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**Géométrie analytique/*Analytic geometry***

# Non subanalyticity of sub-Riemannian Martinet spheres

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**Abstract.** Consider the sub-Riemannian Martinet structure $(M,\Delta,g)$ where $M=\mathbb{R}^3$, $\Delta=\mathrm{Ker}(dz-\frac{y^2}{2}dx)$ and $g$ is the general gradated metric of order $0$ : $g=(1+\alpha y)^2dx^2+(1+\beta x+\gamma y)^2dy^2$. We prove that if $\alpha\neq 0$ then the sub-Riemannian spheres $S(0,r)$ with small radii are not subanalytic. © Académie des Sciences/Elsevier, Paris

***Non sous-analyticité des sphères sous-Riemanniennes de Martinet***

**Résumé.** *Considérons la structure sous-Riemannienne de Martinet* $(M,\Delta,g)$ *où* $M=\mathbb{R}^3$, $\Delta=\mathrm{Ker}(dz-\frac{y^2}{2}dx)$ *et* $g$ *est la métrique générale graduée d'ordre* $0$ *:* $g=(1+\alpha y)^2dx^2+(1+\beta x+\gamma y)^2dy^2$. *On montre que si* $\alpha\neq 0$, *les sphères sous-Riemanniennes* $S(0,r)$ *de petit rayon ne sont pas sous-analytiques.* © Académie des Sciences/Elsevier, Paris

## Version française abrégée

Une *structure sous-Riemannienne* $(M,\Delta,g)$ est constituée d'une variété $M$ de dimension finie, d'une distribution $\Delta$ sur $M$, et d'une métrique $g$ sur $\Delta$. Elle est dite analytique si tous les objets sont analytiques. Une courbe $\gamma$ sur $M$ est dite *horizontale* si elle est tangente en tout point à la distribution $\Delta$. Soient $x_0,x_1$ des points de $M$, et $T>0$. Le *problème sous-Riemannien* est de déterminer une courbe horizontale joignant $x_0$ à $x_1$ en temps $T$ et minimisant la longueur (au sens de la métrique $g$). La *distance sous-Riemannienne* $d_{SR}(x_0,x_1)$ entre $x_0$ et $x_1$ est la borne inférieure des longueurs des chemins horizontaux joignant $x_0$ à $x_1$ en temps $T$. La *sphère sous-Riemannienne* $S(x_0,r)$ centrée en $x_0$, de rayon $r$, est l'ensemble des points $x$ de $M$ tels que $d_{SR}(x_0,x)=r$.

Reformulons le problème sous-Riemannien dans le cadre de la théorie du contrôle optimal. Notre point de vue étant local, on peut supposer que $M=\mathbb{R}^n$ et $x_0=0$. Supposons que la distribution $\Delta$ est engendrée par $m$ champs de vecteurs $X_1,\dots,X_m$. Sans perte de généralité, on peut supposer que $T=1$. Le problème sous-Riemannien est alors équivalent au problème de *contrôle optimal* pour le système (voir par exemple [5]) :

$$\dot{x}(t)=\sum_{i=1}^{m}u_i(t)X_i(x(t)),\ x(0)=0$$

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**Note présentée par ???**

où la fonction $u=(u_1,\dots,u_m)$, appelée le *contrôle*, doit minimiser la *longueur*

$$l(u)=\int_0^1\sqrt{g(\dot{x}(t),\dot{x}(t))}dt$$

L'application $E\ :u\in L^2([0,1],\mathbb{R}^{\mathrm{m}})\longmapsto x_u(1)\in\mathbb{R}^{\mathrm{n}}$, qui à un contrôle $u$ associe l'extrémité de la solution correspondante $x_u$, est appelée *application entrée/sortie.* C'est une application lisse.

Si un contrôle $u$ minimise la longueur $l$, alors il existe un *multiplicateur de Lagrange* $(\psi,\psi^0)\in\mathbb{R}^{\mathrm{n}}\times\mathbb{R}$ défini à scalaire près tel que

$$\psi.dE(u)=-\psi^0dl(u)$$

Si $\psi^0\neq 0$, le contrôle (ou la trajectoire associée) est dit *normal* ; sinon il est dit *anormal.* Un contrôle anormal est une singularité de l'application entrée/sortie.

