OPTIMAL CONTROL WITH STATE CONSTRAINTS AND THE SPACE SHUTTLE RE-ENTRY PROBLEM

B. BONNARD*, L. FAUBOURG*, G. LAUNAY*, AND E. TRÉLAT†

Abstract. In this article, we initialize the analysis under generic assumptions of the small *time optimal synthesis* for single input systems with *state constraints.* We use geometric methods to evaluate *the small time reachable set* and necessary optimality conditions. Our work is motivated by the *optimal control of the atmospheric arc for the re-entry of a space shuttle*, where the vehicle is subject to constraints on the thermal flux and on the normal acceleration. A *multiple shooting technique* is finally applied to compute the optimal longitudinal arc.

Key words. Optimal control with state constraints, Minimum principle, Control of the atmospheric arc, Multiple shooting techniques

AMS subject classifications. 49K15, 70M20,49M15

1. Introduction.

The objective of this article is to initialize the classification of the local closed loop time optimal control for the single input affine systems: $\dot{q} = X(q) + uY(q)$, where $q \in \mathbb{R}^2$ or $\mathbb{R}^3$, $|u| \leq 1$, with *state constraints*: $c(q) \leq 0$. This analysis is motivated by the optimal control of the *atmospheric arc for the re-entry of the space shuttle* where the cost is the total amount of thermal flux and where the constraints are on the thermal flux and on the normal acceleration, the control being the angle of bank. The system is modeled by an equation of the form:

$$m\ddot{q} = \overrightarrow{K}(q) + \overrightarrow{D}(q) + \overrightarrow{L}(q) \; , \tag{1.1}$$

where m is the vehicle mass, q is the position in an inertial frame, $\overrightarrow{K}$ is the standard *Keplerian force*, $\overrightarrow{D}$ is the *drag force* and $\overrightarrow{L}$ is the *lift force* controlled by the bank angle. The target T is a point in an Earth fixed frame. For reasons explained later, we shall express the dynamics in the Earth fixed frame which is rotating with respect to the inertial frame and consequently we have additional *Coriolis and centripetal forces* and the system is more complicated.

Pioneering necessary optimality conditions concerning the *optimality status of a boundary arc* and *junction or reflection* with the boundary are a consequence of Weierstrass theory (see [3]) applied to Riemannian theory with obstacles. The author compares the length of a reference arc with adjacent arcs to deduce geometric necessary conditions. This approach was generalized by Pontryagin and his co-authors [23] to obtain a *minimum principle* under regularity assumptions on the constraints. A general minimum principle based on Kuhn-Tucker theorem and non smooth analysis is presented in [15]. In these principles, the adjoint vector p dual to the state vector q can suffer discontinuities at contact points with the boundary of the domain or in the boundary. Following the works of [16] and [19], these discontinuities can be specified if we assume that the system is single input and if *the order of the constraint* is constant, the order being by definition the first integer such that the control u appears explicitly in the time derivative of the constraint $t \mapsto c(q(t))$ evaluated along a boundary arc of the system. For the space shuttle re-entry problem, the constraints

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are of order 2 and although the system is in dimension 6, from [4] we can mainly restrict our analysis to a subsystem in dimension 3, called the *longitudinal subsystem*, controlling the altitude, the modulus of the relative speed and the flight path angle.

The evaluation of the small time reachable set and its boundary which can be parametrized by the minimum principle with application to the optimal synthesis was a research program initialized by Sussmann [27] for planar system and pursued in dimension 3 by [18], see also [6] for problems with a target of co-dimension one. The objective of this article is mainly to outline such an analysis in the case of optimal control with state constraints. Here the geometry is different and we must classify up to changes of coordinates triplets (X, Y, c) using the order of the constraints. We make direct evaluation of the reachable set for the constrained system, using normal forms. One of the main problem is to characterize the *optimality status of a boundary arc*. We get under suitable generic assumption necessary and sufficient conditions which are compared with the necessary conditions of the minimum principle.

Our geometric work, completed by the preliminary study by [7], is finally applied to the re-entry problem. A quasi-optimal trajectory, consisting of concatenation of bang and boundary arcs, is given and the exact trajectory corresponding to the boundary conditions imposed by [12], is computed using a *multiple shooting algorithm* and numerical simulations.

2. Generalities.

2.1. Definitions. We consider a smooth (C^∞ or C^ω) single input affine system

$$\dot{q} = X(q) + uY(q) \ , \tag{2.1}$$

with $|u| \leq 1$, $q \in U \subset \mathbb{R}^n$ with state constraint $c(q) \leq 0$ where $c : \mathbb{R}^n \to \mathbb{R}$ and the time optimal problem with fixed boundary conditions: $q(0) = q_0$, $q(T) = q_1$. We denote $(\mathcal{P}_0)$ the minimization problem which can be embedded in the one parameter family of problems $(\mathcal{P}_\alpha)$, where the constraint is $c(q) \leq \alpha$, α being small enough, α is an *homotopy parameter*.

If f is a function and Z a vector field, it acts on f by the *Lie derivative* $Zf = \frac{\partial f}{\partial q}(q)Z(q)$ and if Z_1, Z_2 are two vector fields, the *Lie bracket* is computed with the convention: $[X, Y] = X \circ Y - Y \circ X$, that is in local coordinates:

$$[X, Y](q) = \frac{\partial Y}{\partial q}(q)X(q) - \frac{\partial X}{\partial q}(q)Y(q)$$

The *generic order* of the constraint is the integer m such that:

$$Yc = YXc = \ldots = YX^{m-2}c = 0 \ \text{ and } \ YX^{m-1}c \neq 0$$

A *boundary arc* $t \mapsto \gamma_b(t)$ is an arc (not reduced to a point) of the system contained in $c = 0$. If the order is m, a boundary arc and the associated feedback control can be generically computed by differentiating m times the mapping $t \mapsto c(q(t))$ and solving with respect to u the linear equation:

$$c^{(m)} = X^m c + uYX^{m-1}c = 0$$

A boundary arc is contained in

$$c = \dot{c} = \ldots = c^{(m-1)} = 0 \ ,$$

and the constraint $c = 0$ is called *primary* and the constraints $\dot{c} = \ldots = c^{(m-1)} = 0$ are called *secondary*. We denote

$$u_b = -\frac{X^m c}{YX^{m-1}c}$$

the boundary feedback control.

2.2. Assumptions C**.** Let $t \mapsto \gamma_b(t)$, $t \in [0, T]$ be a boundary arc associated to u_b. We need to introduce the following assumptions:

C_1. $YX^{m-1}c_{|\gamma_b} \neq 0$ where m is the order of the constraint.

C_2. $|u_b| < 1$ for $t \in]0, T[$, i.e. the boundary control is admissible.

C_3. $|u_b| < 1$ for $t \in [0, T]$, i.e. the boundary control is not saturating.

2.3. A minimum principle with state constraints. We recall the necessary conditions due to [16] and [19] that we shall use in our study. Consider the single input affine system (2.1), $\dot{q} = X(q) + uY(q)$, $q \in U \subset \mathbb{R}^n$, $|u| \leq 1$, and a cost to be minimized of the form:

$$J(u) = G(q(T)) \ ,$$

where the transfer time is fixed and q satisfies the constraint

$$c(q) \leq 0 \ ,$$

and the boundary conditions are

$$q(0) = 0, \quad \chi(q(T)) = 0$$

with $\chi = (\chi_1, \ldots, \chi_k)$ and $k \leq n$.

2.3.1. Statement of the necessary optimality conditions. Assume that $t \mapsto q(t)$, $t \in [0, T]$ is a piecewise smooth optimal solution which hits the boundary $c(q) = 0$ at times t_{2i-1}, $i = 1, 2, \ldots, M$ and leaves the boundary at times t_{2i}, $i = 1, 2, \ldots, M$ and moreover assume that along each boundary arc , Assumptions C_1 and C_2 are satisfied at contact or junction times. Define the Hamiltonian by

$$H(q, p, u, \eta) = \langle p, X + uY \rangle + \eta c \ ,$$

where $\langle , \rangle$ is the standard scalar product, p is the adjoint vector and η is the Lagrange multiplier of the constraint. The necessary optimality conditions are:

1. There exists $t \mapsto \eta(t) \geq 0$, a real number η_0 and $\tau \in \mathbb{R}^k$ such that the adjoint vector satisfies:

$$\dot{p} = -p\left(\frac{\partial X}{\partial q} + u\frac{\partial Y}{\partial q}\right) - \eta\frac{\partial c}{\partial q} \quad \text{a.e.} \tag{2.2}$$

$$p(T) = \eta_0\frac{\partial G}{\partial q}(q(T)) + \tau\frac{\partial \chi}{\partial q}(q(T)) \tag{2.3}$$

2. The mapping $t \mapsto \eta(t)$ is continuous along the boundary arc and satisfies:

$$\eta(t)c(q(t)) = 0 \ , \quad \forall t \in [0, T].$$

3. At a contact or a junction time t_i with the boundary, we have

$$H(t_i+) = H(t_i-) \tag{2.4}$$

$$p(t_i+) = p(t_i-) - \nu_i \frac{\partial c}{\partial q}(q(t_i)) \ , \quad \nu_i \geq 0 \tag{2.5}$$

4. The optimal control minimizes almost everywhere the Hamiltonian:

$$H(q(t), p(t), u(t), \eta(t)) = \min_{|v| \leq 1} H(q(t), p(t), v, \eta(t)) \tag{2.6}$$

2.3.2. Remarks.

1. In this minimum principle only the primary constraint $c = 0$ is penalized. Other choices are possible using the secondary constraint, see [10] and [23].
2. A minimum principle without assumption on the order is stated in [15] and the adjoint vector is given by:

$$p(t) = -\int p(s) \left(\frac{\partial X}{\partial q}(q(s)) + u(s) \frac{\partial Y}{\partial q}(q(s)) \right) ds - \sum \int \frac{\partial c}{\partial q}(q(s)) d\mu_i$$

   where the μ_i are non negative regular measure supported by $c = 0$. With the assumption of constant order, the measures on the boundary arcs take the form: $d\mu_i = \eta_i dt$ where η_i is C^0. If Assumption C_1 is not satisfied, we can have at a non generic point $YX^{m-1}c = 0$ and η_i can explode.

2.3.3. Application to the time optimal control problem. In the time minimizing problem the transfer time T is not fixed. We reparametrize the trajectories on $[0, 1]$ by setting: $s = t/T, \ z = T$. The problem is to minimize $t(1)$ for the extended system:

$$\frac{dq}{ds} = (X + uY)z, \ \frac{dt}{ds} = z, \ \frac{dz}{ds} = 0$$

The transversality conditions imply:

$$p_t \geq 0 \text{ at } s = 1 \text{ and } p_z = 0 \text{ for } s = 0, 1 \ .$$

The adjoint system decomposes into:

$$\frac{dp}{ds} = -p \left(\frac{\partial X}{\partial q} + u \frac{\partial Y}{\partial q} \right) - \eta \frac{\partial c}{\partial q}$$

$$\frac{dp_t}{ds} = 0 \qquad \frac{dp_z}{ds} = -p(X + uY) - p_t$$

and moreover:

$$M = \min_{|v| \leq 1} H = 0 \tag{2.7}$$

If we reparametrize by t and replace η by η/z and M by M/z, we get the following:

PROPOSITION 2.1. *The necessary optimal conditions for the time minimal control problem are:*

$$\dot{q} = X + uY \qquad a.e. \tag{2.8}$$

$$\dot{p} = -p \left(\frac{\partial X}{\partial q} + u \frac{\partial Y}{\partial q} \right) - \eta \frac{\partial c}{\partial q} \qquad a.e. \tag{2.9}$$

$$u\langle p, Y \rangle = \min_{|v| \leq 1} v \langle p, Y \rangle \qquad a.e. \tag{2.10}$$

$$M = \min_{|v| \leq 1} \langle p, X + uY \rangle + p_t \equiv 0 \qquad \textit{with } p \neq 0 \tag{2.11}$$

At a contact or a junction with the boundary

$$p(t_i+) = p(t_i-) - \nu_i \frac{\partial c}{\partial q}, \quad \nu_i \geq 0 \tag{2.12}$$

$$p_t \geq 0,\ \eta \geq 0 \ \textit{ with } \eta = 0 \textit{ when } c < 0 \textit{ and } \eta \textit{ is } C^0 \textit{on the boundary } c = 0 \tag{2.13}$$

2.4. Definitions. An extremal is a solution (q, p) of the above equations. It is called *exceptional* if $p_t = 0$. In the non exceptional case we use the normalization $p_t = 1/2$. An extremal arc is called *bang-bang* if it corresponds to a piecewise constant control $u(t) = -sign(\langle p(t), Y(q(t))\rangle)$; an extremal arc of the unconstrained problem is called *singular* if $\langle p(t), Y(q(t))\rangle = 0$. We note $\Phi = \langle p, Y(q)\rangle$ the *switching function* and Σ_s the *switching set* formed by points q where the optimal control is discontinuous.

