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Edgeworth and Walras equilibria of an arbitrage-free exchange economy ⋆

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Summary. In this paper, we first give a direct proof of the existence of Edgeworth equilibria for exchange economies with consumption sets which are (possibly) unbounded below. The key assumption is that the individually rational utility set is compact. It is worth noticing that the statement of this result and its proof do not depend on the dimension or the particular structure of the commodity space. In a second part of the paper, we give conditions under which Edgeworth allocations can be decentralized by continuous prices in a finite dimensional and in an infinite dimensional setting. We then show how these results apply to some finance models.

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1 Introduction

Since Hart (1974) [18], one knows that the existence of equilibrium in exchange economies with unbounded below consumption sets requires some nonarbitrage condition. For exchange economies consisting of a finite number of agents and defined on a finite dimensional commodity space, different variants of such a condition and different concepts of arbitrage have been formulated in [23], [24], [25], [27], [33]. The relations between these conditions are studied in [4], [12], [27]. All in turn imply the compactness of the individually rational utility set[1] when the preferences of agents are derived from utility functions. The few papers dealing with the equilibrium existence in an infinite dimensional setting ([9], [10], [11], [13]) assume the compactness of this set. Cheng [10], Chichilnisky and Heal [11], Dana et al. [13] give also sufficient conditions on the primitives of the economy for this condition to be fulfilled. It is worth noticing that this compactness assumption in the utility space is strictly weaker than any compactness assumption in the allocation space.

This nonarbitrage condition is the central assumption of this paper. In order to model asset markets, we consider an exchange economy consisting of m agents, defined on a vector commodity space. Each agent is given with a (possibly unbounded below) consumption set, a utility function representing his preferences on his consumption set, an initial endowment. Our first concern is a direct proof for such models of the existence of Edgeworth equilibria as classically defined by Aliprantis–Brown–Burkinshaw [1]. Since the set of attainable allocations needs not be bounded, this existence cannot be deduced from Debreu and Scarf's theorem [14] or its extensions to an infinite dimensional setting ([1] and [16]). However, given the nonarbitrage condition, this existence is guaranteed under mild assumptions stated independently of the dimension of the commodity space or its particular structure.

The proof of this result is based on an extension to fuzzy coalitions of Scarf's theorem on the nonemptiness of the core of a nontransferable utility game. The arguments of this preliminary result are inspired by a nice paper of Vohra [32] (see also [30]). The notion of balancedness for such a fuzzy game is borrowed from Florenzano [16]. The preliminary result is then applied to a proof (for any integer r) of the nonemptiness of the core of a fuzzy game appropriately associated to the r-replica of the exchange model. Finally, the existence of Edgeworth equilibria is proved using the compactness of the individually rational utility set.

A direct proof of the existence of Edgeworth equilibria opens a room for using core-equilibrium equivalence theorems for proving the existence of

[1] As defined below, the individually rational utility set, sometimes simply called Utility set, is the set of utility vectors in which every agent receives no less than the utility of his initial endowment and no more than the utility of his consumption in some attainable allocation.

Walras equilibria. The second part of the paper is devoted to some core-equilibrium equivalence theorems and to their consequences for the existence of Walras equilibria in asset market models. Obviously, Edgeworth equilibria exist if equilibrium exists. The philosophy of the converse way we offer in this paper is to clearly separate the mild but central assumptions used in the fixed-point based first argument from the peculiarities of the various models of Finance to which this result can be applied when one looks at the decentralization of Edgeworth allocations.

Recall that the purpose of a core-equilibrium equivalence theorem is to show that Edgeworth allocations can be decentralized as quasiequilibria with continuous prices. Unlike the Edgeworth equilibrium existence theorem, the techniques for obtaining the decentralizing continuous prices differ very much according to the dimension and the structure of the commodity (portfolio) space. In the finite dimensional case, the decentralizing vector price is obtained as a tangent linear functional supporting at the origin the set $\mathrm{co}(\bigcup_i \Gamma_i)$, where Γ_i is the set of preferred net trades of the ith consumer. The same argument works in an infinite dimensional setting if the properties of preferred sets allow to use Hahn-Banach's theorem, that is under an interiority assumption. Moreover, it is also under an interiority assumption that the quasi-equilibrium obtained in this way is nontrivial, hence is a good candidate for equilibrium. Without interiority properties, we assume a vector lattice commodity space with a lattice ordered price space and propose several core-equilibrium equivalence results established using properness assumptions on the preferences of the agents. In all cases, adding the assumptions of the core-equilibrium equivalence theorem to the assumptions of the Edgeworth equilibrium existence result allows to extend most of known Walras equilibrium existence results for financial markets.

