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CONTROLLABILITY OF COUETTE FLOWS

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Abstract. In this article, we investigate the problem of controlling Navier-Stokes equations between two infinite rotating coaxial cylinders. We prove that it is possible to move from a given Couette flow, that is a special stationary solution, to another one, by controlling the rotation velocity of the outer cylinder.

1. **Introduction.** We focus on special stationary flows, called *Couette flows*, of a fluid filling the domain between two infinite rotating coaxial cylinders. These flows have been known for a long time, and correspond to steady-states of the incompressible Navier-Stokes equations with no-slip boundary conditions. They have been studied in view of stability issues; the literature on this problem is immense (see for instance [1, 13] and references therein), and many results concerning bifurcation and/or symmetry breaking have been studied, experimentally and mathematically.

In this paper, our purpose is to control Couette flows, by acting on the rotation of the outer cylinder. The problem is challenging because the control is scalar. Moreover, from the physical point of view, such a control is convenient, because it is easy to realize. However, we stress on the fact that the situation is particular: the control is scalar, but on the other part, Couette flows are special flow regimes of the Navier-Stokes equations. Actually, using uniqueness arguments, we prove that the problem of controlling Couette flows reduces to the problem of controlling a one-dimensional parabolic system with boundary control. This reduction is crucial in our analysis, and explains why a scalar control is sufficient for establishing controllability on Couette flows.

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Statement of the problem. We recall the formulation of the problem (see [1]). Consider a viscous incompressible fluid filling the domain Ω between two concentric rotating infinite cylinders. The flow is described by the Navier-Stokes equations

$$\partial_t v + (v.\nabla)v + \frac{1}{\rho}\nabla p = \nu v, \tag{1a}$$
$$\operatorname{div} v = 0, \tag{1b}$$

where $v \in \mathbb{R}^3$ is the velocity vector of fluid particles, ρ is the (constant) density, p is the pressure, ν is the viscosity. Note that the gravity is incorporated in the pressure term. In cylindrical coordinates, the horizontal cross section of Ω is defined by $R_1 < r < R_2$, and if one writes $v(t) = v(t; r, \theta, z)$ and $v = (v_r, v_\theta, v_z)$ in these coordinates, then the no-slip boundary conditions are

$$v_r(t; R_j, \cdot, \cdot) = v_z(t; R_j, \cdot, \cdot) = 0, \qquad v_\theta(t; R_j, \cdot, \cdot) = R_j\Theta_j, \quad j = 1, 2,$$

where Θ_1 (resp. Θ_2) is the angular velocity of the inner (resp., outer) cylinder, and t denotes the time in some time interval $[0, T]$. Throughout this paper, it is assumed that $\Theta_1 > 0$ and that $\Theta_2 = \Theta_2(t)$ can be freely chosen in $\mathbb{R}$ for every $t \in [0, T]$. Under the assumption $\Theta_1 > 0$, it is possible to achieve a nondimensionalization procedure. Introduce the dimensionless control

$$\omega(t) = \frac{\Theta_2(t)}{\Theta_1},$$

and the dimensionless parameters

$$\eta = \frac{R_1}{R_2}, \qquad \mathcal{R} = \frac{R_1\Theta_1(R_2 - R_1)}{\nu},$$

where $\mathcal{R}$ is called the *Reynolds number*. Then, the system (1) writes

$$\begin{aligned}
\partial_t v &= \triangle v - \mathcal{R}\,(v\cdot\nabla)v - \nabla p, && \text{in } \Omega, && (2a)\\
0 &= \operatorname{div} v, && \text{in } \Omega, && (2b)\\
v_r &= v_z = 0, \quad v_\theta = 1, && \text{at } r = r_1, && (2c)\\
v_r &= v_z = 0, \quad v_\theta = \omega/\eta, && \text{at } r = r_2, && (2d)
\end{aligned}$$

where $r_1 = \eta/(1-\eta)$ and $r_2 = 1/(1-\eta)$. The flow domain Ω and its boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ are given in the new dimensionless cylindrical coordinates by

$$\begin{aligned}
\Omega &= \{(r,\theta,z) \in \mathbb{R}_{\geq 0} \times \mathbb{T}^1 \times \mathbb{R} : \quad r_1 < r < r_2\},\\
\partial\Omega_i &= \{(r,\theta,z) \in \mathbb{R}_{\geq 0} \times \mathbb{T}^1 \times \mathbb{R} : \quad r = r_i\}, \quad i = 1, 2,
\end{aligned}$$

where $\mathbb{T}^1$ denotes the torus $\mathbb{R}/2\pi\mathbb{Z}$.

