The Transcendence we need to compute the Sphere and the Wave Front in Martinet SR-Geometry

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Abstract

Consider a *sub-Riemannian geometry* (U, D, g) where U is a neighborhood of O in $\mathbb{R}^3$, D is a *Martinet type distribution* identified to $Ker\,\omega$, $\omega = dz - \frac{y^2}{2}dx$, $q = (x, y, z)$ and g is a *metric on* D which can be taken in the normal form : $a(q)dx^2 + c(q)dy^2$, $a = 1 + yF(q)$, $c = 1 + G(q)$, $G_{|x=y=0} = 0$. In a previous article we analyzed the *flat case* : $a = c = 1$; we showed that the set of geodesics is integrable using *elliptic integrals* of the *first and second kind* ; moreover we described the sphere and the wave front near the abnormal direction using the *exp-log category*. The objective of this article is to analyze the transcendence we need to compute the sphere and the wave front of small radius in the abnormal direction and globally when we consider the gradated normal form of order 0 : $a = (1 + \alpha y)^2$, $c = (1 + \beta x + \gamma y)^2$, where α, β, γ are real parameters.

1 Preliminaries

Consider the local SR-geometry (U, D, g) where U is a neighborhood of $0 \in \mathbb{R}^3$, D is a Martinet type distribution which can be taken in the normal form $D = \mathrm{Ker}\,\omega$, $\omega = dz - \frac{y^2}{2}dx$ and g is a C^ω metric on D which can be written (see [1]) in the normal form : $a(q)dx^2 + c(q)dy^2$, $a = 1 + yF(q)$, $c = 1 + G(q)$, $G_{|x=y=0} = 0$ and a, c can be expanded in Taylor series using the following weights : x, y of weight 1 and z of weight 3 given by the *privileged coordinates system* at O : $q = (x, y, z)$ (see [9]). Hence we get on *orthonormal basis* :

$$F_1 = \frac{1}{\sqrt{a}}G_1 \quad , \quad G_1 = \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z} \quad , \quad F_2 = \frac{1}{\sqrt{c}}G_2 \quad , \quad G_2 = \frac{\partial}{\partial y}$$

Expanding F_1, F_2 in Taylor series according to the previous weights, and identifying at order p two elements whose Taylor series are the same at order p, we get the following normal forms of order -1 and 0 :

- Normal form of order -1 :

$$g = dx^2 + dy^2 \qquad \text{(flat case)}$$

- Normal form of order 0 :

$$g = (1+\alpha y)^2 dx^2 + (1+\beta x + \gamma y)^2 dy^2 \qquad \alpha, \beta, \gamma \in \mathbb{R}$$

1.1 Geodesics equations

The *energy* minimization problem equivalent to the SR-problem is the *optimal control problem* :

$$\begin{cases} \dfrac{dq}{dt}(t) = \displaystyle\sum_{i=1}^{2} u_i(t) G_i(q(t)) \\ \displaystyle\min_{u(.)} \int_0^T \big(a(q(t))u_1^2(t) + c(q(t))u_2^2(t)\big)\, dt \end{cases}$$

and from *Pontryagin's Maximum Principle* [9] the minimizing solutions are solutions of the following equations :

$$\dot{q} = \frac{\partial H_\nu}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H_\nu}{\partial q} \quad , \quad \frac{\partial H_\nu}{\partial u} = 0 \tag{1}$$

where H_ν is the pseudo-Hamiltonian :

$$H_\nu = \sum_{i=1}^{2} u_i < p, G(q) > -\nu(a u_1^2 + c u_2^2)$$

where ν is a constant normalized to 0 or $1/2$. A solution of the previous equations is called an *extremal* ; when $\nu = 1/2$ (resp. $\nu = 0$) they are said *normal* (resp. *abnormal*) and their projection on the state space are called the *geodesics*. They can be easily computed :

- Abnormal case : If $\nu = 0$, $D = \mathrm{Span}\{G_1, G_2\} = \mathrm{Ker}\,\omega$ and they depend only on the distribution D. If $\omega = dz - \frac{y^2}{2}dx$, they are contained in the plane $y = 0$ called the *Martinet plane* and are the straight-lines : $z = z_0$. In particular the line passing through 0 is given by : $t \longmapsto (\pm t, 0, 0)$ and is called the *abnormal direction*.

- Normal case : For $\nu = 1/2$, with $g = a(q)dx^2 + c(q)dy^2$ we get :

$$H_{1/2} = \sum_{i=1}^{2} u_i G_i(q) - \frac{1}{2}(au_1^2 + cu_2^2)$$

Solving $\frac{\partial H_{1/2}}{\partial u} = 0$ we get :

$$u_1 = \frac{1}{a}(p_x + p_z \frac{y^2}{2}) \quad , \quad u_2 = \frac{p_y}{c}$$

and plugging (u_1, u_2) into $H_{1/2}$ we obtain the Hamilton function :

$$H_n(q,p) = \frac{1}{2}\left[\frac{(p_x + p_z \frac{y^2}{2})^2}{a} + \frac{p_y^2}{c}\right]$$

where $p = (p_x, p_y, p_z)$ and (1) takes the form :

$$\dot{q} = \frac{\partial H_n}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H_n}{\partial q}$$

Another representation is obtained using the frame F_1, F_2 and $F_3 = \frac{\partial}{\partial z}$, and defining $P = (P_1, P_2, P_3)$ with $P_i =< p, F_i(q) >$, i.e. $P_1 = \frac{p_x + p_z \frac{y^2}{2}}{\sqrt{a}}$, $P_2 = \frac{p_y}{\sqrt{c}}$, $P_3 = p_z$. The Hamiltonian takes the form : $H_n = \frac{1}{2}(P_1^2 + P_2^2)$. Assuming g not depending on z (*isoperimetric situation*) the normal extremals are solutions of the following equations :

$$\begin{aligned}
\dot{x} &= \frac{1}{a}\left(p_x + p_z \frac{y^2}{2}\right) \\
\dot{y} &= \frac{p_y}{c} \\
\dot{z} &= \frac{y^2}{2a}\left(p_x + p_z \frac{y^2}{2}\right) \\
\dot{p_x} &= \frac{p_y^2 c_x}{2c^2} + \frac{\left(p_x + p_z \frac{y^2}{2}\right)^2}{2a^2} a_x \\
\dot{p_y} &= \frac{p_y^2 c_y}{2c^2} + \frac{\left(p_x + p_z \frac{y^2}{2}\right)^2}{2a^2} a_y - \frac{\left(p_x + p_z \frac{y^2}{2}\right)}{a} p_z y \\
\dot{p_z} &= 0
\end{aligned} \qquad (2)$$

which takes the following form in (q,P) coordinates :

$$\begin{aligned}
\dot{x} &= \frac{P_1}{\sqrt{a}} \\
\dot{y} &= \frac{P_2}{\sqrt{c}} \\
\dot{z} &= \frac{y^2}{2}\frac{P_1}{\sqrt{a}} \\
\dot{P}_1 &= \frac{P_2}{\sqrt{a}\sqrt{c}}\left(yP_3 - \frac{a_y}{2\sqrt{a}}P_1 + \frac{c_x}{2\sqrt{c}}P_2\right) \\
\dot{P}_2 &= -\frac{P_1}{\sqrt{a}\sqrt{c}}\left(yP_3 - \frac{a_y}{2\sqrt{a}}P_1 + \frac{c_x}{2\sqrt{c}}P_2\right) \\
\dot{P}_3 &= 0
\end{aligned} \tag{3}$$

1.2 Sphere and wave front

Let $r>0$. The *wave front* $W(0,r)$ at 0 is the end-points of geodesics with SR-length r starting from 0 ; the *sphere* $S(0,r)$ is the end-points of *minimizing geodesics* of length r and starting from 0. We are interested in the *local* problem near 0, hence we choose r *small enough* ; in this case using Filippov existence theorem about minimizers we have : $S(0,r)\subset W(0,r)$.
The *exponential mapping* $\exp_0$ is defined as follows. Consider a solution (q,p) with $q(0)=0$ corresponding to the Hamiltonian H_n, and *parametrized by arc-length* : $H_n=1/2$. We set $\exp_0 : (p(0),t)\longmapsto q(t)$.

Integrability problem : Two basic questions to compute the sphere and the wave front are the following :

- Question 1 : Are the geodesics equations (2) *integrable by quadratures* ?
- Question 2 : If the geodesics equations are integrable by quadratures, what kind of functions do we need to make the computations : *elementary functions* (exp, log, cos, sin, ...), *elliptic functions* (cn, sn, dn, E, K, ...) or others ?
  In particular if we can parametrize the solutions with no more transcendence than elliptic functions, the sphere and the wave front can be represented using minimal computations with packages of Mathematica or Maple.

We make in this article a complete analysis concerning those two problems with the gradated normal form of order 0 : $g=(1+\alpha y)^2dx^2+(1+\beta x+\gamma y)^2dy^2$.

1.3 The singularity problem and the exp-log category

The general theory (see for instance [2]) tells us that the abnormal geodesic : $t\longmapsto(\pm t,0,0)$ is a global minimizer if its length is small enough ; hence its end-points of length r, r small, given by $(\pm r, 0, 0)$ belongs to the sphere $S(0, r)$. Near those end-points the sphere has *singularities wich do not belong in general to the analytic category*. In particular this will cause *numerical problems* to compute the sphere near those points, even in the *'integrable' case*. An objective of this article is to indicate how to deal with this problem in the 'integrable' case ; we compute *converging asymptotic expansions in the exp-log category*, which is the extension of sub-analytic functions by the exp-log functions (see [7]). We give the scale of the asymptotic expansions.

1.4 General research program

More generally the results presented in this article fit in a general research program to explain the role of abnormal minimizers in SR-geometry on the transcendence of the sphere. The main lines of this program are the following :

1. Prove that the SR-sphere is not sub-analytic if there exists abnormal minimizers.

2. Prove that the SR-sphere is in the *exp-log category* if the geodesics equations are integrable by quadratures.

3. Investigate if the SR-sphere is pfaffian in the general case.

This article gives the main lines of the proof of the two first propositions in the SR-Martinet integrable case. The third difficult problem is briefly discussed in section 4.