The paper is organized as follows: in Section 2, we prove the preliminary result. Section 3 contains the main result of the paper, the Edgeworth equilibrium existence result for an economy with (possibly) unbounded below consumption sets. Section 4 is devoted to decentralization results and to their consequences for the existence of Walras equilibria. Section 5 indicates how these results apply to some known models of Finance.

2 A preliminary nonemptiness theorem for the core of a fuzzy game

Let $M = \{1, \ldots, m\}$ be a finite set of players and $T = [0,1]^m \setminus \{0\}$. An element $t \in T$ is interpreted as a *fuzzy coalition*, that is, a vector $t = (t_i)_{i=1}^m$ of rates of participation to the coalition t for the different players.

We consider in this section m-person *fuzzy games* defined by $(\mathcal{T}, V)$ where $\mathcal{T}$ is a *finite* subset of T containing $\bar{1} = (1, \ldots, 1)$ and the canonical base (e^i) of R^m and $V : \mathcal{T} \to R^m$ is a nonempty-valued correspondence. For a fuzzy coalition $t \in \mathcal{T}$, let us denote

$$\mathrm{supp}\, t = \{i \in M \mid t_i > 0\}$$

the set of agents who participate in this coalition.

Definition 2.1 *The fuzzy core of the m-person fuzzy game* $(\mathcal{T}, V)$ *is the set*

$$\mathcal{C}(\mathcal{T}, V) = \{v \in V(\bar{1}) \mid \nexists t \in \mathcal{T} \text{ and } u \in V(t) \text{ s.t. } v_i < u_i, \forall i \in \operatorname{supp} t\}.$$

Consider the following set

$$\Delta^{\mathcal{T}} = \{\lambda = (\lambda_t)_{t \in \mathcal{T}} \mid \lambda_t \geq 0 \text{ and } \sum_{t \in \mathcal{T}} \lambda_t t = \bar{1}\}.$$

It is easily seen that $\Delta^{\mathcal{T}}$ is nonempty.

Definition 2.2 *A m-person fuzzy game* $(\mathcal{T}, V)$ *is said to be balanced whenever for every* $\lambda \in \Delta^{\mathcal{T}}$,

$$\bigcap_{\{t \in \mathcal{T} \mid \lambda_t > 0\}} V(t) \subset V(\bar{1}).$$

The following theorem extends Scarf's theorem [29] as stated by Aliprantis–Brown–Burkinshaw [1]. Some ideas of the proof are due to R. Vohra [32] (see also Shapley and Vohra [30]). This section is devoted to its proof.

Theorem 2.1 *If* $\mathcal{T}$ *is as above and if* $(\mathcal{T}, V)$ *is a balanced m-person fuzzy game such that*

a) each $V(t)$ *is closed,*
b) each $V(t)$ *is comprehensive from below, i.e.,* $u \leq v$ *and* $v \in V(t)$ *imply* $u \in V(t)$,
c) $u \in R^m$, $v \in V(t)$ *and* $u_i = v_i$ $\forall i \in \operatorname{supp} t$ *imply* $u \in V(t)$,
d) for each $t \in \mathcal{T}$ *there exists* $c_t \in R$, *such that* $v \in V(t)$ *implies* $v_i \leq c_t$ *for all* $i \in \operatorname{supp} t$,

then

$$\mathcal{C}(\mathcal{T}, V) \neq \emptyset.$$

Proof. Each $V(t)$ is comprehensive from below. So for each $t \in \mathcal{T}$, there exists $a_t \in R^m$ such that $0 \in \operatorname{int}(a_t + V(t))$. If $a = \vee_{t \in \mathcal{T}} a_t$, it is obvious that $a + V$ satisfies the properties *a), b), c), d)* and that $\mathcal{C}(\mathcal{T}, a + V) = a + \mathcal{C}(\mathcal{T}, V)$. Hence, without loss of generality, we can (and we will) assume that $0 \in \operatorname{int} V(t)$ for each $t \in \mathcal{T}$.

