CONJUGATE TIMES FOR SMOOTH SINGULAR TRAJECTORIES AND BANG-BANG EXTREMALS

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Abstract: In this paper we discuss the problem of estimating conjugate times along smooth singular or bang-bang extremals. For smooth extremals conjugate times can be defined in the generic case by using the intrinsic second order derivative or the exponential mapping. An algorithm is given which was implemented in the SR-case to compute the caustic [1] or in the recent applied problems [5],[9]. We investigate briefly the problem of using this algorithm in the bang-bang case by smoothing the corners of the extremals.

Keywords: Conjugate time, intrinsic second order derivative, exponential mapping, bang-bang control.

1. INTRODUCTION

Consider a control system

$$\dot{q} = f(q, u), \qquad q(0) = q_0 \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is smooth, $q_0 \in \mathbb{R}^n$, and the set of admissible controls $\mathcal{U}$ is the set a essentially bounded functions $u : [0, T(u)] \to \Omega \subset \mathbb{R}^m$. We consider the time minimal control problem.

The maximum principle (see [10]) provides necessary conditions for a trajectory to be optimal. Consequently having selected a reference extremal, i.e a trajectory $q(.)$ associated with a control $u(.)$ satisfying the necessary conditions of the maximum principle, it is natural to ask wether this trajectory is actually optimal or not.

This paper [1] describes methods to analyze local optimality first for smooth singular extremals, i.e. when $u(.)$ is smooth, and then for bang-bang extremals, i.e when $u(.)$ is piecewise constant. In the second case, the times t when $u(t)$ is not smooth are called switching times.

For smooth extremals, the first time when the so-called second order intrinsic derivative has a vanishing eigenvalue, called the first conjugate time, is the time when the reference singular trajectory loses local optimality when the set of controls is endowed with the L^∞-topology, see Section 2.3. In an equivalent approach, the first conjugate time is the first time when the so-called

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exponential mapping is not of full rank. This provides a way to compute numerically conjugate points, see Section 2.4.

For bang-bang extremals, the loss of local optimality in the L^1-topology, also called conjugate time, is again related to the vanishing of an eigenvalue of a quadratic form, obtained by allowing variation on the switching times. We present in the last section new ideas to estimate first conjugate time in this case.

2. CONJUGATE TIME FOR SMOOTH TRAJECTORIES

2.1 Intrinsic second order derivative

Definition 1. Let $T > 0$ and $q_0 \in \mathbb{R}^n$ fixed. The *end-point mapping at time T of system (1) is the mapping*

$$E^T \; : \begin{array}{l} \mathcal{U} \longrightarrow \mathbb{R}^n \\ u \longmapsto x_u(T) \end{array}$$

where x_u is the trajectory associated to u.

If we endow the set of inputs with the L^∞-topology, it is well known that the end-point mapping is C^∞ and the successive derivatives can be computed as follows. Take a reference control $u \in L^\infty[0,T]$ and assume that the corresponding trajectory, denoted in short $q(t)$, is defined on the whole $[0,T]$. Then the end-point mapping, E^T with $q(0) = q_0$ is defined on a neighborhood V of u. Let $q(t) + \xi(t)$ be the trajectory associated to $u + v \in V$. Since f is smooth we have:

$$f(q+\xi, u+v) = f(q,u) + \frac{\partial f}{\partial q}(q,u)\xi + \frac{\partial f}{\partial u}(q,u)v + \frac{\partial^2 f}{\partial q \partial u}(q,u)(\xi,v) + \frac{1}{2}\frac{\partial^2 f}{\partial q^2}(q,u)(\xi,\xi) + \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(q,u)(v,v) + \dots$$

Moreover we have:

$$\dot{q} + \dot{\xi} = f(q+\xi, u+v)$$

and ξ can be written as $\delta_1 q + \delta_2 q + \dots$, where $\delta_1 q$ is linear in v, $\delta_2 q$ is quadratic in v, ... and are given by identification as:

$$\dot{\delta_1 q} = \frac{\partial f}{\partial q}(q,u)\delta_1 q + \frac{\partial f}{\partial u}(q,u)v \tag{2}$$

$$\dot{\delta_2 q} = \frac{\partial f}{\partial q}(q,u)\delta_2 q + \frac{\partial^2 f}{\partial q \partial u}(q,u)(\delta_1 q, v) + \frac{1}{2}\frac{\partial^2 f}{\partial q^2}(q,u)(\delta_1 q, \delta_1 q) + \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(q,u)(v,v) \tag{3}$$

where $\delta_1 q(0) = \delta_2 q(0) = 0$ if the initial point is fixed to $q(0) = q_0$.