This research program is parallel to a research program of Agrachev-Sarychev to prove that the SR-sphere is sub-analytic if there exists no abnormal minimizers, see [3].

2 The integrability problem

2.1 Isoperimetric situation

Since the metric is not depending on z, the z-coordinate is a *cyclic coordinate* for the Hamilton function $H_n = \frac{1}{2}(P_1^2 + P_2^2)$; hence p_z is a *first integral* and the integrability of equations (2) can be reduced to the integrability of the vector field :

$$
\begin{aligned}
\dot{x} &= \frac{1}{a}\left(p_x + p_z\frac{y^2}{2}\right) \\
\dot{y} &= \frac{p_y}{c} \\
\dot{p_x} &= \frac{p_y^2 c_x}{2c^2} + \frac{\left(p_x + p_z\frac{y^2}{2}\right)^2}{2a^2}a_x \\
\dot{p_y} &= \frac{p_y^2 c_y}{2c^2} + \frac{\left(p_x + p_z\frac{y^2}{2}\right)^2}{2a^2}a_y - \frac{\left(p_x + p_z\frac{y^2}{2}\right)}{a}p_z y
\end{aligned}
\tag{4}
$$

with $p_z = \lambda$ constant. The geodesics corresponding to $\lambda = 0$ are called *exceptional*. They have a geometric interpretation. If we denote by g_R the Riemannian metric $a(x,y)dx^2 + c(x,y)dy^2$ induced by g on the plane (x,y) identified to the quotient space : $\mathbb{R}^3/_{\frac{\partial}{\partial z}}$, the trajectories of (4) with $\lambda = 0$ in the plane (x,y) are the *geodesics of the Riemannian metric*.

2.2 Metrics of the form : $g = a(y)dx^2 + c(y)dy^2$

In this case H_n is not depending on x and x is a *cyclic coordinate* ; therefore p_x is a first integral ; another first integral is the Hamiltonian H_n. Hence the system has three first integrals : p_x, p_z and H_n , with commuting Poisson brackets. Therefore the system is integrable by quadratures.
We proceed as follows. If we parametrize the geodesics by arc-length, we get : $H_n = 1/2$ and the equation :

$$P_1^2 + P_2^2 = 1 \tag{5}$$

with $P_1 = \frac{p_x + p_z \frac{y^2}{2}}{\sqrt{a}}$, $P_2 = \frac{p_y}{\sqrt{c}}$, where p_x, p_z are constant, is called the *characteristic equation* ; it can be written :

$$(\sqrt{c}\,\dot{y})^2 + \left(\frac{p_x + p_z \frac{y^2}{2}}{\sqrt{a}} \right)^2 = 1 \tag{6}$$

Using the time $d\eta = \frac{dt}{\sqrt{c}}$ it can be rewritten :

$$\left(\frac{dy}{d\eta} \right)^2 + \left(\frac{p_x + p_z \frac{y^2}{2}}{\sqrt{a}} \right)^2 = 1$$

It corresponds to the *evolution of a particle* of $\mathbb{R}$ of *mass* 2, *whose energy is* 1, with *potential field* : $V(y) = P_1^2(y)$.

2.3 The general gradated case of order 0 : $g = (1+\alpha y)^2 dx^2 + (1 + \beta x + \gamma y)^2 dy^2$

If we parametrize the geodesics by arc-length, we can set : $P_1 = \cos\theta$, $P_2 = \sin\theta$. Moreover if $P_3 = p_z = \lambda$ and $\theta \neq k\pi$ we get the geodesics equations in *cylindric coordinates* :

$$\begin{aligned}
\dot{x} &= \frac{\cos\theta}{\sqrt{a}} \\
\dot{y} &= \frac{\sin\theta}{\sqrt{c}} \\
\dot{z} &= \frac{y^2}{2}\frac{\cos\theta}{\sqrt{a}} \\
\dot{\theta} &= -\frac{1}{\sqrt{a}\sqrt{c}}\left[y\lambda - \frac{a_y}{2\sqrt{a}}\cos\theta + \frac{c_x}{2\sqrt{c}}\sin\theta\right]
\end{aligned} \tag{7}$$

and the last equation can be written :

$$\dot{\theta} = -\frac{1}{\sqrt{a}\sqrt{c}}(y\lambda - \alpha\cos\theta + \beta\sin\theta)$$

Making the following change of parametrization : $\sqrt{a}\sqrt{c}\frac{d}{dt} = \frac{d}{d\tau}$ and denoting by $'$ the derivative with respect to τ we get :

$$\begin{aligned}
x' &= \cos\theta(1 + \beta x + \gamma y) \\
y' &= \sin\theta(1 + \alpha y) \\
z' &= \frac{y^2}{2}\cos\theta(1 + \beta x + \gamma y) \\
\theta' &= -(y\lambda - \alpha\cos\theta + \beta\sin\theta)
\end{aligned} \tag{8}$$

The vector field can be *projected* on space (y, θ).

Asymptotic integrability The parameters α, β, γ are given by the metric. The exponential mapping is defined on the cylinder (θ, λ) and the relevant behavior is when $|\lambda| \to +\infty$. Hence we shall assume :

Assumption : $|\lambda| \gg \alpha, \beta, \gamma$.
Moreover we make the analysis for $\lambda > 0$, the case $\lambda < 0$ being similar.

Consider the projection of the equations on the plane (y, θ) :

$$\begin{aligned}
y' &= \sin\theta(1 + \alpha y) \\
\theta' &= -(y\lambda - \alpha\cos\theta + \beta\sin\theta)
\end{aligned} \tag{9}$$

The singular points localized near 0 are given by : $\theta = 0$, $y = \frac{\alpha}{\lambda}$ and $\theta = \pi$, $y = -\frac{\alpha}{\lambda}$, where $y \to 0$ when $\lambda \to +\infty$.
Differentiating the second equation and simplifying we get :

$$\theta'' + \lambda\sin\theta + \alpha^2\sin\theta\cos\theta - \alpha\beta\sin^2\theta + \beta\cos\theta\,\theta' = 0 \tag{10}$$

By setting $ds = \sqrt{\lambda}\, d\tau$ we get the equation :

$$\frac{d^2\theta}{ds^2} + \sin\theta + \varepsilon\beta\cos\theta\frac{d\theta}{ds} + \varepsilon^2\alpha\sin\theta(\alpha\cos\theta - \beta\sin\theta) = 0 \qquad (11)$$

where $\varepsilon = \frac{1}{\sqrt{\lambda}}$ is a *small parameter*, and the remaining equations are :

$$\frac{dx}{ds} = \varepsilon\cos\theta(1 + \beta x + \gamma y)$$
$$\frac{dz}{ds} = \varepsilon\frac{y^2}{2}\cos\theta(1 + \beta x + \gamma y)$$

and y is given by the second equation of (9).
For $\varepsilon = \frac{1}{\sqrt{\lambda}}$, the equation (11) defines a *one-dimensional foliation* $(\mathcal{F})$ on the *cylinder* $(\mathrm{e}^{i\theta}, \frac{d\theta}{ds})$.

Local analysis The foliation $(\mathcal{F})$ has two fixed singular points corresponding to $M_1 : (\theta = 0, \theta' = 0)$ and $M_2 : (\theta = \pi, \theta' = 0)$. The behaviors near those two points *can be studied by linearization* of :

$$\begin{aligned} \dot\theta &= v \\ \dot v &= -(\sin\theta + \varepsilon\beta\cos\theta\, v + \varepsilon^2\alpha\sin\theta(\alpha\cos\theta - \beta\sin\theta)) \end{aligned}$$

We get :

- Near M_1. The linearized system is :

  $$\begin{aligned} \dot\theta &= v \\ \dot v &= -(\theta(1 + \varepsilon^2\alpha^2) + \varepsilon\beta v) \end{aligned}$$

  and the eigenvalues are the complex numbers :

  $$\sigma_\pm = \frac{-\varepsilon\beta \pm 2i\sqrt{1 + \varepsilon^2(\alpha^2 - \frac{\beta^2}{4})}}{2}$$

  In particular for $\beta \neq 0$, the point M_1 is a *focus*.

- Near M_2. We set $\psi = \theta - \pi$ and the linearized system is :

  $$\begin{aligned} \dot\psi &= v \\ \dot v &= -(-\psi - \varepsilon\beta v + \varepsilon^2\alpha^2\psi) \end{aligned}$$

  and the eigenvalues are the two real numbers :

  $$\eta_\pm = \frac{\varepsilon\beta \pm 2\sqrt{1 + \varepsilon^2(\frac{\beta^2}{4} - \alpha^2)}}{2}$$

  and the point M_2 is a *saddle*.

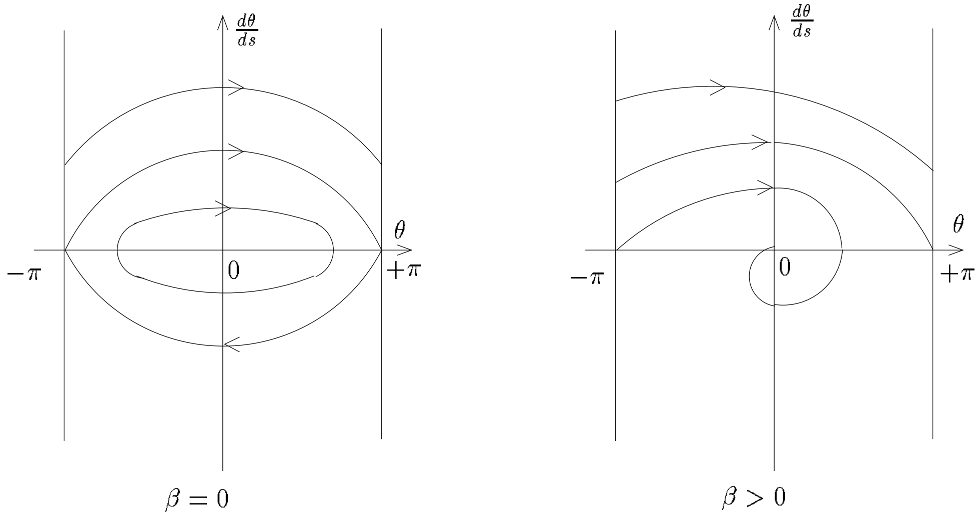


Figure 1: The conservative and the dissipative cases.