Next, fix some constant $c > 0$ such that for each $t \in \mathcal{T}$ and each $v \in V(t)$ we have $v_i < c$ for all $i \in \operatorname{supp} t$, and then consider the set

$$W = \Big(\bigcup_{t \in \mathcal{T}} V(t)\Big) \bigcap \big(\,]-\infty,\ c]^m\big).$$

Clearly, the set W is closed, comprehensive from below and contains 0 in its interior. Let ∂W denote the boundary of W.

Claim 2.1. If $v \in \partial W \cap R^m_+$ and $v_i = 0$ for some i, then $v_j = c$ holds for some j.

Proof of Claim 2.1. To see this, assume that $v_i = 0$ and $v_j < c$ holds for each j. Since $0 \in \operatorname{int} V(e^i)$, there exists some $u \in V(e^i)$ with $0 < u_i < c$. From Property c), we see that the vector x defined by $x_j = c$ for $j \neq i$ and $x_i = u_i$ belongs to $V(e^i)$ (and hence to $W \cap R^m_+$) and satisfies $v \ll x$. This implies $v \in \operatorname{int} W$, a contradiction. □

Let Δ be the unit-simplex of R^m.

Claim 2.2. For each $s \in \Delta$, there exists a unique $\alpha > 0$ (depending on s) such that $\alpha s \in \partial W \cap R^m_+$.

Proof of Claim 2.2. Let $s \in \Delta$. We first prove that there exists at most one α such that $\alpha s \in \partial W \cap R^m_+$. Indeed, let αs and βs be elements of $\partial W \cap R^m_+$ such that $\alpha > \beta > 0$. If $s_i > 0$ holds for each i, then $\alpha s_i > \beta s_i$ holds for each i and so βs is an interior point of W, a contradiction. On the other hand, if $s_i = 0$ holds for some i, then by Claim 2.1 there exists some j such that $\beta s_j = c$ and so that $\alpha s_j > c$, a contradiction. Moreover, let $\alpha = \sup\{\beta \mid \beta s \in W \cap R^m_+\}$. From $0 \in \operatorname{int} W$, we deduce that $\alpha > 0$. From the definition of W, we deduce that α is finite. Since, W is closed it follows that $\alpha s \in \partial W \cap R^m_+$. □

Thus, a function $f : \Delta \longrightarrow \partial W \cap R^m_+$ can be defined by formula

$$f(s) = \alpha s \text{ where } \alpha = \sup\{\beta \in R_+ \mid \beta s \in W \cap R^m_+\}.$$

Claim 2.3. f is continuous.

Proof of Claim 2.3. It suffices to show that f has a closed graph. Let us consider a sequence $(s^n, f(s^n))$ in $\Delta \times (\partial W \cap R^m_+)$ that converges to (s, y). Write $f(s^n) = \alpha^n s^n \in \partial W \cap R^m_+$. Then $\alpha^n = \|\alpha^n s^n\|_1 = \|f(s^n)\|_1 \longrightarrow \|y\|_1$, and hence $f(s^n) = \alpha^n s^n \longrightarrow \|y\|_1 s$. By uniqueness of the limit $y = \|y\|_1 s$. Since $\partial W \cap R^m_+$ is a closed set, it follows from Claim 2.2 that $f(s) = y$. Consequently, f has a closed graph. □

Define a correspondence $\Psi : \Delta \to \Delta$ by

$$\Psi(s) = \left\{ \frac{t}{\|t\|_1} \mid t \in \mathcal{T} \text{ and } f(s) \in V(t) \right\}$$

Claim 2.4. Ψ is nonempty-valued and has a closed graph.

Proof of Claim 2.4. Since $f(s) \in W$, it follows immediately that $\Psi(s)$ is a nonempty subset of Δ. Furthermore, let us assume $s^n \to s$, $y^n \to y$ and $y^n \in \Psi(s^n)$. Since the range of Ψ is a finite set, there exists some n_0 such that for all $n \geq n_0$, $y^n = y$ and $f(s^n) \in \bigcup_{\{t \in \mathcal{T} \mid \frac{t}{\|t\|_1} = y\}} V(t)$. Since $\{t \in \mathcal{T} \mid \frac{t}{\|t\|_1} = y\}$ is a finite set, passing to a subsequence if necessary, we can assume that $f(s^n) \in V(t_0)$ for some $t_0 \in \{t \in \mathcal{T} \mid \frac{t}{\|t\|_1} = y\}$. Since f is continuous and $V(t_0)$ is a closed set, we deduce that $f(s) \in V(t_0)$ and consequently $y \in \Psi(s)$. □