Hence $\delta_1 q$ and $\delta_2 q$ can be computed integrating linear differential equations along the reference control trajectory. Then the Fréchet first derivative of E^T at u:

$$E_u^{T'}.v = \delta_1 q(T) \tag{4}$$

Definition 2. We call singular a control $u(t)$, $0 \leq t \leq T$ with trajectory defined on $[0,T]$ such that the end-point mapping is singular at u, that is the Fréchet derivative of the end-point mapping is not surjective when evaluated on u.

The singular control is of corank 1 if the there exists a unique vector ψ (up to a scalar) such that $\langle \psi, E_u^{T'}(v) \rangle = 0$, $\forall v \in L^\infty$.

Remark 3. If a control u is singular on $[0,T]$ then it is singular on $[0,t]$ for all $t \in]0,T]$.

Definition 4. Let u be a singular control of corank 1 with response $q(t)$ defined on $[0,T]$. Let $N \subset L^\infty[0,T]$ be the kernel of $E_u^{T'}$ evaluated at u and let $\psi \in \mathbb{R}^n \backslash \{0\}$ be the unique vector (up to a scalar) such that $\langle \psi, E_u^{T'}(v) \rangle = 0$, $\forall v \in L^\infty$. The intrinsic second order derivative associated to (q,p,u) is the quadratic form defined on N by:

$$E_u^{T''} : v \in N \mapsto \langle \psi, d^2 E_u^T(v,v) \rangle$$

In practice, the intrinsic second order derivative can be computed as follows:

$$E_u^{T''} : v \in N \mapsto \langle \psi, \delta_2 q(T) \rangle$$

where $\delta_2 q$ is the second variation defined by (3).

2.2 Exponential mapping

Theorem 5. (Maximum principle [10]). Consider the time minimal control problem with boundary conditions: $q(0) = q_0$, $q(T) = q_1$. If u is optimal with response $q(t)$ defined on $[0,T]$, then there exists $t \mapsto p(t) \in \mathbb{R}^n \backslash \{0\}$ absolutely continuous on $[0,T]$ such that the following conditions hold a.e. on $[0,T]$:

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{5}$$

$$H(q(t),p(t),u(t)) = \max_{v \in \Omega} H(q(t),p(t),v) \tag{6}$$

where $H = \langle p, f(q,u) \rangle$. Moreover

$$H(q(t),p(t),u(t)) \quad \text{is constant a.e on } [0,T] \text{ and non negative} \tag{7}$$

Definition 6. A solution of (5)-(7) is called an extremal. The extremal is said normal if $H(q,p,u) > 0$ and abnormal if $H(q,p,u) = 0$ along the extremal.

Remark 7. Let us assume Ω to be open. Then the maximum condition (6) may be rewritten

$$\frac{\partial H}{\partial u} = 0 ,$$

and all extremals are singular.

Let $z = (q, p)$ be a reference extremal associated to a singular control u. We assume that $q(.)$ is one-to-one and we need generic hypotheses:
Assumptions:

(H_1) The system is real analytic.
(H_2) The co-dimension of the singularity is one.
(H_3) The strong Legendre condition $\partial^2 H/\partial u^2 < 0$ is satisfied along the reference extremal.
(H_4) The reference extremal is normal.

Notations: Using the implicit function theorem and (H_3), the extremal control can be computed as a dynamic feedback $\hat{u}(q, p)$ solving $\partial H/\partial u = 0$. Let us denote $\hat{H}(q, p) = H(q, p, \hat{u}(q, p))$ the Hamiltonian function. Then under the previous assumptions our reference extremal is embedded into a family of extremals solutions of the Hamiltonian differential equation associated to $\hat{H}$.

Let $q(t, q_0, t_0)$, $p(t, q_0, p_0)$ denote such an extremal starting at time $t = 0$ from q_0, p_0 and defined on $[0, T(p_0)[$. By homogeneity, we can assume that p_0 belongs to the unit sphere and from (H_4), $\hat{H} > 0$. Hence we can restrict p_0 to $\mathbb{P}^{n-1}$, the projective space in $\mathbb{R}^n$.