Denoting the partial derivatives with respect to t, r, θ and z by ∂_t, ∂_r, ∂_θ and ∂_z, respectively, (2a) and (2b) read in cylindrical coordinates

$$\begin{aligned}
\partial_t v_r &= \triangle v_r - \frac{2}{r^2}\partial_\theta v_\theta - \frac{v_r}{r^2} - \partial_r p - \mathcal{R}\left[v_r\partial_r v_r + \frac{v_\theta}{r}\partial_\theta v_r + v_z\partial_z v_r - \frac{v_\theta^2}{r}\right],\\
\partial_t v_\theta &= \triangle v_\theta - \frac{2}{r^2}\partial_\theta v_r - \frac{v_\theta}{r^2} - \frac{1}{r}\partial_\theta p - \mathcal{R}\left[v_r\partial_r v_\theta + \frac{v_\theta}{r}\partial_\theta v_\theta + v_z\partial_z v_\theta - \frac{v_r v_\theta}{r}\right],\\
\partial_t v_z &= \triangle v_z - \partial_z p - \mathcal{R}\left[v_r\partial_r v_z + \frac{v_\theta}{r}\partial_\theta v_z + v_z\partial_z v_z\right],\\
0 &= \frac{1}{r}v_r + \partial_r v_r + \frac{1}{r}\partial_\theta v_\theta + \frac{1}{r}\partial_z v_z,
\end{aligned}$$

with $\triangle = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2$.

Definition 1. For every $\alpha \in \mathbb{R}$, we define the *Couette flow* $(\bar{v}^\alpha, \bar{p}^\alpha) : \Omega \to \mathbb{R}^3 \times \mathbb{R}$ by

$$\bar{v}^\alpha(r, \cdot, \cdot) = (0, \bar{v}_\theta^\alpha(r), 0)^T, \qquad \bar{p}^\alpha(r, \cdot, \cdot) = \mathcal{R} \int_{r_1}^{r} \frac{(\bar{v}_\theta^\alpha(s))^2}{s} \, ds,$$

with

$$\bar{v}_\theta^\alpha(r) = A(\alpha) r + B(\alpha) \frac{1}{r},$$

and

$$A(\alpha) = \frac{\alpha - \eta^2}{\eta(1+\eta)}, \qquad B(\alpha) = \frac{\eta(1-\alpha)}{(1-\eta)(1-\eta^2)}.$$

It is easy to verify that, for every fixed $\alpha \in \mathbb{R}$, the Couette flow $(\bar{v}^\alpha, \bar{p}^\alpha)$ is a steady-state solution of (2) for the constant control $\omega(t) \equiv \alpha$. Moreover, one can show that, for $\mathcal{R}$ sufficiently small with respect to α, $(\bar{v}^\alpha, \bar{p}^\alpha)$ is the unique steady-state solution, whereas, for $\mathcal{R}$ sufficiently large, there are steady-state solutions which are axi-symmetric and periodic in z and which differ from $(\bar{v}^\alpha, \bar{p}^\alpha)$ (see e.g. [1] or [13, p. 232]), such as Taylor vortex flow, wavy vortex flow, etc.

In the present paper, we prove that it is possible to steer the system (2) from any Couette flow $(\bar{v}^\alpha, \bar{p}^\alpha)$ to any other one by rotating the outer cylinder. Since the proof is based on a stability property of the flow, the control has to be applied over a sufficiently large time interval.

2. **Controllability results.**

2.1. **Periodic function spaces.**

Since the domain Ω is unbounded, we assume that the velocity v and the pressure p are periodic in z with some period $h > 0$ (see e.g. [1] or [13, Ch.II.4]). Then, Ω is identified to

$$\Omega_h = \{(r, \theta, z) \in \Omega : \ -h/2 \leq z \leq h/2\}.$$

Let $L^2(\Omega_h)$ be the usual Lebesgue space of square-integrable functions, endowed with the scalar product

$$(\phi.\psi)_{L^2(\Omega_h)} = \int_{r_1}^{r_2} \int_0^{2\pi} \int_{-h/2}^{h/2} r \, \phi(r, \theta, z) \psi(r, \theta, z) dz \, d\theta \, dr.$$

Define $L^2_h(\Omega)$ as the closure of the set of continuous, h-periodic in z, functions on Ω with respect to the norm induced by the scalar product

$$(\phi.\psi)_{L^2_h(\Omega)} = (\phi|_{\Omega_h} . \psi|_{\Omega_h})_{L^2(\Omega_h)}.$$

Furthermore, introduce

$$H_h(\Omega) = \{v \in [L^2_h(\Omega)]^3 : \quad \operatorname{div} v = 0, \quad v.\nu|_{\partial\Omega} = 0\},$$

endowed with the scalar product of $[L^2_h(\Omega)]^3$ (see [13] for the precise meaning of the divergence and the trace with respect to the outer normal vector ν). The subindices of scalar products will be frequently omitted.

