$F_r = \frac{q}{|q|}\frac{\partial}{\partial \dot{q}} \quad F_{\mathrm{or}} = F_c \wedge F_r.$$

In both cases, we get a 2D-model by setting the component of the control on F_c to zero.

The Lie algebraic analysis of the orbit transfer is performed on the 2D-single-input affine systems $\dot{x} = F_0 + uF_1$ defined by each frame: $F_1 = F_t$ (tangential thrust only) and $F_1 = F_{\mathrm{or}}$ (orthoradial). In the first case, the 2D-elliptic domain is controllable and the analysis consists in studying singular and regular extremals. Here, any singular extremal is of order 2 (see [1]) and the classification is done according to Legendre-Clebsch condition: we only have elliptic extremals, locally time maximizing [3]. Regular extremals are classified by the order of their contact with $\{H_1 = 0\}$ (where $H_1 = \langle p, F_1 \rangle$ is the Hamiltonian lift of F_1, and p the adjoint state): in the fold case (order 2 contact, see [1]), only elliptic and parabolic points are allowed, and the control is bang-bang (no Fuller phenomenon). As for the second case, namely the orthoradial thrust, the 2D-elliptic

domain is still controllable, but the analysis is more intricate. A first result is that all exceptional singular extremals are of order 2: they are locally time minimizing and provide a feedback of the 2D-system. The complete multi-input equation can be analyzed in both frames as a subriemannian system with drift,

$$\dot{x} = F_0 + \sum_{i=1}^{m} u_i F_i,\ \ |u| \leq 1$$

provided we drop momentarily the mass equation (2). We have smooth regular extremals, and the problem is to classify the contacts with $\{\Phi = 0\}$ where $\Phi = (H_1, \ldots, H_m)$ is the switching function (again, the H_i's are the Hamiltonian lifts of the corresponding vector fields). The Pontryagin maximum principle parameterizes the extremals, $u = \Phi/|\Phi|$ outside the switch surface, and we make a singularity resolution in dimension $m = 2$. In particular, when the distribution spanned by the F_i's is involutive, it is possible to connect smooth extremals with the control rotating instanteanously of an angle π, and to give the associated nilpotent model.

Finally, the question of the (local) optimality status of the previous extremals is addressed by means of second order conditions. The concept of conjugate and, more generally, focal point is recalled, first for smooth regular extremals: from a computational point of view, if z is such an extremal, if $z(T) \in M_1^{\perp}$ is the end-point transversality condition, one has to evaluate along the extremal the dimension of the set of Jacobi curves solution to the linearized system with linearized end-point condition, $\delta z(T) \in T_{z(T)} M_1^{\perp}$. Numerically, the rank is checked by computing an appropriate determinant, and the first zero gives the first conjugate time t_{1c}: before t_{1c}, the trajectory is $\mathscr{C}^0$-locally time minimizing among all admissible trajectories, whereas after t_{1c} it is not even an L^{∞}-local minimizer with respect to the control [3, 5]. An example of a such a computation is detailed for a 3D-transfer around the Earth. The case of singular extremals of single-input affine systems is also treated. Two algorithms are proposed, one using the Goh transformation [3], the other being intrinsic [2]. The second method is used to check the optimality status of exceptional singular trajectories of the 2D-orbit transfer model.

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