GLOBAL STEADY-STATE CONTROLLABILITY OF 1-D SEMILINEAR HEAT EQUATIONS

JEAN-MICHEL CORON AND EMMANUEL TRÉLAT*

Abstract. We investigate the problem of exact boundary controllability of semilinear one-dimensional heat equations. We prove that it is possible to move from any steady-state to any other one by means of a boundary control, provided that they are in the same connected component of the set of steady-states. The proof is based on an effective feedback stabilization procedure which is implemented.

Key words. Heat equation, controllability, pole shifting, Lyapunov functional.

AMS subject classifications. 93B05, 93C20, 35B37.

1. Introduction.

1.1. Statement of the main result. Let $L>0$ fixed and $f:\mathbb{R}\to\mathbb{R}$ be a function of class C^2. Let us consider the boundary control system

$$\begin{cases} \dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2} + f(y), \\ y(t,0)=0,\ y(t,L)=u(t), \end{cases} \tag{1.1}$$

where the state is $y(t,.):[0,L]\to\mathbb{R}$ and the control is $u(t)\in\mathbb{R}$.

Concerning the global controllability problem, one of the main results [5] asserts that if f is globally lipschitzian then this control system is approximately globally controllable, see also [11] for exact controllability. When f is superlinear the situation is still widely open, in particular because of possible blowing up. Indeed it is well known that if $yf(y)>0$ as $y\neq 0$ then blow-up phenomena may occur for the Cauchy problem

$$\begin{cases} \dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2} + f(y), \\ y(t,0)=0,\ y(t,L)=0, \\ y(0,x)=y_0(x). \end{cases} \tag{1.2}$$

For example if $f(y)=y^3$ then for numerous initial data there exists $T>0$ such that the unique solution to the previous Cauchy problem is well defined on $[0,T)\times[0,L]$ and satisfies

$$\lim_{t\to T}\|y(t,.)\|_{L^\infty(0,L)} = +\infty,$$

see for instance [1, 8, 2, 12, 14, 15, 18] and references therein.

One may ask if, acting on the boundary of $[0,L]$, one could avoid the blow-up phenomenon. Actually the answer to this question is negative in general, see [7]: for some nonlinear functions f satisfying

$$|f(y)| \sim |y|\log^p(1+|y|) \quad \text{as } |y|\to+\infty,$$

*Université Paris-Sud, Laboratoire ANEDP, Mathématique, UMR 8628, Bât. 425, 91405 Orsay cedex, France (Jean-Michel.Coron@math.u-psud.fr, Emmanuel.Trelat@math.u-psud.fr).

with $p > 2$, and for any time $T > 0$, there exist initial data which lead to blow-up before time T, whatever the control function u is. Notice however that if

$$|f(y)| = \mathrm{o}\left(|y| \log^{3/2}(1+|y|)\right) \quad \text{as } |y| \to +\infty,$$

then the blow-up (which could occur in the absence of control) can be avoided by means of boundary control, see [7].

Nevertheless in the first case where the blow-up phenomenon cannot be compensated by means of boundary control the situation is not completely desperate. In fact as we shall see in this paper, we can move from any given steady-state to any other one belonging to the same connected component of the set of steady-states. More precisely let us define the notion of steady-state.

DEFINITION 1.1. *A function $y \in C^2([0,L])$ is a steady-state of the control system (1.1) if*

$$\frac{d^2y}{dx^2} + f(y) = 0, \; y(0) = 0.$$

We denote by $\mathcal{S}$ the set of steady-states, endowed with the C^2 topology.

Let us also introduce the Banach space

$$Y_T = \Big\{ y(t,x), (t,x) \in (0,T) \times (0,L) \; / \; y \in L^2(0,T,W^{2,2}(0,L)) \\ \text{and } \frac{\partial y}{\partial t} \in L^2((0,T)\times(0,L)) \Big\} \tag{1.3}$$

endowed with the norm

$$\|y\|_{Y_T} = \|y\|_{L^2(0,T,W^{2,2}(0,L))} + \left\| \frac{\partial y}{\partial t} \right\|_{L^2((0,T)\times(0,L))}.$$

Notice that Y_T is continuously imbedded in $L^\infty((0,T)\times(0,L))$.

The main result of the paper is the following.

THEOREM 1.2. *Let y_0 and y_1 be two steady-states belonging to a same connected component of $\mathcal{S}$. There exist a time $T > 0$ and a control function $u \in L^2(0,T)$ such that the solution $y(t,x)$ in Y_T of*

$$\begin{cases} \dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2} + f(y), \\ y(t,0) = 0, \; y(t,L) = u(t), \\ y(0,x) = y_0(x), \end{cases} \tag{1.4}$$

satisfies $y(T,.) = y_1(.)$.

REMARK 1.3. *In fact we prove the following result: for all neighborhood V of y_1 in H^1-topology, there exists a positive real number ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ there exist a control function $u \in H^1(0, 1/\varepsilon)$ such that the solution $y(t,x)$ in Y_T of the Cauchy-Dirichlet problem (1.4) satisfies $y(1/\varepsilon, .) \in V$.*

In the proof of this result, which represents the main part of the paper, we give an explicit construction of the control u in a feedback-type form, and of a Lyapunov functional. We stress that the procedure is effective and consists actually in solving a stabilization problem in finite dimension. Indeed in order to construct u we need to compute only a finite number of quantities related to an Hilbertian expansion of *the solution. The procedure has been implemented numerically, and simulations are presented in the last section of the paper.*

REMARK 1.4. *For any $T > 0$ and $u \in L^2(0,T)$ there is at most one solution of (1.4) in the Banach space Y_T.*

REMARK 1.5. *This is a (partial) global exact controllability result. The time needed in our proof is large, but on the other hand there are indeed cases where the time T of controllability cannot be taken arbitrarily small. For instance in the case where $f(y) = -y^3$, any solution of (1.4) starting from 0 satisfies the inequality*

$$\int_0^L (L-x)^4 y(T,x)^2 dx \leq 8LT,$$

and hence if $y_0 = 0$ a minimal time is needed to reach a given $y_1 \neq 0$. This result is due to Bamberger [10], see also [9, Lemma 2.1].

REMARK 1.6. *In Section 3 we prove that if y_0 and y_1 belong to distinct connected components of $\mathcal{S}$, then it is actually impossible to move either from y_0 to y_1 or from y_1 to y_0, whatever the time and the control are. In the same section we also investigate the connectedness of the set $\mathcal{S}$ of steady-states.*

REMARK 1.7. *The result of Th. 1.2 may be achieved directly by using repeatedly a local exact controllability theorem, see [9, Th. 4.4] or [11, Th. 3.3]. Here we present a new controllability strategy, based on a feedback stabilization procedure, which is more effective. It is clear also that this approach may be applied to other problems, without requiring controllability of the linearized system around an equilibrium, see [3].*

1.2. The idea of the proof. The method we shall use to prove Th. 1.2 is stemming from classical Lyapunov stability theory together with quasi-static deformation theory. For sake of simplicity we explain it in finite dimension. Let us consider in $\mathbb{R}^n$ a general control system of the form

$$\dot{y}(t) = g(y(t), u(t)), \tag{1.5}$$

where $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class C^1, $u(t) \in \mathcal{U}$, and $\mathcal{U}$ denotes the set of measurable essentially bounded admissible controls. Let $y_0, y_1 \in \mathbb{R}^n$ be two equilibrium points of system (1.5), that is

$$g(y_i, u_i) = 0, \ \ i = 0, 1,$$

for some $u_0, u_1 \in \mathbb{R}^m$. We assume that (y_0, u_0) and (y_1, u_1) belong to the same connected component of the zero set of g in $\mathbb{R}^n \times \mathbb{R}^m$. Our aim is to steer the system from y_0 to y_1 in some (large) time $T > 0$. The method splits into four steps:

First step. Construct a C^1-path $(\bar{y}(\tau), \bar{u}(\tau))$, with $\tau \in [0,1]$, connecting (y_0, u_0) to (y_1, u_1) and such that

$$\forall \tau \in [0,1] \quad g(\bar{y}(\tau), \bar{u}(\tau)) = 0.$$

Of course this path is not in general solution of system (1.5), but if $\varepsilon > 0$ is small enough then the C^1-path $(y^\varepsilon, u^\varepsilon)$

$$\begin{array}{rcl} [0, 1/\varepsilon] & \to & \mathbb{R}^n \times \mathbb{R}^m \\ t & \mapsto & (y^\varepsilon(t), u^\varepsilon(t)) = (\bar{y}(\varepsilon t), \bar{u}(\varepsilon t)) \end{array}$$

is “almost” a solution of system (1.5). Indeed

$$\|\dot{y}^\varepsilon - g(y^\varepsilon, u^\varepsilon)\| = \mathrm{O}(\varepsilon).$$

