

Morse-Sard type results in sub-Riemannian geometry

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Abstract

Let (M, Δ, g) be a sub-Riemannian manifold and $x_0 \in M$. Assuming that Chow's condition holds and that M endowed with the sub-Riemannian distance is complete, we prove that there exists a dense subset N_1 of M such that for every point x of N_1 , there is a unique minimizing path steering x_0 to x , this trajectory admitting a normal extremal lift. If the distribution Δ is everywhere of corank one, we prove the existence of a subset N_2 of M of full Lebesgue measure such that for every point x of N_2 , there exists a minimizing path steering x_0 to x which admits a normal extremal lift, is nonsingular, and the point x is not conjugate to x_0 . In particular, the image of the sub-Riemannian exponential mapping is dense in M , and in the case of corank one is of full Lebesgue measure in M .

1 Introduction and main results

The following general definition of a sub-Riemannian distance (also called Carnot-Carathéodory distance) is due to [3]. Let M be a connected smooth n -dimensional manifold, m an integer such that $1 \leq m \leq n$, and f_1, \dots, f_m be smooth vector fields on the manifold M . For all $x \in M$ and $v \in T_x M$, set

$$g(x, v) := \inf \left\{ \sum_{i=1}^m u_i^2 \mid u_1, \dots, u_m \in \mathbb{R}, \sum_{i=1}^m u_i f_i(x) = v \right\}.$$

Then $g(x, \cdot)$ is a positive definite quadratic form on the subspace of $T_x M$ spanned by $f_1(x), \dots, f_m(x)$. Outside this subspace we set $g(x, v) = +\infty$. The form g is called *sub-Riemannian metric* associated to the m -tuple of vector fields (f_1, \dots, f_m) . Let $\mathcal{AC}([0, 1], M)$ denote the set of absolutely continuous paths in M defined on $[0, 1]$, we define the *length* of $\gamma \in \mathcal{AC}([0, 1], M)$ as

$$l(\gamma) := \int_0^1 \sqrt{g(\gamma(t), \dot{\gamma}(t))} dt.$$

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We say that *Chow's condition* holds if the Lie algebra spanned by the vector fields f_1, \dots, f_m , is equal to the tangent space $T_x M$ at every point x of M . It is well-known that under this condition any two points of M can be joined by an absolutely continuous path with finite length.

The *sub-Riemannian distance* associated to the m -tuple of vector fields (f_1, \dots, f_m) , between two points x_0, x_1 in M , is defined as

$$d_{SR}(x_0, x_1) := \inf \{l(\gamma) \mid \gamma \in \mathcal{AC}([0, 1], M), \gamma(0) = x_0, \gamma(1) = x_1\}.$$

The *sub-Riemannian sphere* $S_{SR}(x_0, r)$ (resp. the *sub-Riemannian ball* $B_{SR}(x_0, r)$) centered at x_0 with radius r as the set of points $x \in M$ such that $d_{SR}(x_0, x) = r$ (resp. $d_{SR}(x_0, x) < r$). A path $\gamma \in \mathcal{AC}([0, 1], M)$ is said to be *minimizing* if it realizes the sub-Riemannian distance between its extremities.

Remark 1.1. If Chow's condition holds, then:

- the topology defined by the sub-Riemannian distance d_{SR} coincides with the original topology of M ,
- sufficiently near points can be joined by a minimizing path,
- if the manifold M is moreover a complete metric space for the sub-Riemannian distance d_{SR} , then any two points can be joined by a minimizing path.

Consider on the other part the differential system on the tangent bundle TM of M

$$\dot{x}(t) = \sum_{i=1}^m u_i(t) f_i(x(t)) \quad \text{a.e. on } [0, 1], \quad (1)$$

where the function $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$, called *control function*, belongs to $L^2([0, 1], \mathbb{R}^m)$. Let $x_0 \in \mathbb{R}^n$, and let \mathcal{U} denote the (open) subset of $L^2([0, 1], \mathbb{R}^m)$ such that the solution of (1) starting at x_0 and associated to a control $u(\cdot) \in \mathcal{U}$ is well-defined on $[0, 1]$. The mapping

$$\begin{aligned} E_{x_0} : \mathcal{U} &\longrightarrow \mathbb{R}^n \\ u(\cdot) &\longmapsto x(1), \end{aligned}$$

which to a control $u(\cdot)$ associates the extremity $x(1)$ of the corresponding solution $x(\cdot)$ of (1) starting at x_0 , is called *end-point mapping* at the point x_0 ; it is a smooth mapping. The trajectory $x(\cdot)$ is said to be *singular* if the associated control $u(\cdot)$ is a singular point of the end-point mapping (*i.e.* if the Fréchet derivative of E_{x_0} at $u(\cdot)$ is not onto); it is *minimizing* if it realizes the sub-Riemannian distance between its extremities.