2.5. Computation of singular controls. We have:

LEMMA 2.2. *Let $\Phi(t) = \langle p(t), Y(q(t))\rangle$ be the switching function evaluated along a smooth extremal $z(t) = (p(t), q(t))$ of the unconstrained problem, then:*

$$\begin{aligned}
&\dot{\Phi}(t) = \langle p(t), [X, Y](q(t))\rangle \\
&\ddot{\Phi}(t) = \langle p(t), [X, [X, Y]](q(t))\rangle + u(t)\langle p(t), [Y, [X, Y]](q(t))\rangle
\end{aligned}$$

COROLLARY 2.3. *A singular extremal $(p(t), q(t))$ satisfies:*

$$\begin{aligned}
\langle p(t), Y(q(t))\rangle = \langle p(t), [X, Y](q(t))\rangle = 0 \qquad & a.e \\
\langle p(t), [X, [X, Y]](q(t))\rangle + u(t)\langle p(t), [Y, [X, Y]](q(t))\rangle = 0 \quad & a.e.
\end{aligned}$$

2.6. Geometric computations of the multipliers (η, ν_i) and the junction conditions. One of the main contributions of [19] is to determine the multipliers (η, ν_i) together with the analysis of the junction conditions. This is based on the concept of order and is related to the classification of extremals. We shall establish now these relations when the orders are $m = 1$ and $m = 2$. Also we make the computation geometric, that is related to iterated Lie brackets of (X, Y) acting on the constraint mapping c.

2.6.1. The case $m = 1$. For first order constraint we have:

LEMMA 2.4. *Assume the order $m = 1$, then:*

1. *Along the boundary, $\eta = \dfrac{\langle p, [X, Y](q)\rangle}{(Yc)(q)}$.*
2. *Assume the control discontinuous at the contact or entrance-exit of a bang arc with the boundary, then we have $\nu_i = 0$.*

Proof. Along the boundary, $\Phi = \langle p, Y\rangle$=0 and differentiating we get:

$$0 = \dot{\Phi} = \langle p, [X, Y](q)\rangle - \eta Yc(q)$$

and $Yc \neq 0$ since the boundary arc is of order 1. Hence we get 1.

Let us prove 2. We set $a = Xc$ and $b = Yc$. Hence $\dot{c} = a + ub$. Let Q be a contact point of a bang-bang extremal $t \mapsto q(t)$ with the boundary at time t_i. Let $\epsilon > 0$ small enough. We have:

$$c(q(t_i - \epsilon)) < 0\ , \qquad c(q(t_i + \epsilon)) < 0\ .$$

taking the limit when ϵ tends to 0, we get:

$$(a+bu)_{t_i-} \geq 0 \ , \qquad (a+bu)_{t_i+} \leq 0 \ .$$

Hence making the difference, it follows:

$$b(q(t_i))\,(u(t_i-)-u(t_i+)) \geq 0 \ . \tag{2.14}$$

Assume for instance $b(q(t_i)) > 0$. Hence $u(t_i-) - u(t_i+) > 0$ since $u(t_i-) \neq u(t_i+)$. From the minimum principle we must have:

$$\Phi(t_i-) \leq 0 \ , \qquad \Phi(t_i+) \geq 0$$

from (2.12) we have:

$$\Phi(t_i+) = \Phi(t_i-) - \nu_i b(q(t_i)) \tag{2.15}$$

and we deduce $\nu_i b(q(t_i)) \leq 0$. From the minimum principle $\nu_i \geq 0$. Consequently, if $\nu_i \neq 0$ we have $b(q(t_i)) \leq 0$ and this contradicts our assumption. The case $b(q(t_i)) < 0$ is similar. The discussion is similar at a junction point with a boundary arc. □

2.6.2. The case m=2. For second order constraint we have:

LEMMA 2.5. *Assume the order $m=2$ then:*

1. *Along a boundary arc:*

$$\eta = \frac{\langle p, [X,[X,Y]](q)\rangle + u_b \langle p, [Y,[X,Y]](q)\rangle}{([X,Y]c)(q)}$$

2. *At a contact or entrance-exit point:*

$$\Phi(t_i+) = \Phi(t_i-)$$

3. *At an entry point:*

$$\nu_i = \frac{\dot{\Phi}(t_i-)}{([X,Y]c)(q(t_i))} \ ,$$

and at an exit point:

$$\nu_i = -\frac{\dot{\Phi}(t_i+)}{([X,Y]c)(q(t_i))} \ ,$$

The proof is similar to the proof of Lemma 2.4.

Remark. Hence at a entry (resp. exit) point, ν_i is determined by the extremal before reaching (resp. after leaving) the constraint. The multiplier η is determined by (q,p) along the constraint.

2.7. Small time reachable set, normality and conjugate points along bang-bang extremals.

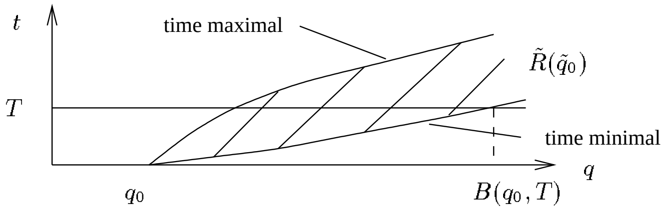


FIG. 1. *Small time reachable set*

2.7.1. Definitions. Consider a system of the form $\dot{q} = X + uY$, $|u| \leq 1$. Let $q(0) = q_0$ be fixed and denote $q(t, q_0, u)$ the solution corresponding to $u(.)$ and starting at time $t = 0$ from q_0. We denote by $R(q_0) = \bigcup_{u,\, t \text{ small enough}} q(t, q_0, u)$ the *small time reachable set.* The time extended system is the system $\dot{q} = X + uY$, $\dot{q}^0 = 1$. Let us note $\tilde{q} = (q, q^0)$ ant let $\tilde{R}(\tilde{q}_0)$ be the small time reachable set for the extended system, with $\tilde{q}_0 = (q_0, 0)$. We note $\tilde{B}(\tilde{q}_0)$ the boundary of $\tilde{R}(\tilde{q}_0)$ which contains both time minimal and time maximal trajectories of the system, parametrized by the minimum principle. Let $T > 0$, we note $B(q_0, T)$ the extremities of time minimal trajectories with time T, see Fig. 1. The *cut-locus* $C(q_0)$ is the set of points q_1 where there exists two distinct minimizing curves starting from q_0.

We define similarly the small time reachable sets, their boundaries and the cut locus for the constrained problem $c(q) \leq 0$, and they are noted with a subscript b. *The purpose of this article is in particular to compute* $C_b(q_0)$ *and to stratify* $B_b(q_0, T)$ *for systems in dimension* 2 *and* 3 *under generic assumptions.*

2.7.2. Normality and conjugate points along bang-bang extremals for the unconstrained system. We recall briefly the concept of normality and conjugate points introduced by [28]. Consider the family of vector fields

$$D = \{X + u_0 Y \ , |u_0| \leq 1\}$$

and if $Z \in D$ denotes $\exp tZ$ the one parameter group of Z. Fix q_0 and $T > 0$ and if $Z_1, \ldots, Z_m \in D$ denotes by φ the mapping,

$$\varphi(t_1, \ldots, t_m) = \exp t_m Z_m \circ \ldots \circ \exp t_1 Z_1(q_0), \ \sum_{i=1}^{m} t_i = T.$$

The point q_1 is said (resp. *quasi*) *normally accessible* from q_0 in time T if there exists such a mapping which satisfies $\varphi(\bar{t}) = q_1$ for some $\bar{t} = (\bar{t}_1, \ldots, \bar{t}_m)$, with $\bar{t}_i > 0$ and such that φ is a submersion at $\bar{t}$ (resp. open mapping). In both cases the image covers a neighborhood of q_1 and hence the corresponding trajectory is not time minimal, nor time maximal. Let $(z(.), q(.))$ be a bang-bang extremal defined on $[0, T]$. If n is the dimension of the state, assume that z has n switchings $0 \leq t_1 < \ldots < t_n = T$ on $[0, T]$. The corresponding points $q(t_1), q(t_n)$ of the trajectory are called *conjugate points.* Hence $\langle p(t_i), Y(q(t_i))\rangle = 0$, for $i = 1, \ldots, n$. Let Y_i^* be the vector $Y(q(t_i))$ transported from $q(t_i)$ to $q(t_n)$ by the flow of Z. Then we have $\langle p(t_n), Y_i^*\rangle = 0$, for $i = 1, \ldots, n$ and since $p(t_n)$ is non zero, the vectors Y_i^* are linearly dependent. The resulting relation between the switching times is called a *conjugate point relation.*

2.8. Conclusion of this section. The remaining of this article is devoted to the construction of the closed loop time optimal trajectories for a single input affine control system in dimension 2 or 3 with application to the space shuttle re-entry problem. To guide the analysis, a standard result (seee [18]) is the following: Consider a system in $\mathbb{R}^3$, $\dot{q} = X + uY$ and take q_0 such that at q_0 the Lie brackets, X, Y, $[X, Y]$ are linearly independent. Then the small time reachable set $R(q_0)$ has a nice structure. It is homeomorphic to a *convex cone* whose boundary is formed by two surfaces S_1 and S_2 corresponding to bang-bang trajectories with at most one switching, see Fig. 2. The arcs γ_+ and γ_- are the trajectories corresponding to $u = 1$ and $u = -1$ respectively.

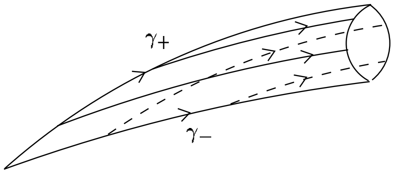


FIG. 2. *Small time reachable set*

3. Small time minimal syntheses for planar systems with state constraints.

3.1. Generalities. We consider a system $\dot{q} = X + uY$, $|u| \leq 1$, $q = (x, y) \in \mathbb{R}^2$ with state constraints $c(q) \leq 0$. We denote by $\omega = pdq$ the *clock form* defined on the set where X, Y are independent by $\omega(X) = 1$ and $\omega(Y) = 0$. The singular trajectories are located on the set $S = \{q \in \mathbb{R}^2,\ det(Y(q), [X, Y](q)) = 0\}$ and the singular control u_s is solution of $\langle p, [X, [X, Y]](q)\rangle + u_s\langle p, [Y, [X, Y]](q)\rangle$. The two form $d\omega$ is zero on the set S.

We take $q_0 \in \{q \in \mathbb{R}^2,\ c(q) = 0\}$ identified to 0. The problem is to determine local optimality status of a boundary arc $t \mapsto \gamma_b(t)$ corresponding to a control u_b and to describe the time minimal syntheses near q_0. The first step is to construct a normal form, assuming the constraint of order one.

LEMMA 3.1. *Assume*

1. *$X(q_0)$, $Y(q_0)$ linearly independent,*
2. *The constraint is of order 1, that is $Yc(q_0) \neq 0$*

Then changing if necessary u into $-u$ there exists a local diffeomorphism preserving $q_0 = 0$ such that the constrained system is

$$\begin{aligned} \dot{x} &= 1 + ya(q) \\ \dot{y} &= b(q) + u\ , \qquad y \leq 0 \end{aligned}$$

Proof. Using a local change of coordinates preserving 0, we can identify Y to $\frac{\partial}{\partial y}$ and the boundary arc to $\gamma_b : t \mapsto (t, 0)$. The admissible space is $y \leq 0$ or $y \geq 0$. Changing if necessary u to $-u$ it can be identified to $y \leq 0$. ☐

3.2. The generic case A_1. In this case we make the additional assumptions:

1. $Y(0), [X, Y](0)$ are linearly independent,
2. the boundary arc is admissible and not saturating at 0.

3.2.1. Local model. Under these assumptions, we have in the previous normal form: $a(0) \neq 0$ and $|b(0)| < 1$. To analyze the optimal synthesis near 0, we set

$a = a(0)$, $b = b(0)$ and the local model is:

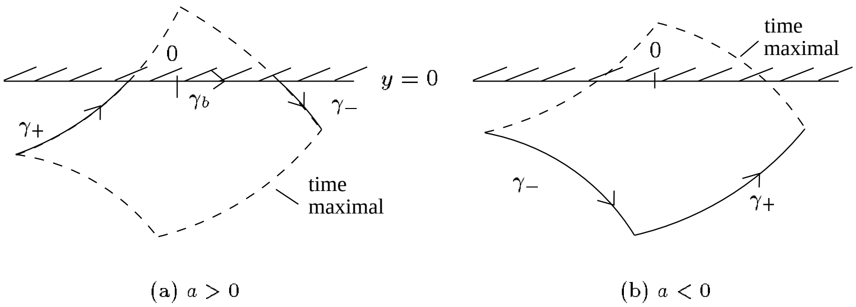
$$\begin{aligned} \dot{x} &= 1 + ya \\ \dot{y} &= b + u\ , \qquad y \leq 0\ . \end{aligned}$$

The clock form is

$$\omega = \frac{dx}{1 + ay} \qquad \text{and} \qquad d\omega = \frac{a}{(1 + ay)^2} dx \wedge dy$$

3.2.2. Local syntheses. First consider the unconstrained case. The small time reachable set for the time extended system is represented on Fig. 2. Its boundary, formed by arc $\gamma_+\gamma_-$ or $\gamma_-\gamma_+$ (where $\gamma_+\gamma_-$ denotes an arc γ_+ followed by an arc γ_-) represents time minimal and time maximal trajectories. They are given by the minimum principle. Considering the model we have two cases: If $a > 0$, then $d\omega > 0$ and each optimal trajectory is of the form $\gamma_+\gamma_-$ ($\gamma_-\gamma_+$ being time maximal). If $a < 0$ then $d\omega < 0$ and each optimal trajectory is of the form $\gamma_-\gamma_+$ ($\gamma_+\gamma_-$ being time maximal).