Second step. This quasi-static trajectory is not in general stable, and thus has to be stabilized. To this aim, introduce the following change of variable:

$$z(t) = y(t) - y^\varepsilon(t),$$
$$v(t) = u(t) - u^\varepsilon(t),$$

where $t \in [0, 1/\varepsilon]$. In the new variables z, v, the control system writes, at least if $\|z(t)\| + \|v(t)\|$ is small enough,

$$\dot{z}(t) = A(\varepsilon t)z(t) + B(\varepsilon t)v(t) + \mathrm{O}(\|z(t)\|^2 + \|v(t)\|^2 + \varepsilon),$$

where $t \in [0, 1/\varepsilon]$, and where

$$A(\tau) = \frac{\partial g}{\partial y}(\bar{y}(\tau), \bar{u}(\tau)) \quad \text{and} \quad B(\tau) = \frac{\partial g}{\partial u}(\bar{y}(\tau), \bar{u}(\tau)),$$

with $\tau = \varepsilon t \in [0, 1]$. Therefore we have to stabilize near the origin a *slowly-varying in time* linear control system; we refer to [13] for this classical theory.

Third step. Under mild controllability assumptions, namely

$$\forall \tau \in [0,1] \quad \mathrm{rank}\left(B(\tau), A(\tau)B(\tau), \ldots, A(\tau)^{n-1}B(\tau)\right) = n$$

(Kalman condition) it is actually possible to stabilize the system by *pole shifting* and to construct a quadratic Lyapunov function. Notice that this does not work in general if the system is not slowly-varying. So if ε is small enough then using this Lyapunov function we infer that $y(1/\varepsilon)$ belongs to some prescribed neighborhood of the target y_1. At this stage, a stabilization result is achieved.

Fourth step. If the system (1.5) is *locally controllable* near the point y_1, we conclude that it is possible to steer the system in finite time from the point $y(1/\varepsilon)$ to the desired target y_1. Usually such a local controllability result is achieved by using an implicit function argument, after proving that the linearized system is controllable.

REMARK 1.8. *The use of quasi-static deformation for the controllability of a nonlinear partial differential control system has already been used in [3]. But note that in [3] the quasi-static trajectory $(y^\varepsilon, u^\varepsilon)$ was stable so it was not necessary to perform steps 2 and 3.*

2. Proof of the main results.

In order to prove Th. 1.2 we shall exactly follow the steps described previously.

2.1. Construction of a path of steady-states.

The following lemma is obvious.

LEMMA 2.1. *Let $\phi_0, \phi_1 \in \mathcal{S}$. Then ϕ_0 and ϕ_1 belong to the same connected component of $\mathcal{S}$ if and only if for any real number α between $\phi_0'(0)$ and $\phi_1'(0)$ the maximal solution of*

$$\frac{d^2y}{dx^2} + f(y) = 0, \;\; y(0) = 0, y'(0) = \alpha,$$

denoted by $y^{\alpha}(.)$*, is defined on* $[0, L]$.

Let now y_0 and y_1 in the same connected component of $\mathcal{S}$. Let us construct in $\mathcal{S}$ a C^1 path $(\bar{y}(\tau, .), \bar{u}(\tau))$, $0 \le \tau \le 1$, joining y_0 to y_1. For each $i = 0, 1$ set

$$\alpha_i = y_i'(0).$$

Then with our previous notations: $y_i(.) = y^{\alpha_i}(.), i = 0, 1$. Now set

$$\bar{y}(\tau, x) = y^{(1-\tau)\alpha_0 + \tau\alpha_1}(x) \quad \text{and} \quad \bar{u}(\tau) = \bar{y}(\tau, L),$$

where $\tau \in [0, 1]$ and $x \in [0, L]$. By construction we have

$$\bar{y}(0, .) = y_0(.), \ \bar{y}(1, .) = y_1(.) \quad \text{and} \quad \bar{u}(0) = \bar{u}(1) = 0,$$

and thus $(\bar{y}(\tau, .), \bar{u}(\tau))$ is a C^1 path in $\mathcal{S}$ connecting y_0 to y_1.

2.2. Reduction of the problem. Let $\varepsilon > 0$. We set, for any $t \in [0, 1/\varepsilon]$ and any $x \in [0, L]$,

$$\begin{aligned} z(t, x) &= y(t, x) - \bar{y}(\varepsilon t, x), \\ v(t) &= u(t) - \bar{u}(\varepsilon t). \end{aligned} \tag{2.1}$$

Then from the definition of $(\bar{y}, \bar{u})$ we infer that z satisfies the initial-boundary problem

$$\begin{cases} z_t = z_{xx} + f'(\bar{y})z + z^2 \displaystyle\int_0^1 (1 - s) f''(\bar{y} + sz) ds - \varepsilon \bar{y}_\tau, \\ z(t, 0) = 0, \ z(t, L) = v(t), \\ z(0, x) = 0. \end{cases} \tag{2.2}$$

Now, in order to deal rather with a Dirichlet-type problem, we set

$$w(t, x) = z(t, x) - \frac{x}{L} v(t), \tag{2.3}$$

and we suppose that the control v is derivable. This leads to the following equation:

$$\begin{cases} w_t = w_{xx} + f'(\bar{y})w + \dfrac{x}{L} f'(\bar{y})v - \dfrac{x}{L} v' + r(\varepsilon, t, x), \\ w(t, 0) = w(t, L) = 0, \\ w(0, x) = -\dfrac{x}{L} v(0), \end{cases} \tag{2.4}$$

where

$$r(\varepsilon, t, x) = -\varepsilon \bar{y}_\tau + \left(w + \frac{x}{L} v \right)^2 \int_0^1 (1 - s) f'' \left(\bar{y} + s(w + \frac{x}{L} v) \right) ds, \tag{2.5}$$

and the next step is to prove that there exist ε small enough and a pair (v, w) solution of (2.4) such that $w(1/\varepsilon, .)$ belongs to some arbitrary neighborhood of 0 in H_0^1-topology. To achieve this we shall construct an appropriate control function and a Lyapunov functional which stabilizes system (2.4) to 0.

In fact as we shall see the control will be chosen in $H^1(0, 1/\varepsilon)$ and such that $v(0) = 0$.

2.3. Construction of a Lyapunov functional. This is the most technical part of the work. In order to motivate what follows, let us first notice that if the residual term r and the control v were equal to zero then Eq. (2.4) would reduce to

$$w_t = w_{xx} + f'(\bar{y})w,$$
$$w(t,0) = w(t,L) = 0.$$

This suggests to introduce the *one-parameter family of linear operators*

$$A(\tau) = \Delta + f'(\bar{y}(\tau,.))Id, \ \ \tau \in [0,1], \tag{2.6}$$

defined on $H^2(0,L) \cap H_0^1(0,L)$. Let $(e_j(\tau,.))_{j\geq 1}$ be an Hilbertian basis of $L^2(0,L)$ of eigenfunctions of $A(\tau)$, such that for each $j \geq 1$ and each $\tau \in [0,1]$,

$$e_j(\tau,.) \in H_0^1(0,L) \cap C^2([0,L]),$$

and let $(\lambda_j(\tau))_{j\geq 1}$ denote the corresponding eigenvalues. A standard application of the minimax principle (see for instance [16]) shows that these eigenfunctions and eigenvalues are C^1 functions of τ. Moreover for each $\tau \in [0,1]$

$$-\infty < \cdots < \lambda_n(\tau) < \cdots < \lambda_1(\tau) \quad \text{and} \quad \lambda_n(\tau) \underset{n\to+\infty}{\longrightarrow} -\infty.$$

From the continuity of the eigenvalues on $[0,1]$, we can define n as the maximal number of eigenvalues taking at least a nonnegative value as $\tau \in [0,1]$, i.e. there exists $\eta > 0$ such that

$$\forall t \in [0,1/\varepsilon] \quad \forall k > n \quad \lambda_k(\varepsilon t) < -\eta < 0. \tag{2.7}$$

REMARK 2.2. *Note that the integer n can be arbitrarily large. For example if $f(y) = y^3$ and if $y_1'(0) \to +\infty$ then $n \to +\infty$.*

We also set, for any $\tau \in [0,1]$ and $x \in [0,L]$,

$$a(\tau,x) = \frac{x}{L}f'(\bar{y}(\tau,x)) \quad \text{and} \quad b(x) = -\frac{x}{L}.$$

In these notations system (2.4) leads to

$$w_t(t,.) = A(\varepsilon t)w(t,.) + a(\varepsilon t,.)v(t) + b(.)v'(t) + r(\varepsilon,t,.). \tag{2.8}$$

Any solution $w(t,.) \in H^2(0,L) \cap H_0^1(0,L)$ of (2.8) can be expanded as series in the eigenfunctions $e_j(\varepsilon t,.)$, convergent in $H_0^1(0,L)$,

$$w(t,.) = \sum_{j=1}^{\infty} w_j(t)e_j(\varepsilon t,.).$$

In fact the w_j's depend on ε and should be called, for example, w_j^ε. For simplicity we omit the index ε, and we shall also omit the index ε for other functions.

