CLASSIFICATION OF LOCAL OPTIMAL SYNTHESES FOR TIME MINIMAL CONTROL PROBLEM WITH STATE CONSTRAINTS

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Abstract: In this article, we describe the analysis under generic assumptions of the small *time minimal syntheses* for single input affine control systems in dimension 3, submitted to *state constraints*. We use geometric methods to evaluate *the small time reachable set* and necessary optimality conditions. Our work is motivated by the *optimal control of the atmospheric arc for the re-entry of a space shuttle*, where the vehicle is subject to constraints on the thermal flux and on the normal acceleration.

Keywords: Optimal control with state constraints, Maximum principle, Control of the atmospheric arc.

1. INTRODUCTION

The objective of this article [1] is to describe the classification of the local closed loop time optimal control for the single input affine systems:

$$\dot{q} = X(q) + uY(q) ,$$

where $\mathbb{R}^3$, $|u| \leq 1$, with *state constraints*: $c(q) \leq 0$.

In classical calculus of variation, necessary conditions concerning the optimality status of a boundary arc and junction or reflection with the boundary have been obtained as a consequence of Weierstrass theory applied to Riemannian theory with obstacles, see [1]. This approach was generalized by Pontryagin and his co-authors [11] to obtain a *minimum principle* under regularity assumptions on the constraints. A general minimum principle based on Kuhn-Tucker theorem and non smooth analysis is presented in [7]. In these principles, the adjoint vector p dual to the state vector q can suffer discontinuities at contact points with the boundary of the domain or in the boundary. Following the works of [8] and [10], these discontinuities can be specified if we assume that the system is single input and if *the order of the constraint* is constant, the order being by definition the first integer such that the control u appears explicitly in the time derivative of the constraint $t \mapsto c(q(t))$ evaluated along a boundary arc of the system.

The evaluation of the small time reachable set and its boundary which can be parametrized by the minimum principle with application to

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the optimal synthesis was a research program initialized by Sussmann [12] for planar system and pursued in dimension 3 by [9], see also [5] for problems with a target of co-dimension one. The objective of this article is mainly to outline such an analysis in the case of optimal control with state constraints in dimension 3, where the constraint has order 2. Here the geometry is different and we must classify up to changes of coordinates triplets (X, Y, c) using the order of the constraints. We make a direct evaluation of the reachable set for the constrained system, using normal forms. One of the main problems is to characterize the *optimality status of a boundary arc*. We get under suitable generic assumptions necessary and sufficient conditions.

This classification has been initialized in [6,2] for system in dimension 2 and 3. The planar case is now clear. In this paper we complete our study in dimension 3.

Our geometric work, completed by the preliminary study by [6], is finally applied to the re-entry problem. A quasi-optimal trajectory, consisting of concatenation of bang and boundary arcs, is given, see also [3].

2. GENERALITIES

2.1 Definitions

We consider the time optimal problem for a smooth (C^∞ or C^ω) single input affine system

$$\dot{q} = X(q) + uY(q) \ , \qquad (1)$$

with $|u| \leq 1$, $q \in U \subset \mathbb{R}^n$ with state constraint $c(q) \leq 0$ where $c : \mathbb{R}^n \to \mathbb{R}$, and fixed boundary conditions: $q(0) = q_0$, $q(T) = q_1$.

The *generic order* of the constraint is the integer m such that:

$$Yc = YXc = \ldots = YX^{m-2}c = 0 \ , \ YX^{m-1}c \neq 0$$

A *boundary arc* $t \mapsto \gamma_b(t)$ is an arc of the system, not reduced to a point, contained in $c = 0$. If the order is m, a boundary arc and the associated feedback control can be generically computed by differentiating m times the mapping $t \mapsto c(q(t))$ and solving with respect to u the linear equation:

$$c^{(m)} = X^m c + uYX^{m-1}c = 0$$

A boundary arc is contained in

$$c = \dot{c} = \ldots = c^{(m-1)} = 0 \ ,$$

and the constraint $c = 0$ is called *primary* and the constraints $\dot{c} = \ldots = c^{(m-1)} = 0$ are called *secondary*. The boundary feedback control is:

$$u_b = -\frac{X^m c}{YX^{m-1}c} \ .$$

2.2 Assumptions

Let $t \mapsto \gamma_b(t)$, $t \in [0, T]$ be a boundary arc associated to u_b. We need to introduce the following assumptions:

C_1. $YX^{m-1}c_{|\gamma_b} \neq 0$ where m is the order of the constraint.