For the constrained case, the same reasoning on the clock form shows that the boundary arc is optimal if and only if $a > 0$. The local optimal synthesis is represented on Fig. 3. We have proved:



(a) $a > 0$ (b) $a < 0$

FIG. 3.

LEMMA 3.2. *In the case A_1 we have:*

1. *For the unconstrained problem, if $a > 0$ an arc $\gamma_+\gamma_-$ is time minimal and an arc $\gamma_-\gamma_+$ is time maximal and conversely if $a < 0$.*
2. *For the constrained problem, a boundary arc is optimal if and only if $a > 0$ and in this case, each optimal trajectory is of the form $\gamma_+\gamma_b\gamma_-$. If $a < 0$ each optimal arc is of the form $\gamma_-\gamma_+$.*

3.2.3. Link with the minimum principle. According to Lemma 2.4, along the boundary $\eta = \dfrac{\langle p, [X,Y](q)\rangle}{(Yc)(q)}$ with $\langle p, Y\rangle = 0$. Hence denoting $p = (p_x, p_y)$ we get $\eta = -ap_x$ and p_x is oriented by $\langle p, X + uY\rangle + p_t = 0$, $p_t \geq 0$. Hence $p_x < 0$. Therefore *sign* $(\eta) =$ *sign* (a) and *the necessary optimality condition is violated if* $a < 0$.

3.3. The singular case B1.

3.3.1. Generalities and local model. If Y and $[X, Y]$ are linearly dependent at 0 then $a(0) = 0$, we assume that the set $S = \{q \in \mathbb{R}^2,\ det(Y(q), [X, Y](q)) = 0\}$ is a simple curve. In Lemma 3.1, we have normalized Y to $\frac{\partial}{\partial y}$ and the boundary arc γ_b to $t \mapsto (t, 0)$. Hence the slope of S at 0 is an invariant. In the small time model, we approximate S by a straight line and the equations of the system are:

$$\begin{aligned} \dot{x} &= 1 + y(ay + bx) \\ \dot{y} &= c + u\ , \qquad y \leq 0 \end{aligned}$$

where the set S is identified to $\{(x, y) \in \mathbb{R}^2,\ 2ay + bx = 0\}$ and we assume $a \neq 0$. Consider the system without state constraint and assume $u \in \mathbb{R}$. From [5] there exists along S a singular arc which can be time minimal or time maximal. The test to distinguish between the two cases is the Legendre-Clebsch condition and we have two cases:

- $a < 0$: the singular arc is time minimal.
- $a > 0$: the singular arc is time maximal.

The singular control u_s which makes S invariant is solution of

$$b(1 + y(ay + bx)) + 2a(c + us) = 0$$

and its value at 0 is

$$u_s = -c - b/2a\ .$$

Taking into account the constraint, the condition of admissibility is $|c + b/2a| \leq 1$. In the previous normalizations the clock form is

$$\omega = \frac{dx}{ay^2 + bxy} \qquad \text{and} \qquad d\omega = \frac{2ay + bx}{(ay^2 + bxy)^2} dx \wedge dy$$

and we have $sign\ (d\omega) = sign\ (2ay + bx)$. We assume that the boundary arc is admissible and non saturating: $|c| < 1$. We have 3 generic cases to analyze. These cases are distinguished by the behavior of the bang-bang extremals for the unconstrained system near the switching surface. Differentiating twice $\Phi = \langle p, Y(q) \rangle$, we have:

$$\begin{aligned} \dot{\Phi} &= \langle p, [X, Y](q) \rangle \\ \ddot{\Phi}_u &= \langle p, [X, [X, Y]](q) + u[Y, [X, Y]](q) \rangle \end{aligned}$$

with $u(t) = -sign\langle p(t), Y(q(t)) \rangle$ and p is oriented by $\langle p, X + uY \rangle \leq 0$. The three generic cases are represented in Fig. 4 and are respectively

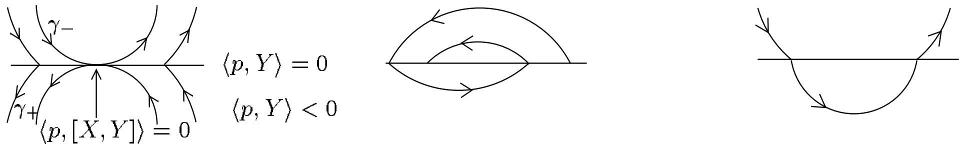
- hyperbolic case (a): $\ddot{\Phi}_+ < 0$ and $\ddot{\Phi}_- > 0$ and $\langle p, Y \rangle = \langle p, [X, Y] \rangle = 0$.
- elliptic case (b): $\ddot{\Phi}_- < 0$ and $\ddot{\Phi}_+ > 0$ and $\langle p, Y \rangle = \langle p, [X, Y] \rangle = 0$.
- parabolic case (c): $\ddot{\Phi}_+$ and $\ddot{\Phi}_-$ have the same sign at $\langle p, Y \rangle = \langle p, [X, Y] \rangle = 0$.

In the hyperbolic case the singular arc is admissible, not saturating and is time minimal. In the elliptic case it is admissible, not saturating and time maximal. In the parabolic case , $|u_s| > 1$ and the singular arc is not admissible. In the sequel we analyze the three cases.

3.3.2. Hyperbolic case. $a < 0$, $|c + b/2a| < 1$, $|c| < 1$ and $b \neq 0$.

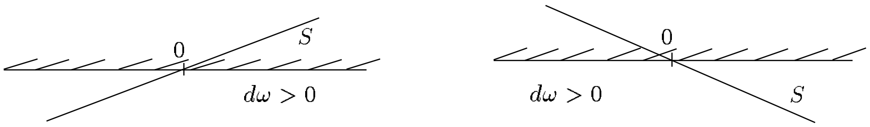
We have two cases according to the slope at 0 of the singular arc, see Fig. 5.

The analysis is similar in both cases and we only study the case $b > 0$ in details. For the unconstrained problem, the singular arc is optimal and each optimal trajectory



(a) (b) (c)

FIG. 4.



(a) $b > 0$ (b) $b < 0$

FIG. 5.

has at most two switchings and the local optimal synthesis is of the form $\gamma_\pm\gamma_s\gamma_\pm$. For the constrained problem, using the clock form together with Stokes theorem leads to optimality of the boundary arc for $x \geq 0$ and non optimality for $x < 0$. The optimal synthesis to join two points of the boundary is then of the form $\gamma_-\gamma_s\gamma_b$. Each optimal curve in a neighborhood of 0 has at most 3 switchings and the local optimal synthesis is of the form $\gamma_\pm\gamma_s\gamma_b\gamma_\pm$. The situations are represented on Fig. 6 and is summarized in the following lemma.

LEMMA 3.3. *Under our assumption, in the hyperbolic case each small time optimal trajectories has at most* 3 *switchings. Moreover:*

1. *If* $b > 0$*, a boundary arc is optimal if and only if* $x \geq 0$ *and each optimal arc is of the form* $\gamma_\pm\gamma_s\gamma_b\gamma_\pm$.
2. *If* $b < 0$*, a boundary arc is optimal if and only if* $x \leq 0$ *and each optimal arc is of the form* $\gamma_\pm\gamma_b\gamma_s\gamma_\pm$

(a) $b > 0$ (b) $b < 0$

FIG. 6.

3.3.3. Elliptic case. $a > 0$, $|c + b/2a| < 1$, $|c| < 1$ and $b \neq 0$. We have again two cases according to the sign of the slope of the singular arc. Both cases are distinguished by the optimality status of the boundary arc, 0 being excluded, see Fig. 7. We shall only study the case $b > 0$ in details.

(a) $b > 0$ (b) $b < 0$

FIG. 7.

For the optimal problem without state constraint the situation is the following. Each optimal control is bang-bang with at most one switching. This is proved by the concept of conjugate points, see [27]. Indeed take near 0 a reference extremal with two switchings. It can be embedded into a one parameter family of extremals with the same initial point, represented on Fig. 8, which reflects on the switching locus.

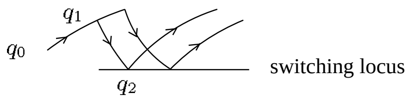
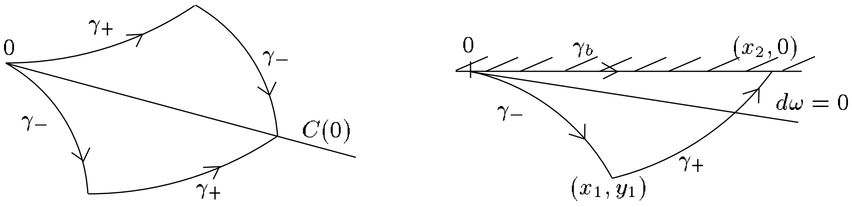


FIG. 8.

Optimality status is lost after the second switching point which is conjugated. In particular there exists a cut locus $C(0)$ for optimal trajectory starting from 0, see Fig. 9(a).



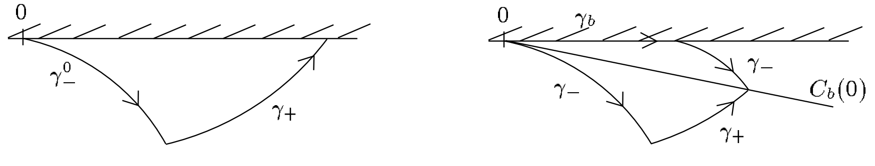
(a) (b)

FIG. 9.

This has the following consequence for the problem with state constraints. Let γ_-^0 be the arc associated to $u = -1$ starting from 0 and consider an arc $\gamma_-^0\gamma_+$ joining 0 to the boundary in this admissible domain, see Fig. 9(b). Then it intersects $d\omega = 0$ and it can be optimal or not. Using the model, we compare the time along the boundary arc γ_b and the time along $\gamma_-^0\gamma_+$ to decide about optimality. A straightforward computation gives us:

LEMMA 3.4. *Consider the elliptic case, with* $b > 0$. *If* $\frac{b}{2a} > \frac{1-c^2}{c+3}$ *then the boundary arc starting from* 0 *is optimal. If* $\frac{b}{2a} < \frac{1-c^2}{c+3}$ *the optimal policy to join* 0 *to a nearby point of the boundary is* $\gamma_-^0\gamma_+$. *The small time reachable set from* 0 *is represented on Fig. 10.*

If the boundary arc is admissible, there exists a cut-locus $C_b(0)$.



(a) $\frac{b}{2a} < \frac{1-c^2}{c+3}$ (b) $\frac{b}{2a} > \frac{1-c^2}{c+3}$

FIG. 10.

3.3.4. Link with the minimum principle. In the hyperbolic and elliptic case the condition $\eta \geq 0$ along a boundary arc of the minimum principle detects the optimality outside 0, since $\eta = -bxp_x$. To decide at 0, we must use second order condition, taking the clock form. In the elliptic case, the bifurcation between the two cases can be obtained by the following standard argument. For an extremal with a non trivial boundary arc, and according to Lemma 2.4, the adjoint vector is continuous at the junction and hence $\langle p, Y\rangle = 0$, which determines p (using homogeneity) prior to the junction. This gives us a switching curve K for extremal curves with a boundary arc, passing through 0. This curve can be computed and we get $y = 2(c+1)\frac{b/2a}{b/2a-(c+1)}x$. Depending on the slope, an extremal can reflect or cross this locus, see Fig. 11. The critical value is when the slope of γ_-^0 is equal to the slope of K that is $\frac{b}{2a} = \frac{1-c^2}{c+3}$. *Hence optimality of the boundary arc corresponds to the crossing case.* This is coherent with [1], see also Fig. 8.

(a) (b)

FIG. 11.

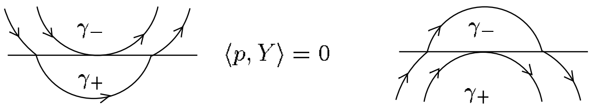
3.3.5. The parabolic case. This case arises when the singular control is not admissible. The unconstrained case is easy to analyze and the optimal synthesis follows from the classification of extremals. Indeed, in the parabolic case, differentiating twice the switching function Φ we have:

$$\ddot{\Phi} = \langle p, [X,[X,Y]] + u[Y,[X,Y]]\rangle$$

and $\ddot{\Phi}$ has the same sign for $u = +1$ and $u = -1$ and we have two situations represented on Fig. 12. Computing using the normal form we get with $\langle p, Y\rangle = 0$, that is $p_y = 0$:

$$\ddot{\Phi} = -p_x(b + 2a(c+u))$$

Hence for the unconstrained problem, if $b + 2a(c \pm 1) > 0$ the optimal solution is of the form $\gamma_-\gamma_+\gamma_-$ and if $b + 2a(c \pm 1) < 0$, the optimal solutions are of the form $\gamma_+\gamma_-\gamma_+$ and the length of the intermediate arc is determined by the first switching point.



(a) $\ddot{\Phi} > 0$ (b) $\ddot{\Phi} < 0$

FIG. 12.