In what follows we are going to move, by means of an appropriate *feedback control*, the n first eigenvalues of the operator A, without moving the others, in order to make all eigenvalues negative. This pole shifting process is the first part of the stabilization procedure, see [17, p. 711].

For any $\tau \in [0,1]$ let $\pi_1(\tau)$ denote the orthogonal projection onto the subspace of $L^2(0,L)$ spanned by $e_1(\tau,.),\dots,e_n(\tau,.)$, and let

$$w^1(t)=\pi_1(\varepsilon t)w(t,.)=\sum_{j=1}^n w_j(t)e_j(\varepsilon t,.). \tag{2.9}$$

It is clear that for any τ the operators $\pi_1(\tau)$ and $A(\tau)$ commute, and moreover for any $y\in L^2(0,L)$ we have

$$\pi_1'(\tau)y=\sum_{j=1}^n \langle y,e_j(\tau,.)\rangle_{L^2(0,L)}\frac{\partial e_j}{\partial\tau}(\tau,.)+\sum_{j=1}^n\left\langle y,\frac{\partial e_j}{\partial\tau}(\tau,.)\right\rangle_{L^2(0,L)}e_j(\tau,.).$$

Hence derivating Eq. (2.9) with respect to t we get

$$\sum_{j=1}^n w_j'(t)e_j(\varepsilon t,.)=\pi_1(\varepsilon t)w_t(t,.)+\varepsilon\sum_{j=1}^n\left\langle w(t,.),\frac{\partial e_j}{\partial\tau}(\varepsilon t,.)\right\rangle_{L^2(0,L)}e_j(\varepsilon t,.).$$

On the other part

$$A(\varepsilon t)w^1(t)=\sum_{j=1}^n\lambda_j(\varepsilon t)w_j(t)e_j(\varepsilon t,.),$$

and thus Eq. (2.8) yields

$$\begin{aligned}\sum_{j=1}^n w_j'(t)e_j(\varepsilon t,.)=&\sum_{j=1}^n\lambda_j(\varepsilon t)w_j(t)e_j(\varepsilon t,.)+\pi_1(\varepsilon t)a(\varepsilon t,.)v(t)\\&+\pi_1(\varepsilon t)b(.)v'(t)+r^1(\varepsilon,t,.),\end{aligned} \tag{2.10}$$

where

$$r^1(\varepsilon,t,.)=\pi_1(\varepsilon t)r(\varepsilon,t,.)+\varepsilon\sum_{j=1}^n\left\langle w,\frac{\partial e_j}{\partial\tau}(\varepsilon t,.)\right\rangle_{L^2(0,L)}e_j(\varepsilon t,.). \tag{2.11}$$

Let us set an upper bound to the residual term r^1. First, it is not difficult to check that there exists a constant C such that if $|v(t)|$ and $\|w(t,.)\|_{L^\infty(0,L)}$ are less than 1 then the inequality

$$\|r(\varepsilon,t,.)\|_{L^\infty(0,L)}\le C(\varepsilon+v(t)^2+\|w(t,.)\|^2_{L^\infty(0,L)})$$

holds, where r is defined by (2.5). Therefore we get easily

$$\|r^1(\varepsilon,t,.)\|_{L^\infty(0,L)}\le C_1(\varepsilon+v(t)^2+\|w(t,.)\|^2_{L^\infty(0,L)}).$$

Moreover since $H^1(0,L)$ is continuously imbedded in $C^0([0,L])$, we can assert that there exists a constant C_2 such that if $|v(t)|$ and $\|w(t,.)\|_{L^\infty(0,L)}$ are less than 1 then

$$\|r^1(\varepsilon,t,.)\|_{L^\infty(0,L)}\le C_2(\varepsilon+v(t)^2+\|w(t,.)\|^2_{H_0^1(0,L)}). \tag{2.12}$$

Now projecting Eq. (2.10) on each $e_i, i=1\dots n$, one comes to

$$w_i'(t)=\lambda_i(\varepsilon t)w_i(t)+a_i(\varepsilon t)v(t)+b_i(\varepsilon t)v'(t)+r_i^1(\varepsilon,t),\quad i=1\dots n, \tag{2.13}$$

where

$$\begin{aligned} r_i^1(\varepsilon,t) &= \langle r^1(\varepsilon,t,.), e_i(\varepsilon t,.)\rangle_{L^2(0,L)}, \\ a_i(\varepsilon t) &= \langle a(\varepsilon t,.), e_i(\varepsilon t,.)\rangle_{L^2(0,L)} = \frac{1}{L}\int_0^L x f'(\bar{y}(\varepsilon t,x)) e_i(\varepsilon t,x)dx, \\ b_i(\varepsilon t) &= \langle b(.), e_i(\varepsilon t,.)\rangle_{L^2(0,L)} = -\frac{1}{L}\int_0^L x e_i(\varepsilon t,x)dx. \end{aligned} \tag{2.14}$$

The n equations (2.13) form a differential system controlled by v, v'. Set

$$\alpha(t) = v'(t), \tag{2.15}$$

and consider now $v(t)$ as a state and $\alpha(t)$ as a control. Then the former finite dimensional system may be rewritten as

$$\begin{cases} v' = \alpha, \\ w_1' = \lambda_1 w_1 + a_1 v + b_1\alpha + r_1^1, \\ \quad\vdots \\ w_n' = \lambda_n w_n + a_n v + b_n \alpha + r_n^1. \end{cases} \tag{2.16}$$

If we introduce the matrix notations

$$X_1(t) = \begin{pmatrix} v(t) \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix}, \ R_1(\varepsilon,t) = \begin{pmatrix} 0 \\ r_1^1(\varepsilon,t) \\ \vdots \\ r_n^1(\varepsilon,t) \end{pmatrix},$$

$$A_1(\tau) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1(\tau) & \lambda_1(\tau) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n(\tau) & 0 & \cdots & \lambda_n(\tau) \end{pmatrix}, \ B_1(\tau) = \begin{pmatrix} 1 \\ b_1(\tau) \\ \vdots \\ b_n(\tau) \end{pmatrix},$$

then equations (2.16) yield the *finite dimensional linear control system*

$$X_1'(t) = A_1(\varepsilon t)X_1(t) + B_1(\varepsilon t)\alpha(t) + R_1(\varepsilon,t). \tag{2.17}$$

Let us now prove the following lemma.

LEMMA 2.3. *For each $\tau \in [0,1]$ the pair $(A_1(\tau), B_1(\tau))$ satisfies the Kalman condition, i.e.*

$$\textit{rank}\left(B_1(\tau), A_1(\tau)B_1(\tau), \ldots, A_1(\tau)^{n-1}B_1(\tau)\right) = n. \tag{2.18}$$

Proof. Let $\tau \in [0,1]$ be fixed. We compute directly

$$\det\left(B_1, A_1B_1, \ldots, A_1^{n-1}B_1\right) = \prod_{j=1}^{n}(a_j + \lambda_j b_j)\ \mathrm{VdM}(\lambda_1, \ldots, \lambda_n), \tag{2.19}$$

where $\mathrm{VdM}(\lambda_1, \ldots, \lambda_n)$ is a Van der Monde determinant, and thus is never equal to zero since the $\lambda_i(\tau), i = 1 \ldots n$, are distinct, for any $\tau \in [0,1]$. On the other part, using the fact that each $e_j(\tau, .)$ is an eigenfunction of $A(\tau)$ and belongs to $H_0^1(0,L)$, we compute

$$\begin{aligned} a_j(\tau) + \lambda_j(\tau) b_j(\tau) &= \frac{1}{L}\int_0^L x\left(f'(\bar{y}(\tau,x)) e_j(\tau,x) - \lambda_j(\tau) e_j(\tau,x)\right) dx \\ &= -\frac{1}{L}\int_0^L x \frac{\partial^2 e_j}{\partial x^2}(\tau,x) dx \\ &= -\frac{\partial e_j}{\partial x}(\tau, L). \end{aligned}$$

But this quantity is never equal to zero since $e_j(\tau, L) = 0$ and $e_j(\tau, .)$ is a nontrivial solution of a linear second-order scalar differential equation. Therefore the determinant (2.19) is never equal to zero and we are done. □

It is a standard fact that the Kalman condition (2.18) implies a *pole shifting* result, and we get the following corollary, see [13].