C_2. $|u_b| < 1$ for $t \in]0, T[$, i.e. the boundary control is admissible.

C_3. $|u_b| < 1$ for $t \in [0, T]$, i.e. the boundary control is not saturating.

2.3 A maximum principle with state constraints

We recall the necessary conditions due to [8] and [10] that we shall use in our study. Consider the single input affine system (1), $\dot{q} = X(q) + uY(q)$, $q \in U \subset \mathbb{R}^n, |u| \leq 1$, and consider the time minimal problem to transfer q_0 to q_1.

Assume that $t \mapsto q(t)$, $t \in [0, T]$ is a piecewise smooth time minimal solution which hits the boundary $c(q) = 0$ at times t_{2i-1}, $i = 1, 2, \ldots, M$ and leaves the boundary at times t_{2i}, $i = 1, 2, \ldots, M$ and moreover assume that along each boundary arc , Assumptions C_1 and C_2 are satisfied at contact or junction times. Define the Hamiltonian by

$$H(q, p, u, \eta) = \langle p, X + uY \rangle + \eta c \ ,$$

where $\langle , \rangle$ is the standard scalar product, p is the adjoint vector and η is the Lagrange multiplier of the constraint. The necessary optimality conditions are:

(1) There exists $t \mapsto \eta(t) \leq 0$, such that the adjoint vector satisfies:

$$\dot{p} = -p\left(\frac{\partial X}{\partial q} + u\frac{\partial Y}{\partial q}\right) - \eta\frac{\partial c}{\partial q} \ \text{a.e.}$$

(2) The mapping $t \mapsto \eta(t)$ is continuous along the boundary arc and satisfies:

$$\eta(t)c(q(t)) = 0 \ , \quad \forall t \in [0, T].$$

(3) At a contact or a junction time t_i with the boundary, we have

$$H(t_i+) = H(t_i-)$$
$$p(t_i+) = p(t_i-) - \nu_i\frac{\partial c}{\partial q}(q(t_i)), \ \nu_i \leq 0$$

(4) The optimal control maximizes almost everywhere the Hamiltonian:

$$H(q(), p(), u(), \eta()) = \max_{|v| \leq 1} H(q(), p(), v, \eta())$$

(5) H is constant and nonnegative along $q(t)$.

2.4 Definitions

Definition 1. An extremal is a solution (q, p) of the above equations. It is called *exceptional* if $H = 0$. An extremal arc is called *bang-bang* if it corresponds to a piecewise constant control $u(t) = sign(\langle p(t), Y(q(t))\rangle)$; an extremal arc of the unconstrained problem is called *singular* if $\langle p(t), Y(q(t))\rangle = 0$. We note $\Phi = \langle p, Y(q)\rangle$ the *switching function* and Σ_s the *switching set* formed by points q where the optimal control is discontinuous.

2.5 Computation of singular controls

We have:

Lemma 2. Let $\Phi(t) = \langle p(t), Y(q(t))\rangle$ be the switching function evaluated along a smooth extremal $z(t) = (p(t), q(t))$ of the unconstrained problem, then:

$$\dot\Phi = \langle p, [X, Y](q)\rangle$$
$$\ddot\Phi = \langle p, [X, [X, Y]](q)\rangle + u\langle p, [Y, [X, Y]](q)\rangle$$

Corollary 3. A singular extremal $(p(t), q(t))$ satisfies almost every where :

$$\langle p, Y(q)\rangle = \langle p, [X, Y](q)\rangle = 0$$
$$\langle p, [X, [X, Y]](q)\rangle + u\langle p, [Y, [X, Y]](q)\rangle = 0$$

2.6 Geometric computations of the multipliers (η, ν_i) and the junction conditions

One of the main contributions of [10] is to determine the multipliers (η, ν_i) together with the analysis of the junction conditions. This is based on the concept of order and is related to the classification of extremals. We review these relations when the order is $m = 2$. Also we make the computation geometric, that is related to iterated Lie brackets of (X, Y) acting on the constraint mapping c.