Now we define the function $g : \Delta \times \Delta \to \Delta$ by

$$g_i(s, \mu) = \frac{s_i + (\mu_i - \frac{1}{m})^+}{1 + \sum_{j=1}^m (\mu_j - \frac{1}{m})^+}$$

where, as usual, $r^+ = \max\{r, 0\}$ for each real number r.
Clearly, g is a continuous function. Finally, we consider the correspondence $\Phi : \Delta \times \Delta \to \Delta \times \Delta$ defined by

$$\Phi(s, \mu) = \{g(s, \mu)\} \times \mathrm{co}\Psi(s).$$

Note that Φ is nonempty and convex-valued and has a closed graph. Thus by Kakutani's fixed point theorem, Φ has a fixed point $(\bar{s}, \bar{\mu})$. That is,

$$\bar{s} = g(\bar{s}, \bar{\mu}) \text{ and } \bar{\mu} \in \mathrm{co}\Psi(\bar{s}).$$

In other words,

$$\bar{s}_i = \frac{\bar{s}_i + (\bar{\mu}_i - \frac{1}{m})^+}{1 + \sum_{j=1}^m (\bar{\mu}_j - \frac{1}{m})^+} \ , \ i \in M \tag{2.1}$$

and there exist $\mathcal{T}' \subset \mathcal{T}$, $(a_t)_{t \in \mathcal{T}'} \in R^{\mathcal{T}'}$, with $a_t > 0$ for each $t \in \mathcal{T}'$ and $\sum_{t \in \mathcal{T}'} a_t = 1$, such that

$$\bar{\mu} = \sum_{t \in \mathcal{T}'} a_t \frac{t}{\|t\|_1} \ , \ f(\bar{s}) \in V(t) \ \forall t \in \mathcal{T}' \tag{2.2}$$

Claim 2.5. For all $i \in M$, $\bar{\mu}_i = \frac{1}{m}$.

Proof of Claim 2.5. Suppose that it is not true. Recalling that $\bar{\mu} \in \Delta$, it follows from (2.1) that $\sum_{j=1}^m (\bar{\mu}_j - \frac{1}{m})^+ > 0$ Then, the sets

$$I = \{i \in M \mid \bar{s}_i > 0\} = \{i \in M \mid \bar{\mu}_i > \frac{1}{m}\}$$

and

$$J = \{i \in M \mid \bar{s}_i = 0\} = \{i \in M \mid \bar{\mu}_i \le \frac{1}{m}\}$$

are both nonempty. Indeed, from $\sum_{i=1}^m (\bar{\mu}_i - \frac{1}{m})^+ > 0$, it follows that $(\bar{\mu}_i - \frac{1}{m})^+ > 0$ for some i. On the other hand, if $\bar{\mu}_i > \frac{1}{m}$ for each i, then $\sum_{i=1}^m \bar{\mu}_i > 1$, a contradiction. Clearly, for all $j \in J$ we have $s_j = 0$ hence $f(s)_j = 0$. From (2.2), for all $i \in I$, there exists $t \in \mathcal{T}'$ such that $t_i > 0$, $f(\bar{s}) \in V(t)$, hence $f(\bar{s})_i < c$, which, together with $J \neq \emptyset$, contradicts Claim 2.1. □

Now, let us consider $\lambda \in R^T$ such that

$$\lambda_t = \begin{cases} \frac{m a_t}{\|t\|_1} & \text{if } t \in \mathcal{T}', \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\sum_{t \in \mathcal{T}} \lambda_t t = \bar{1}$, and therefore $\lambda \in \Delta^{\mathcal{T}}$.