COROLLARY 2.4. *For each $\tau \in [0,1]$ there exist scalars $k_0(\tau), \ldots, k_n(\tau)$ such that, if we denote*

$$K_1(\tau) = (k_0(\tau), \ldots, k_n(\tau)),$$

then the matrix $A_1(\tau) + B_1(\tau)K_1(\tau)$ admits -1 as an eigenvalue with order $n+1$.

Moreover there exists a C^1 application $\tau \mapsto P(\tau)$ on $[0,1]$, where $P(\tau)$ is a $(n+1) \times (n+1)$ symmetric positive definite matrix, such that the identity

$$P(\tau)\left(A_1(\tau) + B_1(\tau)K_1(\tau)\right) + {}^t\!\left(A_1(\tau) + B_1(\tau)K_1(\tau)\right) P(\tau) = -I \tag{2.20}$$

holds for any $\tau \in [0,1]$.

We are now able to construct a control Lyapunov functional in order to stabilize system (2.8). Let $c > 0$ to be chosen later. For any $t \in [0, 1/\varepsilon]$, $v \in \mathbb{R}$ and $w \in H^2(0,L) \cap H_0^1(0,L)$ we set

$$V(t,v,w) = c\, {}^tX_1(t) P(\varepsilon t) X_1(t) - \frac{1}{2}\langle w, A(\varepsilon t) w\rangle_{L^2(0,L)}, \tag{2.21}$$

where $X_1(t)$ denotes the matrix vector in $\mathbb{R}^{n+1}$

$$X_1(t) = \begin{pmatrix} v \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix}$$

and

$$w_i(t) = \langle w, e_i(\varepsilon t, .)\rangle_{L^2(0,L)}.$$

In particular we have

$$V(t,v,w) = c\, {}^tX_1(t) P(\varepsilon t) X_1(t) - \frac{1}{2}\sum_{j=1}^{\infty} \lambda_j(\varepsilon t) w_j(t)^2. \tag{2.22}$$

In what follows we will repeatedly use the equivalence of norms in finite dimension. The following notation will thus happen to be useful.

Notation. *Let Λ be a set and let $\Delta = \{(\varepsilon, t) \ / \ 0 < \varepsilon \leq 1, \ 0 \leq t \leq 1/\varepsilon\}$. Let F_1, F_2 be two real functions defined on $\Delta \times \Lambda$. The notation $F_1 \lesssim F_2$ means that $F_2 \geq 0$ and that there exists a positive constant C such that*

$$F_1(\varepsilon, t, \lambda) \leq C F_2(\varepsilon, t, \lambda) \quad \forall (\varepsilon, t) \in \Delta, \quad \forall \lambda \in \Lambda.$$

We say that $F_1 \sim F_2$ if both $F_1 \lesssim F_2$ and $F_2 \lesssim F_1$. Moreover, if F_3 is a real function defined on $\Delta \times \Lambda$, and if $\theta \in [0, +\infty)$, the notation $F_1 \lesssim F_2$ for $F_3 \leq \theta$ means that $F_2 \geq 0$ and that there exists a positive constant C such that

$$\forall (\varepsilon, t) \in \Delta \quad \forall \lambda \in \Lambda \quad (F_3(\varepsilon, t, \lambda) \leq \theta) \Rightarrow (F_1(\varepsilon, t, \lambda) \leq C F_2(\varepsilon, t, \lambda)).$$

For simplicity, when the set Λ is clear from the context it will not be given explicitly.

Let $\| \ \|_2$ denote the euclidian norm in $\mathrm{I\!R}^{n+1}$. Since $P(\tau)$ is symmetric positive definite, we can write (with $\Lambda = \mathrm{I\!R} \times (\mathrm{H}^2(0,\mathrm{L}) \cap \mathrm{H}^1_0(0,\mathrm{L}))$)

$${}^t X_1(t) P(\varepsilon t) X_1(t) \sim \|X_1(t)\|_2^2 = v^2 + \sum_{j=1}^{n} w_j(t)^2.$$

From (2.7) we know that, except the n first ones, the eigenvalues of A are all negative, less than $-\eta < 0$. By continuity the n first eigenvalues are bounded as $\tau \in [0,1]$ and thus we can assert that if c is large enough in the definition of V then

$$V(t, v, w) \sim \|X_1(t)\|_2^2 - \sum_{j=n+1}^{\infty} \lambda_j(\varepsilon t) w_j(t)^2, \tag{2.23}$$

where $t \in [0, 1/\varepsilon]$. In particular $V(t, ., .)$ is positive definite. Let us further prove the following lemma.

LEMMA 2.5. *The equivalence*

$$V(t, v, w) \sim v^2 + \|w\|^2_{H^1_0(0,L)} \tag{2.24}$$

holds with $\Lambda = \{(v, w) \ / \ v \in \mathrm{I\!R}, w \in H^2(0, L) \cap H^1_0(0, L)\}$, where $\|w\|_{H^1_0(0,L)} = \|w_x\|_{L^2(0,L)}$. Moreover

$$V(t, v, w) \lesssim \|X_1(t)\|_2^2 + \|Aw\|^2_{L^2(0,L)}. \tag{2.25}$$

Proof. Any $w \in H^2(0, L) \cap H^1_0(0, L)$ can be expanded as series in the eigenfunctions of $A(\varepsilon t)$, convergent in $H^1_0(0, L)$,

$$w(.) = \sum_i w_i(t) e_i(\varepsilon t, .).$$

Hence

$$\|w\|^2_{H^1_0(0,L)} = \sum_{i,j} w_i(t) w_j(t) \int_0^L e_{ix}(\varepsilon t, x) e_{jx}(\varepsilon t, x) dx.$$

Integrating by parts, and using the definition of e_j, we compute

$$\int_0^L e_{ix}(\varepsilon t,x)e_{jx}(\varepsilon t,x)dx = \int_0^L f'(\bar{y}(\varepsilon t,x))e_i(\varepsilon t,x)e_j(\varepsilon t,x)dx - \lambda_j\delta_{ij},$$

and thus

$$\|w\|^2_{H^1_0(0,L)} = \int_0^L f'(\bar{y}(\varepsilon t,x))w(x)^2dx - \sum_{j=1}^{\infty}\lambda_j(\varepsilon t)w_j(t)^2.$$

Therefore, since $f'(\bar{y})$ is bounded on $[0,1/\varepsilon]\times[0,L]$, uniformly in $\varepsilon\in(0,1]$,

$$\|w\|^2_{H^1_0(0,L)} \lesssim \|w\|^2_{L^2(0,L)} - \sum_{j=n+1}^{\infty}\lambda_j(\varepsilon t)w_j(t)^2 \lesssim V(t,v,w).$$

Conversely, we have

$$\begin{aligned}-\sum_{j=n+1}^{\infty}\lambda_j(\varepsilon t)w_j(t)^2 &= \|w\|^2_{H^1_0(0,L)} - \int_0^L f'(\bar{y}(\varepsilon t,x))w(x)^2dx + \sum_{j=1}^{n}\lambda_j(\varepsilon t)w_j(t)^2\\ &\lesssim \|w\|^2_{H^1_0(0,L)} + \|w\|^2_{L^2(0,L)}\\ &\lesssim \|w\|^2_{H^1_0(0,L)},\end{aligned}$$

and we conclude easily using (2.23) that (2.24) holds.