We analyze the optimal synthesis for the problem with state constraints. We have four cases, all similar. Consider for instance the case $a > 0$, $b > 0$. Since $|c| < 1$ we have $c + 1 > 0$ and the optimal law for the problem without state constraint is of the form $\gamma_-\gamma_+\gamma_-$. The optimal status of the boundary arc, 0 excluded, is represented on Fig. 13 and the boundary arc is non optimal if $x < 0$ and optimal if $x > 0$. Moreover

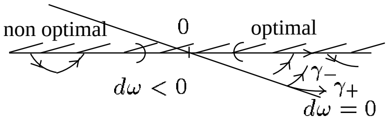


FIG. 13.

the boundary arc γ_b^0 starting from 0 is optimal because the sector $d\omega > 0$ is positively invariant. The optimal synthesis follows from the minimum principle and Lemma 2.4.

LEMMA 3.5. *Consider the parabolic case and assume for instance $a > 0$, $b > 0$. Then*

- *for the unconstrained case each optimal policy is of the form $\gamma_-\gamma_+\gamma_-$;*
- *for the constrained case each optimal policy is of the form $\gamma_-\gamma_+\gamma_b\gamma_-$.*

3.4. The saturated case B_2. We analyze now the generic saturated case. Hence we assume that the boundary control u_b is saturated at 0, that is $u_b = \pm 1$. We suppose that there exists no singular arc that gets through 0, that is a does not vanish at 0 in our model of Lemma 3.1. Consequently, denoting $a(0) = a$ and $b(0) = b$, the local model is

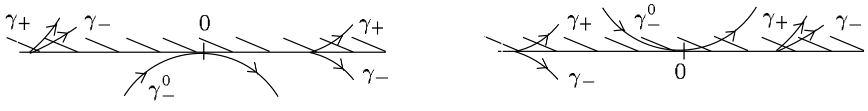
$$\begin{aligned} \dot{x} &= 1 + ya \\ \dot{y} &= b + u + cx + dy\ , \qquad y \le 0 \end{aligned}$$

where a, b, c are constants and $b = \pm 1$. Moreover we assume $c \neq 0$. The various cases are easy to analyze and we shall discuss in details the case $a > 0$, $b = +1$. For the unconstrained problem, an optimal arc is of the form $\gamma_+\gamma_-$ and for the constrained problem the saturating control is $u_b = -1$. We have two cases distinguished by the sign of c, see Fig. 14.

If $c < 0$ (resp. $c > 0$) the boundary arc is admissible only for $x \geq 0$ (resp. $x \leq 0$). The local time minimal synthesis follows easily and is represented on Fig. 15.

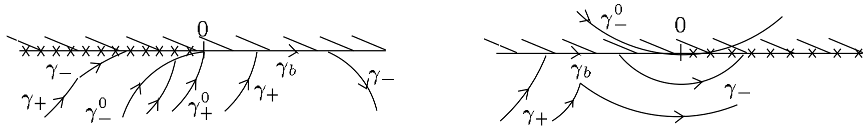
LEMMA 3.6. *Assume we are in the saturated case with $a > 0$, $b = 1$. Then*

1. *If $c < 0$, each local optimal arc has at most 3 switchings and is of the form $\gamma_+\gamma_-\gamma_b\gamma_-$.*
2. *If $c > 0$, each local optimal arc has at most 2 switchings and is of the form $\gamma_+\gamma_b\gamma_-$.*



(a) $c > 0$ (b) $c < 0$

FIG. 14.



(a) $c > 0$ (b) $c < 0$

FIG. 15.

The analysis is similar in the other cases and the syntheses are represented on Fig. 16.

4. Small time minimal synthesis for system in dimension 3 with state constraints.

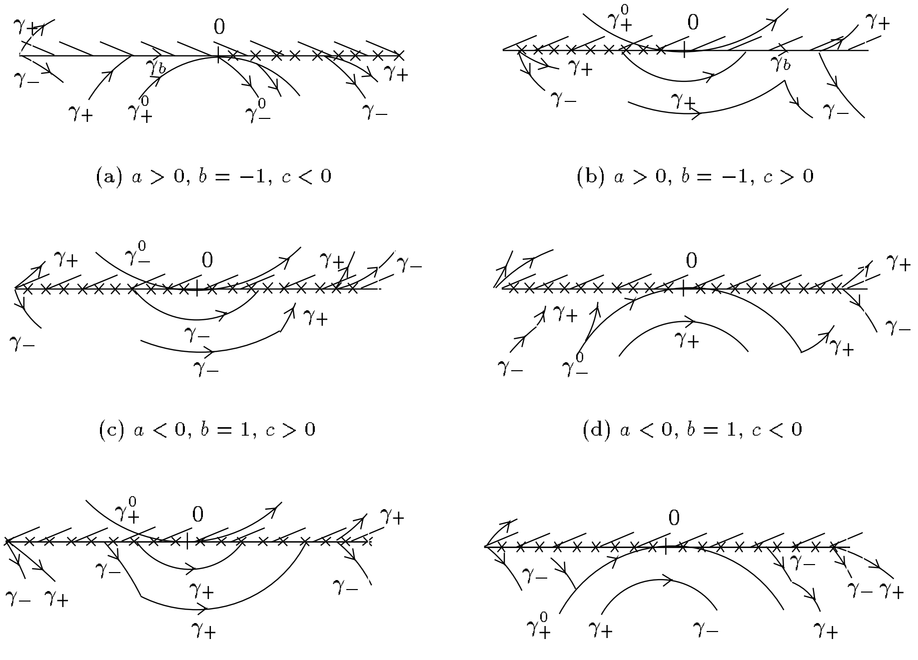
4.1. Preliminaries. We consider a system of the form $\dot{q} = X + uY$, $c(q) \leq 0$, $|u| \leq 1$, with $q = (x, y, z) \in \mathbb{R}^3$. The objective of this section is to initialize the classification of the optimal syntheses near a point q_0, identified to 0, on the boundary of the domain. Consider first the unconstrained case and assume that X, Y, $[X, Y]$ are linearly independent at q_0. The small time reachable set $R(q_0)$ is represented on Fig. 2, and its boundary is formed by the two surfaces S_1 and S_2 respective extremities of arcs of the form $\gamma_-\gamma_+$ and $\gamma_+\gamma_-$. To construct optimal trajectories we must analyze the boundary of the small time reachable set for the time extended system. Its structure is described in [18], under generic assumptions. We proceed as follows. Differentiating twice the switching function $\Phi = \langle p(t), Y(q(t))\rangle$ we get:

$$\dot{\Phi}(t) = \langle p(t), [X, Y](q(t))\rangle \ ,$$
$$\ddot{\Phi}(t) = \langle p(t), [X + uY, [X, Y]](q(t))\rangle \ .$$

If $\langle p(t), [Y, [X, Y]](q(t))\rangle$ is not vanishing we can solve $\ddot{\Phi}(t) = 0$ to compute the singular control:

$$u_s = -\frac{\langle p, [X, [X, Y]](q)\rangle}{\langle p, [Y, [X, Y]](q)\rangle}$$

If Y and $[X, Y]$ are independent, p can be eliminated by homogeneity and u_s computed as a feedback control. Introducing $D = det(Y, [X, Y], [Y, [X, Y]])$ and $D' = det(Y, [X, Y], [X, [X, Y]])$, we get $D'(q) + u_s D(q) = 0$. Hence in dimension 3 through each generic point there is a singular direction. Moreover, as in the planar case, the Legendre-Clebsh condition allows to distinguish between slow and fast directions in



(a) $a > 0$, $b = -1$, $c < 0$ (b) $a > 0$, $b = -1$, $c > 0$

(c) $a < 0$, $b = 1$, $c > 0$ (d) $a < 0$, $b = 1$, $c < 0$

(e) $a < 0$, $b = -1$, $c > 0$ (f) $a < 0$, $b = -1$, $c < 0$

FIG. 16.

the non exceptional case where X, Y, $[X,Y]$ are not collinear. We have two cases, see [5].

- *Case 1*: If X and $[[X,Y],Y]$ are on the opposite side with respect to the plane generated by $Y,[X,Y]$, then the singular arc is locally time optimal if $u \in \mathbb{R}$.
- *Case 2*: On the opposite, if X and $[[X,Y],Y]$ are in the same side, the singular arc is locally time maximal.

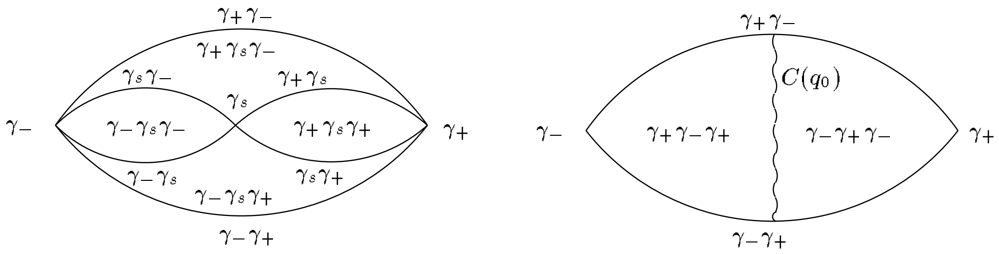
In the two cases, the constraint $|u_s| \leq 1$ is not taken into account and the singular control can be strictly admissible if $|u_s| < 1$, saturating if $|u_s| = 1$ at q_0, or non admissible if $|u_s| > 1$. We have 3 generic cases. Assume X, Y, $[X,Y]$ not collinear and let p oriented with the convention of the minimum principle: $\langle p(t), X + uY\rangle \leq 0$. Let t be a switching time of a bang-bang extremal: $\Phi = \langle p(t), Y(q(t))\rangle = 0$. It is called of order one if $\dot{\Phi}(t) = \langle p(t), [X,Y](q(t))\rangle \neq 0$ and of order two if $\dot{\Phi}(t) = 0$ but $\ddot{\Phi}(t) = \langle p(t), [X + uY, [X,Y]](q(t))\rangle \neq 0$ for $u = \pm 1$. The classification of extremals near a point of order two is similar to the planar case, see Fig. 4. We have three cases:

- parabolic case: $\ddot{\Phi}_\pm$ have the same sign.
- elliptic case: $\ddot{\Phi}_+ > 0$ and $\ddot{\Phi}_- < 0$.
- hyperbolic case: $\ddot{\Phi}_+ < 0$ and $\ddot{\Phi}_- > 0$.

In both hyperbolic and parabolic cases, the local time optimal syntheses are obtained by using only the first order conditions from the minimum principle and hence from extremality, together with Legendre-Clebsch condition in the hyperbolic case. More precisely we have:

LEMMA 4.1. *In the hyperbolic or parabolic cases, each extremal policy is locally time optimal. In the hyperbolic case each optimal policy is of the form* $\gamma_\pm\gamma_s\gamma_\pm$. *In the parabolic case, each optimal policy is bang-bang with at most two switchings.*

The set $B(q_0, T)$ describing the time minimal policy at fixed time is homeomorphic to a closed disk, whose boundary is formed by extremities of arcs $\gamma_-\gamma_+$ and $\gamma_+\gamma_-$ with length T and the stratification in the hyperbolic case is represented on Fig. 17(a).



(a) hyperbolic (b) elliptic

FIG. 17.

In the elliptic case, the situation is more complicated because there exist a cut-locus. The analysis is related to the following crucial result based on the concept of conjugate points defined in Section 2.7.2 due to [28].

LEMMA 4.2. *Consider system* $\dot{q} = X + uY$, $|u| \leq 1, q \in \mathbb{R}^3$. *Let* q_0 *be a point such that each of the two triplets* Y, $[X, Y]$, $[X - Y, [X, Y]]$ *and* Y, $[X, Y]$, $[X + Y, [X, Y]]$ *is linearly independent at* q_0. *Then near* q_0 *each bang-bang locally time optimal trajectory has at most two switchings.*

The local time optimal policy at fixed time is represented on Fig. 17(b). There exists a *cut locus* $C(q_0)$ *whose extremities are conjugate points on the boundary of the reachable set.*

We shall now analyze the constrained case. If the order of the constraint is one, the situation is similar to the planar case, analyzed in Section 3. *Hence we shall assume that the constraint is of order* 2. *The case corresponding to the space shuttle problem is the parabolic case which is considered first.*

4.2. Geometric normal form in the constrained parabolic case and local synthesis. For the unconstrained problem the situation is clear in the parabolic case. Indeed X, Y, $[X, Y]$ form a frame near q_0 and writing:

$$[X \pm Y, [X, Y]] = aX + bY + c[X, Y],$$

the synthesis depends only upon the sign of a at q_0. The small time reachable set is bounded by the surfaces formed by arcs $\gamma_-\gamma_+$ and $\gamma_+\gamma_-$. Each interior point can be reached by an arc $\gamma_-\gamma_+\gamma_-$ and an arc $\gamma_+\gamma_-\gamma_+$. According to Fig. 12, if $a < 0$ the time minimal policy is $\gamma_-\gamma_+\gamma_-$ and the time maximal policy is $\gamma_+\gamma_-\gamma_+$ and the opposite if $a > 0$. To construct the optimal synthesis one can use a *nilpotent model* where all Lie brackets of length greater than 4 are 0. In particular the existence of singular direction is irrelevant in the analysis and a model where $[Y, [X, Y]]$ is zero can be taken. This situation is called the *geometric model.* A similar model is constructed next taking into account the constraints, which are assumed of order 2. Moreover we shall first suppose that C_1, C_3 are satisfied along a boundary arc γ_b, that is $YXc \neq 0$ along γ_b and the boundary control is admissible and not saturating. We have the following:

LEMMA 4.3. *Under our assumptions, a local geometric model in the parabolic case is:*

$$\begin{aligned}
\dot{x} &= a_1 x + a_3 z \\
\dot{y} &= 1 + b_1 x + b_3 z \\
\dot{z} &= (c+u) + c_1 x + c_2 y + c_3 z, \qquad |u| \leq 1
\end{aligned}$$

with $a_3 > 0$, *where the constraint is* $x \leq 0$ *and the boundary arc is identified to* $\gamma_b : t \mapsto (0,t,0)$ *Moreover we have* $[X,Y] = -a_3 \frac{\partial}{\partial x} - b_3 \frac{\partial}{\partial y}$, $[Y,[X,Y]] = 0$, $[X,[X,Y]] = (a_1 a_3 + a_3 c_3)\frac{\partial}{\partial x} + (a_3 b_1 + b_3 c_3)\frac{\partial}{\partial y} + (a_3 c_1 + b_3 c_2 + c_3^2)\frac{\partial}{\partial z}$, *and* $[X,[X,Y]] = a\ X \ \ mod\{Y,[X,Y]\}$, *with* $a = a_3 b_1 - a_1 b_3 \neq 0$. *If the boundary arc is admissible and not saturating we have* $|c| < 1$. *Moreover* $a_3 = -[X,Y]c$.