Remark 1.2. A sub-Riemannian manifold is often defined as a triple (M, Δ, g) , where M is a n -dimensional manifold, Δ is a distribution of rank $m \leq n$, and g is a Riemannian metric on Δ . If the vector fields (f_1, \dots, f_m) are everywhere linearly independent, then controlled paths solutions of (1) coincide with absolutely continuous paths tangent to the distribution Δ , where

$$\Delta(x) = \text{Span} \{f_1(x), \dots, f_m(x)\},$$

for all $x \in M$. These paths are said Δ -horizontal.

On the other part, for $x_0 \in M$, let $\Omega(x_0, \Delta)$ be the set of Δ -horizontal paths starting from x_0 whose derivative is square integrable for the metric g (and hence for any Riemannian metric on Δ). Endowed with the H^1 -topology, $\Omega(x_0, \Delta)$ inherits of a Hilbert manifold structure, see [4]. For $(x_0, x_1) \in M \times M$, let $\Omega(x_0, x_1, \Delta)$ be the subset of paths $x(\cdot) \in \Omega(x_0, \Delta)$ such that $x(1) = x_1$. The set $\Omega(x_0, x_1, \Delta)$ is a submanifold of $\Omega(x_0, \Delta)$ in a neighborhood of any nonsingular path, but might fail to be a (global) manifold due to the possible existence of singular paths.

Let x_0 and x_1 in M . The sub-Riemannian problem of determining a minimizing trajectory steering x_0 to x_1 can be easily seen (up to reparametrization, and using the Cauchy-Schwarz inequality) to be equivalent to the *optimal control problem* of finding a control $u(\cdot) \in \mathcal{U}$ such that the solution of the control system (1) steers x_0 to x_1 in time 1, and minimizes the *cost function*

$$C(u(\cdot)) := \int_0^1 \sum_{i=1}^m u_i(t)^2 dt. \quad (2)$$

If a control $u(\cdot)$ associated to a trajectory $x(\cdot)$ such that $x(0) = x_0$ is optimal, then there exists a nontrivial *Lagrange multiplier* $(\psi, \psi^0) \in T_{x(1)}^*M \times \mathbb{R}$ such that

$$\psi \cdot dE_{x_0}(u(\cdot)) = -\psi^0 dC(u(\cdot)), \quad (3)$$

where $dE_{x_0}(u(\cdot))$ (resp. $dC(u(\cdot))$) denotes the Fréchet derivative of E_{x_0} (resp. C) at the point $u(\cdot)$. The well-known Pontryagin maximum principle (see [8]) parametrizes this condition and asserts that the trajectory $x(\cdot)$ is the projection of an *extremal*, that is a quadruple $(x(\cdot), p(\cdot), p^0, u(\cdot))$, solution of the constrained Hamiltonian system

$$\begin{aligned} \dot{x}(t) &= \frac{\partial H}{\partial p}(x(t), p(t), p^0, u(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), p^0, u(t)), \\ \frac{\partial H}{\partial u}(x(t), p(t), p^0, u(t)) &= 0, \end{aligned}$$

almost everywhere on $[0, 1]$, where

$$H(x, p, p^0, u) := \langle p, \sum_{i=1}^m u_i f_i(x) \rangle + p^0 \sum_{i=1}^m u_i^2$$

is the *Hamiltonian* of the optimal control problem, $p(\cdot)$ (called *adjoint vector*) is an absolutely continuous mapping on $[0, 1]$ such that $p(t) \in T_{x(t)}^*M$, and p^0 is a real nonpositive constant. Moreover there holds

$$(p(1), p^0) = (\psi, \psi^0), \quad (4)$$

up to a multiplying scalar. If $p^0 < 0$ then the extremal is said to be *normal*, and in this case it is normalized to $p^0 = -1/2$. If $p^0 = 0$ then the extremal is said to be *abnormal*.

Remark 1.3. Any singular trajectory is the projection of an abnormal extremal, and conversely.

Furthermore, a singular trajectory is said to be *strict* (or *strictly singular*) if it does not admit a normal extremal lift; equivalently in that case we say that its abnormal extremal lift is *strictly abnormal*.

The *sub-Riemannian wave-front* $W_{SR}(x_0, r)$ centered at x_0 and with radius r is defined as the set of end-points $x(1)$, where $(x(\cdot), p(\cdot), p^0, u(\cdot))$ is an extremal such that $x(0) = x_0$ and $C(u(\cdot)) = r^2$. Under Chow's condition, it is clear from Remark 1.1 that $S_{SR}(x_0, r)$ is a subset of $W_{SR}(x_0, r)$.

Using the previous normalization, controls associated to normal extremals can be computed as

$$u_i(t) = \langle p(t), f_i(x(t)) \rangle, \quad i = 1, \dots, m.$$

Hence normal extremals are solutions of the Hamiltonian system

$$\dot{x}(t) = \frac{\partial H_1}{\partial p}(x(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H_1}{\partial x}(x(t), p(t)), \quad (5)$$

where

$$H_1(x, p) = \frac{1}{2} \sum_{i=1}^m \langle p, f_i(x) \rangle^2.$$

Notice that $H_1(x(t), p(t))$ is constant along each normal extremal and that the length of the path $x(\cdot)$ equals $(2H_1(x(0), p(0)))^{1/2}$. Actually, given a point x_0 of M , the differential system (5) has a well-defined smooth solution on $[0, 1]$ such that $x(0) = x_0$ and $p(0) = p_0$, for $p_0 \in U$, where U is a connected open subset of $T_{x_0}^*M$. In what follows, the point x_0 is fixed.