2.2. **Perturbation with respect to a path of Couette flows.** Let $(\bar{v}^a, \bar{p}^a)$ and $(\bar{v}^b, \bar{p}^b)$ be two (possibly equal) Couette flows, and a, b be real numbers. For $\tau \in [0, 1]$, define the path of Couette flows

$$\bar{\omega}(\tau) = a + \tau(b - a), \qquad \bar{v}(\tau) = \bar{v}^{\bar{\omega}(\tau)}, \qquad \bar{p}(\tau) = \bar{p}^{\bar{\omega}(\tau)}.$$

For $\varepsilon > 0$, we introduce *perturbation coordinates* along the path,

$$u(t) = v(t) - \bar{v}(\varepsilon t), \quad q(t) = p(t) - \bar{p}(\varepsilon t), \quad \gamma(t) = \omega(t) - \bar{\omega}(\varepsilon t),$$

for $t \in [0, 1/\varepsilon]$. The reason for introducing a small parameter $\varepsilon > 0$ will appear to be clear later. Note that

$$\triangle \bar{v}(\varepsilon t) - \mathcal{R}(\bar{v}(\varepsilon t).\nabla)\bar{v}(\varepsilon t) - \nabla \bar{p}(\varepsilon t) = 0,$$

for every $t \in [0, 1/\varepsilon]$. Hence, in the new coordinates, the system (2) writes

$$\begin{aligned} \partial_t u(t) = {} & \triangle u(t) - \mathcal{R}[(u(t).\nabla)\bar{v}(\varepsilon t) + (\bar{v}(\varepsilon t).\nabla)u(t)] \\ & - \mathcal{R}(u(t).\nabla)u(t) - \nabla q - \varepsilon \partial_\tau \bar{v}(\varepsilon t) && \text{in } [0, 1/\varepsilon] \times \Omega, \quad \text{(4a)} \\ 0 = {} & \operatorname{div} u(t) && \text{in } [0, 1/\varepsilon] \times \Omega, \quad \text{(4b)} \\ u(t) = {} & 0 && \text{on } [0, 1/\varepsilon] \times \partial\Omega_1, \quad \text{(4c)} \\ u(t) = {} & (0, \gamma(t)/\eta, 0)^T && \text{on } [0, 1/\varepsilon] \times \partial\Omega_2, \quad \text{(4d)} \\ (u(0), q(0)) = {} & (u^0, q^0) = (v^0 - \bar{v}^a, p^0 - \bar{p}^a). && \text{(4e)} \end{aligned}$$

In the following, the initial conditions (u^0, q^0) are assumed to be *compatible*, in the sense that

$$\triangle q^0 = \operatorname{div} \left(\triangle u^0 - \mathcal{R}[(u^0.\nabla)\bar{v}^a + (\bar{v}^a.\nabla)u^0 + (u^0.\nabla)u^0] \right).$$

Note that

$$\partial_\tau \bar{v}(\varepsilon t) = (b - a)g(r),$$

where $g(r) = (0, g_\theta(r), 0)^T$, and

$$g_\theta(r) = \frac{r}{\eta(1+\eta)} - \frac{\eta}{(1-\eta)(1-\eta^2)r}.$$

2.3. **Functional analytic framework.** We next recall how equations (4) for the perturbation of the path $(\bar{v}(\varepsilon t), \bar{p}(\varepsilon t), \bar{\omega}(\varepsilon t))$ can be written, in $H_h(\Omega)$, as

$$\partial_t u(t) = L(\varepsilon t)u(t) + N(u(t)) - \varepsilon(b - a)g,$$

whenever $\gamma \equiv 0$ (see [1, 13]).

The space $H_h(\Omega)$ is the orthogonal supplement, in $[L_h^2(\Omega)]^3$, of the space $\{\nabla q : q \in [H_h^1(\Omega)]^3\}$, where

$$H_h^1(\Omega) = \{v \in L_h^2(\Omega) : \quad \nabla v \in [L_h^2(\Omega)]^3\}.$$