Dans [1] il est prouvé que s'il n'existe pas de trajectoire anormale minimisante non triviale partant de 0 alors la distance sous-Riemannienne à 0, $d_{SR}(0,.)$, est sous-analytique en dehors de 0, et donc les sphères sous-Riemanniennes $S(0,r)$ de petit rayon sont sous-analytiques. Cette situation est générique pour des distributions de rang $\geq 3$, voir [3]. Cependant les anormales minimisantes apparaissent génériquement pour des distributions de rang 2, et on conjecture que dans ce cas les sphères sous-Riemanniennes ne sont jamais sous-analytiques. Remarquons que dans [3] les auteurs prouvent que l'existence d'une anormale minimisante est responsable de la non sous-analyticité du germe en 0 de $d_{SR}(0,.)$. Ici la conjecture concerne les sphères sous-Riemanniennes.

Cette conjecture a été prouvée dans [2] dans le cas dit de *Martinet plat*, mais il n'est pas générique. Ici on montre que cette conjecture est vraie pour une métrique générale graduée d'ordre 0. Le cas Martinet est le cas sous-Riemannien le plus simple pour lequel une anormale minimisante apparaît génériquement. La situation est la suivante. Considérons dans $\mathbb{R}^3$ la distribution $\Delta=\mathrm{Ker}(dz-\frac{y^2}{2}dx)$ ; elle est engendrée par les deux champs de vecteurs

$$X=\frac{\partial}{\partial x}+\frac{y^2}{2}\frac{\partial}{\partial z},\quad Y=\frac{\partial}{\partial y}$$

Les variables $x,y$ sont de poids 1, et $z$ est de poids 3 (voir [5]). Soit $g$ une métrique analytique sur $\Delta$ ; d'après [2] une forme normale générale graduée d'ordre 0 est :

$$g=(1+\alpha y)^2dx^2+(1+\beta x+\gamma y)^2dy^2$$

où $\alpha,\beta,\gamma\in\mathbb{R}$. C'est une perturbation du cas plat où $\alpha=\beta=\gamma=0$. L'unique direction anormale correspond à l'axe $x$ ; elle est minimisante [4].

THÉORÈME 1. – *Si $\alpha\neq 0$ alors les sphères sous-Riemanniennes $S(0,r)$ de petit rayon ne sont pas sous-analytiques.*

# 1. Introduction

## 1.1. Sub-Riemannian geometry

A *sub-Riemannian structure* $(M,\Delta,g)$ is a triple, where $M$ is a finite dimensional manifold, $\Delta$ is a distribution on $M$, and $g$ is a metric on $\Delta$. It is said analytic if all objects are analytic. A curve $\gamma$ on $M$ is called *horizontal* if it is tangent at any point to the distribution $\Delta$. Let $x_0,x_1$ in $M$, and $T > 0$. The *sub-Riemannian problem* is to determine an horizontal curve joining $x_0$ and $x_1$ in time $T$ and minimizing the length (in the sense of the metric $g$). The *sub-Riemannian distance* $d_{SR}(x_0, x_1)$ between $x_0$ and $x_1$ is the infimum of lengths of horizontal paths joining $x_0$ and $x_1$ in time $T$. The *sub-Riemannian sphere* $S(x_0, r)$ centered at $x_0$, with radius $r$, is the set of points $x$ in $M$ such that $d_{SR}(x_0, x) = r$.