Lemma 4. Assume that the constraint has order $m = 2$ then:

(1) Along a boundary arc:
$$\eta = \frac{\langle p, [X, [X, Y]](q)\rangle + u_b\langle p, [Y, [X, Y]](q)\rangle}{([X, Y]c)(q)}$$
(2) At a contact or entrance-exit point:
$$\Phi(t_i+) = \Phi(t_i-)$$
(3) At an entry point:
$$\nu_i = \frac{\dot\Phi(t_i-)}{([X, Y]c)(q(t_i))},$$
and at an exit point:
$$\nu_i = -\frac{\dot\Phi(t_i+)}{([X, Y]c)(q(t_i))},$$

The remaining of this article is devoted to the construction of the closed loop time optimal trajectories under generic assumptions for a single input affine control system in dimension 3 with application to the space shuttle re-entry problem.

3. SMALL TIME MINIMAL SYNTHESIS FOR SYSTEM IN DIMENSION 3 WITH STATE CONSTRAINTS

3.1 The non constraint case

We consider a system of the form $\dot q = X + uY$, $c(q) \le 0$, $|u| \le 1$, with $q = (x, y, z) \in \mathbb{R}^3$. The aim of this section is to give the classification of the optimal syntheses near a point q_0, identified to 0, on the boundary of the domain. Consider first the unconstrained case and assume that X, Y, $[X, Y]$ are linearly independent at q_0. To construct optimal trajectories we must analyze the boundary of the small time reachable set for the time extended system. Its structure is described in [9], under generic assumptions.

If $\langle p(t), [Y, [X, Y]](q(t))\rangle$ is not vanishing we can solve $\ddot\Phi(t) = 0$ to compute the singular control:

$$u_s = -\frac{\langle p, [X, [X, Y]](q)\rangle}{\langle p, [Y, [X, Y]](q)\rangle}$$

We have two cases, see [4].

- *Case 1*: If X and $[[X, Y], Y]$ are on the opposite side with respect to the plane generated by $Y, [X, Y]$, then the singular arc is locally time optimal if $u \in \mathbb{R}$.
- *Case 2*: On the opposite, if X and $[[X, Y], Y]$ are in the same side, the singular arc is locally time maximal.

In the two cases, the constraint $|u_s| \le 1$ is not taken into account and the singular control can be strictly admissible if $|u_s| < 1$, saturating if $|u_s| = 1$ at q_0, or non admissible if $|u_s| > 1$. We have 3 generic cases:

- parabolic case: $\ddot\Phi_\pm$ have the same sign.
- elliptic case: $\ddot\Phi_+ > 0$ and $\ddot\Phi_- < 0$.
- hyperbolic case: $\ddot\Phi_+ < 0$ and $\ddot\Phi_- > 0$.

In both hyperbolic and parabolic cases, the local time optimal syntheses are obtained by using only the first order conditions from the minimum principle and hence from extremality, together with Legendre-Clebsch condition in the hyperbolic case. More precisely we have:

Lemma 5. In the hyperbolic or parabolic cases, each extremal policy is locally time optimal. In the hyperbolic case each optimal policy is of the form $\gamma_\pm\gamma_s\gamma_\pm$, where γ_+ (resp. γ_-, resp. γ_s) denotes a solution of the system associated to the control $u = 1$ (resp. $u = -1$, resp. $u = u_s$). In the parabolic case each optimal policy is bang-bang with at most two switchings and has the form $\gamma_-\gamma_+\gamma_-$ or $\gamma_+\gamma_-\gamma_+$ depending on the configuration of the Lie Brackets.

In the elliptic case, the situation is more intricate because there exist a cut-locus, i.e. a set of points which can be reached by at least two optimal trajectories.

Lemma 6. In the elliptic case, each optimal policy has the form $\gamma_-\gamma_+\gamma_-$ and $\gamma_+\gamma_-\gamma_+$. Moreover there exists a cut locus where two optimal trajectories of the previous form have the same duration.

We shall now analyze the constrained case for a constraint of order 2, for the above three generic cases. Notice that both parabolic and hyperbolic cases have been already treated in [2]. Let us also underline that our application to the space shuttle re-entry problem, corresponds to the parabolic case.