Let Π_0 denote the orthogonal projection from $[L_h^2(\Omega)]^3$ onto $H_h(Q)$. This projection is used for eliminating the pressure term ∇q in (4) by incorporating the condition $\operatorname{div} u = 0$ and a part of the boundary conditions in $H_h(Q)$. Then, the linear operator $L(\tau)$ and the quadratic operator N are defined by

$$\begin{aligned} L(\tau)u &= \Pi_0(\triangle u - \mathcal{R}((u.\nabla)\bar{v}(\tau) + (\bar{v}(\tau).\nabla)u)), \\ N(u) &= -\mathcal{R}\Pi_0(u.\nabla)u, \end{aligned}$$

for every $\tau \in [0, 1]$, and the domain of $L(\tau)$ is defined by

$$D_h = \{u \in H_h(\Omega) : \quad u \in [H_h^2(\Omega)]^3, \quad u|_{\partial\Omega} = 0\},$$

where $H^2_h(Q)$ is the space of functions belonging, up to their second derivative, to $L^2_h(\Omega)$. It is well known (see [1, 8, 9, 10, 11, 12, 13]) that the operator $L(\tau)$ depends analytically on the parameter $\tau \in [0,1]$, and on the parameters $\mathcal{R}, \eta, \gamma(t)$, that, for every $\tau \in [0,1]$, $L(\tau)$ is the generator of an analytic and compact semigroup $(S(\tau,t))_{t\geq 0}$, and that the quadratic operator N is continuous from D_h to the space K_h defined by

$$K_h = \{v \in H_h(\Omega) : \quad v \in [H^1_h(\Omega)]^3\}.$$

From the Sobolev embedding theorem, there exists $c_1 > 0$ such that

$$\|N(u)\|_{K_h} \leq c_1 \|u\|^2_{D_h}, \tag{5}$$

for every $u \in D_h$. On the other part, there exists $c_2 > 0$ such that

$$\|S(\tau,t)\|_{\mathcal{L}(K_h,D_h)} \leq \frac{c_2}{t^{3/4}}, \tag{6}$$

for every $\tau \in [0,1]$ and every $t > 0$ (see [1, 7]). Hence, the integral formulation of the Cauchy problem (4) writes

$$u(t) = S(\varepsilon t,t)u^0 + \int_0^t S(\varepsilon t, t-s)N(u(s))\,ds - \varepsilon(b-a)\int_0^t S(\varepsilon t,t-s)g\,ds. \tag{7}$$

2.4. **Approximate controllability of Couette flows.** Our first main result is the following.

Theorem 1. *For all Couette flows $(\bar{v}^\alpha, \bar{p}^\alpha)$ and $(\bar{v}^\beta, \bar{p}^\beta)$, $\alpha, \beta \in \mathbb{R}$, there exist $c > 0$ and $\varepsilon_0 > 0$, such that, for every $\varepsilon \in (0, \varepsilon_0)$, there exists a unique solution*

$$v \in \{v : \, v \in L^2(0,1/\varepsilon;[H^3_h(\Omega)]^3),\ \partial_t v \in L^2(0,1/\varepsilon;[H^1_h(\Omega)]^3),\ \operatorname{div} v = 0\},$$
$$p \in L^2(0,1/\varepsilon; H^2_h(\Omega)),$$

of (2), associated with the control

$$\omega(t) = \alpha + \varepsilon t(\beta - \alpha). \tag{8}$$

Moreover, this solution is of the form

$$v(t;r,\theta,z) = (0, v_\theta(t;r), 0)^T, \quad p(t;r,\theta,z) = p(t;r),$$

and satisfies

$$\|v(1/\varepsilon) - \bar{v}^\beta\|_{[H^2_h(\Omega)]^3} + \|p(1/\varepsilon) - \bar{p}^\beta\|_{H^2_h(\Omega)} < c\,\varepsilon. \tag{9}$$

Proof. The uniqueness property will be proved *a posteriori*, and thus, we assume, in a first step, that uniqueness holds. Then, consider the solution $(u(\cdot), q(\cdot))$ of (4), starting from $(u(0), q(0)) = (0,0)$. Since the system (4) is invariant with respect to translations along the z-axis, reflections $z \to -z$, and rotations about the z-axis, it follows, by uniqueness, that the solution $(u(\cdot), q(\cdot))$ enjoys all previous symmetry properties, i.e., is of the form

$$u(t;r,\theta,z) = (u_r(t;r), u_\theta(t;r), u_z(t;r))^T, \qquad q(t) = q(t;r).$$