### 1.2. Necessary optimality conditions

In order to determine optimality conditions for an horizontal curve, we shall reformulate the sub-Riemannian problem in optimal control theory. Our point of view is local, hence we can suppose that $M = \mathbb{R}^n$ et $x_0 = 0$. Assume that the distribution $\Delta$ is spanned by $m$ vector fields $X_1, \ldots, X_m$. Without loss of generality, we may suppose that $T = 1$. Then the sub-Riemannian problem is equivalent to the *optimal control problem* for the system (see for instance [5]) :

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) X_i(x(t)),\ x(0) = 0$$

where the function $u = (u_1, \ldots, u_m)$, called the *control*, has to minimize the *length*

$$l(u) = \int_0^1 \sqrt{g(\dot{x}(t), \dot{x}(t))}dt$$

The mapping $E\ : u \in L^2([0,1], \mathbb{R}^m) \longmapsto x_u(1) \in \mathbb{R}^n$, which to a control $u$ associates the extremity of the corresponding solution $x_u$, is called *end-point mapping*. It is smooth.

If a control $u$ minimizes the length $l$, then there exists a *Lagrange multiplier* $(\psi, \psi^0) \in \mathbb{R}^n \times \mathbb{R}$ defined up to a scalar such that

$$\psi.dE(u) = -\psi^0 dl(u)$$

Moreover from the *Pontryagin's Maximum Principle* (see [9]) the trajectory $x$ corresponding to this minimizing control $u$ is the projection of an *extremal*, i.e. a solution of the Hamiltonian system :

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u),\ \dot{p} = -\frac{\partial H}{\partial x}(x, p, u),\ \frac{\partial H}{\partial u}(x, p, u) = 0$$

where $H(x, p, u) = < p, \sum_{i=1}^{m} u_i X_i(x) > + p^0 g(\dot{x}(t), \dot{x}(t))$ is the Hamiltonian function, $p$ is an absolutely continuous function on $[0, 1]$ called *adjoint vector*, and $p^0$ is a constant. Moreover we have, up to a scalar : $(p(1), p^0) = (\psi, \psi^0)$. If $p^0 \neq 0$, the extremal is said to be *normal*, and we can normalize to $p^0 = -\frac{1}{2}$ ; otherwise it is said *abnormal*. An abnormal control is a singularity of the end-point mapping. An abnormal trajectory is said to be *strict* if it is not the projection of a normal extremal.

### 1.3. Abnormal minimizers and subanalyticity

In [1] it is proved that if there exists no (non trivial) abnormal minimizer starting from 0 then the sub-Riemannian distance to 0, $d_{SR}(0, .)$, is subanalytic outside 0, and hence sub-Riemannian spheres $S(0, r)$ with small radii are subanalytic. This happens generically for distributions of rank $\geq 3$, see [3]. However abnormal minimizers may appear generically for rank 2 distributions, and we conjecture that in this case sub-Riemannian spheres are never subanalytic. Note that in [3] authors prove that the existence of an abnormal minimizer is responsible for non subanalyticity of the germ at 0 of $d_{SR}(0,.)$. Here the conjecture concerns sub-Riemannian spheres.

This conjecture was proved in [2] in the so-called *Martinet flat case*, but this case is not generic. Here we prove that this conjecture is true for a general gradated metric of order 0. The Martinet case is the easier sub-Riemannian case where an abnormal minimizer appears generically. The situation is the following. Consider in $\mathbb{R}^3$ the distribution $\Delta = \mathrm{Ker}(dz - \frac{y^2}{2}dx)$ ; it is spanned by the two vector fields

$$X = \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}$$

Variables $x, y$ have weight 1, and $z$ has weight 3 (see [5]). Let $g$ be an analytic metric on $\Delta$ ; from [2] a general gradated normal form of order 0 is :

$$g = adx^2 + cdy^2$$

where $a = (1+\alpha y)^2, c = (1+\beta x+\gamma y)^2$, and $\alpha, \beta, \gamma$ are real parameters. It is a perturbation of the flat case where $\alpha = \beta = \gamma = 0$. There exists an unique abnormal direction, corresponding to the axis $x$ ; it is minimizing, see [4]. Moreover it is *strict* if and only if $\alpha \neq 0$ (see [2]).