Definition 8. Fix q_0 and define the *exp mapping* by $\exp_{q_0}$:

$$\begin{array}{llll} [0, T(p_0)[& \times \; \mathbb{P}^{n-1} & \rightarrow \mathbb{R}^n & \\ (t & , \quad p_0) & \mapsto \exp_{q_0}(t, p_0) = q(t, q_0, p_0) \end{array}$$

2.3 Conjugate time

The following definition of a conjugate time is given in [11]:

Definition 9. Having selected a singular extremal (q, p, u) of corank 1, a value of the parameter T such that the quadratic form $E_u^{T''}$ has a zero eigenvalue is called a conjugate time along the reference trajectory. The first conjugate time, if it exists, is denoted by t_{1c}.

It is well known (see [11]) that assumption (H_3) implies that, if T is small enough then all eigenvalues of $E_u^{T''}$ are positive.

The concept of conjugate time provides conditions for local optimality. More precisely we have:

Proposition 10. ([11]). Let (q, u) be a reference trajectory and suppose that assumptions (H_1-H_4) are satisfied. Then (q, u) is locally optimal on $[0, t_{1c}[$ in the L^∞-topology, and is no more optimal beyond t_{1c}.

The crucial result from [11] and [6] is the following:

Theorem 11. If assumptions (H_1-H_4) are satisfied, the exp mapping is submersive on $[0, t_{1c}[$ and is not of full rank at $t = t_{1c}$.

This theorem gives a practical way to compute numerically conjugate times as developed in the next subsection.

2.4 Algorithm to compute conjugate times

Our aim is to estimate the determinant of the differential of the *exp* mapping along a reference extremal $z = (q, p)$ satisfying assumptions (H_1-H_4). From the definition of the exp mapping the differential $d\exp_{q_0}(t, p_0) : \mathbb{R} \times T_{p_0}\mathbb{P}^{n-1} \rightarrow \mathbb{R}^n$ is the application

$$(\delta t, \delta p_0) \mapsto \left(\frac{\partial \exp_{q_0}}{\partial t}(t, p_0)\delta t, \frac{\partial \exp_{q_0}}{\partial p_0}(t, p_0)\delta p_0 \right) .$$

Moreover, we have

$$\frac{\partial \exp_{q_0}}{\partial t}(t, p_0) = f(q(t), \hat{u}(z(t)))$$

and the tangent space to $\mathbb{P}^{n-1}$ at p_0 is given by:

$$T_{p_0}\mathbb{P}^{n-1} = \{\delta p_0 \in \mathbb{R}^n, \; \langle p_0, \delta p_0 \rangle = 0\}$$

Let $(\psi_1, \ldots, \psi_{n-1})$ be a basis of $T_{p_0}\mathbb{P}^{n-1}$ and let us introduce the variational equation associated to $\hat{H}$ along the reference extremal $z = (q, p)$:

$$\dot{\delta z} = \frac{\partial \hat{H}}{\partial z}(z(t))\delta z \; . \tag{8}$$

It follows from Theorem 11 that:

Proposition 12. If assumptions (H_1-H_4) are satisfied, the first conjugate time t_{1c} is the first time at which the determinant

$$C(t) = det(\delta q_1(t), \ldots, \delta q_{n-1}(t), f(q(t), \hat{u}(z(t))))$$

vanishes, where $\delta z_i(t) = (\delta q_i(t), \delta p_i(t))$ is the solution of the variational equation (8) with the initial conditions $\delta q_i(0) = 0$ and $\delta p_i(0) = \psi_i$, $i = 1, \ldots, n-1$.

This algorithm can be easily implemented, see [9] for other details and [5], [1] for applications.

3. CONJUGATE TIME FOR BANG-BANG EXTREMALS

Theorem 11 relates the computation of conjugate times to a property of the extremal flow. If the reference extremal is not smooth we cannot in general differentiate the end-point mapping in the L^1-topology but conjugate times for this topology can be still computed using the extremal flow, see for instance the theory of envelopes of [12] for broken extremals. In this section we investigate the problem of estimating conjugate times for broken extremals. We consider the time minimal problem for a single input affine control system.

$$\dot{q} = X(q) + uY(q), \quad |u| \leq 1 \tag{9}$$

According to the maximum principle it is known that optimal controls are concatenations of bang-bang arcs, i.e. arcs associated to controls $u = \pm 1$, and singular arcs. In the following we only focus on bang-bang extremals.

3.1 Second order condition for local optimality

We first recall a necessary condition for a reference bang-bang trajectory to be optimal, see [2,3].