Definition 1.1. The smooth mapping

$$\begin{aligned} \exp_{x_0} : U &\longrightarrow M \\ p_0 &\longmapsto x(1) \end{aligned}$$

where $(x(\cdot), p(\cdot))$ is the solution of the system (5) such that $x(0) = x_0$ and $p(0) = p_0$, is called *exponential mapping* at the point x_0 .

The exponential mapping parametrizes normal extremals. Notice that every minimizing trajectory steering x_0 to a point of $M \setminus \exp_{x_0}(U)$ is necessarily strictly singular.

Remark 1.4. Using notations of Definition 1.1, it is easy to see by reparametrization that $x(t) = \exp_{x_0}(tp_0)$, for all $t \in [0, 1]$.

Remark 1.5. For all $p_0 \in U$ such that $H_1(x_0, p_0) = \frac{r^2}{2}$, one has $\exp_{x_0}(p_0) \in W_{SR}(x_0, r)$. The space of normal extremals with length r is parametrized by the manifold $U_r = U \cap H_1^{-1}(\frac{r^2}{2})$, which is diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}^{n-m}$ if the distribution Δ has rank m at x_0 .

A point $x \in \exp_{x_0}(U)$ is said *conjugate* to x_0 if it is a critical value of the mapping \exp_{x_0} , *i.e.* if there exists $p_0 \in U$ such that $x = \exp_{x_0}(p_0)$ and the differential $d\exp_{x_0}(p_0)$ is not onto. The *conjugate locus*, denoted by $\mathcal{C}(x_0)$, is defined as the set of all points conjugate to x_0 .

Remark 1.6. By Sard Theorem applied to the mapping \exp_{x_0} , it is clear that the conjugate locus $\mathcal{C}(x_0)$ has Lebesgue measure zero in M .

Remark 1.7. Let $x \in \exp_{x_0}(U)$, and $p_0 \in U$ such that $x = \exp_{x_0}(p_0)$. We denote by $(x(\cdot, p_0), p(\cdot, p_0), -\frac{1}{2}, u(\cdot, p_0))$ the associated normal extremal. Then we have

$$\exp_{x_0}(p_0) = E_{x_0}(u(\cdot, p_0)).$$

Therefore if x is not conjugate to x_0 then the control $u(\cdot, p_0)$ is nonsingular. In particular, the set of endpoints of nonstrictly singular trajectories starting from x_0 has Lebesgue measure zero in M .

Remark 1.8. With notations of the previous remark, if x is not conjugate to x_0 then the path $x(\cdot) := x(\cdot, p_0)$ associated to the control $u(\cdot) := u(\cdot, p_0)$ admits a unique normal extremal lift. Indeed if it had two distinct normal extremals lifts $(x(\cdot), p_1(\cdot), -\frac{1}{2}, u(\cdot))$ and $(x(\cdot), p_2(\cdot), -\frac{1}{2}, u(\cdot))$, then the extremal $(x(\cdot), p_1(\cdot) - p_2(\cdot), 0, u(\cdot))$ would be an abnormal extremal lift of the path $x(\cdot)$, which is a contradiction since $u(\cdot)$ is nonsingular.

In the present paper we prove the two following theorems.

Theorem 1.1. *Suppose Chow's condition holds, and that the manifold M is complete for the sub-Riemannian distance d_{SR} . There exists a dense subset N_1 of M such that, for every point $x \in N_1$, there is a unique minimizing path joining x_0 to x ; moreover this trajectory admits a normal extremal lift. In particular the image $\exp_{x_0}(U)$ of the exponential mapping is dense in M .*

For all $x \in M$, let $\Delta(x) := \text{Span} \{f_1(x), \dots, f_m(x)\}$, and let μ denote the Lebesgue measure on M . Regarding the previous result, one can wonder whether almost every point of M belongs to $\exp_{x_0}(U)$. The following result gives a positive answer in the case of a corank-one distribution.

Theorem 1.2. *Suppose Chow's condition holds, and that the manifold M is complete for the sub-Riemannian distance d_{SR} . If the distribution Δ is everywhere of corank one, then there exists a subset N_2 of M of full Lebesgue measure such that, for every point $x \in N_2$, there exists a minimizing path joining x_0 to x and having a normal extremal lift. Moreover this trajectory is nonsingular, and x is not conjugate to x_0 . In particular, the set $\exp_{x_0}(U)$ is of full measure in M , *i.e.* $\mu(M \setminus \exp_{x_0}(U)) = 0$.*

The next two sections are devoted to the proof of the latter results. In a last section we discuss some consequences and open problems.