On the other part, notice that

$$\|w\|^2_{H^1_0(0,L)} \lesssim \sum_{j=n+1}^{\infty}|\lambda_j|w_j^2 + \sum_{j=1}^{n}w_j^2.$$

Therefore, using (2.7),

$$\|w\|^2_{H^1_0(0,L)} \lesssim \sum_{j=1}^{n}w_j^2 + \sum_{j=1}^{\infty}\lambda_j^2w_j^2 = \sum_{j=1}^{n}w_j^2 + \|Aw\|^2_{L^2(0,L)},$$

and hence the estimate (2.25) follows. □

Let now $(v(t),w(t,.))$ denote a solution of (2.8) in which we choose the control in the feedback form suggested from Corollary 2.4, namely

$$\alpha(t) = K_1(\varepsilon t)X_1(t),$$

such that $v(0)=0$ and $w(0,.)=0$, i.e. $(v(t),w(t,.))$ satisfies

$$w_t = Aw + av + bK_1(\varepsilon t)X_1(t) + r, \quad v'(t) = K_1(\varepsilon t)X_1(t), \tag{2.26}$$

$$v(0)=0, \quad w(0,x)=0. \tag{2.27}$$

We set

$$V_1(t) = V(t,v(t),w(t,.)) = c\,{}^tX_1(t)P(\varepsilon t)X_1(t) - \frac{1}{2}\langle w(t,.),A(\varepsilon t)w(t,.)\rangle_{L^2(0,L)}.$$

Let us compute $V_1'(t)$ and state a differential inequality satisfied by V_1. We have

$$\begin{aligned} V_1'(t) \;=\; & c\left({}^tX_1'(t)P(\varepsilon t)X_1(t) + {}^tX_1(t)P(\varepsilon t)X_1'(t)\right) + \varepsilon c\, {}^tX_1(t)P'(\varepsilon t)X_1(t) \\ & - \frac{1}{2}\langle w_t(t,.), A(\varepsilon t)w(t,.)\rangle_{L^2(0,L)} - \frac{1}{2}\langle w(t,.), A(\varepsilon t)w_t(t,.)\rangle_{L^2(0,L)} \\ & - \frac{1}{2}\varepsilon\langle w(t,.), A'(\varepsilon t)w(t,.)\rangle_{L^2(0,L)}. \end{aligned} \tag{2.28}$$

Notice the following facts:

- From (2.17) and (2.20) we infer

$$ {}^tX_1'PX_1 + {}^tX_1PX_1' = -\|X_1\|_2^2 + {}^tR_1PX_1 + {}^tX_1PR_1. $$

- The operator A is selfadjoint in L^2, hence

$$ \langle w, Aw_t\rangle_{L^2(0,L)} = \langle Aw, w_t\rangle_{L^2(0,L)}. $$

- Eq. (2.26) leads to

$$\begin{aligned} \langle Aw, w_t\rangle_{L^2(0,L)} &= \langle Aw, Aw + av + bK_1X_1 + r\rangle_{L^2(0,L)} \\ &= \|Aw\|_{L^2(0,L)} + \langle Aw, a\rangle_{L^2(0,L)}v + \langle Aw, b\rangle_{L^2(0,L)}K_1X_1 + \langle Aw, r\rangle_{L^2(0,L)}. \end{aligned}$$

- From the definition of $A(\tau)$, we have

$$ A'(\tau) = f''(\bar{y}(\tau,.))\bar{y}_\tau(\tau,.)Id, $$

  and thus

$$ \langle w(t,.), A'(\varepsilon t)w(t,.)\rangle_{L^2(0,L)} = \langle w(t,.), f''(\bar{y}(\varepsilon t,.))\bar{y}_\tau(\varepsilon t,.)w(t,.)\rangle_{L^2(0,L)}. $$

Therefore, turning back to Eq. (2.28),

$$\begin{aligned} V_1' = & - c\,\|X_1\|_2^2 - \|Aw\|_{L^2(0,L)}^2 - \langle Aw, a\rangle_{L^2(0,L)}v - \langle Aw, b\rangle_{L^2(0,L)}K_1X_1 \\ & - \langle Aw, r\rangle_{L^2(0,L)} + \varepsilon c\, {}^tX_1P'X_1 + c\left({}^tR_1PX_1 + {}^tX_1PR_1\right) - \frac{1}{2}\varepsilon\langle w, f''(\bar{y})\bar{y}_\tau w\rangle_{L^2(0,L)}. \end{aligned} \tag{2.29}$$

Let us set an upper bound to the terms of second line of Eq. (2.29):

- From Corollary 2.4, the application $\tau \mapsto P'(\tau)$ is bounded on $[0,1]$, hence

$$ \left|\varepsilon c {}^tX_1P'X_1\right| \lesssim \varepsilon\|X_1\|_2^2 \lesssim \varepsilon V_1. $$

- Inequality (2.12) yields, for $|v(t)| + \|w(t,.)\|_{L^\infty(0,L)} \leq 1$,

$$ \|R_1(\varepsilon,t)\|_{L^\infty(0,L)} \lesssim \varepsilon + v(t)^2 + \|w(t,.)\|_{H_0^1(0,L)}^2, $$

  and thus, still for for $|v(t)| + \|w(t,.)\|_{L^\infty(0,L)} \leq 1$,

$$\begin{aligned} {}^tR_1PX_1 + {}^tX_1PR_1 &\lesssim \|X_1\|_2\left(\varepsilon + v^2 + \|w\|_{H_0^1(0,L)}^2\right) \\ &\lesssim \sqrt{V_1}\,(\varepsilon + V_1) = \varepsilon\sqrt{V_1} + V_1^{3/2}. \end{aligned}$$

- Since f is of class C^2 we can assert

$$\varepsilon \left| \frac{1}{2} \langle w, f''(\bar{y}) \bar{y}_\tau w \rangle_{L^2(0,L)} \right| \lesssim \varepsilon \|w\|^2_{L^2(0,L)} \lesssim \varepsilon \|w\|^2_{H^1_0(0,L)} \lesssim \varepsilon V_1.$$

- The term $\langle Aw, r \rangle_{L^2(0,L)}$ is the most difficult to handle. Using (2.5), write

$$\begin{aligned} \langle Aw, r \rangle_{L^2(0,L)} &= \left\langle Aw, -\varepsilon \bar{y}_\tau + \left(w + \frac{x}{L} v\right)^2 \int_0^1 (1-s) f''\left(\bar{y} + s(w + \frac{x}{L} v)\right) ds \right\rangle_{L^2(0,L)} \\ &= \left\langle Aw, \left(w + \frac{x}{L} v\right)^2 \int_0^1 (1-s) f''\left(\bar{y} + s(w + \frac{x}{L} v)\right) ds \right\rangle_{L^2(0,L)} \\ &\quad - \varepsilon \langle Aw, \bar{y}_\tau \rangle_{L^2(0,L)}. \end{aligned}$$

First, we clearly have

$$|\varepsilon \langle Aw, \bar{y}_\tau \rangle_{L^2(0,L)}| \lesssim \varepsilon \|Aw\|_{L^2(0,L)}.$$

Let us now deal with the integral term. First of all, using the continuous embedding of H^1_0 in C^0, we estimate

$$\begin{aligned} \left\| \left(w(t,x) + \frac{x}{L} v(t)\right)^2 \right\|_{L^\infty(0,L)} &\lesssim \|w(t,.)\|^2_{L^\infty(0,L)} + v(t)^2 \\ &\lesssim \|w(t,.)\|^2_{H^1_0(0,L)} + \|X_1(t)\|_2^2 \\ &\lesssim V_1(t) \end{aligned}$$

since $V_1 \sim \|w\|^2_{H^1_0(0,L)} + \|X_1\|_2^2$. For $|v(t)| + \|w(t,.)\|_{L^\infty(0,L)} \leq 1$, one has

$$\left\langle Aw, \left(w + \frac{x}{L} v\right)^2 \int_0^1 (1-s) f''\left(\bar{y} + s(w + \frac{x}{L} v)\right) ds \right\rangle_{L^2(0,L)} \lesssim \|Aw\|_{L^2(0,L)} V_1,$$

and we arrive at the estimate

$$\left| \langle Aw, r \rangle_{L^2(0,L)} \right| \lesssim \varepsilon \|Aw\|_{L^2(0,L)} + \|Aw\|_{L^2(0,L)} V_1,$$

for $|v(t)| + \|w(t,.)\|_{L^\infty(0,L)} \leq 1$.