THEOREM 1. – *If $\alpha \neq 0$ (i.e. in the strict case), the sub-Riemannian spheres $S(0,r)$ with small radii are not subanalytic.*

The main idea of the proof is the following. The interest of the previous metric of order 0 is all in the fact that the problem, formulated in the cotangent space of $\mathbb{R}^3$, projects onto the phase plane of a *one-parameter pendulum.* This fact was already used in [2], [6]. In [2], in order to prove that the sphere in the Martinet flat case is not subanalytic, the authors make a direct computation of the sphere. But this is impossible in the general case because equations of extremals given by the Maximum Principle *are no more integrable.* Actually such computations lead to asymptotic expansions of the sphere near the abnormal direction (see [6]), but these expansions are not precise enough to check non subanalyticity. The new idea here is rather *indirect* and consists in using Lagrange multipliers : then proving the theorem amounts to estimating precisely a *first return time near a saddle point* of this one-parameter pendulum and proving that it is not subanalytic.

## 2. Proof of Theorem 1

Consider for the system $\dot{q} = uX(q) + vY(q), q(0) = 0$, the abnormal control $u = -r, v = 0$. It is always minimizing, see [4]. In order to prove the theorem we shall prove that *the intersection of the sphere $S(0,r)$ with the plane $y = 0$ is not a subanalytic curve near the abnormal direction $(Ox)$.* Let $A = (-r, 0, 0)$ be the extremity of the abnormal extremal associated to $u = -r, v = 0$ ; it belongs to the sub-Riemannian sphere $S(0,r)$ with radius $r$. Consider near $A$ the curve $\mathcal{C} = S(0,r) \cap (y=0) \cap (z \leq 0)$. If $\alpha \neq 0$, i.e. if the abnormal is *strict*, this curve is tangent at $A$ to $z = 0$ with a contact of order 2 (see [6] and [11] for a general result). Consider the set of Lagrange multipliers $(\psi(q), \psi^0(q))$ associated to points $q \in \mathcal{C}\backslash\{A\}$. Since these points are reached by normal minimizers, and since Lagrange multipliers are defined up to a scalar, we may suppose that their component $\psi^0$ is equal to $-1/2$. On the other part we easily get from the definition of Lagrange multipliers that vectors $(\psi_x(q), \psi_z(q))$, $q \in \mathcal{C}$ (where $\psi_x, \psi_z$ denote respectively the $x$ and $z$-components of $\psi$), are normal to the curve $\mathcal{C}$ in the plane $y = 0$. Hence to prove that the germ at $A$ of the curve $\mathcal{C}$ is not subanalytic, it is enough to prove that the set $\mathcal{L} = \{(\psi_x(q), \psi_z(q)) \ / \ q \in \mathcal{C}\}$ is not subanalytic.

The method is the following. First note that $\psi(q)$ is equal to $p(1)$ where $p(.)$ is the adjoint vector associated to the normal minimizer steering 0 to $q$. Hence to compute Lagrange multipliers we have to estimate adjoint vectors at final time $t = 1$. The Maximum Principle gives us a parametrization of normal extremals and leads to the following differential system in the Martinet case :

$$\begin{gathered}\dot{x} = \frac{1}{a}(p_x + p_z\frac{y^2}{2}),\quad \dot{y} = \frac{p_y}{c},\quad \dot{z} = \frac{y^2}{2a}(p_x + p_z\frac{y^2}{2})\\ \dot{p_x} = \frac{p_y^2 c_x}{2c^2} + \frac{\left(p_x + p_z\frac{y^2}{2}\right)^2}{2a^2}a_x,\quad \dot{p_y} = \frac{p_y^2 c_y}{2c^2} + \frac{\left(p_x + p_z\frac{y^2}{2}\right)^2}{2a^2}a_y - \frac{\left(p_x + p_z\frac{y^2}{2}\right)}{a}p_z y,\quad \dot{p_z} = 0\end{gathered} \tag{1}$$

We investigate the Lagrange multipliers $(p_x(1), p_z(1))$ as $y(1) = 0$ and $z(1) \leq 0$. Note that $p_z$ is constant along an extremal. Near the point $A$, it happens that $p_z \to +\infty$ ; it is a phenomenon of *non-properness* due to the existence of an abnormal minimizer, see [11]. From [5], extremal controls satisfy $au^2 + cv^2 = r^2$. Then we can set : $r\cos\theta = \sqrt{a}u = \frac{p_x + \frac{y^2}{2}p_z}{\sqrt{a}}$, $r\sin\theta = \sqrt{c}v = \frac{p_y}{\sqrt{c}}$. In particular we introduce $\theta$ which is an angular coordinate of the adjoint vector. This (*analytic*) cylindric change of coordinates on the adjoint vector is classical in sub-Riemannian geometry.