Integrability properties of $\mathcal{F}$

- Near M_1 We must distinguish between two cases :

  Case $\beta \neq 0$: M_1 is a focus. The equation can be *linearized in the real analytic category* ; hence the system can be locally integrated using the following elementary functions : exp, sin, cos. This does not mean that the system can be (even locally) integrated by quadratures. In particular *a focus does not admit any continuous first integral.*

  Case $\beta = 0$: In this case M_1 is a *center*. The global solution will be given later.

- Near M_2 The integrability problem is much more complex because M_2 is a *saddle*. The *formal linearization* depends upon the *resonant situation*: $\eta_+/\eta_- \in \mathbb{Q}$, and the *non resonant situation* : $\eta_+/\eta_- \notin \mathbb{Q}$; but *in both cases there exists a formal first integral* (see [6], and the discussion of section 4).
  The analytic integrability requires some extra work but we can conjecture :

  Conjecture : For $\beta \neq 0$, there exists ε such that the saddle is not integrable in the real analytic category.

Global discussion For $\beta = 0$, the foliation $(\mathcal{F})$ is described by :

$$\theta'' + \sin\theta + \varepsilon^2\alpha^2 \sin\theta\cos\theta = 0 \qquad (12)$$

where θ' is the derivative with respect to s. This equation is integrable and is indeed a standard equation from elasticity theory, see [14], [4]. It can be integrated as follows. Multiplying by θ' we get :

$$\theta''\theta' + \sin\theta\,\theta' + \varepsilon^2\alpha^2\sin\theta\cos\theta\,\theta' = 0$$

Integrating we obtain :

$$\frac{1}{2}({\theta'}^2(s) - {\theta'}^2(0)) = \cos\theta(s) - \cos\theta(0) + \frac{\varepsilon^2\alpha^2}{2}(\cos^2\theta(s) - \cos^2\theta(0)) \qquad (13)$$

Remark In our problem $\frac{d\theta}{ds}_{|s=0}$ is computed using : $q(0) = (x(0), y(0), z(0)) = 0$.

We observe that θ can be integrated with *only one quadrature* using equation (13). Hence we have :

Proposition 2.1. *The foliation $(\mathcal{F})$ is integrable in the C^0-category if and only if $\beta = 0$. In this case it is integrable in the C^ω-category. The condition $\beta = 0$ is equivalent to the fact that θ can be integrated using one quadrature.*

Integration in the case $\beta = 0$

This case is called the *conservative case* and the Hamiltonian H_n has two cyclic coordinates : x and z, and the geodesics equations have the three first integrals : p_x, p_z and $H_n = \frac{1}{2}(P_1^2 + P_2^2)$ whose Poisson brackets are zero. The angle θ can be computed using one quadrature and the same is true for y using the relation between y and θ coming from the equation $p_x =$ constant.
Using the analogy with the pendulum where the derivative of the angle can be represented with the Jacobi functions cn and dn (see [10]), we can compute y using the same Jacobi functions. This comes from the following analysis. The *characteristic equation* (6) can be written using the parametrization $d\tau = \frac{dt}{\sqrt{a}\sqrt{c}}$:

$$\left(\frac{dy}{d\tau}\right)^2 + \left(p_x + p_z\frac{y^2}{2}\right)^2 = a \qquad (14)$$

where $a = (1 + \alpha y)^2$.
Hence setting : $F(y) = (1+\alpha y)^2 - \left(p_x + p_z\frac{y^2}{2}\right)^2$, we observe that F is a quartic which can be written as F_1F_2 with :

$$F_1 = (1 + \alpha y) - \left(p_x + p_z\frac{y^2}{2}\right) \quad , \quad F_2 = (1 + \alpha y) + \left(p_x + p_z\frac{y^2}{2}\right)$$

and we can write :

$$F(y) = \left(2m^2 - \frac{\lambda}{2}\left(y - \frac{\alpha}{\lambda}\right)^2\right)\left(2m'' + \frac{\lambda}{2}\left(y + \frac{\alpha}{\lambda}\right)^2\right)$$

where $2m^2 = 1 - p_x + \frac{\alpha^2}{2\lambda}$, $2m'' = 1 + p_x - \frac{\alpha^2}{2\lambda}$ and $m^2 + m'' = 1$.
We have $p_x = \cos\theta(0)$, hence $|p_x| \leqslant 1$. Then $m^2 > 0$ if $\alpha \neq 0$; $m^2 > 0$ if $\alpha = 0$

and $\theta(0) \neq 2n\pi$.
If we set :

$$\eta = \frac{\sqrt{\lambda}\,y}{2m} - \frac{\alpha}{2m\sqrt{\lambda}} \quad , \quad \bar{\eta} = \frac{\sqrt{\lambda}\,y}{2m} + \frac{\alpha}{2m\sqrt{\lambda}}$$

we can write :

$$F(y) = 4m^2(1-\eta^2)(m'' + m^2\bar{\eta}^2) \tag{15}$$

and F is a quartic whose roots on $\mathbb{C}$ are $\eta = \pm 1$, $\bar{\eta} = \pm\frac{\sqrt{m''}}{m}$. The case $m'' = 0$ is called *critical* and it corresponds to a double root for F.

Lemma 2.2. *If $\alpha \neq 0$ in the gradated normal form of order 0, there exist geodesics starting from 0 which are critical.*

Geometric interpretation If $\alpha = 0$, the geodesics can be integrated like in the flat case studied in [2] : $m'' = {k'}^2 = \sqrt{1-k^2}$, where k is the modulus of the elliptic functions, and k' is the complementary of the modulus. When $p_x \to -1$, $k' \to 0$, then y behaves like a sech. In the $(\theta, \dot{\theta})$ projection, the system has a *saddle connection* and the projections of the geodesics tend to the *separatrix.*
When $\alpha \neq 0$, the separatrix is the projection of a geodesic starting from 0.
The role of the parameter α is to make the separatrix and hence some rotating trajectories of the pendulum as projection of geodesics starting from 0.

Normal form The characteristic equation can be normalized using a classical method, see [10,p.55]. We proceed as follows ; F is factorized into F_1F_2 and we consider the *pencil* $F_1 + \nu F_2$ of two quadratic forms.
If $\alpha \neq 0$, there exists two distinct real numbers ν_1, ν_2 such that $F_1 + \nu F_2$ is a *perfect square* : $K_1(y-p)^2, K_2(y-q)^2$.
Using the homographic transformation :

$$u = \frac{y-p}{y-q} \tag{16}$$

the characteristic equation can be written in the normal form :

$$\frac{dy}{\sqrt{F(y)}} = \frac{(p-q)^{-1}du}{\sqrt{(A_1u^2+B_1)(A_2u^2+B_2)}} \tag{17}$$

Excepted the critical case $m'' = 0$, the solution in the u-coordinate can be computed as follows :

- if the quartic F admits two real roots, u can be parametrized using the cn Jacobi function ;
- if the quartic F admits four real roots, u can be parametrized using the dn Jacobi function.

If $\alpha = 0$, the analysis is simplier ; indeed $F(y)$ can be written :

$$F(y) = 4k^2(1-\eta^2)(k'^2 + k^2\eta^2)$$

where $\eta = \frac{\sqrt{\lambda}y}{2k}$ and η can be computed using only the cn function.
Hence we have proved the following :

Proposition 2.3. *We have two cases :*

(i) If $\alpha = 0$, $y = \frac{2k}{\sqrt{\lambda}}\eta$ where η is the cn Jacobi function.

(ii) If $\alpha \neq 0$, y is generically the image by an homography of the cn or dn Jacobi function.

Geometric interpretation If $\alpha = 0$, the motion of y is a cn whose amplitude is $\frac{2k}{\sqrt{\lambda}}$; in particular the motion is symmetric with respect to $y = 0$ and the amplitude tends to 0 when λ tends to the infinity.
If $\alpha \neq 0$, we can expand the homography : $y = \frac{uq-p}{u-1}$ near $u = 0$. The motion of y is no more symmetric with respect to $y = 0$; there is a constant term in the expansion. Hence y can generically be approximated for u small enough by a *shift* plus a cn or dn motion.

Integrating x or z Both $x(\tau)$ and $z(\tau)$ can be computed using only one integral. The integrand is a polynomic function of y. Moreover y can be expanded into a power series in u. Hence the transcendence we need to compute x or z is given by primitives of the form :

$$J_m = \int \mathrm{cn}^m u du \quad , \quad K_m = \int \mathrm{dn}^m u\, du$$

Those primitives are computed by recurrence in [10, p.87]. It involves a new transcendence : the *Jacobi epsilon function* $E(u,k)$ defined by :

$$E(u,k) = \int_0^u dn^2(v,k)\, dv$$

This function was already needed in the flat case, see [2].

Arc-length parameter To recover the length parameter we use the formula: $dt = (1+\alpha y)(1+\gamma y)d\tau$. As previously y can be computed as a power series in u ; hence it can be evaluated using the same primitives J_m and K_m.

2.4 Application : computation of conjugate points

One interesting and non trivial application of the previous parametrizations is the computation of the conjugate points ; they are solutions of the equation :

$$\frac{\partial x}{\partial \lambda}\frac{\partial y}{\partial \theta_0} - \frac{\partial y}{\partial \lambda}\frac{\partial x}{\partial \theta_0} = 0$$

where $\theta_0 = \theta(0)$ and x, y are the two first components of a normal geodesic. This equation was used in the flat case to evaluate the conjugate points.