In these conditions, the system (4) can be written, in cylindric coordinates,

$$\partial_t u_r = \partial_r^2 u_r + \frac{1}{r}\partial_r u_r - \frac{u_r}{r^2} - \partial_r q - \mathcal{R}\left(u_r \partial_r u_r - 2\frac{\bar{v}_\theta}{r}u_\theta - \frac{u_\theta^2}{r}\right), \tag{10a}$$

$$\begin{aligned}\partial_t u_\theta = \partial_r^2 u_\theta + \frac{1}{r}\partial_r u_\theta - \frac{u_\theta}{r^2} \\ - \mathcal{R}\left(u_r \partial_r \bar{v}_\theta - u_r \partial_r u_\theta + \frac{u_r \bar{v}_\theta}{r} + \frac{u_r u_\theta}{r}\right) - \varepsilon(b-a)g_\theta,\end{aligned} \tag{10b}$$

$$\partial_t u_z = \partial_r^2 u_z + \frac{1}{r}\partial_r u_z - \mathcal{R} u_r \partial_r u_z, \tag{10c}$$

the zero divergence condition (4b) reduces to

$$0 = \frac{1}{r}\partial_r(r u_r), \tag{10d}$$

the boundary conditions are

$$u(t) = 0 \text{ on } \partial\Omega_1, \tag{10e}$$

$$u(t) = (0, \gamma(t)/\eta, 0)^T \text{ on } \partial\Omega_2, \tag{10f}$$

and the initial condition is

$$(u(0), q(0)) = (0, 0). \tag{10g}$$

From (10d) and (10e), one first gets $u_r \equiv 0$, and hence, (10c) reduces to

$$\partial_t u_z = \partial_r^2 u_z + \frac{1}{r}\partial_r u_z.$$

Since $u_z(0) = 0$, and $u_z(t; r_1) = u_z(t; r_1) = 0$, this yields $u_z \equiv 0$. Finally, the system (10) reduces to the one dimensional parabolic system

$$\partial_t u_\theta(t;r) = \partial_r^2 u_\theta(t;r) + \frac{1}{r}\partial_r u_\theta(t;r) - \frac{1}{r^2}u_\theta(t;r) - \varepsilon(b-a)g_\theta(r), \tag{11a}$$

$$u_\theta(t; r_1) = 0, \qquad u_\theta(t; r_2) = \gamma(t)/\eta, \tag{11b}$$

$$u_\theta(0; r) = 0. \tag{11c}$$

Note that the pressure is reconstructed by solving

$$\partial_r q(t;r) = \mathcal{R}\left(\frac{u_\theta^2(t;r)}{r} + \frac{2}{r}u_\theta(t;r)\bar{v}_\theta(t;r)\right). \tag{12}$$

We next prove the exponential stability of the system (11), for the control $\gamma \equiv 0$. Let $L^2_\theta(r_1, r_2)$ denote the space of measurable functions $\phi : [r_1, r_2] \to \mathbb{R}$ such that

$$\int_{r_1}^{r_2} r\phi(r)^2 dr < \infty.$$

Endowed with the scalar product

$$(\phi.\psi)_{L^2_\theta} = \int_{r_1}^{r_2} r\, \phi(r)\psi(r)\, dr,$$

$L^2_\theta(r_1, r_2)$ is a Hilbert space. The Sobolev spaces $H^1_\theta(r_1, r_2)$ and $H^2_\theta(r_1, r_2)$ are defined similarly. Note that, for functions u of the form $u = (0, u_\theta(r), 0)^T \in [L^2(\Omega_h)]^3$,

$$\|u\|_{[L^2_h(\Omega)]^3} = \sqrt{2\pi h}\|u_\theta\|_{L^2_\theta(r_1,r_2)}.$$

The system (11) can be written as

$$\partial_t u_\theta(t) = A u_\theta(t) - \varepsilon(\beta-\alpha) g_\theta, \quad u_\theta(0)=0,$$

where the operator $A : D(A) \to L^2_\theta(r_1, r_2)$ is defined by

$$A = \frac{1}{r}\partial_r(r\partial_r) - \frac{1}{r^2}, \tag{13}$$

on the domain

$$D(A) = \{\phi \in H^2_\theta(r_1,r_2) : \quad \phi(r_1)=\phi(r_2)=0\}.$$

It is easy to verify that A is selfadjoint and negative. Moreover, A is an operator of Sturm-Liouville type, and thus, has a compact resolvent (see e.g. [4, p. 180]). Consequently, A is the infinitesimal generator of an analytic semigroup $(S_\theta(t))_{t\geq 0}$ of negative type. Hence, there exists $\lambda > 0$ such that, for every $t \geq 0$

$$\|S_\theta(t)\|_{\mathcal{L}(H^1_\theta(r_1,r_2),H^2_\theta(r_1,r_2))} \leq \frac{e^{-\lambda t}}{\sqrt{t}},$$