Theorem 13. Let $(q(.), p(.))$, be a bang-bang extremal trajectory of system (9) on $[0, T]$ and let $u(t)$ be the corresponding control with k switching times $0 < \tau_1 < \ldots < \tau_k < T$. Denote by ν the value of u in $(0, \tau_1)$. Assume that $p(.)$ is uniquely defined (up to a positive scalar) and let $p_0 = p(0)$. Define:

$$\begin{aligned} h_0 &= f + \nu g \\ h_i &= e^{\tau_1 ad(f+\nu g)} \circ e^{(\tau_2-\tau_1)ad(f-\nu g)} \circ \ldots \\ &\quad \circ e^{(\tau_i - \tau_{i_1})ad(f-(-1)^i \nu g)}(f - (-1)^i \nu g) \\ &\quad i = 1 \ldots k \end{aligned}$$

Consider the quadratic form

$$Q(\alpha) = \sum_{0 \leq i < j \leq k} \alpha_i \alpha_j \langle p_0, [h_i, h_j](q(0)) \rangle, \tag{10}$$

defined on the space

$$\{\alpha = (\alpha_0, \ldots, \alpha_k) \in \mathbb{R}^{k+1}, \quad \text{such that} \\ \sum_{i=0}^{k} \alpha_i = 0, \quad \sum_{i=0}^{k} \alpha_i h_i(q(0)) = 0\} .$$

If q is optimal on $[0, T]$ in the L^1-topology then Q is nonnegative.

Moreover if the switching point is normal [8] (i.e. if it satisfies the so-called strong Legendre condition in the bang-bang case) and if the quadratic form Q is positive definite, then the trajectory is optimal.

This result gives a practical algorithm to evaluate the conjugate times by testing the quadratic form.

There are numerous reasons to believe that a common algorithm exists to evaluate the conjugate times for both singular and bang-bang extremals. Next we present an attempt to provide such an algorithm by smoothing the bang-bang extremals.

3.2 A tentative algorithm to compute conjugate times for bang-bang extremals by smoothing the corners

This section is heuristic and is based on the behaviours of the extremals in the Earth re-entry problem, see [5].

3.2.1. SR-systems with drift Let X, Y_1, Y_2 be vector fields on $\mathbb{R}^n$ and consider a time minimal problem for a system of the form:

$$\dot{q} = X(q) + u_1 Y_1 + u_2 Y_2 \quad , \tag{11}$$

where $u_1 = \cos\mu$, $u_2 = \sin\mu$ and the associated convexified problem: $u_1^2 + u_2^2 \leq 1$. The hamiltonian is $H = \langle p, u_1 Y_1 + u_2 Y_2 \rangle$ and maximizing H over $u_1^2 + u_2^2 = 1$ yields outside the switching surface Σ: $\langle p, Y_1 \rangle = \langle p, Y_2 \rangle = 0$:

$$\begin{aligned} u_1 &= \frac{\langle p, Y_1 \rangle}{\sqrt{\langle p, Y_1 \rangle^2 + \langle p, Y_2 \rangle^2}} \\ u_2 &= \frac{\langle p, Y_2 \rangle}{\sqrt{\langle p, Y_1 \rangle^2 + \langle p, Y_2 \rangle^2}} \end{aligned} \tag{12}$$

Such extremals are called of order 0 and they correspond to smooth singular trajectories of the associated end-point mapping. Singular trajectories associated to the end-point mapping where u_1 and u_2 are indepedent are extremals of the convexified problem if $u_1^2 + u_2^2 \leq 1$ and are solutions of: $\langle p, Y_1 \rangle = \langle p, Y_2 \rangle = 0$. Differentiating the switching function $\Phi = (\Phi_1, \Phi_2)$, $\Phi_i = \langle p, Y_i \rangle$ one gets :

$$\begin{aligned} \langle p, [Y_1, X] \rangle + u_1 \langle p, [Y_1, Y_2] \rangle = 0, \\ \langle p, [Y_2, X] \rangle + u_2 \langle p, [Y_2, Y_1] \rangle = 0. \end{aligned}$$

The existence of a switching surface or such trajectories is governing the behaviors of the smooth extremals of the original system and allows the concatenation of smooth extremals to build bang-bang extremals. The classification is complicated and is an open problem in singularity theory.