Now proving that the set $\mathcal{L}$ is not subanalytic amounts to exhibiting a non subanalytic relation between $\theta(1)$ and $p_z$ as $p_z \to +\infty$. We proceed as follows. Reparametrizing with $ds = \frac{r\sqrt{p_z}}{\sqrt{a}\sqrt{c}}dt$, first note that, due to the particular form of the metric, the previous equations (1) project onto a *pendulum equation* :

$$\frac{d^2\theta}{ds^2} + \sin\theta + \frac{\alpha^2}{p_z}\sin\theta\cos\theta - \frac{\alpha\beta}{p_z}\sin^2\theta + \frac{\beta}{\sqrt{p_z}}\cos\theta\frac{d\theta}{ds} = 0 \tag{2}$$

It is a *pendulum perturbed* by the small parameter $\varepsilon = \frac{1}{\sqrt{p_z}}$. In this pendulum representation the plane $y = 0$ projects onto the curve $\Sigma$ :

$$\frac{d\theta}{ds} = \frac{\alpha}{\sqrt{p_z}}\cos\theta - \frac{\beta}{\sqrt{p_z}}\sin\theta \tag{3}$$

and moreover the abnormal direction projects onto the *saddle point* $\theta = -\pi$, $\frac{d\theta}{ds} = 0$. Near this saddle point we shall now compute a *first return time* $t_f$ for the section $\Sigma$, in function of $\theta(t_f)$ and of the parameter $p_z$. Then, imposing $t_f = 1$ shall give a relation between $p_z$ and $\theta(1)$, and we shall prove that this relation is not subanalytic.

*Remark 2.1.* – In the pendulum representation the branch $\mathcal{C}$ corresponds to a *local* computation near a saddle point, whereas the branch $S(0,r) \cap (y = 0) \cap (z \geq 0)$ corresponds to a *global* computation of return mapping near the separatrices of the pendulum. The interest of this branch $\mathcal{C}$ is all in the fact that computations are *localized near a saddle*, hence we shall use *normal forms* according to the following Lemma.

LEMMA 2.1. – *Let* $(X_\varepsilon)_{0\leq\varepsilon<1}$ *be a one-parameter analytic family of vector fields in* $\mathbb{R}^2$ *near* 0 *such that for any* $\varepsilon$ : $X_\varepsilon(0) = 0$, *and let* $(u, v)$ *denote standard coordinates in* $\mathbb{R}^2$. *Let* $\mu(\varepsilon)$ *and* $\nu(\varepsilon)$ *be the eigenvalues of* $dX_\varepsilon(0)$. *We suppose that* $\mu(0) = -1$ *and* $\nu(0) = 1$. *Then there exists an analytic germ at* 0 *from* $\mathbb{R}^3$ *to* $\mathbb{R}^2$ *:*

$$\varphi\ : (u, v, \varepsilon) \longmapsto (u_1, v_1) = \varphi(u, v, \varepsilon) \qquad \text{with } \varphi(0, 0, \varepsilon) = 0$$

*such that in the new coordinates* $(u_1, v_1)$ *the system* $(\dot{u}, \dot{v}) = X_\varepsilon(u, v)$ *can be written near* 0 *as :*

$$\dot{u}_1 = \mu(\varepsilon)u_1(1 + \mathrm{o}(\varepsilon)),\ \dot{u}_2 = \nu(\varepsilon)u_2(1 + \mathrm{o}(\varepsilon)) \qquad \text{as } \varepsilon \to 0$$