(cf. [12]). Then the solution u_θ of (11) satisfies

$$\begin{aligned}
\|u_\theta(1/\varepsilon)\|_{H^2_\theta(r_1,r_2)} &\leq \left\| \varepsilon(\beta-\alpha)\int_0^{1/\varepsilon} S_\theta(1/\varepsilon - s) g_\theta(r)\,ds \right\|_{H^2_\theta(r_1,r_2)} \\
&\leq \varepsilon\,|\beta-\alpha| \|g_\theta\|_{H^1_\theta(r_1,r_2)} \int_0^{1/\varepsilon} \frac{e^{-\lambda s}}{s}\,ds \\
&\leq \varepsilon\,|\beta-\alpha| \|g_\theta\|_{H^1_\theta(r_1,r_2)} (2 + \frac{2}{\lambda}).
\end{aligned}$$

Using (12), one estimates

$$|\partial_r^2 q| \leq \frac{2}{r_1}|u_\theta||\partial_r u_\theta| + \frac{1}{r_1^2}|u_\theta^2| + \frac{2}{r_1^2}|u_\theta||\bar{v}_\theta| + \frac{2}{r_1}|\partial_r u_\theta||\bar{v}_\theta| + \frac{2}{r_1}|u_\theta||\partial_r \bar{v}_\theta|.$$

Applying the inequality $|ab| \leq (a^2+b^2)/2$, one obtains

$$\|q(1/\varepsilon)\|^2_{H^2_\theta(r_1,r_2)} \leq c_3(\|u_\theta(1/\varepsilon)\|^2_{L^2_\theta(r_1,r_2)} + \|u_\theta(1/\varepsilon)\|^2_{H^1_\theta(r_1,r_2)}),$$

with some constant $c_3 > 0$. Hence, the estimate (9) follows.

It remains to prove the uniqueness argument. To this aim, consider the general integral formulation (7). Then, using (5) and (6), one has

$$\begin{aligned}
\|u(t)\|_{D_h} &\leq c_1 c_2 \int_0^t \frac{1}{(t-s)^{3/4}} \|u(s)\|^2_{D_h}\,ds + c_2 \int_0^t \frac{1}{(t-s)^{3/4}} \varepsilon|\beta-\alpha| \|g\|_{D_h}\,ds \\
&\leq c_1 c_2 \int_0^t \frac{1}{(t-s)^{3/4}} \|u(s)\|^2_{D_h}\,ds + 4\varepsilon c_2 |\beta-\alpha| t^{1/4} \|g\|_{D_h}.
\end{aligned}$$

For $T > 0$, this inequality yields

$$\|u\|_{C(0,T;D_h)} \leq 4c_1c_2T^{1/4}\|u\|^2_{C(0,T;D_h)} + 4\varepsilon c_2|\beta-\alpha|\|g\|_{D_h}T^{1/4}. \tag{14}$$

If we assume that $\|u\|_{C(0,T;D_h)} \leq \delta$, then the right-hand side of (14) is estimated by

$$M = 4c_2T^{1/4}(c_1\delta^2 + \varepsilon|\beta-\alpha|\|g\|_{D_h}).$$

To get the conclusion of the theorem, we have to impose that, if $0 < \varepsilon < \varepsilon_0$, $T = 1/\varepsilon$, $\delta = c\varepsilon^{3/4}$, then

$$M \leq c\,\varepsilon^{3/4},$$

where $\varepsilon_0 > 0$ and $c > 0$ have to be chosen.

For $T = 1/\varepsilon$ and $\delta = c\varepsilon^{3/4}$, one has

$$\begin{aligned} M &= 4c_2(c_1c^2\varepsilon^{1/2} + |\beta - \alpha|\|g\|_{D_h})\varepsilon^{3/4} \\ &\leq 4c_2(c_1c^2\varepsilon_0^{1/2} + |\beta - \alpha|\|g\|_{D_h})\varepsilon^{3/4}. \end{aligned}$$

If we choose

$$c = 2c_2|\beta - \alpha|\|g\|_{D_h},$$

and

$$\varepsilon_0 = \frac{1}{(2c_1c_2c)^2},$$

then one has $M \leq c\,\varepsilon^{3/4}$. The uniqueness property then follows from a standard fixed point argument in the space $C(0, T; D_h)$ (see [5, 11, 13]). □

Remark 1. Note that the use of such a quasi-static deformation has already been used in [2] for shallow-water controllability issues, and in [3] for 1-D heat equations stabilization issues. In the latter reference, the system is not stable along the path, and a stabilization procedure has been performed.