Claim 2.6. $f(\bar{s}) \in \mathcal{C}(\mathcal{T}, V)$

Proof of Claim 2.6. Since $\lambda \in \Delta^{\mathcal{T}}$ we have

$$\bigcap_{t \in \mathcal{T}'} V(t) = \bigcap_{\{t \in \mathcal{T} \mid \lambda_t > 0\}} V(t) \subset V(\bar{1}).$$

Hence $f(\bar{s}) \in V(\bar{1})$. Suppose that there exists $t \in \mathcal{T}$ and $v \in V(t)$ such that $f(\bar{s})_i < v_i$ for all $i \in \operatorname{supp} t$. Let u be such that $u_i = v_i$ for all $i \in \operatorname{supp} t$ and $u_i = c$ otherwise. It follows from Property *c*) that $u \in V(t) \subset W$. But $f(\bar{s}) \ll u$ contradicts the fact that $f(\bar{s}) \in \partial W \cap R^m_+$. Therefore $f(\bar{s}) \in \mathcal{C}(\mathcal{T}, V)$, which ends the proof of theorem 2.1. □

3 Application to the existence of Edgeworth equilibria of an arbitrage-free exchange economy

3.1 Definitions

In order to apply the previous theorem, we consider an exchange economy defined on a commodity vector space L and recall some definitions. $M = \{1, \dots, m\}$ is the set of consumers. Each consumer i is described by a consumption set $X_i \subset L$, an initial endowment $e_i \in X_i$, and a preference relation which is represented by a utility function $u_i : X_i \to R$. We normalize the utility functions by requiring $u_i(e_i) = 0$. To summarize, the economy $\mathcal{E}$ is a collection

$$\mathcal{E} = \big((X_i, u_i, e_i)_{i \in M}\big).$$

Let $\mathcal{A}(\mathcal{E})$ be the set of all *attainable allocations* of the economy $\mathcal{E}$, that is:

$$\mathcal{A}(\mathcal{E}) = \{x = (x_i)_{i \in M} \in \prod_{i \in M} X_i \mid \sum_{i \in M} x_i = \sum_{i \in M} e_i\}.$$

Let also $\mathcal{M} = 2^M \setminus \{\emptyset\}$ be the family of all coalitions of consumers. The allocation $x \in \mathcal{A}(\mathcal{E})$ is improved upon by the coalition $S \in \mathcal{M}$ if there exists $(x'_i)_{i \in S} \in \prod_{i \in S} X_i$ satisfying $\sum_{i \in S} x'_i = \sum_{i \in S} e_i$ and such that $u_i(x_i) < u_i(x'_i)$ for every $i \in S$. The core of the economy $\mathcal{E}$, denoted by $\mathcal{C}(\mathcal{E})$, is defined as the set of all allocations $x \in \mathcal{A}(\mathcal{E})$ which are improved upon by no coalition. Finally, following Aliprantis–Brown–Burkinshaw [1], $x \in \mathcal{A}(\mathcal{E})$ is said to be an *Edgeworth equilibrium* if, for every integer $r \geq 1$, the r-repetition of x belongs to the core of the r-fold replica of $\mathcal{E}$. We will denote by $\mathcal{C}^E(\mathcal{E})$ the set of all Edgeworth equilibria of $\mathcal{E}$.

For each integer $r \geq 1$, using the notations of the previous section, if

$$\mathcal{T}^r = \{t \in T \mid rt_i \in \{0, \dots, r\}, \forall i \in M\},$$

let us define $\mathcal{C}^r(\mathcal{E})$ as the set of all attainable allocations $x \in \mathcal{A}(\mathcal{E})$ such that there exist no $t \in \mathcal{T}^r$ and no $x^t \in \prod_{i \in \operatorname{supp} t} X_i$ such that

$$\sum_{i \in \operatorname{supp} t} t_i x_i^t = \sum_{i \in \operatorname{supp} t} t_i e_i \text{ and } \forall i \in \operatorname{supp} t,\ u_i(x_i) < u_i(x_i^t).$$

As it is easily seen and proved in Florenzano [16], under convexity assumptions on preferences and consumption sets, $\mathcal{C}^E(\mathcal{E}) = \bigcap_{r\geq 1} \mathcal{C}^r(\mathcal{E})$. In other words, $\mathcal{C}^E(\mathcal{E})$ is the set of all $x \in \mathcal{A}(\mathcal{E})$ such that there exist no $t = (t_i)_{i\in M} \in T$, with rational rates of participation, and no $x^t \in \prod_{i\in \operatorname{supp} t} X_i$ such that

$$\sum_{i\in \operatorname{supp} t} t_i x_i^t = \sum_{i\in \operatorname{supp} t} t_i e_i \text{ and } \forall i \in \operatorname{supp} t,\ u_i(x_i) < u_i(x_i^t).$$