Let us also estimate the terms of the principal part of Eq. (2.29). We clearly have

$$\left| \langle Aw, a \rangle_{L^2(0,L)} v \right| \leq \frac{1}{4} \|Aw\|^2_{L^2(0,L)} + \|a\|^2_{L^2(0,L)} \|X_1\|_2^2,$$

and

$$\left| \langle Aw, b \rangle_{L^2(0,L)} K_1 X_1 \right| \leq \frac{1}{4} \|Aw\|^2_{L^2(0,L)} + M \|X_1\|_2^2,$$

where

$$M = \|b\|_{L^2(0,L)} \max \left\{ \sum_{i=0}^n k_i(\tau)^2 / \tau \in [0,1] \right\}.$$

Hence, concerning the principal part of Eq. (2.29), we first get

$$\begin{aligned} &- c\|X_1\|_2^2 - \|Aw\|^2_{L^2(0,L)} - \langle Aw, a \rangle_{L^2(0,L)} v - \langle Aw, b \rangle_{L^2(0,L)} K_1 X_1 \\ &\leq -c_1 \|X_1\|_2^2 - \frac{1}{2} \|Aw\|^2_{L^2(0,L)}, \end{aligned}$$

where $c_1 = c - \|a\|^2_{L^2(0,L)} - M$. We choose c so that $c_1 > 0$.

The previous estimates and Eq. (2.29) now yield, for $|v(t)| + \|w(t,.)\|_{L^\infty(0,L)} \leq 1$,

$$V_1' + \|X_1\|_2^2 + \|Aw\|^2_{L^2(0,L)} \lesssim \varepsilon\sqrt{V_1} + V_1^{3/2} + \varepsilon\|Aw\|_{L^2(0,L)} + V_1\|Aw\|_{L^2(0,L)}. \quad (2.30)$$

Note that, for every $\theta \in (0, +\infty)$,

$$\begin{aligned}
\varepsilon\sqrt{V_1} &\leq \frac{\theta}{2}V_1 + \frac{1}{2\theta}\epsilon^2,\\
\varepsilon\|Aw\|_{L^2(0,L)} &\leq \frac{\theta}{2}\|Aw\|^2_{L^2(0,L)} + \frac{1}{2\theta}\varepsilon^2,\\
V_1\|Aw\|_{L^2(0,L)} &\leq \frac{\theta}{2}\|Aw\|^2_{L^2(0,L)} + \frac{1}{2\theta}V_1^2.
\end{aligned}$$

Hence, taking $\theta > 0$ small enough, we get, using (2.25) and (2.30), the existence of $\sigma > 0$ and of $\rho \in (0, \sigma]$ such that, for every $\varepsilon \in (0, 1]$ and for every $t \in [0, 1/\varepsilon]$ such that $V_1(t) \leq \rho$,

$$V_1'(t) \leq \sigma\varepsilon^2.$$

Hence, since $V_1(0) = 0$, we get, if $\varepsilon \in (0, \rho/\sigma]$, that

$$V_1(t) \leq \sigma\varepsilon, \quad \forall t \in [0, 1/\varepsilon],$$

and in particular

$$V_1\left(\frac{1}{\varepsilon}\right) \leq \sigma\varepsilon.$$

Coming back to definitions (2.1) and (2.3), we have proved

$$\left\| y\left(\frac{1}{\varepsilon}, .\right) - y_1(.)\right\|_{H^1(0,L)} \leq \gamma\varepsilon, \quad (2.31)$$

where $y_1(.) = \bar{y}(1, .)$ is the final target and γ is a positive constant which does not depend on $\varepsilon \in (0, \rho/\sigma]$. This concludes the third step, and thus the proof of the stabilization part of Th. 1.2 (see Remark 1.3).

2.4. End of the proof. The last step consists in solving a local exact controllability result: from the previous section $y(\frac{1}{\varepsilon}, .)$ belongs to an arbitrarily small neighborhood of $y_1(.)$ in H^1-topology if ε is small enough, and our aim is now to construct a trajectory $q(t, x)$ solution of the control system steering $y(\frac{1}{\varepsilon}, .)$ to $y_1(.)$ in some time $T > 0$ (for instance $T = 1$), i.e.

$$\begin{cases}
q_t = q_{xx} + f(q),\\
q(t, 0) = 0, \ q(t, L) = u(t),\\
q(0, x) = y\left(\dfrac{1}{\varepsilon}, x\right), \ q(T, x) = y_1(x).
\end{cases}$$

Existence of such a solution q is given by [11, Th. 3.3]. Actually in [11] the function f is assumed to be globally Lipschitzian, but the local result we need here readily follows from the proofs and the estimates contained in this paper. Indeed, let $T>0$ and let $\tilde{f}$ be a globally Lipschitzian mapping such that

$$\tilde{f}(s)=f(s), \quad \forall s\in[-\|y_1\|_{L^\infty}-1,\|y_1\|_{L^\infty}+1]. \tag{2.32}$$

From the proof of [11, Th. 3.3], we get the existence of $\mu>0$ such that there exists $z\in Y_T$ satisfying

$$\begin{cases} z_t=z_{xx}+\tilde{f}(z+y_1)-\tilde{f}(y_1),\\ z(t,0)=0,\\ z(0,x)=y\left(\dfrac{1}{\varepsilon},x\right)-y_1(x),\ z(T,x)=0\end{cases}$$

and the estimate

$$\|z\|_{Y_T}\le\mu\left\|y\left(\frac{1}{\varepsilon},\cdot\right)-y_1(\cdot)\right\|_{H^1(0,L)}, \tag{2.33}$$

which leads, with $q=z+\tilde{y}_1$, to

$$\begin{cases} q_t=q_{xx}+\tilde{f}(q),\\ q(t,0)=0,\\ q(0,x)=y\left(\dfrac{1}{\varepsilon},x\right),\ q(T,x)=y_1(x)\end{cases}$$

and

$$\|q-\tilde{y}_1\|_{Y_T}\le\mu\left\|y\left(\frac{1}{\varepsilon},\cdot\right)-y_1(\cdot)\right\|_{H^1(0,L)}, \tag{2.34}$$

where $\tilde{y}_1(t,x):=y_1(x)$. From (2.33) and (2.34), we get

$$\|q-\tilde{y}_1\|_{L^\infty((0,T)\times(0,L))}\le 1 \tag{2.35}$$

for $\|y(1/\varepsilon,\cdot)-y_1(\cdot)\|_{H^1(0,L)}$ small enough. From (2.32) and (2.35), we infer that $\tilde{f}(q)=f(q)$, which ends the proof.

3. Controllability versus connectedness. Let us first give some sufficient conditions ensuring connectedness of $\mathcal{S}$.

PROPOSITION 3.1. *In each of the following cases the set of steady-states $\mathcal{S}$ is connected:*

- *The function F defined as*

$$F(y)=\int_0^y f(s)ds$$

  satisfies the asymptotic condition

$$F(y)\underset{|y|\to+\infty}{\longrightarrow}+\infty.$$

- *For any $\alpha>0$ the indefinite integral*

$$\int\frac{dy}{\sqrt{\alpha-F(y)}}$$

  diverges in $-\infty$ and in $+\infty$ (if it makes sense).

- *The function f is odd, i.e. for any $y \in \mathbb{R}$*

$$f(-y) = -f(y).$$

REMARK 3.2. *Notice that, contrarily to the two first cases of the proposition, in the third case blow-up phenomena may occur, nevertheless the set of steady-states is connected.*

On the other part we have the following result.

PROPOSITION 3.3. *If y_0 and y_1 belong to distinct connected components of $\mathcal{S}$, then it is not possible to move either from y_0 to y_1, or from y_1 to y_0, whatever the control $u \in L^2(0,T)$ and the time T are.*

REMARK 3.4. *If y_0 and y_1 are both periodic then they are in the same connected component.*

In order to prove these two propositions, let us first notice some general facts about the maximal solutions of the scalar differential equation

$$y''(x) + f(y(x)) = 0, \; y(0) = 0. \tag{3.1}$$

LEMMA 3.5.

- *Any solution of (3.1) satisfies on its maximal interval of definition the conservation law*

$$y'(x)^2 + 2F(y(x)) = y'(0)^2. \tag{3.2}$$

- *Any solution of (3.1) such that y' vanishes at least at two distinct points is actually periodic.*
- *The phase portrait in the plane (y, y') of the associated differential system*

$$y' = z, \; z' = -f(y),$$

is symmetric with respect to the y-axis, and moreover all singular points of the system are located on this axis.