2 Proof of Theorem 1.1

2.1 The proximal sub-differential

Let M be a smooth manifold of dimension n and Ω be an open subset of M . Let $f : \Omega \rightarrow \mathbb{R}$ be a continuous function on Ω ; we call *proximal sub-differential* of the function f at the point $x \in \Omega$ the subset of T_x^*M defined by

$$\partial_P f(x) := \{d\phi(x) \mid \phi \in C^\infty(M) \text{ and } f - \phi \text{ attains a local minimum at } x\}.$$

Note that since every local C^∞ function can be extended to a C^∞ function on M , the proximal sub-differential of f at x depends only on the local behavior of the function f near x . In addition, remark that $\partial_P f(x)$ is a convex subset of T_x^*M which may be empty; for instance the proximal sub-differential of the real function $t \mapsto -|t|$ at $t = 0$ is empty.

Remark 2.1. Notice that when $M = \mathbb{R}^n$, a vector ζ belongs to the proximal sub-differential of f at a point x if and only if there exists σ and $\delta > 0$ such that

$$f(y) - f(x) + \sigma\|y - x\|^2 \geq \langle \zeta, y - x \rangle, \quad \forall y \in x + \delta B.$$

This is the usual definition of proximal sub-differentials in Hilbert spaces; we refer the reader to [6] for further details on that subject.

In fact, an immediate application of the smooth variational principle of Borwein-Preiss (see [5]) implies the following result.

Theorem 2.1. *The proximal sub-differential of a continuous function $f : \Omega \rightarrow \mathbb{R}$ is nonempty on a dense subset of Ω .*

The proximal sub-differential of f defines a multivalued mapping from Ω into the cotangent bundle T^*M . It is said to be locally bounded on Ω if for each $x \in \Omega$ there exists a neighborhood \mathcal{V} of x such that $\partial_P f(\mathcal{V})$ is relatively compact in T^*M . The following result is standard.

Proposition 2.2. *The function f is Lipschitz continuous on Ω if and only if the proximal sub-differentials of f are locally bounded on Ω .*

Remark 2.2. Notice that the Fréchet (or viscosity) sub-differential of f at x , defined by

$$D^- f(x) := \{d\phi(x) \mid \phi \in C^1(M) \text{ and } f - \phi \text{ attains a local minimum at } x\},$$

is larger than the proximal sub-differential, but in fact both notions coincide locally; we refer the reader to [6, Prop. 4.5 p. 138, Prop. 4.12 p. 142] for a precise statement.

To conclude this preliminary section, we remark that there exists a complete calculus of proximal sub-differentials, one that extends all the theorems of the usual smooth calculus, see [6].

2.2 Application to the proof of Theorem 1.1

In what follows we denote $e(\cdot) := d_{SR}(x_0, \cdot)^2$.

Proposition 2.3. *Let $x \in M$ such that $\partial_P e(x) \neq \emptyset$. Then there exists a unique minimizing path $x(\cdot)$ joining x_0 to x . Moreover for every $\zeta \in \partial_P e(x)$, the path $x(\cdot)$ admits a normal extremal lift $(x(\cdot), p(\cdot), -\frac{1}{2}, u(\cdot))$ such that $p(1) = \frac{1}{2}\zeta$.*

Proof. We adopt the following notation: for every control $u(\cdot) \in \mathcal{U}$, we denote by $x_u(\cdot)$ the trajectory solution of (1) associated to the control $u(\cdot)$ and such that $x_u(0) = x_0$. Let $x \in M$ and $\zeta \in \partial_P e(x)$. We first prove that every minimizing path steering x_0 to x admits a normal extremal lift such that $p(1) = \frac{1}{2}\zeta$. Let $u(\cdot) \in \mathcal{U}$ be an optimal control such that the associated trajectory $x_u(\cdot)$ joins x_0 to x ; there holds

$$e(x) = \int_0^1 \sum_{i=1}^m u_i(t)^2 dt.$$

On the other hand, since $\zeta \in \partial_P e(x)$, there exists a function ϕ of class C^∞ with $d\phi(x) = \zeta$ and such that $e - \phi$ attains a local minimum at x . Thus there exists a neighborhood \mathcal{V} of $u(\cdot)$, contained in \mathcal{U} , such that

$$e(x) \leq e(x_v(1)) - \phi(x_v(1)) + \phi(x),$$

for every control $v(\cdot) \in \mathcal{V}$. Moreover it can be easily seen by definition of the distance function, that

$$e(x_v(1)) \leq \int_0^1 \sum_{i=1}^m v_i(t)^2 dt.$$

Therefore we obtain

$$e(x) \leq \int_0^1 \sum_{i=1}^m v_i(t)^2 dt - \phi(x_v(1)) + \phi(x),$$

for every control $v(\cdot) \in \mathcal{V}$. In particular, this means that $u(\cdot)$ is a solution of the minimization problem

$$\min_{v \in \mathcal{V}} \left(\int_0^1 \sum_{i=1}^m v_i(t)^2 dt - \phi(x_v(1)) + \phi(x) \right).$$

Hence $u(\cdot)$ is a critical point of the function

$$v(\cdot) \in \mathcal{V} \mapsto C(v(\cdot)) - \phi(E_{x_0}(v(\cdot))) + \phi(x),$$

and thus

$$dC(u(\cdot)) - \zeta \cdot dE_{x_0}(u(\cdot)) = 0.$$

This leads to the existence of a normal extremal lift $(x_u(\cdot), p_u(\cdot), -\frac{1}{2}, u(\cdot))$ such that $(x_u(1), p_u(1)) = (x, \frac{1}{2}\zeta)$. In particular, uniqueness of a minimizing path joining x_0 to x follows. \square

Th. 1.1 is a straightforward consequence of Prop. 2.3 together with Th. 2.1.