2.5. **Exact controllability of Couette flows.**

Corollary 1. *Let* $(\bar{v}^\alpha, \bar{p}^\alpha)$ *and* $(\bar{v}^\beta, \bar{p}^\beta)$, $\alpha, \beta \in \mathbb{R}$, *be two Couette flows. There exist a time* $T > 0$ *and a control* $\omega \in L^2(0, T)$ *such that the (unique) solution*

$$\begin{aligned} &v \in \{v :\ v \in L^2(0, T; [H_h^2(\Omega)]^3,\ \partial_t v \in L^2(0, T; [L_h^2(\Omega)]^3),\ \operatorname{div} v = 0\}, \\ &p \in L^2(0, T; H_h^1(\Omega)), \end{aligned}$$

of (2), starting from $(v(0), p(0)) = (\bar{v}^\alpha, \bar{p}^\alpha)$, *satisfies*

$$(v(T), p(T)) = (\bar{v}^\beta, \bar{p}^\beta).$$

Remark 2. The time T of controllability is large. This requirement is necessary in the proof (see the definition of ε_0) in order to ensure the *uniqueness* of the solution. On the other part, the uniqueness property is essential to reduce the problem of controllability (2) to the problem of controllability of a one-dimensional parabolic control system.

Proof. We first write the Navier-Stokes equations (2) in the neighborhood of $(\bar{v}^\beta, \bar{p}^\beta)$. Set

$$u(t) = v(t) - \bar{v}^\beta, \qquad q(t) = p(t) - \bar{p}^\beta, \qquad \gamma(t) = \omega(t) - \beta.$$

Then,

$$\begin{aligned} \partial_t u(t) &= \triangle u(t) - \mathcal{R}[(u(t).\nabla)\bar{v}(\varepsilon t) + (\bar{v}(\varepsilon t).\nabla)u(t)] \\ &\quad - \mathcal{R}(u(t).\nabla)u(t) - \nabla q && \text{in } \Omega, && \text{(15a)} \\ 0 &= \operatorname{div} u(t) && \text{in } \Omega, && \text{(15b)} \\ u(t) &= 0 && \text{on } \partial\Omega_1, && \text{(15c)} \\ u(t) &= (0, \gamma(t)/\eta, 0)^T && \text{on } \partial\Omega_2, && \text{(15d)} \end{aligned}$$

and

$$(u(0), q(0)) = (v(1/\varepsilon) - \bar{v}^\beta, p(1/\varepsilon) - \bar{p}^\beta). \qquad \text{(15e)}$$

Actually, there holds moreover

$$u_r(0) = u_z(0) = 0.$$

Hence, *if* the existence and uniqueness of the solution of (15) is ensured, then u_r and u_z are identically equal to zero, and (15) reduces to

$$\begin{aligned} \partial_t u_\theta(t;r) &= \partial_r^2 u_\theta(t;r) + \frac{1}{r}\partial_r u_\theta(t;r) - \frac{1}{r^2}u_\theta(t;r), \\ u_\theta(t;r_1) = 0, &\qquad u_\theta(t;r_2) = \gamma(t)/\eta, \\ u_\theta(0;r) &= v_\theta(1/\varepsilon, r) - \bar{v}_\theta^\beta(r). \end{aligned}$$

Then, the conclusion follows from [6, Corollary 2.1, p. 897]. Indeed, from this result, there exists a control $\gamma \in L^2(0,T')$ (where T' is a positive real number) such that the solution u_θ of (16), which belongs to

$$\{y = y(t;r): \quad q \in L^2(0,T';H^2(r_1,r_2)), \quad \partial_t y \in L^2((0,T')\times(r_1,r_2))\},$$

satisfies $u_\theta(T') = 0$.

Moreover, it follows from the proof of [6] that, if ε_0 is small enough, then the control γ can be chosen such that the norms of u_θ and of γ remain small. Existence and uniqueness for the complete problem (15) then follow from a standard argument (see [5, 11, 13]). □

Remark 3. The uniqueness of weak solutions of the nonstationary Navier-Stokes equations for arbitrarily large data and on arbitrarily large time intervals is an open problem (see e.g. [13]). Note that *if* this problem has a positive answer, then one could steer system (2) from any $(\bar{v}^\alpha, \bar{p}^\alpha)$ exactly to any $(\bar{v}^\beta, \bar{p}^\beta)$ in arbitrarily short time. In fact, system (4) with $a = b = \beta$ and initial conditions $(u^0, q^0) = (\bar{v}^\alpha - \bar{v}^\beta, \bar{p}^\alpha - \bar{p}^\beta)$ would reduce to (11) independently of the choice of control and time interval.

Note, however, that the controllability result in [6] is not constructive, since its proof is based on a fixed point argument.

3. **Numerical simulations.** In this section, we present numerical simulations with *Matlab*. Setting $\alpha = -10$, $\beta = 50$, $\eta = 0.5$ and $\mathcal{R} = 1$, we aim to steer (2) from $(\bar{v}^\alpha, \bar{p}^\alpha)$ to $(\bar{v}^\beta, \bar{p}^\beta)$ by applying the control (8) with different choices of $\varepsilon > 0$.