3.2 The constrained parabolic case

For the unconstrained problem the situation is clear in the parabolic case. Indeed X, Y, $[X,Y]$ form a frame near q_0 and writing:

$$[X \pm Y, [X,Y]] = a_\pm X + b_\pm Y + c_\pm [X,Y],$$

with $a_\pm \neq 0$, the synthesis depends only upon the sign of $\delta = a_+a_-$ at q_0. The small time reachable set is bounded by the surfaces formed by arcs $\gamma_-\gamma_+$ and $\gamma_+\gamma_-$. Each interior point can be reached by an arc $\gamma_-\gamma_+\gamma_-$ and an arc $\gamma_+\gamma_-\gamma_+$. If $\delta < 0$ the time minimal policy is $\gamma_-\gamma_+\gamma_-$ and the time maximal policy is $\gamma_+\gamma_-\gamma_+$ and the opposite if $\delta > 0$. To construct the optimal synthesis one can use a *nilpotent model* where all Lie brackets of length greater than 4 are 0. In particular the existence of singular direction is irrelevant in the analysis and a model where $[Y,[X,Y]]$ is zero can be taken. This situation is called the *geometric model.* A similar model is constructed next taking into account the constraints, which are assumed of order 2. Moreover we shall first suppose that C_1, C_3 are satisfied along a boundary arc γ_b, that is $YXc \neq 0$ along γ_b and the boundary control is admissible and not saturating. We have the following:

Lemma 7. Under our assumptions, a local geometric model in the parabolic case is:

$$\dot{x} = a_1 x + a_3 z$$
$$\dot{y} = 1 + b_1 x + b_3 z$$
$$\dot{z} = (c+u) + c_1 x + c_2 y + c_3 z, \qquad |u| \leq 1$$

with $a_3 > 0$, where the constraint is $x \leq 0$ and the boundary arc is identified to $\gamma_b : t \mapsto (0,t,0)$.

If the boundary arc is admissible and not saturating we have $|c| < 1$.

Theorem 8. Consider the time minimization problem for the system: $\dot{q} = X(q) + uY(q)$, $q \in \mathbb{R}^3, |u| \leq 1$ with the constraint $c(q) \leq 0$. Let $q_0 \in \{c = 0\}$ and assume the following:

(1) At q_0, X, Y and $[X,Y]$ form a frame and
$$[X \pm Y, [X,Y]] = a_\pm X + b_\pm Y + c_\pm [X,Y]$$
with $a_+a_- < 0$.

(2) The constraint is of order 2 and Assumptions C_1 and C_3 are satisfied at q_0.

Then the boundary arc through q_0 is small time optimal if and only if the arc γ_- is contained in the non admissible domain $c \geq 0$. In this case the local time minimal synthesis with a boundary arc is of the form $\gamma_-\gamma_+^T\gamma_b\gamma_+^T\gamma_-$, where γ_+ (resp. γ_-, resp. γ_b) denotes solution of the system associated to the control $u = 1$ (resp. $u = -1$, resp. $u = u_b$) and γ_+^T are arcs tangent to the boundary arc.

3.3 Connecting two constraints of order 2 in the parabolic case

If there are two constraints in a small neighborhood of a point q_0, one needs to describe the transition between the two constraints. Hence we give a geometric normal form to analyze such a transition together with the optimal strategy.

Lemma 9. Consider the control system $\dot{q} = X + uY$, $|u| \leq 1$, $q \in \mathbb{R}^3$ with two distinct constraints $c_i(q) \leq 0$, $i = 1, 2$. Take a small neighborhood U of 0 containing subarcs of both γ_b^1 and γ_b^2, and assume that γ_b^1 hits the boundary $c_2 = 0$. Assume also that Lie Brackets conditions of Lemma 7 are satisfied. Then there exists a geometric model of the form:

$$\dot{x} = a_1 x + a_3 z$$
$$\dot{y} = 1 + b_1 x + b_3 z$$
$$\dot{z} = c + u + c_1' x + c_2' y + c_3' z$$

where the constraints are given by $c_1(q) = x$ and $c_2(q) = x + \varepsilon y$, with ε small.