3 Proof of Theorem 1.2

3.1 The limiting sub-differential

Let M be a smooth manifold of dimension n and Ω be an open subset of M . Let $f : \Omega \rightarrow \mathbb{R}$ be a continuous function on Ω ; we call *limiting sub-differential* of the function f at the point $x \in \Omega$ the subset of T_x^*M defined by

$$\partial_L f(x) := \{\lim \zeta_n \mid \zeta_n \in \partial_P f(x_n), x_n \rightarrow x\}.$$

As the proximal sub-differential, the limiting sub-differential of f at x depends only on the local behavior of f near x . Moreover by construction, $\partial_L f(x)$ is a closed subset of T_x^*M which contains $\partial_P f(x)$, which is not necessarily convex and which may be empty. In some situations, the limiting sub-differential of f at x can be proven to be nonempty; the result is as follows.

Proposition 3.1. *Let $x \in \Omega$. If there exists a Lipschitz continuous ϕ defined in a neighborhood of x such that $f - \phi$ attains a local minimum at x , then $\partial_L f(x)$ is nonempty.*

Proof. Without loss of generality, we can assume to be in \mathbb{R}^n . By assumption, the function $f - \phi$ attains a local minimum at x ; this implies that $0 \in \partial_L(f - \phi)(x)$. By the sum rule on limiting sub-differentials (see [6, Proposition 10.1 p. 62]), the function $-\phi$ being Lipschitz continuous, there holds

$$\partial_L(f - \phi)(x) \subset \partial_L f(x) + \partial_L(-\phi)(x),$$

and hence $\partial_L f(x)$ is necessarily nonempty. \square

This proposition will be the key result to prove Th. 1.2. Notice that there exist some continuous functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $n \geq 2$, such that their limiting sub-differential is empty on a subset of positive Lebesgue measure. However if $n = 1$, it can be proven that the limiting sub-differential of any continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is nonempty almost everywhere. Our proof of Th. 1.2 for corank-one distributions is in some way related to this latter result, but is not a consequence of it.

3.2 Application to the proof of Theorem 1.2

In what follows, we denote $e(\cdot) := d_{SR}(x_0, \cdot)^2$.

Proposition 3.2. *Let $x \in M$ such that $\partial_L e(x) \neq \emptyset$ and let $\zeta \in \partial_L e(x)$. Then there exists a minimizing trajectory joining x_0 to x which admits a normal extremal lift $(x(\cdot), p(\cdot), -\frac{1}{2}, u(\cdot))$ such that $p(1) = \frac{1}{2}\zeta$.*

Proof. By definition of the limiting sub-differential, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of points in M converging to x and a sequence $(\zeta_n)_{n \in \mathbb{N}} \in \partial_P e(x_n)$ such that $\lim \zeta_n = \zeta$. For each integer n , we denote by $u_n(\cdot)$ a minimizing control joining x_0 to x_n , and by $x_{u_n}(\cdot)$ its associated trajectory. From Prop.

2.3, for each integer n , we know that $x_{u_n}(\cdot)$ admits a normal extremal lift $(x_{u_n}(\cdot), p_{u_n}(\cdot), -\frac{1}{2}, u_n(\cdot))$ such that $p_{u_n}(1) = \frac{1}{2}\zeta_n$. Since the sub-Riemannian distance is continuous, the sequence of controls $(u_n(\cdot))_{n \in \mathbb{N}}$ is clearly bounded in $L^2([0, 1], \mathbb{R}^m)$, and then up to a subsequence, it converges towards an element $u(\cdot)$ for the weak L^2 -topology. As a consequence, since the end-point mapping E_{x_0} is continuous for the weak L^2 -topology (see [9] for a proof), we deduce, passing to the limit, that $E_{x_0}(u(\cdot)) = x$. Furthermore, up to a subsequence the sequence $(x_{u_n}(\cdot))_{n \in \mathbb{N}}$ converges uniformly towards a minimizing path $x_u(\cdot)$. This implies that the sequence $(p_{u_n}(\cdot))_{n \in \mathbb{N}}$ converges uniformly towards some $p_u(\cdot)$, where $p_u(\cdot)$ is an adjoint vector associated to the trajectory $x_u(\cdot)$, and $p_u(1) = \frac{1}{2} \lim_{n \rightarrow \infty} \zeta_n$. Finally the quadruple $(x_u(\cdot), p_u(\cdot), -\frac{1}{2}, u(\cdot))$ is a normal extremal lift of $x_u(\cdot)$. \square

Analogously to Th. 2.1, we have the following result.