Figure 1 shows the evolution of the velocity $v_\theta(t,\cdot)$, of the pressure $p(t,\cdot)$ and of its respective perturbations $u_\theta(t,\cdot)$ and $q(t,\cdot)$ for $\varepsilon = 1$. Figure 2 shows the corresponding results for $\varepsilon = 0.1$.

In each case, the initial states are represented by dashed-dotted lines, terminal states at $T = 1/\varepsilon$ by black solid lines, and intermediate states by gray solid lines. The dotted lines indicate the desired terminal velocity profile $\bar{v}^\beta(\cdot)$ and pressure profile $\bar{p}^\beta(\cdot)$, respectively.

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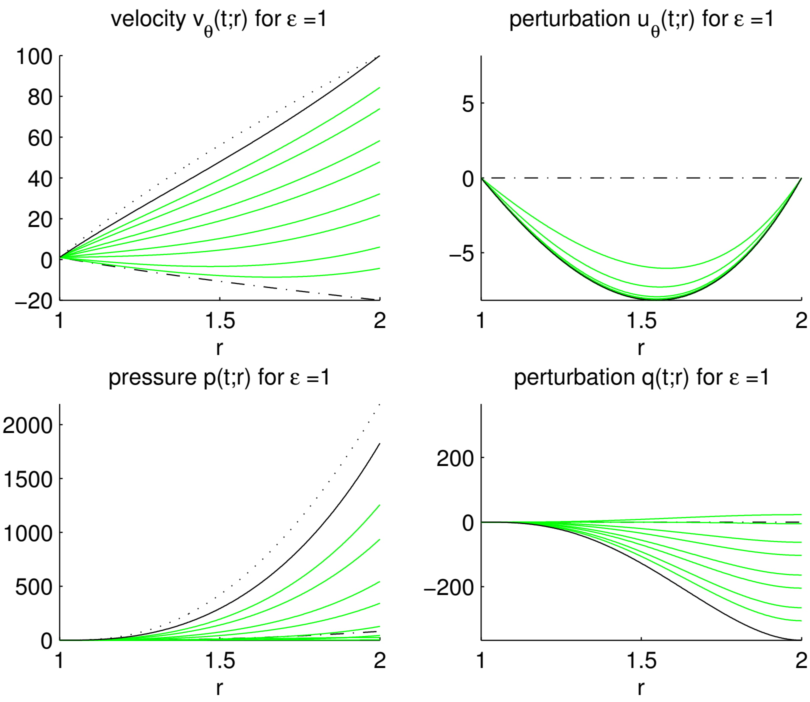

FIGURE 1. Simulation results for $\varepsilon = 1$ and $t \in [0,1]$.

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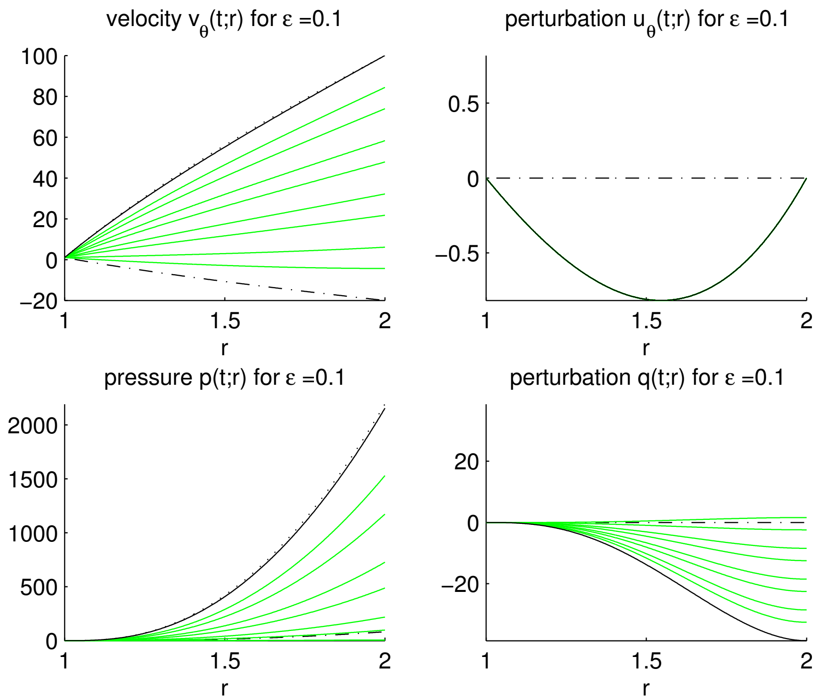

FIGURE 2. Simulation results for $\varepsilon = 0.1$ and $t \in [0, 10]$.