3 Transcendence of the sphere and the wave front near the abnormal line. The exp-log category.

3.1 The geometric framework

In order to study the structure of the sphere or wave front near the abnormal direction it is convenient to consider the following *traces* :

$$\widetilde{S}(0,r) = S(0,r) \cap \{y=0\} \quad \text{and} \quad \widetilde{W}(0,r) = W(0,r) \cap \{y=0\}$$

This leads to the important concept of *return mapping* :

Definition 3.1. *Let $e : t \in [0,T] \longmapsto (x(t), y(t), z(t))$ be a normal geodesic parametrized by arc-length. If $y(t) \not\equiv 0$ we can define : $0 < t_1 < \cdots < t_N \leqslant T$ as the times corresponding to $y(t_i) = 0$. The first* return mapping *is :*

$$R_1 : (\lambda, \theta(0)) \longmapsto (x(t_1), z(t_1))$$

and more generally the n-th mapping is the map :

$$R_n : (\lambda, \theta(0)) \longmapsto (x(t_n), z(t_n))$$

If the length is fixed at r, we observe that $\widetilde{W}$ is the union of the image of the return mapping with $x = \pm r$, $z = 0$.
The following proposition is straightforward, see[4] :

Proposition 3.2. *For each $n \geqslant 1$, the return mapping R_n is not proper.*

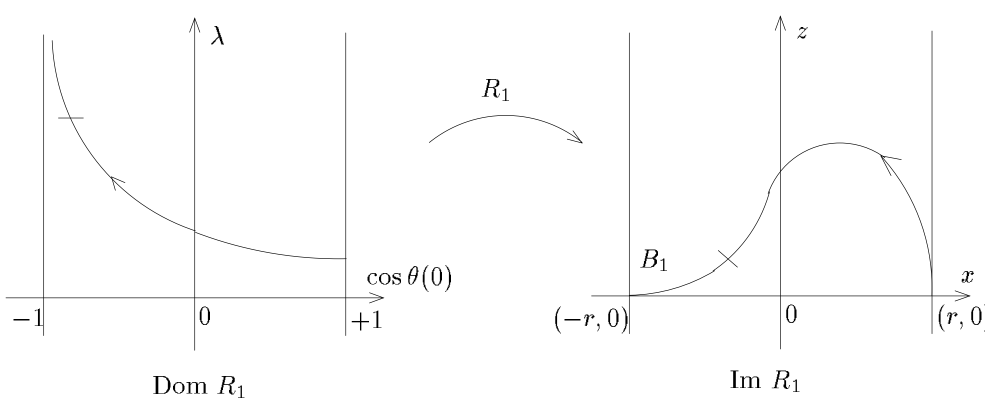


Figure 2: The first return mapping in the flat case.

3.2 Formulas in the conservative case

If the metric g is not depending on x, it is convenient to use the following formulas from [8]. We introduce :

$$\sigma = \begin{cases} \text{sign } \dot{y}(0) & \text{if } \dot{y}(0) \neq 0 \\ \text{sign } \ddot{y}(0) & \text{if } \dot{y}(0) = 0 \end{cases}$$

If the motion of y is periodic with period $\mathcal{P}$, we set :

$$y_+ = \max_{t\in[0,\mathcal{P}]} y(t) \quad , \quad y_- = \min_{t\in[0,\mathcal{P}]} y(t)$$

Parametrizing the geodesics by y we must solve the equations :

$$\frac{dx}{dy} = \frac{\sqrt{c}}{\sqrt{a}}\frac{P_1}{P_2} \quad , \quad \frac{dz}{dy} = \frac{y^2}{2}\frac{\sqrt{c}}{\sqrt{a}}\frac{P_1}{P_2} \quad , \quad dt = \frac{\sqrt{c}}{P_2}dy$$

where $P_2(y) = \sigma\sqrt{1-P_1^2(y)}$ for $t \in [0, t_1]$.
If $y(T) = 0$ for $T = t_N$ we get the formulas :

- <u>N odd</u>

$$\begin{aligned} x(T) &= 2\int_0^{y_\sigma} \sigma\frac{\sqrt{c}}{\sqrt{a}}\frac{P_1(y)}{\sqrt{1-P_1^2(y)}}dy + (N-1)\int_{y_-}^{y_+}\frac{\sqrt{c}}{\sqrt{a}}\frac{P_1(y)}{\sqrt{1-P_1^2(y)}}\,dy \\ z(T) &= \int_0^{y_\sigma} \sigma\frac{\sqrt{c}}{\sqrt{a}}\frac{y^2P_1(y)}{\sqrt{1-P_1^2(y)}}\,dy + (N-1)\int_{y_-}^{y_+}\frac{\sqrt{c}}{2\sqrt{a}}\frac{y^2P_1(y)}{\sqrt{1-P_1^2(y)}}\,dy \end{aligned} \tag{18}$$

- <u>N even</u>

$$\begin{aligned} x(T) &= N\int_{y_-}^{y_+}\frac{\sqrt{c}}{\sqrt{a}}\frac{P_1(y)}{\sqrt{1-P_1^2(y)}}\,dy \\ z(T) &= N\int_{y_-}^{y_+}\frac{\sqrt{c}}{2\sqrt{a}}\frac{y^2P_1(y)}{\sqrt{1-P_1^2(y)}}\,dy \end{aligned} \tag{19}$$

and the period is given by :

$$\mathcal{P} = 2\int_{y_-}^{y_+}\frac{\sqrt{c}}{\sqrt{1-P_1^2(y)}}\,dy \tag{20}$$

3.3 Computations in the flat case

The basis of the general algorithm to compute the image of the return mapping is coming from the flat case where $g = dx^2 + dy^2$. The algorithm is the following.

Both sets $\widetilde{S}(0,r)$ and $\widetilde{W}(0,r)$ are symmetric with respect to 0 and we can assume $z>0$. From [2] the image of R_1 in $z>0$ is parametrized by :

$$\begin{aligned} x(k,\lambda) &= -t + \frac{4E}{\sqrt{\lambda}} \\ z(k,\lambda) &= \frac{4}{3\lambda^{3/2}}[(2k^2-1)E + {k'}^2 K] \end{aligned} \tag{21}$$

where K and E are the *complete elliptic integrals* with modulus $k = \sqrt{\frac{1-p_x}{2}}$, $p_x = \cos\theta(0)$, $k' = \sqrt{1-k^2}$ and $\theta(0) \in [-\pi, 0]$:

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad , \quad E = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\, d\theta$$

and the period $\mathcal{P} = \frac{4}{\sqrt{\lambda}}K(k)$.

Both parameters λ, k are related when we fix the length to r :

$$t = r = \frac{2K}{\sqrt{\lambda}} \tag{22}$$

Hence the image of R_1 in $z>0$ is given by :

$$\begin{aligned} x &= -r + 2r\frac{E}{K} \\ z &= \frac{r^3}{6K^3}\left[(2k^2-1)E + {k'}^2 K\right] \end{aligned} \tag{23}$$

It is a parametric curve parametrized by k'. It is semi-analytic excepted when $\theta(0) \longrightarrow -\pi$ and $k \longrightarrow 1^-$. This can be seen using the following expansions for E, K when $k' \longrightarrow 0$:

$$\begin{aligned} E &= u_1({k'}^2)\ln\frac{4}{k'} + u_2({k'}^2) \\ K &= u_3({k'}^2)\ln\frac{4}{k'} + u_4({k'}^2) \end{aligned}$$

where the u_i's are analytic functions and moreover :

$$\begin{aligned} u_1({k'}^2) &= \tfrac{k'^2}{2} + o({k'}^3) & \qquad u_2({k'}^2) &= 1 - \tfrac{k'^2}{4} + o({k'}^3) \\ u_3({k'}^2) &= 1 + \tfrac{k'^2}{4} + o({k'}^3) & \qquad u_4({k'}^2) &= -\tfrac{k'^2}{4} + o({k'}^3) \end{aligned}$$

In particular both E and K have a *logarithmic singularity* when $k' \longrightarrow 0$ and hence, using [11], the branch of (23) near $x = -r$, denoted by B_1, can be computed in the *exp-log category* by eliminating k'. More precisely the algorithm is the following :

Let $X = \frac{x+r}{2r}$, $Z = \frac{z}{r^3}$. We get :

$$X = \frac{E}{K} = \frac{u_1(k'^2)\ln\frac{4}{k'} + u_2(k'^2)}{u_3(k'^2)\ln\frac{4}{k'} + u_4(k'^2)} \tag{24}$$

$$Z = \frac{1}{6K^3}\left[(2k^2-1)E + k'^2 K\right] \tag{25}$$

Then :

Step 1 : 'Compactification' If we introduce :

$$X_1 = k' \quad , \quad X_2 = \frac{1}{\ln\frac{4}{k'}}$$

we have : $X_1, X_2 \longrightarrow 0$ when $k' \longrightarrow 0^+$ and both X and Z are *analytic functions* of X_1 and X_2.

Step 2 : 'Finding equivalents' An easy computation using (24) shows the following :

$$X_1 \sim 4\mathrm{e}^{-\frac{1}{X}} \quad , \quad X_2 \sim X \qquad \text{when } X \longrightarrow 0^+$$

and we can write :

$$X_1 = 4\mathrm{e}^{-\frac{1}{X}}(1+Y_1(X)) \quad , \quad X_2 = X(1+Y_2(X))$$

where $Y_1, Y_2 \longrightarrow 0$ when $X \longrightarrow 0^+$
Both Y_1 and Y_2 *can be compared* and a computation gives us :

$$Y_2 = X\,A_1(X, Y_1) \quad , \quad Y_1 \sim \frac{Y_2}{X} \qquad \text{when } X \longrightarrow 0^+$$

where A_1 is a germ of analytic function at 0.

Step 3 : 'Solving equation (24) in the analytic category' The equation (24) can be solved in the variables Y_1, X_1, X_2 by using the *implicit function theorem in the analytic category* and the computations show the following :

$$Y_1 = A_2\left(X, \frac{\mathrm{e}^{-\frac{1}{X}}}{X}\right)$$

where A_2 is a germ at 0 of an analytic function.
Using this relation we end with :

$$Z = F\left(X, \frac{\mathrm{e}^{-\frac{1}{X}}}{X}\right)$$

where F is a germ at 0 of an analytic function.

N.B. If we use only the fact that the u_i's functions are analytic with respect to k' we get a scale $\frac{\mathrm{e}^{-\frac{1}{X}}}{X^2}$.