Proposition 3.3. *If the distribution is everywhere of corank one, then $\partial_L e(x) \neq \emptyset$ for almost every $x \in M$.*

Proof. In what follows, our point of view being local, we can assume to work in \mathbb{R}^n . Denote by P the set of points x of M such that

$$\liminf_{y \rightarrow x} \frac{e(y) - e(x)}{\|y - x\|} = -\infty.$$

We have $M = P \cup P^c$, where P^c denotes the complement of the set P in M . Note that if $x \in P^c$ then there exists $\alpha \in \mathbb{R}$ such that $\liminf_{y \rightarrow x} \frac{e(y) - e(x)}{\|y - x\|} = \alpha$, which means that there exists a neighborhood \mathcal{V} of x such that

$$e(y) \geq e(x) + (\alpha - 1)\|y - x\|, \quad \forall y \in \mathcal{V}.$$

We infer that the function e has a Lipschitz continuous support function at x and hence from Prop. 3.1 that $\partial_L e(x)$ is nonempty. The rest of the proof is devoted to show that the set $\partial_L e(x)$ is nonempty for almost every point $x \in P$. We argue by contradiction: denote by A the subset of P where the limiting sub-differential of f is empty, and suppose that $\mu(A) > 0$.

For all $x \in M$, let $\nu(x)$ denote a vector of $T_x M$ transverse to the distribution $\Delta(x)$. We may assume the vector field $\nu(\cdot)$ to be smooth on M . Let us consider integral curves of the differential system

$$\dot{y}(t) = \nu(y(t)). \tag{6}$$

From Fubini's theorem, there exists an interval $I \subset \mathbb{R}$ and an integral curve $(y(t))_{t \in I}$ of (6) such that the set

$$T := \{t \in I \mid y(t) \in A\},$$

satisfies $\lambda(T) > 0$, where λ denotes the Lebesgue measure on \mathbb{R} . We are going to prove that some $\bar{t} \in I$ is the limit of local minima of the function $e(\cdot)$ restricted to the curve $y(t)$. To this aim we need different lemmas.

Lemma 3.4. *For all $x \in M$, there exist a neighborhood \mathcal{V}_x of x in M , a neighborhood U_x of 0 in T_x^*M , and a submanifold D_x of codimension 1 in M , such that*

$$\mathcal{V}_x \cap D_x \subset \exp_x(U_x).$$

Proof. Clearly the mapping \exp_x is smooth on its domain of definition, and its differential at 0, denoted $d\exp_x(0)$, can be computed as

$$d\exp_x(0) \cdot \delta p_0 = \delta x(1),$$

where $(\delta x(\cdot), \delta p(\cdot))$ is the solution of the linearized system of system (5) at the equilibrium point $(x, 0)$, such that $\delta x(0) = 0$ and $\delta p(0) = \delta p_0$. This linearized system writes

$$\delta \dot{x}(t) = \sum_{i=1}^{n-1} \langle \delta p(t), f_i(x) \rangle f_i(x), \quad \delta \dot{p}(t) = 0,$$

and thus $\delta p(t)$ is constant, equal to δp_0 , whence

$$\delta x(1) = \sum_{i=1}^{n-1} \langle \delta p_0, f_i(x) \rangle f_i(x). \quad (7)$$

Therefore the mapping \exp_x has rank $n - 1$ at the point 0, and the conclusion follows. \square

For each $x \in M$, let $(p_i^*(x))_{i=1, \dots, n}$ denote the dual basis in T_x^*M of the basis $(f_1(x), \dots, f_{n-1}(x), \nu(x))$ in T_xM . We define the mapping $\Phi : I \times O \rightarrow M$, where O is a neighborhood of 0 in \mathbb{R}^{n-1} , by the formula

$$\Phi(t, \alpha_1, \dots, \alpha_{n-1}) := \exp_{y(t)} \left(\sum_{i=1}^{n-1} \alpha_i p_i^*(y(t)) \right).$$

Using (7), it is quite easy to see that, for all $t_0 \in I$, the mapping Φ is a local diffeomorphism at $(t_0, 0)$. Thus the following lemma is straightforward.

Lemma 3.5. *Let $t_0 \in T$. There exist a neighborhood \mathcal{V} of $y(t_0)$ in M and a smooth function $\rho : \mathcal{V} \rightarrow I$ such that for every $z \in \mathcal{V}$, one has $z \in D_{y(\rho(z))}$, and such that for every $t \in T$ with $y(t) \in \mathcal{V}$, there holds $\rho(y(t)) = t$. Moreover, there exists a real number $\delta > 0$ such that*

$$|e(z) - e(y(\rho(z)))| \leq \delta \|z - y(\rho(z))\|. \quad (8)$$

for all $z \in \mathcal{V}$.