Proof. We give the details of the normalizations.

Normalization 1. Since $Y(0) \neq 0$, we identify locally Y to $\partial/\partial z$. The local diffeomorphisms $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ preserving 0 and Y satisfy: $\frac{\partial \varphi_1}{\partial z} = \frac{\partial \varphi_2}{\partial z} = 0$ and $\frac{\partial \varphi_3}{\partial z} = 1$. Since the constraint is of order 2, $Yc = 0$ near 0 and Y is tangent to all surfaces $c = \alpha$, α small enough, hence $\frac{\partial c}{\partial z} = 0$.

Normalization 2. Since c is not depending on z, using a local diffeomorphism preserving 0 and $Y = \frac{\partial}{\partial z}$, we can identify the constraint to $c = x$. Then the system can be written: $\dot{x} = X_1(q)$, $\dot{y} = X_2(q)$, $\dot{z} = X_3(q) + u$, and $x \leq 0$. The secondary constraint is $\dot{x} = 0$, and by assumption a boundary arc γ_b is contained in $x = \dot{x} = 0$ and passing through 0. In the parabolic case the affine approximation is sufficient for our analysis and the geometric model is:

$$\begin{aligned}
\dot{x} &= a_1 x + a_2 y + a_3 z \ , \\
\dot{y} &= b_0 + b_1 x + b_2 y + b_3 z \ , \\
\dot{z} &= c_0 + c_1 x + c_2 y + c_3 z + u \ ,
\end{aligned}$$

where γ_b is approximated by the straight line: $x = 0$, $a_2 y + a_3 z = 0$.

Normalization 3. Finally we normalize the boundary as follows. In the plane $x = 0$, making a transformation of the form: $z' = \alpha y + z$, we can normalize the boundary arc to $x = z = 0$. Using a diffeomorphism $y' = \varphi(y)$, the boundary arc can be parametrized as $\gamma_b : t \mapsto (0,t,0)$. The normal form follows, changing if necessary u to $-u$, and hence permuting the arcs γ_+ and γ_-. □

THEOREM 4.4. *Consider the time mimization problem for the system:* $\dot{q} = X(q) + uY(q)$, $q \in \mathbb{R}^3, |u| \leq 1$ *with the constraint* $c(q) \leq 0$. *Let* $q_0 \in \{c = 0\}$ *and assume the following:*

1. *At* q_0, X, Y *and* $[X,Y]$ *form a frame and* $[X \pm Y, [X,Y]](q_0) = aX(q_0) + bY(q_0) + c[X,Y](q_0)$, *with* $a < 0$.
2. *The constraint is of order* 2 *and Assumptions* C_1 *and* C_3 *are satisfied at* q_0.

Then the boundary arc through q_0 *is small time optimal if and only if the arc* γ_- *is contained in the non admissible domain* $c \geq 0$. *In this case the local time minimal synthesis with a boundary arc is of the form* $\gamma_- \gamma_+^T \gamma_b \gamma_+^T \gamma_-$, *where* γ_+^T *are arcs tangent to the boundary arc.*

Proof. The proof is straightforward and can be done using a simple reasoning visualized on the normal form. In this case $q_0 = 0$, the boundary arc is identified to $t \mapsto (0,t,0)$ and due to $a_3 > 0$, arcs tangent to γ_b corresponding to $u = \pm 1$, are contained in $c \leq 0$ if $u = -1$ and in $c \geq 0$ if $u = +1$. Let B be a point of the boundary arc γ_b, for small enough $B = (0, y_0, 0)$. If $u = \pm 1$, we have the following approximations for arcs initiating from B:

$$\begin{aligned} x(t) &= a_3(c_0 + c_2 y_0 + u)t^2/2 + o(t^2) \\ z(t) &= (c_0 + c_2 y_0 + u)t + o(t) \ . \end{aligned}$$

The projections in the plane (x,z) of the arcs $\gamma_-\gamma_+\gamma_-$ and $\gamma_+\gamma_-\gamma_+$ joining 0 to B are *loops* denoted $\tilde{\gamma}_-\tilde{\gamma}_+\tilde{\gamma}_-$ and $\tilde{\gamma}_+\tilde{\gamma}_-\tilde{\gamma}_+$ represented on Fig. 18. The loops $\tilde{\gamma}_-\tilde{\gamma}_+\tilde{\gamma}_-$

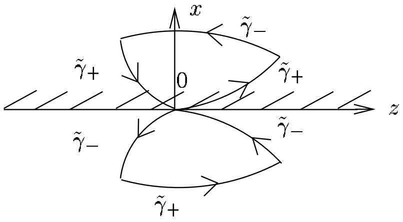


FIG. 18.

(resp. $\tilde{\gamma}_+\tilde{\gamma}_-\tilde{\gamma}_+$) are contained in $x \leq 0$ (resp. $x \geq 0$). We can now achieve the proof. Taking the original system, if the arc $\gamma_-\gamma_+\gamma_-$ joining 0 to B which is the optimal policy for the unconstrained problem is contained in $c \leq 0$, it is time minimal for the constrained case and the boundary arc is not optimal. On the opposite, we can join 0 to B by an arc $\gamma_+\gamma_-\gamma_+$ in $c \leq 0$, but this arc is time maximal. Hence clearly the boundary arc γ_b is optimal.

In this case the optimal synthesis follows easily. Indeed take two points $B_1 < 0 < B_2$ of the boundary arc and consider the arcs $\gamma_-\gamma_+$ arriving at B_1 and $\gamma_+\gamma_-$ departing from B_2, this gives us the local optimal synthesis represented on Fig. 19. □

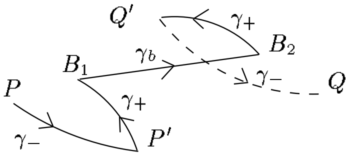


FIG. 19.

4.2.1. Connection with the minimum principle and geometric interpretation.

From Lemma 2.5 we have

$$\eta = \frac{\langle p, [X,[X,Y]](q)\rangle + u_b \langle p, [Y,[X,Y]](q)\rangle}{([X,Y]c)(q)}$$

where we can assume $[Y,[X,Y]] \equiv 0$. Moreover $[X,[X,Y]] = aX + bY + c[X,Y]$ and $\langle p, Y\rangle = \langle p, [X,Y]\rangle = 0$ along the boundary, hence we have:

$$\eta = \frac{a\langle p, X(q)\rangle}{([X,Y]c)(q)}$$

In the normal form $[X,Y]c = -a_3 < 0$ and $\langle p, X\rangle < 0$ by extremality. Hence the necessary condition $\eta \geq 0$ tells us $a \geq 0$. In this case $\gamma_+\gamma_-\gamma_+$ is the optimal policy of the non constrained problem and from Fig. 18 it is contained in $x \geq 0$. *Hence the necessary optimality condition $\eta \geq 0$ is violated if $a < 0$ in the normal form, when the boundary arc is not optimal.*

Observe that also the jump ν_1 and ν_2 of the adjoint vectors at the entry point B_1 and at the exit point B_2 are given by the extremals arcs of the non constrained problem joining respectively P to B_1 and B_2 to Q. This will be used later in the multiple shooting algorithm.

4.3. Connecting two constraints of order 2 in the parabolic case. If there are two constraints in a small neighborhood of a point q_0, one needs to describe the transition between the two constraints. Hence we give a geometric normal form to analyze such a transition together with the optimal strategy.

PROPOSITION 4.5. *Consider the control system $\dot{q} = X + uY$, $|u| \leq 1$, $q \in \mathbb{R}^3$ with two distinct constraints $c_i(q) \leq 0$, $i = 1, 2$. Assume that Assumptions 1 and 2 of Theorem 4.4 are satisfied for both constrained system and denote γ_b^1, γ_b^2 the respective boundary arcs. Moreover assume that the boundary arcs are optimal. Take a small neighborhood U of 0 containing subarcs of both γ_b^1 and γ_b^2, and assume that γ_b^1 hits the boundary $c_2 = 0$. Then there exists a geometric model of the form:*

$$\begin{aligned}
\dot{x} &= a_1 x + a_3 z \\
\dot{y} &= 1 + b_1 x + b_3 z \\
\dot{z} &= c + u + c_1' x + c_2' y + c_3' z
\end{aligned}$$

where the constraints are given by $c_1(q) = x$ and $c_2(q) = x + \varepsilon y$, with ε small. Moreover the optimal policy near $q_0 = 0$ with boundary arcs are of the form $\gamma_+\gamma_-^T\gamma_b^1\gamma_-^T\gamma_b^2\gamma_-^T\gamma_+$ where the intermediate arc γ_-^T is the only arc tangent to both constraints.

Proof. The first constrained system is normalized by Lemma 4.3 as:

$$\begin{aligned}
\dot{x} &= a_1 x + a_3 z \\
\dot{y} &= 1 + b_1 x + b_3 z \\
\dot{z} &= (c + u) + c_1' x + c_2' y + c_3' z, \qquad |u| \leq 1
\end{aligned}$$

and $c_1(q) = x \leq 0$. Since the constraint c_2 is of order 2, we have $\dfrac{\partial c_2}{\partial z} = 0$ and we can set $c_2(q) = d_1 x + d_2 y$, at first order. Observe that $c_2 = c_1$ if and only if $d_2 = 0$. Hence proceeding by perturbation and assuming that the arc hits the boundary of $c_2 \leq 0$, we can set:

$$c_2(q) = x + \varepsilon y, \text{ where } \varepsilon > 0 \text{ is small}.$$

The arcs γ_- tangent γ_b^1 identified to $t \mapsto (0, t, 0)$ are approximated by

$$\begin{aligned}
x(t) &= a_3(c_0 + c_2 y_0 + u)t^2/2 + o(t^2) \\
y(t) &= (t + y(0)) + o(t) \\
z(t) &= (c + c_2 y_0 + u)t + o(t) ,
\end{aligned}$$

and to make the connection with γ_b^2 at time t, we must have:

$$x(t) + \varepsilon y(t) = 0 , \qquad \dot{x}(t) + \varepsilon \dot{y}(t) = 0$$

This gives us two conditions which geometrically means that we must construct near 0 an arc γ_- tangent to both constraints. The equations have near 0 a unique solution parametrized by $(0, y(0), 0)$, where $y(0) < 0$ is the exit point of γ_b^1 and t the time to reach the arc γ_b^2. The estimates are

$$y(0) \sim \frac{\varepsilon}{2a_3(c+u)} \quad , \quad t \sim -\frac{\varepsilon}{a_3(c+u)} \ .$$

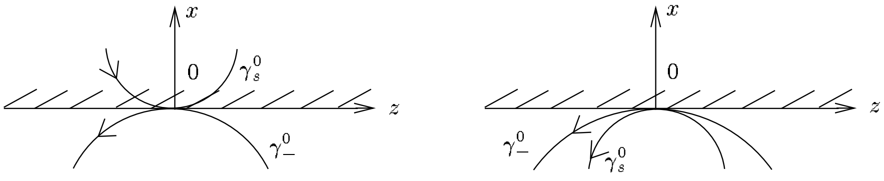
Also observe that from the practical point of view, that the construction can be extended on each domain where there exists an unique arc γ_- tangent to the constraints. □

4.4. The constrained hyperbolic case. Let γ_s^0 be the singular arc through $q_0 = 0$ and let u_s^0 be the associated singular control. We assume that γ_s^0 is not tangent to the boundary arc. Changing if necessary u into $-u$, we can assume that the arc γ_-^0 through 0 is contained in the admissible domain identified to $x \leq 0$ and the model is:

$$\begin{aligned} \dot{x} &= a_1 x + a_3 z + \dots \\ \dot{y} &= 1 + b_1 x + b_3 z + \dots \\ \dot{z} &= (c+u) + \dots , \qquad x \leq 0, \qquad a_3 > 0 \end{aligned}$$

Where $t \mapsto (0, t, 0)$ is the boundary arc.

we have two situations represented by projection in the plane (x, z) on Fig. 20. The case 20(a) corresponds to $c + u_s^0 > 0$ and the case 20(b) to $c + u_s^0 < 0$.