The proof of these facts is obvious. Now the key lemma to prove Prop. 3.1 and 3.3 is the following.

LEMMA 3.6. *Let y_0 and y_1 two steady-states, extended on their maximal interval of definition as solutions of (3.1), belonging to distinct connected components of $\mathcal{S}$, such that $y_0'(0) < y_1'(0)$. Then there exists $l \in (0, L]$ and $\bar{y} \in C^2([0,l))$ solution of (3.1) such that either*

$$\bar{y}(x) \underset{x \to l}{\longrightarrow} +\infty, \tag{3.3}$$

or

$$\bar{y}(x) \underset{x \to l}{\longrightarrow} -\infty. \tag{3.4}$$

In the first case we have moreover, see Fig. 3.1,

1. *$y_0(x) < \bar{y}(x)$ for any $x \in [0, l)$,*
2. *$y_0'(0) < \bar{y}'(0) < y_1'(0)$, $|y_1'(0)| < |\bar{y}'(0)|$, and $\bar{y}'(0) < 0$,*
3. *$\#\{x \in [0, l) \;/\; \bar{y}(x) = y_1(x)\} = 1$,*
4. *y_0 is not periodic, and $y_1(x)$ does not tend to $-\infty$ as x tends to b, where (a, b) denotes the maximal interval of definition of y_1.*

In the second case we have the symmetric situation, see Fig. 3.1,

1. $y_1(x) > \bar{y}(x)$ *for any* $x \in [0, l)$,
2. $y_0'(0) < \bar{y}'(0) < y_1'(0)$, $|y_0'(0)| < |\bar{y}'(0)|$, *and* $\bar{y}'(0) > 0$,
3. $\#\{x \in [0, l) \ / \ \bar{y}(x) = y_0(x)\} = 1$,
4. y_1 *is not periodic, and* $y_0(x)$ *does not tend to* $+\infty$ *as* x *tends to* b, *where* (a, b) *denotes the maximal interval of definition of* y_0.

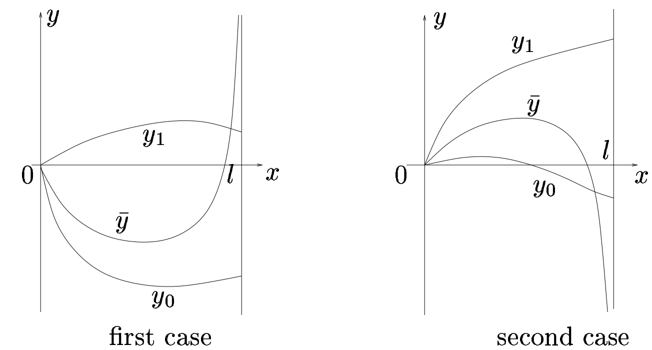

FIG. 3.1. *Existence of an explosive solution.*

Proof. [Proof of Lemma 3.6] Clearly one of the two cases (3.3) and (3.4) occurs, with moreover $y_0'(0) < \bar{y}'(0) < y_1'(0)$. Assume we are in the first case, and let us first prove the second of the four properties claimed in that case. To proceed we have to distinguish between three possibilities:

- First case: y_1 is monotonic on $[0, l]$. In this case the conservation law (3.2) immediately implies that $|y_1'(0)| < |\bar{y}'(0)|$.
- Second case: y_1 is not monotonic on $[0, l]$, and is not periodic on its maximal interval (a, b). That is, y_1' vanishes exactly once on (a, b). The only non obvious case occurs when $y_1'(0) > 0$. But then, since the phase portrait is symmetric with respect to the y-axis, either $y_1(x)$ tends to $-\infty$ as x tends to a and b, or $y_1(x)$ tends to a finite limit which corresponds to a singular point on the phase portrait, see Fig. 3.2. In both cases it is clear on the phase portrait that $\bar{y}(x)$ may tend to $+\infty$ as x tends to l only if $|y_1'(0)| < |\bar{y}'(0)|$.

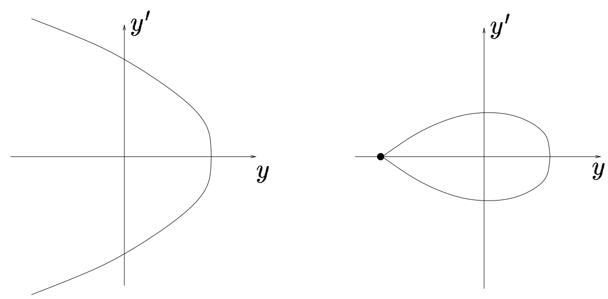

FIG. 3.2. *Behavior of* $(y_1(x), y_1'(x))$ *in the phase plane.*

- Third case: y_1 is periodic (i.e. y_1' vanishes at least two times). Again in this case the phase portrait implies immediately the desired inequality.

Before proving that $\bar{y}'(0) < 0$, let us prove the third point. The only non obvious case occurs when y_1 is periodic and $\bar{y}$ is not monotonic on $[0, l)$. Notice that $\bar{y}'$ vanishes only once (if not $\bar{y}$ would be periodic), and thus it decreases on an interval $[0, x_0]$

and increases on $[x_0, l)$. Now on the one part the function y_1 cannot intersect $\bar{y}$ on the interval $[0, x_0]$, for this would contradict the conservation law (3.2). On the other part if y_1 would intersect $\bar{y}$ more than once on the interval $[x_0, l)$, then there would be at least three intersections, and again this leads to a contradiction with (3.2). This proves the third point.

Now the inequality $\bar{y}(0) < 0$ is an obvious consequence of (3.2).

Let us now prove that $y_0 < \bar{y}$ on $[0, l)$. The same reasoning as above shows that $\bar{y}$ intersects y_0 at most once (notice that y_0' cannot vanish more than once). But such an intersection would contradict the fact that $\bar{y}(x)$ tends to $+\infty$ as x tends to l.

Finally, the last point of the lemma is proved by observing the phase portrait. □

Proof. [Proof of Prop. 3.3.] Prop. 3.3 follows from Lemma 3.6. Indeed, let us assume for example that we are in the first case of the lemma. Then for any $T > 0$ and $u \in L^2([0, T])$ the solution y of the control system (1.4) satisfies, as long as defined, the inequality

$$y(t, x) \leq \bar{y}(x),$$

see [4] for this application of the classical maximum principle to similar control problems. In particular $y(T, .) \neq y_1(.)$. □

Finally let us prove Prop. 3.1. The only difficult case is to prove that if f is odd then the set $\mathcal{S}$ is connected. In this case the conservation law (3.2) implies that the phase portrait is symmetric with respect to the y-axis and the y'-axis. As a consequence any solution of (3.1), such that y' vanishes at least once, is necessarily periodic.

Now from Lemma 3.6 we know that if y_0 and y_1 are not in the same connected component, then there exists an explosive solution $\bar{y}$ of (3.1), such that $\bar{y}'$ vanishes at least once. Hence $\bar{y}$ must be periodic and we get a contradiction.

EXAMPLE 3.7. *An example where the situation of Prop. 3.3 and Lemma 3.6 occurs is given by*

$$f(y) = y - y^2 - y^3.$$

The graph of y_0, y_1, and an explosive y, and the phase portrait are drawn on Fig. 3.3.

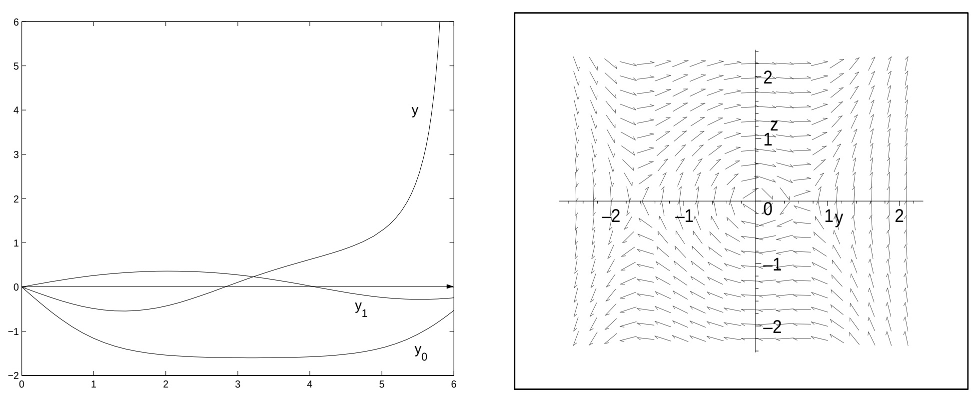

FIG. 3.3. *An example.*

4. Numerical simulations. In this section we present numerical simulations with Matlab for the nonlinear function $f(y) = y^3$. Let $L = 1$; the set $\mathcal{S}$ of steady-states consists of all solutions of class C^2 on $[0,1]$ such that

$$y''(x) + y(x)^3 = 0, \; y(0) = 0 \tag{4.1}$$

It follows from Prop. 3.1 that this set is connected. Let y_0 be identically zero, and let y_1 denote the solution of (4.1) vanishing at $0, 1/2$ and 1, and having no other zero on $[0,1]$, see Fig. 4.1.