Following Aubin [8], the fuzzy core of the economy $\mathcal{E}$, $\mathcal{C}^F(\mathcal{E})$, is the set of all $x \in \mathcal{A}(\mathcal{E})$ such that there exist no $t = (t_i)_{i\in M} \in T$ and no $x^t \in \prod_{i\in \operatorname{supp} t} X_i$ such that

$$\sum_{i\in \operatorname{supp} t} t_i x_i^t = \sum_{i\in \operatorname{supp} t} t_i e_i \text{ and } \forall i \in \operatorname{supp} t,\ u_i(x_i) < u_i(x_i^t).$$

3.2 The existence result

Let us now denote by

$$\mathcal{U} = \{v = (v_i)_{i=1}^m \in R_+^m \mid \exists x \in \mathcal{A}(\mathcal{E}),\ s.t.\ 0 \leq v_i \leq u_i(x_i),\ \forall i \in M\}$$

the *individually rational utility set*[2]. Set also

$$\mathcal{A} = \{x = (x_i)_{i=1}^m \in \mathcal{A}(\mathcal{E}) \mid u_i(x_i) \geq 0,\ \forall i \in M\}.$$

We make on $\mathcal{E}$ the following assumptions:

[A.1] For each i, X_i is convex (with $e_i \in X_i$);
[A.2] For each i, $u_i : X_i \to R$ is quasi-concave;
[A.3] $\mathcal{U}$ is compact.

For a fuzzy coalition $t \in \mathcal{T}^r$, let

$$\mathcal{A}^t(\mathcal{E}) = \{x^t \in \prod_{i\in \operatorname{supp} t} X_i \mid \sum_{i\in \operatorname{supp} t} t_i x_i^t = \sum_{i\in \operatorname{supp} t} t_i e_i\}$$

and

$$U^t = \{v^t \in R_+^{\operatorname{supp} t} \mid \exists x^t \in \mathcal{A}^t(\mathcal{E}),\ s.t.\ 0 \leq v_i \leq u_i(x_i^t),\ \forall i \in \operatorname{supp} t\}.$$

Finally, let

$$V(t) = \overline{(U^t - R_+^{\operatorname{supp} t})} \times R^{M\setminus \operatorname{supp} t},$$

where $\overline{(U^t - R_+^{\operatorname{supp} t})}$ denotes the closure of $(U^t - R_+^{\operatorname{supp} t})$.

Proposition 3.1 *Assume* [A.1]–[A.3]. *Then* $\mathcal{C}(\mathcal{T}^r, V)$ *is a nonempty subset of* $\mathcal{U}$.

[2] Recall that $u_i(e_i) = 0$, $i = 1, \ldots, m$.

Proof. Since $\mathcal{U}$ is compact, there exists $c > 0$ such that $\mathcal{U} \subset]-\infty, c[^m$. For each $t \in \mathcal{T}^r$, let us define

$$V^c(t) = \left(\overline{(\mathcal{U}^t - R_+^{\operatorname{supp} t})} \bigcap ( ]-\infty, c])^{\operatorname{supp} t}\right) \times R^{M \setminus \operatorname{supp} t}$$

We will keep in mind that $V^c(\bar{1}) = V(\bar{1}) = \mathcal{U} - R_+^m$ and that for every i, $V^c(e^i) = V(e^i) = -R_+ \times R^{M \setminus \{i\}}$. We first claim that the m-person fuzzy game $(\mathcal{T}^r, V^c)$ has a nonempty fuzzy core, that is, $\mathcal{C}(\mathcal{T}^r, V^c) \neq \emptyset$.
Clearly, $\mathcal{T}^r$ is a finite subset of T containing $\bar{1} = (1, \ldots, 1)$ and the canonical base (e^i) of R^m. The properties listed in Theorem 2.1 are also trivially satisfied. It suffices to verify that the m-person fuzzy game $(\mathcal{T}^r, V^c)$ is balanced in the sense of Definition 2.2.
To this end, let $\lambda \in \Delta^{\mathcal{T}^r}$ and $v \in \bigcap_{\{t \in \mathcal{T}^r | \lambda_t > 0\}} V^c(t)$. For each integer n and for every $t \in \mathcal{T}^r$ such that $\lambda_t > 0$, there exists $x^{n,t} \in \mathcal{A}^t(\mathcal{E})$ such that $u_i(x_i^{n,t}) \geq 0, \ \forall i \in \operatorname{supp} t$ and

$$v_i \leq u_i(x_i^{n,t}) + \frac{1}{n}, \ \forall i \in \operatorname{supp} t. \tag{3.1}$$