In particular all the singularities corresponding to single-input system:

$$\dot{q} = X(q_1) + u_1 Y(q_1) \tag{13}$$

can be embeded in the above case. Indeed we have:

Lemma 14. Consider the time optimal control problem for a system of the form:

$$\begin{aligned} \dot{q}_1 &= X(q_1) + u_1 Y_1(q_1) \\ \dot{q}_2 &= u_2 Y_2(q_1, q_2) \end{aligned}$$

with $q = (q_1, q_2) \in \mathbb{R}^{n_1+n_2}$. Then the extremals of the problem with free condition on q_2 are the extremals of the system $\dot{q}_1 = X(q_1) + u_1 Y_1(q_1)$. Hence they are concatenations of smooth singular extremals with $u_1 = \cos\mu = \pm 1$.

Of course we cannot a priori use our algorithm to evaluate conjugate point because we are not in the smooth case moreover $\cos\mu$ is singular at ± 1 and the associated singularity is of codimension at least n_1.

3.2.2. smoothing the corners Consider a system of the form $\dot{q} = X(q) + u_1 Y_1(q), |u_1 \leq 1$. To estimate conjugate times along bang-bang extremals the idea is to regularize the extremals by adding an extra term:

$$\dot{q_\varepsilon} = X(q_\varepsilon) + \cos\mu_\varepsilon Y_1(q_\varepsilon) + \varepsilon \sin\mu_\varepsilon Y_2(q_\varepsilon) \ , \tag{14}$$

where the control $\mu \in \mathbb{R}$. For $\varepsilon = 0$, we get the original system and Y_2 is conditioning the regularizing process. We restrict ourselves to extremals of order 0, where $\cos\mu_\varepsilon$ and $\sin\mu_\varepsilon$ are given, according to Eq. 12, by:

$$\cos\mu_\varepsilon = \frac{\langle p_\varepsilon, Y_1\rangle}{\sqrt{\langle p_\varepsilon, Y_1\rangle^2 + \varepsilon^2\langle p_\varepsilon, Y_2\rangle^2}}$$
$$\sin\mu_\varepsilon = \frac{\varepsilon\langle p_\varepsilon, Y_2\rangle}{\sqrt{\langle p_\varepsilon, Y_1\rangle^2 + \varepsilon^2\langle p_\varepsilon, Y_2\rangle^2}}$$

Let $z = (q,p)$ a reference bang-bang extremal of the time optimal control problem (13) associated to the control u and for all ε let $z_\varepsilon = (q_\varepsilon, p_\varepsilon)$ be the extremal of time minimal problem (14) such that

$$q_\varepsilon(0) = q(0), \ p_\varepsilon(0) = p(0) \ .$$

Our conjecture is that there exists Y_2 in general position such that the first conjugate time $t_{1c\varepsilon}$ for the regularized system tends to t_{1c} as ε tends to 0, where t_{1c} is the conjugate time associated to the bang-bang extremal.

First of all we have to choose Y_2, e.g equal to X, such that z_ε avoids the switching surface and is a regular perturbation of z and hence z_ε tends to z uniformly as ε tends to 0.

First conjugate points are points where a quadratic approximation of the end-point mapping becomes open which is a stable property.

3.2.3. Numerical testing In order to check our conjecture, we consider a system in $\mathbb{R}^3$: $\dot{q} = X(q) + u_1 Y_1(q)$, $|u_1| \leq 1$. We recall briefly the evaluation of the small time reachable set, see [7] for details. We make the assumption that $X(q), Y_1(q), [X, Y_1](q)$ form a frame in $\mathbb{R}^3$. we write:

$$[[Y_1, X], X \pm Y_1] = a_\pm X + b_\pm Y_1 + c_\pm [X, Y_1] \quad .$$

We are in the parabolic case if $a_+ a_- > 0$, in the elliptic if $a_+ > 0$ and $a_- < 0$, and in the hyperbolic case if $a_- > 0$ and $a_+ < 0$. The singular controls are given by:

$$\langle p, [[Y_1, X], X] + u_s [[Y_1, X], Y_1]\rangle = 0 \quad .$$

We can restrict our analysis to the elliptic and parabolic case. In the elliptic case, the singular arcs are not optimal and the small time optimal trajectory is bang-bang, with at most two switchings. There exists an asymptotic conjugate locus and cut locus similar in terms of singularity theory to the generic SR-contact case in dimension 3.

In the parabolic case, a singular arc if existing is not admissible and each small time optimal trajectory is bang-bang with at most two switchings. In this case, conjugate time does not exists for small times, contrary to the elliptic case.

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