3.4 Computations in the general conservative case

The algorithm is similar to the flat case using the *integral formulas* of subsection 3.2, but the computations are much more complex. The additional complexity is coming from two phenomena called respectively the *double log* and the *period halfing.*

3.4.1 Double log

In the flat case the relation (22) expressing the fact that the length is fixed to r is trivial. In general this is no longer true and we must solve an equation of the type :

$$x = y \ln y \qquad y \longrightarrow +\infty$$

We set :

$$y = \frac{x}{\ln x}(1 + Y_1(x)) \quad \text{with } Y_1 = o(1)$$

and plugging y into the equation we get, using the implicit function theorem in the analytic category, a relation :

$$Y_1 = A(X_1, X_2)$$

where A is a germ at 0 of an analytic function, and X_1, X_2 represent the *scale factors* :

$$X_1 = \frac{1}{\ln x} \quad , \quad X_2 = \frac{\ln \ln x}{\ln x}$$

3.4.2 The period halfing

In the flat case, the image of R_1 contains only one branch B_1 in the domain $z > 0$ which is not sub-analytic. It corresponds to the limit behavior of the *oscillating* trajectories of the pendulum when $\theta(0) \longrightarrow -\pi$, which tend to the separatrix. When $\alpha \neq 0$ it comes from our analysis that we must consider : on one side, *oscillating trajectories*, where y is parametrized by the cn Jacobi elliptic function ; on the other side, *rotating trajectories*, where y is parametrized by the dn Jacobi elliptic function. It can be interpreted as a *period halfing phenomenon* by using for fixed k the relation : $dn^2 s = k'^2 + k^2 cn^2 s$ and taking $k \longrightarrow 1$. In this case the image of R_1 contains *two branches* B_1 and B_2 which end at $x = -r, z = 0$ and they are *not sub-analytic*. The branch B_1 corresponds to a cn-behavior and the branch B_2 to a dn-behavior. The branch B_2 shrinks to 0 when $\alpha \longrightarrow 0$.

The figure 3 illustrates the role of the parameter α. Indeed imposing $y(0) = 0$ and $y(r) = 0$, this defines a section S given in the space $(\theta, \frac{d\theta}{ds})$ by the equation :

$$\frac{d\theta}{ds} = \varepsilon(\alpha \cos\theta - \beta \sin\theta)$$

The role of the parameter α is to push the separatrix Σ as an admissible trajectory ; hence we get the two non sub-analytic branches B_1 and B_2. This phenomenon is illustrated on figure 3.

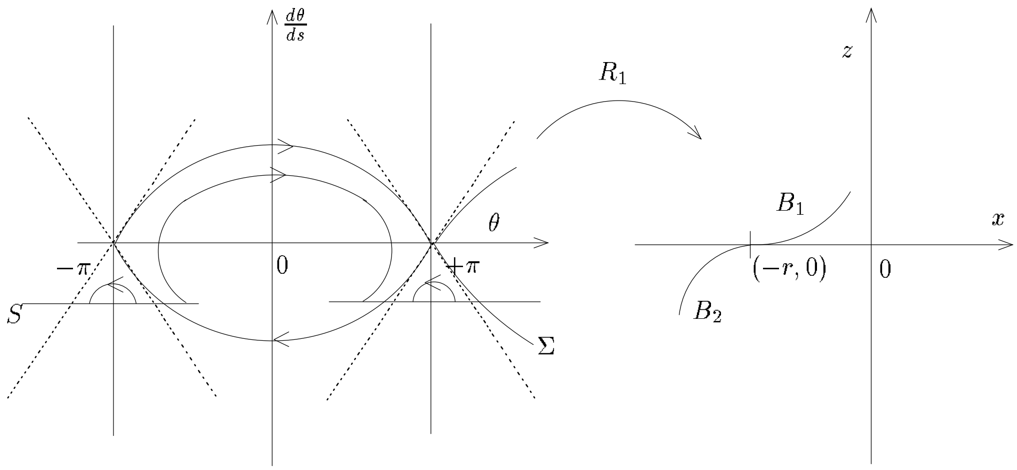


Figure 3: The separatrix Σ

3.5 The algorithm to compute B_1 and the complexity of B_1

The aim is to give the precise transcendence of the branch B_1 in the general conservative case. From now on, $An(.)$ and $An_0(.)$ denote a germ of analytic function at 0, and moreover $An_0(0) = 0$.
Recall the general formulas that give a parametrization of this branch :

$$x(r) = -2\int_0^{y_{-1}} \frac{\sqrt{c}}{\sqrt{a}} \frac{P_1}{\sqrt{1-P_1^2}}\, dy \qquad (26)$$

$$z(r) = -\int_0^{y_{-1}} \frac{\sqrt{c}}{\sqrt{a}} \frac{y^2 P_1}{\sqrt{1-P_1^2}}\, dy \qquad (27)$$

$$r = -2\int_0^{y_{-1}} \sqrt{c}\, \frac{1}{\sqrt{1-P_1^2}}\, dy \qquad (28)$$

where :

$$\begin{cases} a(y) = An(y) = 1 + \alpha y + \alpha' y^2 + \dots \\ c(y) = An(y) = 1 + \gamma y + \dots \end{cases}$$

and :

$$P_1(y) = \frac{p_x + \frac{\lambda}{2} y^2}{\sqrt{a(y)}} = p_x - p_x \frac{\alpha}{2} y + \left(p_x (\frac{3}{8}\alpha^2 - \alpha') + \frac{\lambda}{2} \right) y^2 + \dots$$

and y_{-1} is the negative root of $1 - P_1(y)$ (we will justify it later).

The objective is to express x and z as a parametric curve in the exp-log category and compute the graph in the same category by elimination of the parameter.

3.5.1 Precision on parameters

We study the system near the abnormal direction, so we have :

$$\lambda \longrightarrow +\infty \ , \quad p_x \longrightarrow -1$$

Precision on y_{-1} In the flat case, P_1's graph is a parabola, represented on figure 4.

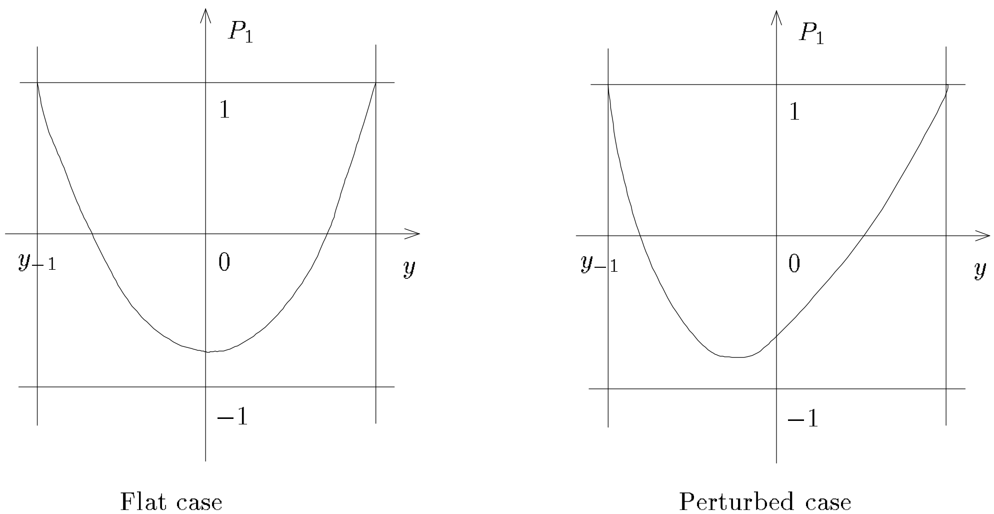


Figure 4: P_1's graph

The general integrable case can be considered as a perturbation of the flat case where the parabola is deformed into a non symmetric graph (see figure 4).

Hence the existence of y_{-1}, as a negative simple root of $P_1 = 1$, is straightforward.

Using the implicit function theorem, we get :

$$y_{-1} = \frac{1}{\sqrt{\lambda}} An\left(p_x, \frac{1}{\sqrt{\lambda}}\right)$$

Moreover, by continuity with the flat case : $y_{-1} \underset{\sqrt{\lambda} \to +\infty}{\sim} -\sqrt{\frac{2}{\lambda}(1 - p_x)}$

Precision on $P_1'(y)$'s roots

Proposition 3.3. *There is in the foregoing domain an unique root S of P_1'. Moreover :*

$$S = \frac{\alpha}{\lambda} An(p_x, \frac{1}{\lambda}) \underset{\lambda \to +\infty}{\sim} \frac{p_x \alpha}{2\lambda} < 0$$

Remark 3.4. $S = 0 \Leftrightarrow \alpha = 0$.

Proof. If λ is large enough, then : $\quad \forall y \quad P_1''(y) > 0$
Moreover : $P_1'\left(-\frac{2}{\sqrt{\lambda}}\right) P_1'\left(\frac{2}{\sqrt{\lambda}}\right) < 0$
We can deduce that P_1' admits an unique root in this domain. The remaining goes as before. □

Remark 3.5. *In the flat case, we have : $S = 0$, $P_1(S) = p_x$.*

This remark leads us to define :

Definition 3.6. $k' = \sqrt{\frac{1+P_1(S)}{2}} \quad , \quad k = \sqrt{\frac{1-P_1(S)}{2}}$

As in the flat case, we have : $k^2 + {k'}^2 = 1$, and then : $k = 1 + An({k'}^2)$. Note that $p_x \longrightarrow -1$, so $k' \longrightarrow 0$.

k' and λ are our new parameters. They are the initial conditions for the geodesics. Our aim is the following : from (27) and (28) we express x and z in terms of k' and λ ; from (28) we get an implicit relation between k' and λ. Solving this implicit equation, we will get λ in terms of k', so that we get x and z in terms of k'. Then the problem is to eliminate the parameter k', to get finally the graph $z(x)$.

As we will see, the previous expansions are in the *exp-log category* (see [11]), e.g. these are analytic expansions in k' and some functions composed of exp and log. Hence the aim is to express the graph $z(x)$ in this category, with a precise scale.

We meet two technical problems :

1. justifying the *convergence of the expansions.*
2. solving algorithmically this problem of elimination of parameter in the *exp-log category.*

The problem of analyticity of the expansions is based on the following :

Proposition 3.7. *Let $f_n(x) = \sum\limits_p a_{n,p} x^p$, $n \in \mathbb{N}$, be a family of entire series that converge for $|x| < 1$. Suppose : $\exists A \ / \ \forall p \quad \sum\limits_n |a_{n,p}| \leqslant A$.*

Then $f(x) = \sum\limits_n f_n(x) = \sum\limits_p (\sum\limits_n a_{n,p}) x^p$ is analytic and converges for $|x| < 1$.