Theorem 10. If assumptions of Lemma 9, and assumptions (1) and (2) of Theorem 8 for both constrained system are satisfied, denote γ_b^1, γ_b^2

the respective boundary arcs. Assume that the boundary arcs are optimal. Then each optimal policy near $q_0 = 0$ with boundary arcs is of the form $\gamma_-\gamma_+^T\gamma_b^1\gamma_+^T\gamma_b^2\gamma_+^T\gamma_-$ where the intermediate arc γ_+^T is the only arc tangent to both constraints.

3.4 The constrained hyperbolic case

In the hyperbolic constrained case, we first describe the small time optimal synthesis starting from a point q_0, identified to 0, on the constraint. For sake of simplicity, we adopt the point of view of [9] and we focus on the so called "free" nilpotent case, where $[X,[X,Y]] = 0$, i.e.

$$\dot{x} = z\ ,\dot{y} = 1 - 2cx - z^2\ ,\dot{z} = c + u \qquad (2)$$

with $|c| < 1$ and the constraint given by $x \le 0$. Then the singular control is $u_s = 0$ and is strictly admissible. The boundary arc is $t \mapsto (0,t,0)$ and is associated with the control $u_b = -c$. The singular arc is not tangent to the boundary arc and we assume that γ_s^0 is admissible, which means $c < 0$.

Lemma 11. Under our assumptions:

- The boundary arc is the optimal trajectory that joins two points $A = (0,a,0)$ and $B(0,b,0)$ of the boundary.
- Each small time optimal trajectory starting from 0 is of the form $\gamma_b\gamma_s^T\gamma_\pm$ or $\gamma_-^0\gamma_s\gamma_\pm$.

We can make the same reasoning backward in time and finally we obtain :

Theorem 12. Consider the hyperbolic case and assume that the singular arc γ_s^0 through $q_0 = 0$ is not tangent to the boundary arc. If γ_s^0 is contained in the non admissible domain, the boundary arc is not optimal. If γ_s^0 is contained in the admissible domain, the boundary arc is locally optimal and the local optimal trajectories with boundary arcs are of the form $\gamma_\pm\gamma_s^T\gamma_b\gamma_s^T\gamma_\pm$.

3.5 The constrained elliptic case

Again for sake of simplicity, we focus on the so-called "free" nilpotent systems, which means that all the Lie brackets of order more than three are equal to 0 and that $[X,[X,Y]]$ is identically 0. Therefore we study the following model:

$$\dot{x} = z,\ \dot{y} = 1 + 2cx + z^2,\ \dot{z} = c + u \qquad (3)$$

with $|c| < 1$ and the constraint is given by $x \le 0$. The singular control is $u_s = 0$ and is strictly admissible. The boundary arc is $t \mapsto (0,t,0)$ and is associated with the boundary control $u_b = -c$. Let us first study the optimal synthesis to join two points of the boundary arc $A = (0,0,0)$ and $B = (0,b,0)$:

Lemma 13. For system (3), the boundary arc is optimal if and only if $c > \frac{1}{3}$. If $c < \frac{1}{3}$, the optimal policy to join two points of the boundary arc is $\gamma_-^T\gamma_+\gamma_-^T$.

Remark 14. The local model can not be used to compare arc γ_b to arc $\gamma_-^T\gamma_+\gamma_-^T$ when $c = \frac{1}{3}$. In this case we have to add higher order terms in system (3).

Proposition 15. For system (3), assume that the boundary arc is optimal ($c > 1/3$), then we have for the constrained problem:

(1) if $c > 1/2$, each optimal trajectory starting from 0 is of the form $\gamma_b\gamma_-\gamma_+$.
(2) if $c < 1/2$, each optimal trajectory starting from 0 is of the form $\gamma_b\gamma_-\gamma_+$, $\gamma_b\gamma_-\gamma_+\gamma_-$ or $\gamma_-\gamma_+\gamma_-$.

As in the hyperbolic case, the reasoning can also be made backwards in time and we can infer the local optimal synthesis.