Define the continuous function $g : I \rightarrow \mathbb{R}$ by $g(t) := e(y(t))$.

Lemma 3.6. *There exists $\bar{t} \in T$ and a sequence $(t_n)_{n \in \mathbb{N}}$ of I converging towards \bar{t} , such that the function g attains a local minimum at t_n , for every integer n .*

Proof. We argue by contradiction. If the conclusion of the lemma does not hold, this means that for every $t \in T$, there exists a neighborhood V_t of t in I on which g is monotonous. In particular g has bounded variations on V_t , and hence g is differentiable almost everywhere in V_t . On the other hand, since the set T has positive Lebesgue measure, there exists $t \in T$ such that $\lambda(V \cap T) > 0$ for any neighborhood V of t in I . Hence, this proves that for such a $t \in T$, the function g is differentiable almost everywhere in V_t which has positive measure. Fix some $s \in V_t$ where g is differentiable. As a consequence, there exists some Lipschitz continuous function $\psi : I \rightarrow \mathbb{R}$ such that

$$\psi(s) = g(s), \text{ and } \psi(s') \leq g(s'), \quad \forall s' \in I.$$

On the other hand, by Lemma 3.5, there exists a neighborhood \mathcal{V} of $y(s)$ in M and a smooth function $\rho : \mathcal{V} \rightarrow I$ which satisfy the property given in the lemma. By (8), we deduce that for any $x \in \mathcal{V}$, there holds

$$\begin{aligned} e(x) &\geq e(y(\rho(x))) - \delta \|x - y(\rho(x))\| \\ &\geq \phi_s(\rho(x)) - \delta \|x - y(\rho(x))\|. \end{aligned}$$

Therefore if we define locally $\phi(x) := -\delta \|x - y(\rho(x))\|$, the function $e - \psi \circ \rho - \phi$ attains a local minimum at $y(s)$. Since $\psi \circ \rho$ and ϕ are Lipschitz continuous, the sum rule on limiting sub-differentials (see [6, Prop. 10.1 p. 62]) implies that

$$0 \in \partial_L(e - \psi \circ \rho - \phi)(y(s)) \subset \partial_L e(y(s)) + \partial_L(-\psi \circ \rho - \phi)(y(s)).$$

Hence there exists $\zeta \in \partial_L e(y(s))$ and $\zeta' \in \partial_L(-\psi \circ \rho - \phi)(y(s))$ such that $0 = \zeta + \zeta'$ Which proves that $\partial_L e(y(s))$ is nonempty and which contradicts the fact that $y(s) \in A$. \square

Lemma 3.7. *There exists some constant $K > 0$ such that for every integer n , the limiting sub-differential $\partial_L e(y(t_n))$ contains an element with norm less than K .*

Proof. By construction of the sequence $(t_n)_{n \in \mathbb{N}}$, for every integer n the function g attains a minimum at t_n . This means that there exists an interval (a_n, b_n) containing t_n such that

$$\forall t \in (a_n, b_n) \quad g(t) \geq g(t_n).$$

On the other hand, by Lemma 3.5, there exists a neighborhood \mathcal{V} of $y(\bar{t})$ such that for n large enough, any x close enough to $y(t_n)$ belongs to $D_{y(\rho(x))}$ where $\rho(x) \in (a_n, b_n)$. By (8), we deduce that for x close enough to $y(t_n)$, there holds

$$\begin{aligned} e(x) &\geq e(y(\rho(x))) - \delta \|x - y(\rho(x))\| \\ &\geq e(y(t_n)) - \delta \|x - y(\rho(x))\|. \end{aligned}$$

Therefore as before, if we define locally $\phi(x) := -\delta \|x - y(\rho(x))\|$, the function $e - \phi$ attains a local minimum at $y(t_n)$. Since ϕ is Lipschitz continuous, the sum rule on limiting sub-differentials implies that

$$0 \in \partial_L(e - \phi)(y(t_n)) \subset \partial_L e(y(t_n)) + \partial_L(-\phi)(y(t_n)).$$

Hence there exists $\zeta \in \partial_L e(y(t_n))$ and $\zeta' \in \partial_L(-\phi)(y(t_n))$ such that $0 = \zeta + \zeta'$. Finally $\|\zeta\| = \|\zeta'\|$ where $\|\zeta'\|$ is less than the Lipschitz constant of the function ϕ . This concludes the proof of the lemma. \square

Returning to the proof of Prop. 3.3, we infer easily that $\partial_L e(y(\bar{t}))$ is nonempty. This yields a contradiction with the fact that $y(\bar{t}) \in A$, and ends the proof of the proposition. \square

Propositions 3.2 and 3.3 imply the existence of a subset N of full Lebesgue measure in M such that, for every $x \in N$, there exists a minimizing trajectory steering x_0 to x and having a normal extremal lift. Let $N_2 := N \setminus \mathcal{C}(x_0)$. It is the set of points $x \in M$ which are not conjugate to x_0 , and such that there exists a minimizing path $x(\cdot)$ joining x_0 to x and having a normal extremal lift. Remark 1.7 implies that the trajectory $x(\cdot)$ is moreover nonsingular. From Remark 1.6 it is clear that N_2 is of full Lebesgue measure in M . This ends the proof of Th. 1.2.