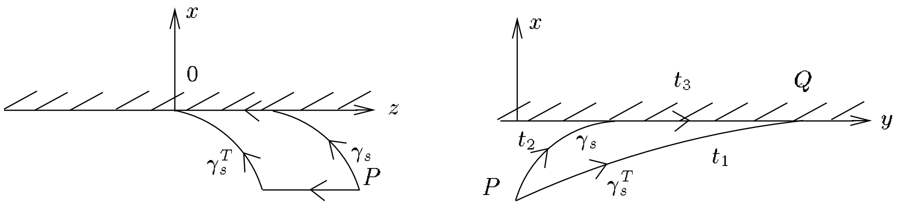
(a) (b)

FIG. 20.

In the first case, the arc γ_s^0 is not admissible and the local synthesis follows easily. Indeed take two points $0 = (0, 0, 0)$ and $B = (0, y, 0)$, with $y > 0$, on the boundary. According to [18], the optimal synthesis to join the two points in the unconstrained cases is $\gamma_- \gamma_s \gamma_-$ and is contained in the domain $x \leq 0$. Hence the boundary arc is not optimal. Moreover a computation similar to the parabolic case proves that the necessary condition $\eta \geq 0$ along the boundary of the minimum principle is violated. Hence we have proved:

LEMMA 4.6. *With our assumption in the hyperbolic case, assume that the singular arc γ_s^0 through 0 is not admissible for the constrained problem. Then the boundary arc is not optimal and the necessary condition $\eta \geq 0$ in the minimum principle is violated.*

Consider now the second case, where γ_s^0 is contained in $x \leq 0$. To simplify we only analyze the *limit case* $|u| \leq M$ when M tends to ∞. According to the analysis of [5], the singular arc is C^0-locally optimal for the unconstrained problem and the optimal synthesis consists in following a singular arc, with jumps at the extremities along the control direction Y identified to $\frac{\partial}{\partial z}$ to match the boundary conditions. To analyze the constrained case, we make the following reasoning. We denote by γ_s^T the singular arcs tangent to the boundary, which are by assumption contained in $x \leq 0$. Let P be a point of γ_s^T which reaches the boundary at Q. And consider the policy represented on Fig. 21, which consists in jumping at P to another arc γ_s, reaching the boundary and then jumping to the boundary arc and following the latter. If $t_1,\ t_2,\ t_3$ are the times represented on Fig. 21, we have $t_2 + t_3 > t_1$ because γ_s^T is optimal for the unconstrained problem. This proves the following:



(a) (b)

FIG. 21.

LEMMA 4.7. *With our assumption in the hyperbolic case, assume that the singular arc through* $q_0 = 0$ *is admissible. Then the boundary arc is optimal and for the limit case, each optimal trajectory with boundary is of the form* $\mathbb{R}Y\gamma_s^T\gamma_b\gamma_s^T\mathbb{R}Y$*, where* $\mathbb{R}Y$ *represents jumps in the control direction to match the boundary conditions.*

To study the case $|u| \leq 1$, we first describe the local time optimal synthesis at time T, T small enough, starting from $q_0 = 0$. For the sake of simplicity we adopt the point of view of [18] and we restrict our attention on the so called "free" nilpotent case, where $[X, [X, Y]] = 0$

$$\dot{x} = z\ ,\dot{y} = 1 - 2cx - z^2\ ,\dot{z} = c + u \tag{4.1}$$

with $|c| < 1$ and the constraint given by $x \leq 0$. Then the singular control is $u_s = 0$ and is strictly admissible. The boundary arc is $t \mapsto (0, t, 0)$ and is associated with the control $u_b = -c$. The singular arc is not tangent to the boundary arc and since we assume that γ_s^0 is admissible (see Fig. 20(a)), that means $c < 0$.

LEMMA 4.8. *Under our assumptions, the boundary arc is the optimal policy to join two points* $A = (0, a, 0)$ *and* $B(0, b, 0)$ *of the boundary.*

Proof. According to [18], extremal trajectories without boundary arc are of the form $\gamma_\pm\gamma_s\gamma_\pm$. None of these extremal trajectories allows a junction between A and B staying in the admissible domain $x < 0$. Hence the boundary arc is optimal. $\square$

To describe the small time optimal syntheses let us define, for T small enough, the sets:

$$\begin{array}{ll} \Gamma_{bs-} & = \{\exp t_3 Z_- \exp t_2 Z_s \exp t_1 Z_b(0), t_1 + t_2 + t_3 = T\} \bigcap \{x < 0\} \\ \Gamma_{bs+} & = \{\exp t_3 Z_+ \exp t_2 Z_s \exp t_1 Z_b(0), t_1 + t_2 + t_3 = T\} \bigcap \{x < 0\} \\ \Gamma_{-s-} & = \{\exp t_3 Z_- \exp t_2 Z_s \exp t_1 Z_-(0), t_1 + t_2 + t_3 = T\} \\ \Gamma_{-s+} & = \{\exp t_3 Z_+ \exp t_2 Z_s \exp t_1 Z_-(0), t_1 + t_2 + t_3 = T\} \bigcap \{x < 0\} \end{array} \tag{4.2}$$

where Z_b, Z_s and $Z_\pm$ corresponds to $X + uY$ with u respectively equal to u_b the boundary control, u_s the singular control and $u = \pm 1$ the regular controls.

LEMMA 4.9. *Under our assumptions, each small time optimal trajectory starting from* 0 *is of the form* $\gamma_b\gamma_s^T\gamma_\pm$ *or* $\gamma_-^0\gamma_s\gamma_\pm$. *Moreover, each of the sets defined in (4.2) is homeomorphic to a disk. Furthermore we have:*

$$\begin{array}{ll}
\Gamma_{bs-}\bigcap\Gamma_{bs+} & =\Gamma_{bs}=\{\exp t_2Z_s\exp t_1Z_b(0), t_1+t_2=T\}\\
\Gamma_{bs-}\bigcap\Gamma_{-s-} & =\Gamma_{s-}=\{\exp t_2Z_-\exp t_1Z_s(0), t_1+t_2=T\}\\
\Gamma_{bs-}\bigcap\Gamma_{-s+} & =\Gamma_s\bigcup\Gamma_-=\{\exp tZ_s(0), t=T\}\bigcup\{\exp tZ_-(0), t=T\}\\
\Gamma_{bs+}\bigcap\Gamma_{-s-} & =\Gamma_s\\
\Gamma_{bs+}\bigcap\Gamma_{-s+} & =\Gamma_{s+}=\{\exp t_2Z_+\exp t_1Z_s(0), t_1+t_2=T\}\bigcap\{x<0\}
\end{array}\qquad(4.3)$$

These relation are described in Fig. 22, by projection in the plane (x, z).

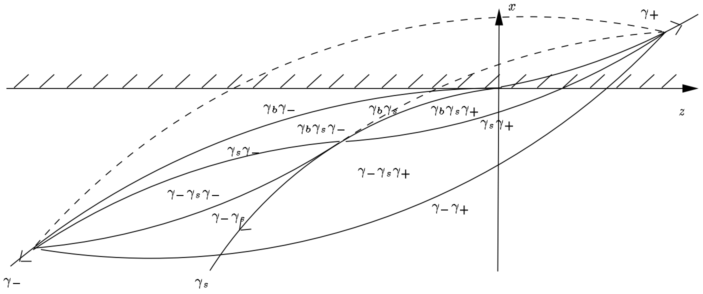


FIG. 22.

Proof. According to [18], extremal trajectories without boundary arc are of the form $\gamma_\pm\gamma_s\gamma_\pm$. In the same way, after leaving a boundary arc an extremal trajectory is of the form $\gamma_\pm\gamma_s\gamma_\pm$. Moreover, an extremal trajectory for the constrained problem can not leave the boundary arc with an arc γ_+ which is not contained in the admissible domain. Let us also prove that if an extremal trajectory leaves a boundary arc with an arc γ_- then there is no more switching. Indeed, writing the necessary conditions of Proposition 2.1 with our system (4.1), we obtain for the adjoint system:

$$\dot{p}_x = 2cp_y - \eta,\ \dot{p}_y = 0,\ \dot{p}_z = -p_x + 2zp_y.$$

Along a boundary arc $p_x = p_y = 0$, $p_y < 0$ and $\eta = 2cp_y > 0$. At the exit time t_1, we have $p_x(t_1+) = p_x(t_1-) - \nu_1 = -\nu_1$, $p_y(t_1+) = p_y(t_1-)$, $p_z(t_1+) = p_z(t_1-) = 0$, with $\nu_1 \geq 0$. Therefore, after leaving the boundary with an arc γ_-, the switching function is $\Phi(t) = -p_y(t-t_1)(t-t_1-\nu_1/p_y)$ and is strictly positive for $t > t_1$. This ends the proof of the first part of the lemma. For the second part of the lemma, straightforward computations prove assertions (4.3). □

Solving the same problem backward in time, we obtain the small time optimal syntheses to join $q_0 = 0$.

LEMMA 4.10. *Under our assumptions, each small time optimal trajectory joining* 0 *is of the form* $\gamma_\pm\gamma_s^T\gamma_b$ *or* $\gamma_\pm\gamma_s\gamma_-^0$.

The small time local synthesis near the boundary follows from Lemma 4.9 and Lemma 4.10. The computations are simplified in the "free" nilpotent case but our reasoning uses only the minimum principle and more generally, we deduce the following:

PROPOSITION 4.11. *Consider the hyperbolic case and assume that the singular arc* γ_s^0 *through* $q_0 = 0$ *not tangent to the boundary arc. If* γ_s^0 *is contained in the non* *admissible domain, the boundary arc is not optimal. If* γ_s^0 *is contained in the admissible domain, the boundary arc is locally optimal and the local optimal trajectories with boundary arcs are of the form* $\gamma_\pm \gamma_s^T \gamma_b \gamma_s^T \gamma_\pm$.

4.5. Conclusion. In order to classify the generic situation in dimension 3, the elliptic case has to be analyzed. This is beyond the scope of this article. Indeed in the elliptic case we must introduce second order conditions to compute *conjugate points* along bang-bang trajectories, *in the constrained case* and make a Morse theory. This will be the purpose of a forthcoming paper.

5. Control of the atmospheric arc, multiple shooting algorithm and numerical results.

5.1. A short review on the multiple shooting algorithm. This numerical algorithm is standard, and a classical reference is [26].
Let us consider a *single-input affine control system* in $\mathbb{R}^n$:

$$\dot{q} = X(q) + uY(q)$$

where the control satisfies the constraint:

$$|u| \leq 1$$

and the state q is submitted to a scalar constraint:

$$c(q) \leq 0$$

The problem is to minimize the cost:

$$J(u) = \int_0^{t_f} \varphi(q(t))dt$$

among all trajectories satisfying $q(0) \in M_0, q(t_f) \in M_1$, where M_0 and M_1 are submanifolds in $\mathbb{R}^n$.
Now applying the *minimum principle* stated in Section 2.3.1 one comes to a *boundary value problem* of the following type:

$$\dot{z}(t) = F(z(t), t) = \begin{cases} F_0(t, z(t)) & \text{if } t_0 \leq t < t_1 \\ F_1(t, z(t)) & \text{if } t_1 \leq t < t_2 \\ \vdots & \\ F_s(t, z(t)) & \text{if } t_s \leq t \leq t_f \end{cases} \tag{5.1}$$

where $z = (q, p) \in \mathbb{R}^{2n}$ and $t_1, t_2, \dots, t_s \in [t_0, t_f]$ are:

- either *switching times*, i.e. times at which the shooting function $\Phi(t) = \langle p(t), Y(q(t)) \rangle$ vanishes, and hence the control $u(t)$ may pass for instance from -1 to $+1$,
- either *junction times*, i.e. times at which the trajectory joins a boundary arc,
- either *contact times*, i.e. times at which the trajectory only touches the boundary.

Moreover the following conditions hold at these points:

$$\forall j \in \{1, \dots, s\} \quad r_j(t_j, z(t_j^-), z(t_j^+)) = 0 \tag{5.2}$$

Among these conditions we have:

- *continuity conditions* on the state and costate at switching points,
- at junction and contact points: *continuity conditions* on the state, *jump conditions* on the costate, and *conditions on the constraint* c.
  For instance if one joins a boundary arc *with order* p then the following p conditions hold at that point:

$$c = \dot{c} = \cdots = c^{(p-1)} = 0$$

Finally, we have the *boundary conditions*:

$$r_{s+1}(t_f, z(t_0), z(t_f)) = 0 \tag{5.3}$$

which contain:

- the initial and final conditions on the state,
- the initial and final conditions on the costate, given by the minimum principle (for instance if the component $q_i(t_0)$ of the state is free then the corresponding component of the costate vector is equal to zero at time t_0),
- if the final time t_f is not fixed then the Hamiltonian vanishes at time t_f.

Remark. A priori t_f is unknown. On the other part in the multiple shooting method *the number* s *of switchings has to be fixed* and must be deduced from a geometric analysis of the problem.