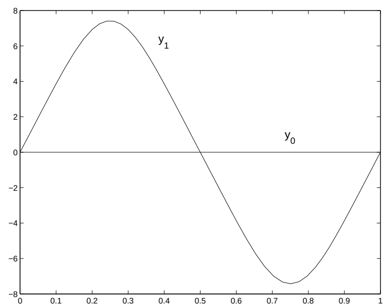

FIG. 4.1. *Definition of the steady-states y_0 and y_1.*

For all $\tau \in [0,1]$ we define the function $\bar{y}(\tau, .)$ on $[0,1]$ as the solution of (4.1) such that

$$\frac{\partial \bar{y}}{\partial x}(\tau, 0) = \tau y_1'(0),$$

and we set $\bar{u}(\tau) = \bar{y}(\tau, 1)$. We then introduce on $H^2(0,1) \cap H_0^1(0,1)$ the one-parameter family of linear operators

$$A(\tau) = \Delta + 3\bar{y}(\tau, .)^2 Id, \;\; \tau \in [0,1].$$

For $\tau = 0$ we have $A(0) = \Delta$, and the eigenvalues and eigenvectors write

$$\lambda_i(0) = -i^2\pi^2, \;\; e_i(0, x) = \sqrt{2} \sin k\pi x.$$

Then, solving by homotopy as $\tau \in [0,1]$ boundary value problems, we compute numerically, using a standard finite difference code implemented in Matlab, the first eigenvalues and associated eigenvectors. In the present example numerical experiments show that only the two first eigenvalues may take positive values as $\tau \in [0,1]$. In other words, with the notations of Section 2.3, one has $n = 2$. Then we achieve a pole placement on the finite dimensional system (2.16) by applying a LQR algorithm, see [13]. Notice that the finite dimensional system corresponding to these two first modes is very unstable: numerically one has $\lambda_1(1) \simeq 89.743$ and $\lambda_2(1) \simeq 82.518$.

Results are drawn on Fig. 4.2, for $\varepsilon = 0.05$ and $\varepsilon = 0.001$. Notice that is ε is too large then the solution blows up.

REFERENCES

[1] J. BEBERNES AND D. EBERLY, *Mathematical problems from combustion theory*, Applied Math. Sciences, 83, Springer-Verlag, New York, 1989.

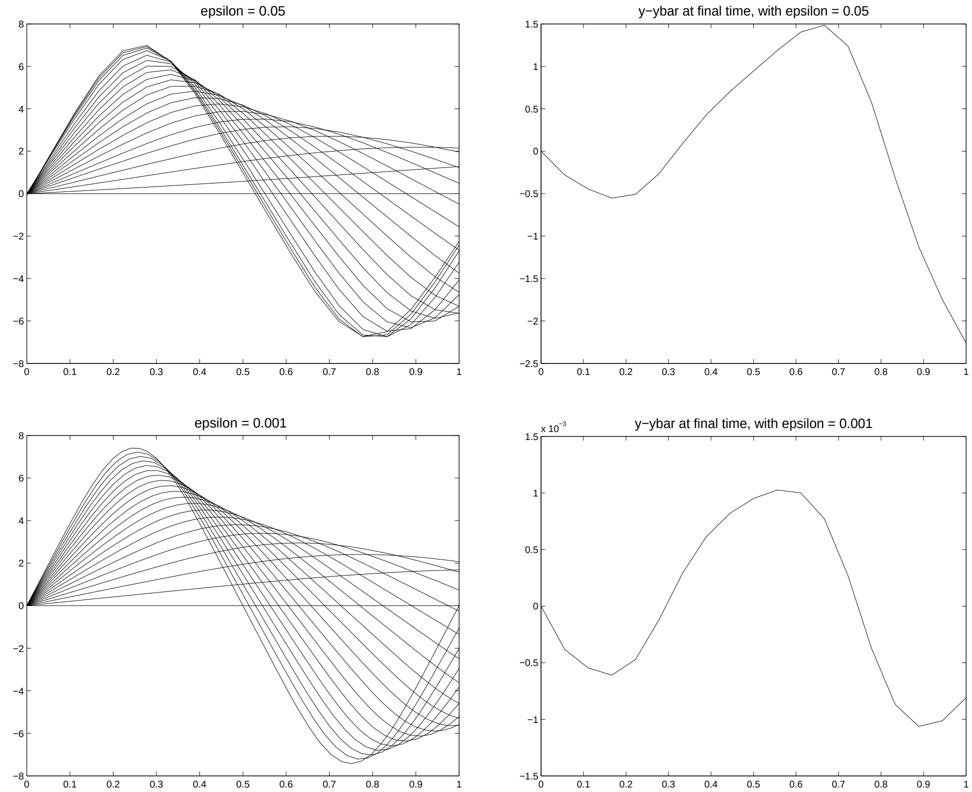

FIG. 4.2. *Simulations results for $y(t,.)$, where $t \in [0, 1/\varepsilon]$.*

[2] T. CAZENAVE AND A. HARAUX, *Introduction aux problèmes d'évolution semi-linéaires*, Math. & Appl., 1, Ellipses, 1990.

[3] J. M. CORON, *Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations*, ESAIM Control Optim. Calc. Var., 8 (2002), pp. 513–554.

[4] J. I. DIAZ, *Obstruction and some approximate controllability results for the Burgers equation and related problems*, Control of partial differential equations and applications, Lecture Notes in Pure and Appl. Math., 174, Dekker, New York, 1996, pp. 63–76.

[5] C. FABRE, J. P. PUEL AND E. ZUAZUA, *Approximate controllability of the semilinear heat equation*, Proc. Roy. Soc. Edinburgh Sect. A, 125 (1995), pp. 31–61.

[6] E. FERNANDEZ-CARA, *Null controllability of the semilinear heat equation*, ESAIM Control Optim. Calc. Var., 2 (1997), pp. 87–103.

[7] E. FERNANDEZ-CARA AND E. ZUAZUA, *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 17, 5 (2000), pp. 583–616.

[8] H. FUJITA, *On the blowing up of solutions to the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, J. Faculty Science, U. of Tokyo, 13 (1966), pp. 109–124.

[9] A. FURSIKOV AND O. YU. IMANUVILOV, *On controllability of certain systems simulating a fluid flow*, Flow Control, IMA, Max D. Gunzburger Editor, Springer-Verlag, 1992, pp. 149–184.

[10] J. HENRY, *Etude de la contrôlabilité de certaines équations paraboliques non linéaires*, Thèse, Paris, 1977.

[11] O. YU. IMANUVILOV, *Controllability of parabolic equations*, Sb. Math., 186, 6 (1995), pp. 879–900.

[12] S. KAPLAN, *On the growth of solutions of quasilinear parabolic equations*, Comm. Pure Appl. Math., 16 (1963), pp. 305–330.

[13] H. K. KHALIL, *Nonlinear Systems*, Macmillan Publishing Company, New York, 1992.

[14] H. A. LEVINE, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$*, Arch. Rat. Mech. Anal., 51 (1973), pp. 371–386.

[15] F. Merle and H. Zaag, *Stability of the blow-up profile for equations of the type* $u_t = \Delta u + |u|^{p-1}u$, Duke Math. J., 86, 1 (1997), pp. 143–195.
[16] M. Reed and B. Simon, *Methods of modern mathematical physics*, Vol. 4, Analysis of operators, Academic Press, 1978.
[17] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions*, SIAM Rev., 20, 4 (1978), pp. 639–739.
[18] H. Zaag, *A remark on the energy blow-up behavior for nonlinear heat equations*, Duke Math. J., 103, 3 (2000), pp. 545–555.