In what follows, we will not detail all these calculations, which would be too long. Note that, to do this, formal computations using Maple packages was very helpful.

We will now express all our parameters in terms of k' and $\frac{1}{\sqrt{\lambda}}$.

Expression of p_x By definition : ${k'}^2 = \frac{1+P_1(S)}{2}$ with $S = \frac{\alpha}{\lambda} An(p_x, \frac{1}{\lambda})$
So : $2{k'}^2 - 1 = P_1(S) = p_x + \frac{1}{\lambda} An(p_x, \frac{1}{\lambda})$
And, using the implicit function theorem, we conclude that :

$$p_x = An\left({k'}^2, \frac{1}{\lambda}\right) \sim -1 \tag{29}$$

Expression of S We get easily :

$$S = \frac{1}{\lambda} An\left({k'}^2, \frac{1}{\lambda}\right) \sim -\frac{\alpha}{2\lambda} \tag{30}$$

Expression of y_{-1} We obtain :

$$y_{-1} = \frac{1}{\sqrt{\lambda}} An\left({k'}^2, \frac{1}{\sqrt{\lambda}}\right) \sim -\frac{2}{\sqrt{\lambda}} \tag{31}$$

3.5.2 Preliminaries before calculating integrals

The aim is to expand analytically all the integrands, so that very simple reference integrals appear, which will give the precise transcendence of the branch.

Expansion of P_1 with the new parameters P_1 appears in all formulas, so it's natural to work on its expression.
Recall that : $P_1(y) = \frac{p_x + \frac{\lambda}{2} y^2}{\sqrt{a(y)}} = An\left(p_x, \frac{1}{\sqrt{\lambda}}, \sqrt{\lambda}\, y\right)$
Let's make a *change of variable* :

$$y = \frac{1}{\sqrt{\lambda}}\left(2k\eta + S\sqrt{\lambda}\right) \tag{32}$$

We get actually, recalling that $P_1'(S) = 0$:

$$P_1(y) = P_1(S) + 2k^2\eta^2\left(1 + \frac{1}{\lambda} F\left(p_x, \frac{1}{\sqrt{\lambda}}\right)\right) + \frac{\eta^3}{\sqrt{\lambda}} An\left({k'}^2, \frac{1}{\sqrt{\lambda}}, \frac{\eta}{\sqrt{\lambda}}\right) \tag{33}$$

with F analytic.

Expansion of $\frac{1}{\sqrt{1-P_1(y)}}$ From (33) we get :

$$1 - P_1(y) = 2k^2 - 2k^2\eta^2\left(1 + \frac{1}{\lambda} An\left(p_x, \frac{1}{\sqrt{\lambda}}\right)\right) - \frac{\eta^3}{\sqrt{\lambda}} An\left({k'}^2, \frac{1}{\sqrt{\lambda}}, \frac{\eta}{\sqrt{\lambda}}\right)$$

Hence, remarking that $k = An({k'}^2)$:

$$\frac{1}{\sqrt{1 - P_1(y)}} = \frac{1}{\sqrt{2}}\left(1 + An\left({k'}^2, \frac{1}{\sqrt{\lambda}}, \eta^2, \frac{\eta}{\sqrt{\lambda}}\right)\right) \tag{34}$$

(valid if $|P_1(y)| < 1$, which will be the case in our integrals...)

Expansion of $\frac{1}{\sqrt{1+P_1(y)}}$ From (33), we get :

$$1+P_1(y)=2k'^2+2k^2\eta^2\left(1+\frac{1}{\lambda}F\left(p_x,\frac{1}{\sqrt{\lambda}}\right)\right)+\frac{\eta^3}{\sqrt{\lambda}}An\left(k'^2,\frac{1}{\sqrt{\lambda}},\frac{\eta}{\sqrt{\lambda}}\right)$$

If we make the change of variable : $\eta=\frac{k'}{k}\frac{1}{1+\frac{1}{\lambda}F\left(p_x,\frac{1}{\sqrt{\lambda}}\right)}u$

Then :

$$\begin{aligned}1+P_1(y) &= 2k'^2\left(1+u^2+\frac{k'u^3}{\sqrt{\lambda}}An\left(k'^2,\frac{1}{\sqrt{\lambda}},\frac{k'u}{\sqrt{\lambda}}\right)\right)\\ &= 2k'^2(1+u^2)\left(1+\frac{k'u^3}{\sqrt{\lambda}(1+u^2)}An\left(k'^2,\frac{1}{\sqrt{\lambda}},\frac{k'u}{\sqrt{\lambda}}\right)\right)\end{aligned}$$

Hence :

$$\frac{1}{\sqrt{1+P_1(y)}}=\frac{1}{k'\sqrt{2}}\frac{1}{\sqrt{1+u^2}}\left(1+An\left(k'^2,\frac{1}{\sqrt{\lambda}},\frac{k'u}{\sqrt{\lambda}},\frac{k'u^3}{\sqrt{\lambda}(1+u^2)}\right)\right) \quad (35)$$

(same remark on the validity of the expansion)

3.5.3 Expansions of the integrals

Reference integrals As we will see later, the following integrals are useful. They will be, in our expansions of x, z, and r, our reference integrals.

Proposition 3.8. *Let $p,i\in\mathbb{N}$. Then :*

$$\begin{aligned}\int_0^x \frac{u^{2i}}{\left(\sqrt{1+u^2}\right)^{2p+1}}\,du &= \frac{1}{\left(\sqrt{1+x^2}\right)^{2p-1}}\Bigg(\mu_0\left(\sqrt{1+x^2}\right)^{2p-1}\ln(x+\sqrt{1+x^2})\\ &\qquad +\mu_1x+\mu_3x^3+\cdots+\mu_{2i-1}x^{2i-1}\Bigg)\\ \int_0^x \frac{u^{2i+1}}{\left(\sqrt{1+u^2}\right)^{2p+1}}\,du &= \mu_1+\frac{1}{\left(\sqrt{1+x^2}\right)^{2p-1}}\left(\mu_0+\mu_2x^2+\cdots+\mu_{2i}x^{2i}\right)\end{aligned}$$

Proof. The proof is elementary. Let $I_{i,p}$ be one of both integrals studied. We have immediately :

$$I_{i+1,p}=I_{i,p-1}-I_{i,p}$$

So it's enough calculating $(I_{i,0})_{i\in\mathbb{N}}$ and $(I_{0,p})_{p\in\mathbb{N}}$ to get all $I_{i,p}$, which is easy. $\square$

Proposition 3.9. *Let $i\in\mathbb{N}$. Then :*

$$\begin{aligned}&\text{If } i\geqslant 1: \quad \int_0^x t^{2i}\sqrt{1+t^2}\,dt &&= \lambda_0\ln(x+\sqrt{1+x^2})+\lambda_1x\sqrt{1+x^2}\\ & &&\quad +(1+x^2)^{3/2}(\mu_1x+\mu_3x^3+\cdots+\mu_{2i-1}x^{2i-1})\\ &\text{If } i=1: \quad \int_0^x t^2\sqrt{1+t^2}\,dt &&= \lambda_0\ln(x+\sqrt{1+x^2})+\lambda_1x\sqrt{1+x^2}\\ &\text{If } i=0: \quad \int_0^x \sqrt{1+t^2}\,dt &&= \lambda_0\ln(x+\sqrt{1+x^2})\end{aligned}$$

Proof. The change of variable : $t = \mathrm{sh}(u)$ leads to the formulas. □

Expansion of the length Recall the formula :

$$r = \int_0^{y_{-1}} \sqrt{c(y)}\, \frac{1}{\sqrt{1 - P_1^2(y)}}\, dy$$

It is an improper integral, because $P_1(y_{-1}) = 1$. Since y_{-1} is a simple root, the integral exists. We see easily that the formal expansions done previously are relevant, and that we can exchange $\int$ and $\sum$, which is not obvious *a priori*.
From our previous calculations, and both changes of variables, we get :

$$r = 2 \int_A^B C\, du$$

with :

$$\begin{aligned}
A &= \frac{1}{k'}\left(-1 + An({k'}^2, \frac{1}{\sqrt{\lambda}})\right)\\
B &= \frac{\alpha}{4\sqrt{\lambda}k'}\left(1 + An({k'}^2, \frac{1}{\sqrt{\lambda}})\right)\\
C &= \sqrt{c} \times 1/\sqrt{1 - P_1} \times 1/\sqrt{1 + P_1} \times dy\\
&= \left(1 + An({k'}^2, \frac{1}{\sqrt{\lambda}}, \frac{k'u}{\sqrt{\lambda}})\right) \times \frac{1}{\sqrt{2}}\left(1 + An({k'}^2, \frac{1}{\sqrt{\lambda}}, {k'}^2u^2, \frac{k'u}{\sqrt{\lambda}})\right) \times\\
&\frac{1}{k'\sqrt{2}}\frac{1}{\sqrt{1+u^2}}\left(1 + An({k'}^2, \frac{1}{\sqrt{\lambda}}, \frac{k'u}{\sqrt{\lambda}}, \frac{k'u^3}{\sqrt{\lambda}(1+u^2)})\right) \times \frac{2k}{\sqrt{\lambda}}k'\left(1 + An({k'}^2, \frac{1}{\sqrt{\lambda}})\right)
\end{aligned}$$

Hence :

$$\begin{aligned}
\frac{r\sqrt{\lambda}}{2} &= \int_A^B \frac{du}{\sqrt{1+u^2}} + \int_A^B \frac{1}{\sqrt{1+u^2}} An({k'}^2, \frac{1}{\sqrt{\lambda}}, \frac{k'u}{\sqrt{\lambda}}, \frac{k'u^3}{\sqrt{\lambda}(1+u^2)}, {k'}^2u^2)\, du\\
&= I_1 + I_2
\end{aligned}$$

Calculation of I_2 I_2 is sum of the following integrals :

$$J_{p,m,n} = \int_A^B \frac{{k'}^{p+m+2n}u^{3p+m+2n}}{\left(\sqrt{\lambda}\right)^{p+m}\left(\sqrt{1+u^2}\right)^{2p+1}}\, du \qquad p, m, n \in \mathbb{N}$$