This Lemma may be proved by adding parameters in the proof of [7], see for instance [8]. To fit in this situation, set $u = \theta + \pi$, $v = \frac{d\theta}{ds}$, and $\varepsilon = \frac{1}{\sqrt{p_z}}$. Then the pendulum equation (2) can be written as the system in $\mathbb{R}^2$ :

$$\frac{du}{ds} = v, \quad \frac{dv}{ds} = \sin u - \alpha^2\varepsilon^2 \sin u \cos u + \alpha\beta\varepsilon^2 \sin^2 u + \beta\varepsilon v \cos u$$

Applying Lemma 2.1, we get in new coordinates $(u_1, v_1)$ :

$$\frac{du_1}{ds} = \mu(\varepsilon)u_1(1 + \mathrm{o}(\varepsilon)), \quad \frac{dv_1}{ds} = \nu(\varepsilon)v_1(1 + \mathrm{o}(\varepsilon))$$

where $\mu(\varepsilon) = 1 + \frac{\beta\varepsilon}{2} + \mathrm{O}(\varepsilon^2)$, $\nu(\varepsilon) = -1 + \frac{\beta\varepsilon}{2} + \mathrm{O}(\varepsilon^2)$, and actually : $u = u_1 + v_1 + \mathrm{o}(\varepsilon)$, $v = u_1 - v_1 + \mathrm{o}(\varepsilon)$. Therefore in the new coordinates the section (3) is $\Sigma\ : v_1 = u_1 + \alpha\varepsilon + \mathrm{o}(\varepsilon)$.

Now the return time $s_f$ can be computed in a standard way :

$$s_f = \int_0^{s_f} ds = \int_{v_1(0)}^{v_1(s_f)} \frac{dv_1}{\nu(\varepsilon)v_1(1 + o(\varepsilon))} = \frac{1}{\nu(\varepsilon)}(1 + o(\varepsilon))\ln\frac{v_1(s_f)}{v_1(0)} \tag{4}$$

On the one part : $v_1(0) = \alpha\varepsilon + \mathrm{o}(\varepsilon)$ ; on the other part we get by integrating the system up to $\mathrm{O}(\varepsilon^{\frac{3}{2}})$ : $\frac{dt}{ds} = \frac{\varepsilon}{r}(1 + \alpha y)(1 + \beta x + \gamma y) = \frac{\varepsilon}{r}\mathrm{e}^{-\beta\varepsilon s} + \mathrm{O}(\varepsilon^{\frac{3}{2}})$. Now claiming that the final $t_f$ is equal to 1, we get :

$$1 = \int_0^{s_f} \frac{dt}{ds} ds = \frac{1 - \mathrm{e}^{-\beta\varepsilon \mathrm{s_f}}}{\beta \mathrm{r}} + \mathrm{O}(\varepsilon^2)$$

And thus : $s_f = -\frac{\ln(1-\beta r)}{\beta\varepsilon} + \mathrm{O}(\varepsilon)$. Putting into (4) we obtain finally :

$$v_1(s_f) = \frac{\alpha\varepsilon}{\sqrt{1 - \beta r}}\mathrm{e}^{\frac{\ln(1-\beta \mathrm{r})}{\beta\varepsilon}}(1 + \mathrm{o}(1)) \qquad \text{as } \varepsilon \to 0$$

In particular $v_1(s_f)$ is not an analytic function in $\varepsilon$ as $\varepsilon \to 0$.

From Lemma 2.1 we know that $v_1(s_f) = An(u(s_f), v(s_f), \varepsilon) \sim \frac{u(s_f) - v(s_f)}{2}$, where $An(.)$ denotes an analytic germ at 0. Moreover, on the section $\Sigma$, we have : $v(s_f) = -\alpha\varepsilon \cos u(s_f) + \beta\varepsilon \sin u(s_f)$. Hence : $v_1(s_f) = An(u(s_f), \varepsilon) = \frac{u(s_f)}{2} + \cdots$. From the Implicit Function Theorem in the analytic class we get : $u(s_f) = An(v_1(s_f), \varepsilon)$. Therefore $u(s_f)$ is not an analytic function in $\varepsilon$, which ends the proof.

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