The multiple shooting method consists in subdividing the interval $[t_0, t_f]$ in N subintervals, where the value of $z(t)$ at the beginning of each subinterval is unknown. More precisely, let $t_0 < \sigma_1 < \cdots < \sigma_k < t_f$ be a *fixed* subdivision of the interval $[t_0, t_f]$. At each point σ_j the function z is *continuous*. We can consider σ_j as a fixed switching point at which the following conditions hold:

$$\begin{cases} z(\sigma_j^+) = z(\sigma_j^-) \\ \sigma_j = \sigma_j^* \ \text{ fixed} \end{cases}$$

Now introduce the *nodes*:

$$\{x_1, \ldots, x_m\} \ = \ \{t_0, t_f\} \cup \{\sigma_1, \ldots, \sigma_k\} \cup \{t_1, \ldots, t_s\} \tag{5.4}$$

We arrive at the following *boundary value problem*:

$$\begin{aligned} &\bullet \ \dot{z}(t) = F(t, z(t)) = \begin{cases} F_1(t, z(t)) & \text{if } x_1 \leq t < x_2 \\ F_2(t, z(t)) & \text{if } x_2 \leq t < x_3 \\ \vdots & \\ F_{m-1}(t, z(t)) & \text{if } x_{m-1} \leq t \leq x_m \end{cases} \\ &\bullet \ \forall j \in \{2, \ldots, m-1\} \quad r_j(x_j, z(x_j^-), z(x_j^+)) = 0 \\ &\bullet \ r_m(x_m, z(x_1), z(x_m)) = 0 \end{aligned} \tag{5.5}$$

where $x_1 = t_0$ is fixed and $x_m = t_f$.

Remark. The stability of the method can be improved by increasing the number of nodes. Indeed the principle of the method is to overcome the unstability of a simple shooting method where the influence of inaccurate initial data can grow exponentially with the length t_f-t_0, see [26].

Set $z_j^+ = z(x_j^+)$, and let $z(t, x_{j-1}, z_{j-1}^+)$ denote the solution of the Cauchy problem:

$$\dot{z}(t) = F(t, z(t)),\ z(x_{j-1}) = z_{j-1}^+$$

We have:

$$z(x_j^-) = z(x_j^-, x_{j-1}, z_{j-1}^+)$$

The interior and boundary conditions can be rewritten as:

$$\begin{aligned} \forall j \in \{2, \dots, m-1\} \quad r_j(x_j, z(x_j^-, x_{j-1}, z_{j-1}^+), z_j^+) &= 0 \\ r_m(x_m, z_1^+, z(x_m^-, x_{m-1}, z_{m-1}^+)) &= 0 \end{aligned} \tag{5.6}$$

Now set:

$$Z = (z_1^+, x_m, z_2^+, x_2, \dots, z_{m-1}^+, x_{m-1})^T \ \in \ \mathbb{R}^{(2n+1)(m-1)}$$

(where $z \in \mathbb{R}^{2n}$). Then the previous conditions hold if:

$$G(Z) \ = \ \begin{pmatrix} r_m(x_m, z_1^+, z(x_m^-, x_{m-1}, z_{m-1}^+)) \\ r_2(x_2, z(x_2^-, x_1, z_1^+), z_2^+) \\ \vdots \\ r_{m-1}(x_m, z(x_{m-1}^-, x_{m-2}, z_{m-2}^+), z_{m-1}^+) \end{pmatrix} \ = \ 0 \tag{5.7}$$

The problem is now reduced to *determine a zero* of the function G which is defined on a space vector whose dimension is proportional to the number of switching points and points of the subdivision. The equation $G = 0$ can be solved iteratively with the help of a Newton type method.

We refer to [11, 17] for more details on numerical methods. Our algorithm is written in *Fortran*, and simulations were lead with *Matlab*.

5.2. The atmospheric re-entry problem.

5.2.1. The model. Let 0 be the center of the planet, $K = NS$ is the axis of rotation, Ω is the angular velocity. We denote by $E = (e_1, e_2, e_3)$ with $e_3 = K$, an inertial frame with center 0. The reference frame is the quasi-inertial frame $R_1 = (I, J, K)$ with origin 0, rotating around K, with angular speed Ω and I is chosen to intersect the Greenwich meridian. Let r_T be the radius of the planet, G the center of mass of the shuttle. We note (r, l, L) the spherical coordinates of G, $r \geq r_T$ being the distance OG, $h = r - r_T$ is the altitude, l is the longitude and L is the latitude. We note $R_1' = (e_r, e_l, e_L)$, a moving frame with center G, where e_r is the local vertical, (e_l, e_L) is the local horizontal plane and e_L is pointing to the north. The spherical coordinates have a singularity at the poles.

Let $\xi : t \mapsto (x(t), y(t), z(t))$ be the trajectory of G measured in the quasi-inertial frame attached to the planet and let $\overrightarrow{v} = \dot{x}I + \dot{y}J + \dot{z}K$ be the relative velocity. The vector $\overrightarrow{v}$ is represented by its modulus v and two angles:

- γ: *path inclination* which is the angle with respect to the horizontal plane,
- χ: *azimut angle* which is the angle of the projection of $\overrightarrow{v}$ in the horizontal plane measured with respect to the axis e_L.

We denote by (i, j, k) the orthonormal frame defined by $i = \overrightarrow{v}/v$, j is the unitary vector in the plane (i, e_r) perpendicular to i and oriented by $j.e_r > 0$ and $k = i \wedge j$.

The system is written in the coordinates $(r, v, \gamma, L, l, \chi)$. The forces acting on the vehicle are the gravitational force $\overrightarrow{P} = m\overrightarrow{g}$ and the aerodynamic force which decomposes into a *drag force* $\overrightarrow{D}$ opposite to the relative velocity and a *lift force* $\overrightarrow{L}$ perpendicular to the velocity. Since $q = (r, L, l)$ are measured in a quasi inertial frame, we have additional Coriolis and centripetal forces. The aerodynamics forces have simple expressions in the frame (i, j, k) and from [20] and [12] the equations of the system are:

$$\begin{aligned}
\frac{dr}{dt} &= v\sin\gamma \\
\frac{dv}{dt} &= -g\sin\gamma - \frac{1}{2}\rho\frac{SC_D}{m}v^2 + \Omega^2 r\cos L(\sin\gamma\cos L - \cos\gamma\sin L\cos\chi) \\
\frac{d\gamma}{dt} &= \cos\gamma\left(-\frac{g}{v} + \frac{v}{r}\right) + \frac{1}{2}\rho\frac{SC_L}{m}v\cos\mu \\
&\quad + 2\Omega\cos L\sin\chi + \Omega^2\frac{r}{v}\cos L(\cos\gamma\cos L + \sin\gamma\sin L\cos\chi) \\
\frac{dL}{dt} &= \frac{v}{r}\cos\gamma\cos\chi \\
\frac{dl}{dt} &= \frac{v}{r}\frac{\cos\gamma\sin\chi}{\cos L} \\
\frac{d\chi}{dt} &= \frac{1}{2}\rho\frac{SC_L}{m}\sin\mu\frac{v}{\cos\gamma} + \frac{v}{r}\cos\gamma\tan L\sin\chi + 2\Omega(\sin L - \tan\gamma\cos L\cos\chi) \\
&\quad + \Omega^2\frac{r}{v}\frac{\sin L\cos L\sin\chi}{\cos\gamma}
\end{aligned} \tag{5.8}$$

where μ is the *bank angle*, S is the reference area and C_L, C_D are respectively the lift and drag coefficients depending upon the *angle of attack* α (incidence) and the Mach number. The air density is ρ and we take an exponential model: $\rho(r) = \rho_0 e^{-\beta r}$. For the atmospheric arc the angle of attack α is held constant and *the control is the bank angle* μ. We set $u_1 = \cos\mu$ and $u_2 = \sin\mu$.

5.2.2. Optimal control. The problem is to steer the vehicle from an initial manifold M_0 to a terminal manifold M_1. To be more precise the terminal time t_f is free and the boundary conditions are given in Table 5.1.

	Initial conditions	Terminal conditions
altitude h	119.82 km	15 km
velocity v	7404.95 $m.s^{-1}$	445 $m.s^{-1}$
flight angle γ	-1.84 deg	free
latitude L	0	10.99 deg
longitude l	free	166.48 deg
azimut χ	free	free

TABLE 5.1
Boundary conditions

The state constraints are of the form $c_i(q) \leq 0$, for $i = 1, 2, 3$ and are:

- constraint on the *thermal flux*:

$$\varphi = c_q\sqrt{\rho}v^3 \leq \varphi^{max} ,$$

where c_q is a constant.

- constraint on the *normal acceleration*

$$\gamma_n = \gamma_{n_0}\rho v^2 \leq \gamma_n^{max} ,$$

- constraint on the *dynamic pressure*

$$\frac{1}{2}\rho v^2 \leq P^{max} .$$

They are approximated on the Fig. 23 in the flight domain, in terms of the *drag* $d = \frac{1}{2}\frac{SC_D}{m}\rho v^2$ and v.

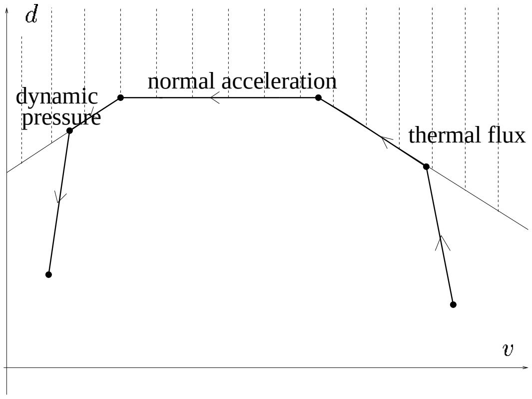


FIG. 23. *Constraints - Harpold and Graves strategy*

The optimal control problem is to minimize the total amount of the thermal flux:

$$J(\mu) = \int_0^{t_f} c_q\sqrt{\rho}v^3 dt \tag{5.9}$$

If we introduce the new time parameter

$$ds = \varphi dt \tag{5.10}$$

our optimal problem is a *time minimizing problem.*

5.2.3. Harpold and Graves strategy [14]. If we use the approximation $\dot{v} = -d$, the cost can be written:

$$J(\mu) = K\int_{v_f}^{v_0} \frac{v^2}{\sqrt{d}}dv, \; K > 0$$

and the optimal strategy is to maximize during the flight the drag d. This is the policy described in [14] which reduces the problem to find a system trajectory to track the boundary of the domain in the following order: thermal flux → normal acceleration → dynamic pressure, see Fig. 23.

5.2.4. Properties and structure of the system. The problem is to minimize time for a system of the form:

$$\frac{dq}{dt} = X(q) + u_1Y_1(q) + u_2Y_2(q) ,$$

where $u_1 = \cos\mu$, $u_2 = \sin\mu$ and $q = (r, v, \gamma, L, l, \chi)$. If we set $q_1 = (r, v, \gamma)$ and $q_2 = (L, l, \chi)$, the system can be decomposed into:

$$\dot{q}_1 = F_1(q_1, u_1) + O(\Omega),\ \dot{q}_2 = F_2(q, u_2)\ .$$

The first system governing the *longitudinal motion* is given by

$$\begin{aligned}
\frac{dr}{dt} &= (v\sin\gamma)\psi \\
\frac{dv}{dt} &= -\left(g\sin\gamma + \frac{1}{2}\rho\frac{SC_D}{m}v^2\right)\psi + o(\Omega) \\
\frac{d\gamma}{dt} &= \left(\cos\gamma\left(-\frac{g}{v} + \frac{v}{r}\right) + \frac{1}{2}\rho\frac{SC_L}{m}v\cos\mu + 2\Omega\cos L\sin\chi\right)\psi + o(\Omega)
\end{aligned} \tag{5.11}$$

and the second system governing the *lateral motion* is:

$$\begin{aligned}
\frac{dL}{dt} &= \left(\frac{v}{r}\cos\gamma\cos\chi\right)\psi \\
\frac{dl}{dt} &= \left(\frac{v}{r}\frac{\cos\gamma\sin\chi}{\cos L}\right)\psi \\
\frac{d\chi}{dt} &= \left(\frac{1}{2}\rho\frac{SC_L}{m}\sin\mu\frac{v}{\cos\gamma} + \frac{v}{r}\cos\gamma\tan L\sin\chi\right)\psi \\
&\quad + 2\Omega(\sin L - \tan\gamma\cos L\cos\chi)\psi + o(\Omega)
\end{aligned} \tag{5.12}$$

where $\psi = 1/\varphi$ and the centripetal force $o(\Omega)$ is neglected.

For the control of the atmospheric arc, one of the main problem in the flight domain is to avoid violation of the constraint thermal flux in the first part of the arc and this requires a careful analysis of the longitudinal motion. Omitting the Coriolis and centripetal terms, (5.11) is a scalar input affine system of the form

$$\dot{q} = X + uY,\ |u| \leq 1,\quad \text{where } q = (r, v, \gamma) \in \mathbb{R}^3\ ,$$

and the state constraints are of the form $c_i(q) \leq 0,\ i = 1, 2, 3$.