Hence from (3.8) and (3.9) we get :

$$J_{p,m,n} = \begin{cases} An\left({k'}^2, \frac{1}{\sqrt{\lambda}}, {k'}^2 \ln\frac{1}{k'}, {k'}^2 \ln\sqrt{\lambda}\right) & \text{if } \alpha \neq 0\\ An\left({k'}^2, \frac{1}{\sqrt{\lambda}}, {k'}^2 \ln\frac{1}{k'}\right) & \text{if } \alpha = 0\end{cases}$$

Remark that k' always appears *squared*. This induces that the expression of z in function of $x+r$ will not contain any $\sqrt{x+r}$.
Moreover, a detailed analysis gives :

$$I_2 = \begin{cases} \ln 2 + An_0\left(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2 \ln\frac{1}{k'}, k'^2 \ln\sqrt{\lambda}\right) & \text{if } \alpha \neq 0 \\ \ln 2 + An_0\left(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2 \ln\frac{1}{k'}\right) & \text{if } \alpha = 0 \end{cases}$$

Calculation of I_1

$$I_1 = \mathrm{Argsh}(B) - \mathrm{Argsh}(A)$$

But, if $X \to +\infty$:

$$\mathrm{Argsh}(X) = \ln(X + \sqrt{1+X^2}) = \ln X + \ln 2 + An_0(\frac{1}{X^2})$$

Hence :

$$I_1 = \begin{cases} 2\ln\frac{1}{k'} - \ln\sqrt{\lambda} + \ln(\alpha) + An_0\left(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2\lambda\right) & \text{if } \alpha \neq 0 \\ \ln\frac{1}{k'} + \ln 2 + An_0\left(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2\lambda\right) & \text{if } \alpha = 0 \end{cases}$$

Remark 3.10. *Here we can state that there is a 'period doubling' if and only if $\alpha \neq 0$.*

We obtain actually an implicit equation in $\sqrt{\lambda}$:

$$\frac{r\sqrt{\lambda}}{2} = \begin{cases} 2\ln\frac{1}{k'} - \ln\sqrt{\lambda} + \ln 2\alpha + An_0(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2\ln\frac{1}{k'}, k'^2\lambda, k'^2\ln\sqrt{\lambda}) & \text{if } \alpha \neq 0 \\ \ln\frac{1}{k'} + 2\ln 2 + An_0(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2\lambda) & \text{if } \alpha = 0 \end{cases}$$

Resolution of the implicit equation From now on, we set : $\boxed{t = k'^2}$.
We must distinguish between two cases :

- Case $\alpha \neq 0$ We have :

$$\frac{r\sqrt{\lambda}}{2} = \ln\frac{1}{t} - \ln(\frac{r\sqrt{\lambda}}{2}) + \ln(r\alpha) + An_0(t, \frac{1}{\sqrt{\lambda}}, t\ln\frac{1}{t}, t\lambda, t\ln\sqrt{\lambda})$$

Easily : $\frac{r\sqrt{\lambda}}{2} \sim \ln\frac{1}{t}$. We set : $\frac{r\sqrt{\lambda}}{2} = \ln\frac{1}{t} + u$. Hence :

$$u = -\ln\ln\frac{1}{t} + \ln(r\alpha) + An_0(t, \frac{1}{\ln\frac{1}{t}}, \frac{u}{\ln\frac{1}{t}}, t\ln\frac{1}{t}, t\ln^2\frac{1}{t}, t\ln\ln\frac{1}{t})$$

Then we set : $u = -\ln\ln\frac{1}{t} + \ln(r\alpha) + v$. We get :

$$v = An_0(t, \frac{1}{\ln\frac{1}{t}}, \frac{\ln\ln\frac{1}{t}}{\ln\frac{1}{t}}, t\ln\frac{1}{t}, t\ln^2\frac{1}{t}, t\ln\ln\frac{1}{t}, v)$$

And the *implicit function theorem* allows us to conclude that :

$$v = An_0(t, \frac{1}{\ln\frac{1}{t}}, \frac{\ln\ln\frac{1}{t}}{\ln\frac{1}{t}}, t\ln\frac{1}{t}, t\ln^2\frac{1}{t}, t\ln\ln\frac{1}{t})$$
$$= An_0(\frac{1}{\ln\frac{1}{t}}, \frac{\ln\ln\frac{1}{t}}{\ln\frac{1}{t}}, t\ln^2\frac{1}{t})$$

ccl : $\boxed{\frac{r\sqrt{\lambda}}{2} = \ln\frac{1}{t} - \ln\ln\frac{1}{t} + \ln(r\alpha) + An_0(\frac{1}{\ln\frac{1}{t}}, \frac{\ln\ln\frac{1}{t}}{\ln\frac{1}{t}}, t\ln^2\frac{1}{t}, v)}$

- Case $\alpha = 0$ We set : $\frac{r\sqrt{\lambda}}{2} = \frac{1}{2}\ln\frac{1}{t} + 2\ln 2 + u$. Hence :

$$u = An_0(t, \frac{1}{\ln\frac{1}{t}}, t\ln\frac{1}{t}, t\ln^2\frac{1}{t}, u)$$

And, in the same way, thanks to the *implicit function theorem*, we obtain :

ccl : $\boxed{\frac{r\sqrt{\lambda}}{2} = \frac{1}{2}\ln\frac{1}{t} + 2\ln 2 + An_0(\frac{1}{\ln\frac{1}{t}}, t\ln^2\frac{1}{t})}$

Expansion of x Recall that :

$$x(r) = -2\int_0^{y_{-1}} \frac{\sqrt{c}}{\sqrt{a}} \frac{P_1}{\sqrt{1-P_1^2}}\, dy$$

Write : $\frac{P_1}{\sqrt{1-P_1^2}} = \frac{1+P_1}{\sqrt{1-P_1^2}} - \frac{1}{\sqrt{1-P_1^2}} = \frac{\sqrt{1+P_1}}{\sqrt{1-P_1}} - \frac{1}{\sqrt{1-P_1^2}}$

And using :

$$r = -2\int_0^{y_{-1}} \sqrt{c}\, \frac{1}{\sqrt{1-P_1^2}}\, dy$$

We get :

$$X = \frac{x+r}{2r} = \frac{1}{r}\int_{y_{-1}}^0 \frac{\sqrt{c}}{\sqrt{a}} \frac{\sqrt{1+P_1}}{\sqrt{1-P_1}}\, dy - \frac{1}{r}\int_{y_{-1}}^0 \left(\frac{\sqrt{c}}{\sqrt{a}} - 1\right) \frac{1}{\sqrt{1-P_1^2}}\, dy$$

Now, with the previous notations, we have :

- $\frac{\sqrt{c}}{\sqrt{a}} = 1 + An({k'}^2, \frac{1}{\sqrt{\lambda}}, \frac{k'u}{\sqrt{\lambda}})$
- $\sqrt{1+P_1} = k'\sqrt{2}\sqrt{1+u^2}\left(1 + An({k'}^2, \frac{1}{\sqrt{\lambda}}, \frac{k'u}{\sqrt{\lambda}}, \frac{k'u^3}{\sqrt{\lambda}(1+u^2)})\right)$

Thus we obtain $X = \frac{x+r}{2r}$ as an analytic sum of integrals of following type :

$$J_{p,n,m} = \int_A^B \frac{{k'}^{p+m+2n+2} u^{3p+m+2n}}{(\sqrt{\lambda})^{p+m+1} (\sqrt{1+u^2})^{2p-1}}\, du$$

From (3.8), we get :

$$J_{p,n,m} = \begin{cases} \frac{1}{\sqrt{\lambda}} An(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2 \ln \frac{1}{k'}, k'^2 \ln \sqrt{\lambda}) & \text{if } \alpha \neq 0 \\ \frac{1}{\sqrt{\lambda}} An(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2 \ln \frac{1}{k'}) & \text{if } \alpha = 0 \end{cases}$$

Knowing $\frac{1}{\sqrt{\lambda}}$, we obtain :

$$\boxed{\begin{array}{llllll} \text{If } \alpha \neq 0 : & X & = & \dfrac{x+r}{2r} & = & \dfrac{1}{\ln \frac{1}{t}} An\left(\dfrac{1}{\ln \frac{1}{t}}, \dfrac{\ln \ln \frac{1}{t}}{\ln \frac{1}{t}}, t \ln^2 \dfrac{1}{t}\right) \\ & & & & = & \dfrac{2}{r\sqrt{\lambda}} + \dfrac{C}{r\lambda} + \mathrm{O}(\dfrac{1}{\lambda^{\frac{3}{2}}}) \\ \text{If } \alpha = 0 : & X & = & \dfrac{x+r}{2r} & = & \dfrac{1}{\ln \frac{1}{t}} An\left(\dfrac{1}{\ln \frac{1}{t}}, t \ln^2 \dfrac{1}{t}\right) \end{array}} \quad (36)$$

Expansion of z Recall that :

$$Z = \frac{z(r)}{r^3} = \int_{y_{-1}}^{0} \frac{\sqrt{c}}{\sqrt{a}} \frac{y^2 P_1}{\sqrt{1 - P_1^2}} \, dy$$

In the same way, we prove :

$$\boxed{\begin{array}{llll} \text{If } \alpha \neq 0 : & Z & = & \dfrac{1}{\ln^3 \frac{1}{t}} An\left(\dfrac{1}{\ln \frac{1}{t}}, \dfrac{\ln \ln \frac{1}{t}}{\ln \frac{1}{t}}, t \ln^2 \dfrac{1}{t}\right) \\ & & = & \dfrac{4}{r^3 \lambda^{\frac{3}{2}}} + \mathrm{o}(\dfrac{1}{\lambda^{\frac{3}{2}}}) \\ \text{If } \alpha = 0 : & Z & = & \dfrac{1}{\ln^3 \frac{1}{t}} An\left(\dfrac{1}{\ln \frac{1}{t}}, t \ln^2 \dfrac{1}{t}\right) \end{array}} \quad (37)$$

Now we have a parametrization $(X(t), Z(t))$, the problem is *to eliminate the parameter* t.