4. CONTROL OF THE ATMOSPHERIC ARC

4.1 The optimal control problem

We consider the dynamic model of a space shuttle in the atmosphere submitted to gravitational force $\overrightarrow{K}$ and aerodynamic forces which split into the drag force $\overrightarrow{D}$ and the lift force $\overrightarrow{L}$:

$$\ddot{q} = \overrightarrow{K} + \overrightarrow{D} + \overrightarrow{L}\ .$$

Choosing a system of coordinates this system lies in $\mathbb{R}^6$ and writes :

$$\dot{q} = X(q) + \cos\mu Y_1(q) + \sin\mu Y_2(q) \qquad (4)$$

The problem is to steer the vehicle from an initial manifold to a terminal manifold without violating constraints, $c_i(q) \le 0$ for $i = 1,2,3$, corresponding to constraints on the thermal flux, the normal acceleration and the dynamic pressure. The optimal control problem consists in minimizing the total amount of the thermal flux φ accumulated during the flight :

$$J(\mu) = \int_0^{t_f} \varphi(q)dt \qquad (5)$$

If we introduce the new time parameter

$$ds = \varphi dt \qquad (6)$$

our optimal problem is a *time minimal problem.*

4.2 Properties and structure of the system

The problem is to minimize time for a system of the form:

$$\frac{dq}{dt} = X(q) + u_1Y_1(q) + u_2Y_2(q)\ ,$$

where $u_1 = \cos\mu,\ u_2 = \sin\mu$. We can decompose the previous system into 2 subsystems (see [3]) as follow :

$$\dot{q}_1 = F_1(q_1, u_1) + O(\Omega),\ \dot{q}_2 = F_2(q, u_2)\ .$$

where Ω is the modulus of the Earth angular velocity. The first subsystem governs the *longitudinal motion* and the second system governs the *lateral motion*. Neglecting the influence of the Earth rotation, we can restrict our attention to the first subsystem which is a scalar input affine system of the form

$$\dot{q} = X + uY,\ |u| \leq 1,\ q = (r, v, \gamma) \in \mathbb{R}^3\ , \qquad (7)$$

where r is the distance between the shuttle and the center of the Earth, v is the modulus of the relative velocity and γ is the path inclination. The vector fields X and Y are:

$$X = \begin{bmatrix} v\sin\gamma \\ -g\sin\gamma - k\rho v^2 \\ \cos\gamma\left(-\frac{g}{v} + \frac{v}{r}\right) \end{bmatrix} \quad Y = \begin{bmatrix} 0 \\ 0 \\ k'\rho v \end{bmatrix},$$

where $\rho(r)$ is the density of the atmosphere and $k,\ k'$ are aerodynamics coefficients. The state constraints are of the form $c_i(q) \leq 0,\ i = 1, 2, 3$ and the boundary conditions are given in Table 1.

	Initial conditions	Terminal conditions
altitude h	119.82 km	15 km
velocity v	7404.95 $m.s^{-1}$	445 $m.s^{-1}$
flight angle γ	-1.84 deg	free

Table 1. Boundary conditions

The following results, coming from computations, are crucial:

Lemma 16. In the flight domain where $\cos\gamma \neq 0$, we have:

(1) $X,\ Y,\ [X, Y]$ are linearly independent.
(2) $[Y, [X, Y]] \in span\{Y,\ [X, Y]\}$.
(3) $[X, [X, Y]](q) = a(q)X(q) + b(q)Y(q) + c(q)[X, Y](q)$ with $a < 0$.

Hence the longitudinal subsystem is parabolic according to our previous classification.

4.3 Application of the classification to the space shuttle problem

Applying Theorems 8 and 10 actually leads to the following result, see [3] for details:

Corollary 17. The optimal trajectory for the problem of the space shuttle has the form:

$$\gamma_-\gamma_+^T\gamma_{\text{flux}}\gamma_+^T\gamma_{\text{acc}}\gamma_+^T$$

where γ_{flux} (resp. γ_{acc}) is an iso-flux (resp. isonormal acceleration) boundary arc, the constraint on the dynamic pressure being not active during the flight.

This is actually the optimal synthesis for the subsystem (7) which governs the longitudinal motion. As to the complete system in dimension 6, it may be proved that for certain boundary conditions, trajectories given in Corollary 17 are still *quasi optimal*, see [2] for more details.

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