5.2.5. Lie bracket properties of the longitudinal motion and constraints. Consider the system $\dot{q} = X + uY,\ q = (x, y, z) \in \mathbb{R}^3$, where

$$\begin{aligned}
X &= \psi\left(v\sin\gamma\frac{\partial}{\partial r} - (g\sin\gamma - k\rho v^2)\frac{\partial}{\partial v} + \cos\gamma\left(-\frac{g}{v} + \frac{v}{r}\right)\frac{\partial}{\partial\gamma}\right)\ , \\
Y &= \psi k'\rho v\frac{\partial}{\partial\gamma}\ ,
\end{aligned}$$

which describes the longitudinal motion when the rotation of the Earth is neglected $(\Omega = 0)$ and $g = g_0/r^2$ is assumed to be constant. The following results, coming from computations, are crucial:

LEMMA 5.1. *In the flight domain where* $\cos\gamma \neq 0$*, we have:*

1. X, Y, $[X, Y]$ *are linearly independent.*
2. $[Y, [X, Y]] \in span\{Y,\ [X, Y]\}$.
3. $[X, [X, Y]](q) = a(q)X(q) + b(q)Y(q) + c(q)[X, Y](q)$ *with* $a < 0$.

LEMMA 5.2. *Assuming* C_D *and* C_L *are constants, the constraints are of order* 2 *and Assumption* C_1*, "YXc does not vanish on the boundary", is satisfied in the flight domain.*

5.2.6. Application of the classification to the space shuttle. The constraints are of order 2 and Assumption C_1 is satisfied. In the part of the flight domain where the boundary arc is admissible and not saturating (Assumption C_3), the arc γ_- is violating the constrained along the boundary. Hence we proved (see Theorem 4.4):

COROLLARY 5.3. *Assume* $\Omega = 0$ *and consider the longitudinal motion in the re-entry problem. Then in the flight domain where Assumption* C_3 *is satisfied, a boundary arc is locally optimal and the small time optimal synthesis with fixed boundary conditions on* (r, v, γ) *is of the form* $\gamma_-\gamma_+^T\gamma_b\gamma_+^T\gamma_-$.

5.3. Extremals of the problem.

Preliminaries. First consider the problem without constraint on the state. The Hamiltonian is:

$$H(q,p,u) = \langle p, X(q)\rangle + u_1\langle p, Y_1(q)\rangle + u_2\langle p, Y_2(q)\rangle + p^0\varphi\ ,$$

where $u = (u_1, u_2)$, $u_1 = \cos\mu$, $u_2 = \sin\mu$ and

- $p = (p_r, p_v, p_\gamma, p_L, p_l, p_\chi)$ is the vector dual to the state q,
- p^0 is the dual component of the flux.

If the trajectories are parametrized by $ds = \varphi(q)dt$, the optimal problem is a time minimization problem. The control domain is $u_1^2 + u_2^2 = 1$ and the optimal control problem is *not convex* and can be relaxed by taking $u_1^2 + u_2^2 \leq 1$ in order to ensure the existence of optimal solutions. According to the minimum principle the optimal controls have to minimize $u \mapsto H(q,p,u)$ over $u_1^2 + u_2^2 = 1$. Hence outside the switching surface Σ: $\langle p, Y_1\rangle = \langle p, Y_2\rangle = 0$, an extremal control is given by:

$$\begin{aligned} u_1 = \cos\mu &= -\frac{\langle p, Y_1\rangle}{\sqrt{\langle p, Y_1\rangle^2 + \langle p, Y_2\rangle^2}} = -\frac{\cos\gamma\, p_\gamma}{\sqrt{\cos^2\gamma\, p_\gamma^2 + p_\chi^2}}, \\ u_2 = \sin\mu &= -\frac{\langle p, Y_2\rangle}{\sqrt{\langle p, Y_1\rangle^2 + \langle p, Y_2\rangle^2}} = -\frac{p_\chi}{\sqrt{\cos^2\gamma\, p_\gamma^2 + p_\chi^2}} \end{aligned} \qquad (5.13)$$

The corresponding extremals are called *regular* and the extremals contained in the switching surface are called *singular*. Due to the existence of singularities, the behavior of regular extremals is complex and the analysis is outlined in [7].

The system parametrized by s can be written:

$$\frac{dq}{ds} = \tilde{X}(q) + u_1\tilde{Y}_1(q) + u_2\tilde{Y}_2(q) \qquad (5.14)$$

where $\tilde{X} = \varphi^{-1}X$, $\tilde{Y}_1 = \varphi^{-1}Y_1$, $\tilde{Y}_2 = \varphi^{-1}Y_2$ and set $\tilde{F}(q,\mu) = \tilde{X}(q) + \cos\mu\,\tilde{Y}_1(q) + \sin\mu\,\tilde{Y}_2(q)$.

Definitions. We denote by $E : u(.) \mapsto q(t, q_0, u)$ the end-point mapping of system (5.14) and $E' : \mu(.) \mapsto q(t, q_0, \mu)$ the end-point mapping associated to $\dot{q} = \tilde{F}(q,\mu)$. Observe that if e is the mapping $\mu(.) \mapsto (\cos\mu, \sin\mu)$, we have: $E' = E \circ e$. If we endow the set of inputs with the L^∞ norm topology, both mapping E and E' are Fréchet differentiable and they have inputs where the Fréchet derivatives are singular, that is not surjective. The following result is standard.

LEMMA 5.4. *The regular extremals are the singularities of* E' *and the singular extremals are the singularities of* E.

Hence regular extremals are the singular trajectories of the system $\dot{q} = \tilde{F}(q,\mu)$. They are depending upon the system (5.14) and the constraint: $u_1^2 + u_2^2 = 1$. The reduced Hamiltonian takes the form

$$\tilde{H}(q,p,\mu) = \langle p, \tilde{F}(q,\mu)\rangle \ .$$

The extremals contained in $\tilde{H} = 0$ are called *exceptional*. The optimality status of singular trajectories in the time optimal control problem can be investigated using the Morse theory developed in [5] under generic assumptions, see also [24], [2] and [30]. This requires the computation of the second order derivative of E' to evaluate the conjugate points. This computation is simplified along singular trajectories corresponding to constant controls. The algorithm is given in [4].

The computations of singular extremals which depend only upon the system can be reduced to a calculation of [7] if we assume $\Omega = 0$. Indeed in this case the axis NS is arbitrary and the system decomposes into a system in dimension 5 of the form $\dot{q}' = \tilde{X}' + u_1\tilde{Y}_1' + u_2\tilde{Y}_2'$ where $q' = (r, v, \gamma, L, \chi)$ and $\dot{l} = F(q')$. From [7] we have:

LEMMA 5.5. *Assume to simplify $g = g_0$. The singular trajectories of $\dot{q}' = \tilde{X}' + u_1\tilde{Y}_1' + u_2\tilde{Y}_2'$ are located at $\chi = k\pi$ with $u_2 = 0$ and are the singular trajectories of single input system $\dot{q}' = \tilde{X}' + u_1\tilde{Y}_1'$.*

The computation in the single input case is standard. Now, making the axis NS vary, we generate all the singular extremals for the full system, assuming $\Omega = 0$. If the Earth rotation is not neglected, the poles are fixed.

5.4. Reduction procedure. In order to implement the multiple shooting algorithm we design a quasi-optimal trajectory based on the following reasoning:

Sling effect: According to our numerical data, every trajectory starting from our initial conditions violates the constraint on the thermal flux if Ω is taken as 0. Hence the Coriolis force which dominates at the beginning the centripetal force has to be used in the first part of the trajectory to compensate the gravitation and to track the boundary arc. Moreover the Coriolis component in the longitudinal motion is given by $F_c = 2\Omega\cos L \sin\chi$ and is maximized for $L = 0$ and $\chi = \pi/2$. Therefore since $L(0) = 0$, we choose $\chi(0) \simeq \pi/2$.

Embedding procedure: if $\Omega = 0$, taking into account the structure of the system, we observe that if we relax the boundary conditions on χ, L and l then the adjoint vector is such that $p_\chi \equiv p_L \equiv p_l \equiv 0$, and the problem is reduced to an optimal control problem for an affine single input control system in dimension 3, describing the longitudinal motion. Hence the boundary arcs are small time optimal and the local optimal synthesis has been computed in Section 4 for fixed boundary conditions. Since at the end γ is free, a quasi-optimal trajectory is of the form:

$$\gamma_- \gamma_+^T \gamma_{\text{flux}} \gamma_+^T \gamma_{\text{acc}} \gamma_+^T \ ,$$

where $\gamma_\pm$ are arcs associated to $u_1 = \cos\mu = \pm 1$, $u_2 = \sin\mu = 0$ and γ_{flux}, γ_{acc} are boundary arcs, corresponding respectively to the constraint on the thermal flux and on the normal acceleration, the constraint on the dynamic pressure being not active. The terminal latitude is adjusted using a small variation of $\chi(0)$ near $\pi/2$.

This strategy is only an *approximation of the optimal policy* for two reasons. First of all, since Ω cannot be neglected in the first part of the trajectory our policy is not extremal. But it can be checked numerically that an extremal policy is such that $|\cos\mu| \simeq 1$, $\sin\mu \simeq 0$ since $|p_\gamma| \gg |p_\chi|$, except during short durations corresponding to switchings. Secondly, the transfer time has to be supposed small enough to ensure optimality. Otherwise we must estimate conjugate and cut points, see [4] for details.

5.5. Conclusion. Having selected such a policy the exact switching times are computed using our multiple shooting algorithm, *implemented without using the extremal system*. This is realized in the next section.

5.6. Numerical simulations and results. Switching times and initial values of latitude, longitude and azimut have to be determined by the multiple shooting method. More precisely:

- The first switching time, from γ_- to γ_+, allows to adjust the entry in the iso-flux phase, which is characterized by $\varphi = \varphi^{\max}, \dot{\varphi} = 0$.
- The third switching time, from γ_{flux} to γ_+, is used to adjust the entry in the iso-normal acceleration phase.
- The fifth switching time, from γ_{acc} to γ_+, permits to adjust the final velocity $v(t_f)$.
- The initial azimut $\chi(0)$ is used to adjust the terminal latitude $L(t_f)$.

On the other part the final time is determined by the final altitude.

Results are drawn on Fig. 24 and Fig .25.

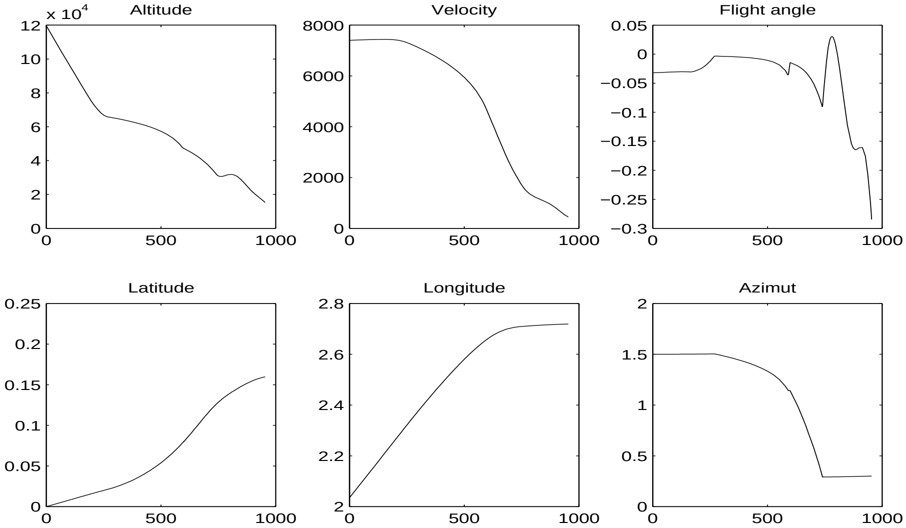


FIG. 24. *State coordinates.*

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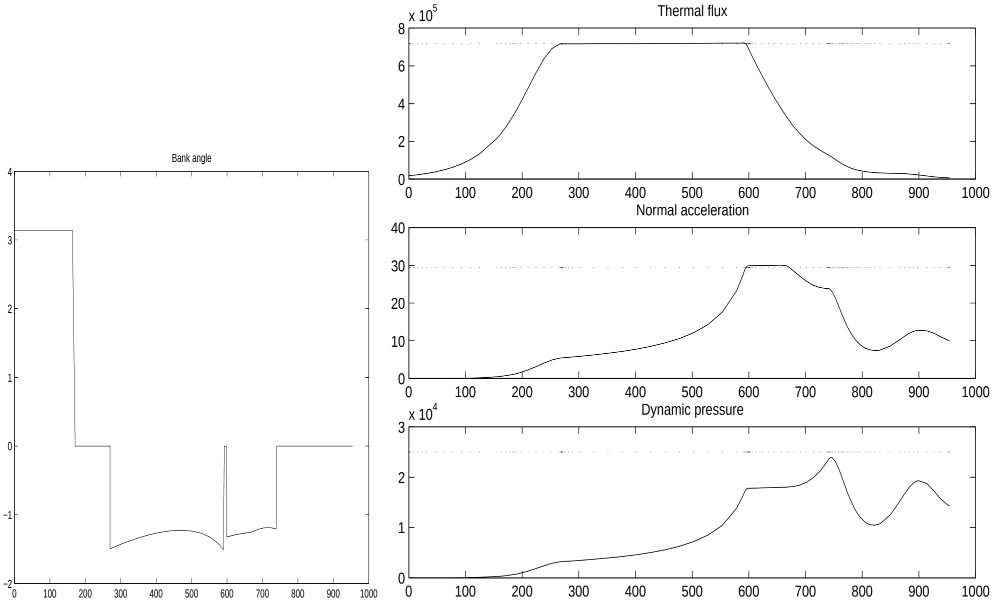

FIG. 25. *Bank angle and state constraints.*

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