4 Consequences and open questions

In what follows, we assume that Chow's condition holds, and that the manifold M is complete for the sub-Riemannian distance. Let $x_0 \in M$ be fixed.

4.1 A formula for the sub-Riemannian distance

From Th. 1.1, there exists a dense subset N_1 of M such that every point of N_1 can be joined from x_0 by a unique minimizing trajectory, which moreover admits a normal extremal lift. This yields the following result.

Corollary 4.1. *For all point $x \in N_1$ one has*

$$d_{SR}(x_0, x) = \inf \left\{ (2H_1(x_0, p))^{1/2} \mid p \in U \text{ s.t. } \exp_{x_0}(p) = x \right\}.$$

Remark 4.1. Actually Th. 1.1 implies that for every $x \in N_1$ there exists a unique $p \in U$ such that the above infimum is attained.

As a consequence, we deduce that the function $g : M \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$g(x) := \inf \left\{ (2H_1(x_0, p))^{1/2} \mid p \in U \text{ s.t. } \exp_{x_0}(p) = x \right\},$$

for all $x \in M$, coincides with the mapping $d_{SR}(x_0, \cdot)$ on a dense subset of the manifold M . In particular, since g is continuous on M , there holds

$$d_{SR}(x_0, x) = \inf \{ \lim g(x_n) \mid x_n \rightarrow x \}$$

for all $x \in M$.

Remark 4.2. If the sub-Riemannian distance to x_0 is Lipschitz continuous outside x_0 , then from Prop 2.2 the limiting sub-differentials of $d_{SR}(x_0, \cdot)$ are always nonempty; hence the set of points x of M such that every minimizing trajectory joining x_0 to x is strictly singular, is empty. The converse is false; a counterexample is given by the so-called Martinet flat case, see [2]. To get a converse statement, the assumption has to be strengthened as follows: if there does not exist any nontrivial singular minimizing trajectory, then $d_{SR}(x_0, \cdot)$ is Lipschitz continuous outside x_0 , see [1].

4.2 On the sub-Riemannian wave-front and sphere

The following result is a direct consequence of Th. 1.1.

Corollary 4.2. *The sub-Riemannian wave-front $W_{SR}(x_0, r)$ is connected, for all $r > 0$.*

Proof. Using notations of Remark 1.5, and from Th. 1.1, we have the inclusions

$$\exp_{x_0}(U_r) \subset W_{SR}(x_0, r) \subset \overline{\exp_{x_0}(U_r)},$$

where U_r is diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}^{n-m}$, and thus is connected. The conclusion follows readily. \square

Proposition 4.3. *If the distribution Δ is everywhere of corank one, then the sub-Riemannian wave-front $W_{SR}(x_0, r)$, and thus the sub-Riemannian sphere $S_{SR}(x_0, r)$, has Lebesgue measure zero, for all $r > 0$.*

Proof. It suffices to notice that the image by a locally Lipschitzian mapping from \mathbb{R}^n to \mathbb{R}^n of a set of zero measure has zero measure, and to apply Th. 1.2. \square

4.3 Sard type conjectures

Let \mathcal{A} (resp. \mathcal{A}_s) denote the set of points x of M such that every minimizing trajectory joining x_0 to x is singular (resp. strictly singular). Obviously $\mathcal{A}_s \subset \mathcal{A}$. Th. 1.1 and 1.2 yield the following result.

Corollary 4.4. *The subset \mathcal{A}_s has an empty interior in M . In the case of a corank-one distribution the subset \mathcal{A} has Lebesgue measure zero in M .*

Let now \mathcal{S} (resp. \mathcal{S}_{min} , resp. $\mathcal{S}_{min}^{strict}$) denote the set of points x of M such that there exists a singular trajectory (resp. a singular minimizing trajectory, resp. a strictly singular minimizing trajectory) steering x_0 to x . Notice that \mathcal{S} is the set of critical values of the end-point-mapping E_{x_0} .

Corollary 4.5. *The set $\mathcal{S}_{min}^{strict}$ has an empty interior in M .*

Let N_3 be the set of points $x \in M$ such that there exists a unique minimizing path $x(\cdot)$ joining x_0 to x , which moreover admits a normal extremal lift, and such that x is not conjugate to x_0 . Notice that from Remark 1.7, the path $x(\cdot)$ is nonsingular. The set N_3 can be proven to be open in M ; we formulate the following conjecture.

Conjecture 4.6. The subset N_3 is of full Lebesgue measure in M . In particular, the set \mathcal{S}_{min} has Lebesgue measure zero in M .

We end the paper with the following open question.

Conjecture 4.7. The end-point mapping satisfies Sard's property, *i.e.* the set \mathcal{S} has Lebesgue measure zero in M .

This conjecture has been formulated and discussed, among others, in [7]. Up to now, it is still open, even in the case of a corank-one distribution.

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