3.5.4 Inversion of the parameter : t in function of X

Set $X = \frac{x+r}{2r}$.
The method to express t in function of X is general in the category of functions in which we work. In our example, X is an analytic function of t and some functions composed of ln and t, which tend to 0 when t tends to 0. So we work in a *sub-class* of the exp-log category (see [11]), denoted by LE. The general theory from [11] tells us that $t = F(X)$ with $F \in$ LE. But this general theorem, whose proof is based on Weierstrass preparation theorem, is not algorithmic. Our problem is more specific : we work with a parametrization with a *specific scale*. In this case we can develop an algorithm to compute precisely F, and thus find the sub-class of LE which is needed to express t as a function of X.

First of all we give the algorithm in the particular case of our example, then we give a general result :

Algorithm Let $u = \frac{1}{\ln \frac{1}{t}}$. Then :

$$\begin{aligned} X &= An\left(u, u\ln\frac{1}{u}, \frac{e^{-\frac{1}{u}}}{u^2}\right) \\ &= u + u^2 \ln\frac{1}{u} + Cu^2 + \mathrm{o}(u^2) \end{aligned} \tag{38}$$

where C is a constant (which can be precisely computed).

We get easily :

$$u = X - X^2 \ln\frac{1}{X} - CX^2 + \mathrm{o}(X^2)$$

which leads us to set :

$$u = X - X^2 \ln\frac{1}{X} - CX^2 + X^2 v$$

Then :

$$\begin{aligned} \frac{1}{u} &= \frac{1}{X} + \ln\frac{1}{X} + C - v + An(X, X\ln^2\frac{1}{X}, X\ln\frac{1}{X}, v) \\ \mathrm{e}^{-\frac{1}{u}} &= \mathrm{e}^{-C} X \mathrm{e}^{-\frac{1}{X}}(1 + vAn(v)) \\ \ln\frac{1}{u} &= \ln\frac{1}{X} + An(X, X\ln\frac{1}{X}, Xv) \end{aligned}$$

Plugging into (38), we get :

$$0 = X^2 v + An(X, X\ln\frac{1}{X}, \frac{\mathrm{e}^{-\frac{1}{X}}}{X}, v)$$

the analytic function being a $\mathrm{o}(X^2)$, which allows us to divide this equation by X^2. Then :

$$0 = v + An(X, X\ln\frac{1}{X}, X\ln^2\frac{1}{X}, X\ln^3\frac{1}{X}, \frac{\mathrm{e}^{-\frac{1}{X}}}{X^3}, v)$$

Applying the *implicit function theorem*, we get :

$$v = An(X, X\ln\frac{1}{X}, X\ln^2\frac{1}{X}, X\ln^3\frac{1}{X}, \frac{\mathrm{e}^{-\frac{1}{X}}}{X^3})$$

and the same goes for u.
Plugging into the expansion of Z, we conclude :

$$\boxed{Z = An(X, X\ln\frac{1}{X}, X\ln^2\frac{1}{X}, X\ln^3\frac{1}{X}, \frac{\mathrm{e}^{-\frac{1}{X}}}{X^3})} \tag{39}$$

Remark 3.11. *we can be more specific on the first terms of the expansion :*

$$Z = \frac{1}{6}X^3 + X^4 An(X) + O(X^4 e^{-\frac{1}{X}}).$$

Our computations have proved the following :

Theorem 3.12. *The sub-Riemannian sphere in the general Martinet conservative case is in the log-exp category.*

3.5.5 Generalization of the algorithm

The previous algorithm can be generalized in the following manner. Our aim is to build a sub-class of the general *log-exp category* (see [11]) with the following functions :

$$h_1(t) = t$$
$$h_2(t) = \ln \frac{1}{t}$$
$$h_3(t) = \mathrm{e}^{-\frac{1}{t}}$$

Notation h^p means $h \times h \times \ldots \times h$ (p times).
$h^{[p]}$ means $h \circ h \circ \ldots \circ h$ (p times).

Definition 3.13. *We set :*

$$\mathcal{E}_1 = \left\{ h_1^p \prod_{0<i_1<i_2<\cdots<i_m} \left(h_2^{[i_k]}\right)^{i'_k} \quad / \ i'_k \in \mathbb{Z}^*, m \in \mathbb{N}, (p = 1 \ and \ i'_1 < 0) \ or \ (p \geqslant 2) \right\}$$

$$\mathcal{I} = \left\{ h_1^p \prod_{0<i_1<i_2<\cdots<i_m} \left(h_2^{[i_k]}\right)^{i'_k} \quad / \ (p \geqslant 1) \ or \ (p = 0 \ and \ i_1 = 1, i'_1 \leqslant -2) \right\}$$

$$\mathcal{E}_2 = \left\{ h_1^p \prod_{0<i_1<i_2<\cdots<i_m} \left(h_2^{[i_k]}\right)^{i'_k} e^{-\frac{1}{f}} \quad / \ p \in \mathbb{Z}, i'_k \in \mathbb{Z}, f \in \mathcal{I} \right\}$$

Proposition 3.14. *Let $F(X_0, X_1, \ldots, X_n)$ be an analytic function near 0, so that : $F \underset{0}{\sim} X_0$.*
Let $X(t) = F(t, f_1(t), \ldots, f_n(t))$ where $f_i \in \mathcal{E}_1 \cup \mathcal{E}_2$.

Let r be the greatest degree of meromorphy in the expressions of the f_i's, i.e. the greatest power to which appears $\frac{1}{t}$ in the expressions of the f_i's.

Let $g_1(t)$ so that : $X(t) = g_1(t) + o(t^{r+1})$ (in fact : $g_1 \in Vect(h_1, \mathcal{E}_1)$)
Then :

1. *$\exists g_2 \in Vect(h_1, \mathcal{E}_1) \ / \ t = g_2(X) + o(X^{r+1})$.*
2. *If we set : $t = g_2(X) + X^{r+1}u$, then u can be computed using the analytic implicit function theorem.*

Remark 3.15. *This algorithm implies not only that t can be expressed as a function of X in the log-exp category ; it gives a precise scale.*

4 The general gradated case of order 0

To investigate the general case a method is *to consider the general case as a perturbation of the integrable case.* This point of view is similar to the one used to solve the 16^{th} Hilbert problem about limit cycles.

We proceed as follows : if $\beta \neq 0$ in the gradated normal form of order 0, the basic second order equation (11) describes a non conservative pendulum. The *asymptotic expansions* of the SR-distance near the abnormal direction can be evaluated by estimating the solutions near the saddle.

Since this saddle is not a priori integrable in the analytic category for any value of the parameter $\varepsilon = \frac{1}{\sqrt{\lambda}}$, we use the procedure of [13, p91] to compute the Poincaré transition map near an hyperbolic saddle point depending on a parameter.

It is based on the *existence of a formal first integral* and uses the following normal form near a saddle :

$$X_\varepsilon \sim x\frac{\partial}{\partial x} + y(-r(\varepsilon) + \sum_{i=1}^{N} \alpha_{i+1}(\varepsilon)(xy)^i)\frac{\partial}{\partial y}$$

where $r(\varepsilon)$ is defined using the linearized system :

$$X_\varepsilon(0) = x\frac{\partial}{\partial x} - r(\varepsilon)y\frac{\partial}{\partial y} \quad , \quad r(\varepsilon) = \left|\frac{\lambda_2(\varepsilon)}{\lambda_1(\varepsilon)}\right|$$

where λ_1, λ_2 are the two eigenvalues of the saddle, and $r(0) = 1$ (flat case) in our situation.

The previous vector field can be integrated by making the following (toric) *blowing-up* : $u = xy, v = x$. This procedure allows to compute asymptotic expansions for the solutions near the saddle. By essence this method will not provide *converging expansions.*

This procedure is based on the use of our normal form. Moreover for computing x and z we require one more integration. Hence we can imagine that the final expansions are converging. Another method which could be used to compute converging expansions is the use of *Briot-Bouquet theory.* This method is the following :

Similarly to the general conservative case, the objective is to express X and Z in terms of k'. To understand precisely the role of the parameter β, one may study the system with the following particular metric :

$$a = 1 \quad , \quad c = (1 + \beta x)^2$$

In this particular case, the general differential system (3) is simplier. Indeed dividing by $\dot{y}$ we obtain :

$$\begin{aligned} \frac{dx}{dy} &= \sigma(1+\beta x)\frac{P_1}{\sqrt{1-P_1^2}} && (40) \\ \frac{dz}{dy} &= \sigma(1+\beta x)\frac{y^2}{2}\frac{P_1}{\sqrt{1-P_1^2}} && (41) \\ \frac{P_1}{dy} &= \lambda y + \varepsilon\sigma\sqrt{1-P_1^2} && (42) \end{aligned}$$

where $\sigma = \text{sign}(\dot{y})$.

Moreover we fix the length to r, hence :

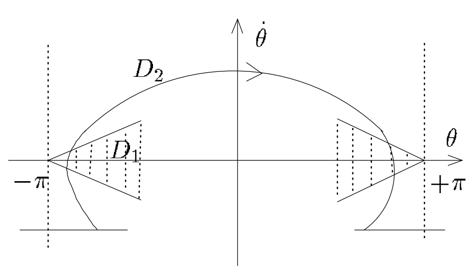
$$r = 2\int_{y_{-1}}^{y_1} (1+\beta x)\frac{P_1}{\sqrt{1-P_1^2}}dy \tag{43}$$

Contrary to the conservative case where P_1 was explicitly given, P_1 is solution of a differential equation.
It seems reasonable to think that one could express P_1 analytically in some class of functions. Indeed plugging in (41) one would express $x(y)$, then the relation (43) would give $\lambda(k')$, and finally it would go as before.In this analysis the key-equation is equation (42) :
If we set : $P_1 = -1 + 2f^2, f(0) = k',$ and $\eta = \sqrt{\lambda}y$, we get :

$$2f\frac{df}{d\eta} = \frac{1}{2}\eta + \varepsilon f\sqrt{1-f^2} = \frac{1}{2}\eta + \varepsilon f + \varepsilon\sum_{n=1}^{\infty} a_n f^{2n+1}$$

This is a *Briot-Bouquet equation*, studied by Boutroux, see [5]. We can expect to get *sectorially converging expansions* of P_1, which could help us to compute expansions of X and Z. More precisely we conjecture that :

- for $D_1 = \{0 \leqslant \eta < k'\}$, P_1 has a convergent Taylor series (which is computable thanks to (42)).
- for $D_2 = \{k' < \eta < 1\}$, P_1 can be analytically expanded in some scale.



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