For each $i \in M$, let

$$x_i^n = \sum_{t \in \mathcal{T}^r} \lambda_t t_i x_i^{n,t}.$$

Since $\sum_{t \in \mathcal{T}^r} \lambda_t t = \bar{1}$, we have for each $i \in M$, $x_i^n \in X_i$ (X_i is convex) and

$$\sum_{i=1}^m x_i^n = \sum_{i=1}^m \sum_{t \in \mathcal{T}^r} \lambda_t t_i x_i^{n,t} = \sum_{t \in \mathcal{T}^r} \lambda_t (\sum_{i \in \operatorname{supp} t} t_i x_i^{n,t})$$

$$= \sum_{t \in \mathcal{T}^r} \lambda_t (\sum_{i \in M} t_i e_i) = \sum_{i=1}^m (\sum_{t \in \mathcal{T}^r} \lambda_t t_i) e_i = \sum_{i=1}^m e_i,$$

which shows that $x^n \in \mathcal{A}(\mathcal{E})$. Now, from relations (3.1) and in view of the definition of x^n and the quasi-concavity of utility functions u_i, we have

$$v_i \leq u_i(x_i^n) + \frac{1}{n}, \ \forall i \in M.$$

Since for every $t \in \mathcal{T}^r$ such that $\lambda_t > 0$, $u_i(x_i^{n,t}) \geq 0 \ \forall i \in \operatorname{supp} t$, we have also $\left(u_i(x_i^n)\right)_{i \in M} \in \mathcal{U}$. Passing to a subsequence if necessary, it follows from the compactness of $\mathcal{U}$ that there exists $x \in \mathcal{A}(\mathcal{E})$ such that

$$v_i \leq \lim_{n \to +\infty} u_i(x_i^n) \leq u_i(x_i), \ \forall i \in M.$$

Hence $v \in V(\bar{1}) = V^c(\bar{1})$, which shows that the game $(\mathcal{T}^r, V^c)$ is balanced. It then follows from Theorem 2.1 that $\mathcal{C}(\mathcal{T}^r, V^c) \neq \emptyset$.

To end the proof, let $v \in \mathcal{C}(\mathcal{T}^r, V^c)$. Note that $v \in V^c(\bar{1}) = V(\bar{1}) = \mathcal{U} - R_+^m$. Moreover, $v \in \mathcal{U}$. Indeed if not, for some i, $\{0\} \times R^{M \setminus \{i\}} \in V(e^i)$ with $0 > v_i$. We now prove by contraposition that $v \in \mathcal{C}(\mathcal{T}^r, V)$. Let us assume on the contrary that there exist $t \in \mathcal{T}^r$ and $u \in V(t)$ such that $v_i < u_i \ \forall i \in \operatorname{supp} t$. Since $v_i < c \ \forall i \in M$, one can find $\lambda \in]0,1[$ such that

$$v_i < \lambda v_i + (1-\lambda)u_i < \min\{c, u_i\} \quad \forall i \in \operatorname{supp} t.$$

Hence $\big(\lambda v_i + (1-\lambda)u_i\big)_{i \in \operatorname{supp} t} \in V^c(t)$. We have got a contradiction. □

Proposition 3.2 *Assume* [A.1]–[A.3] *on* $\mathcal{E}$. *Then* $\bigcap_{r \geq 1} \mathcal{C}(\mathcal{T}^r, V) \neq \emptyset$.

Proof. First, we show that $\mathcal{C}(\mathcal{T}^r, V)$ is closed. Let $v = \lim_{n \to +\infty} v^n$ with $v^n \in \mathcal{C}(\mathcal{T}^r, V)$. Suppose that $v \notin \mathcal{C}(\mathcal{T}^r, V)$. Then there exists $t \in \mathcal{T}^r$ and $u \in V(t)$ such that $v_i < u_i \ \forall i \in \operatorname{supp} t$. But, for n large enough, we have $v_i^n < u_i \ \forall i \in \operatorname{supp} t$, a contradiction. To end the proof, in view of the compactness of $\mathcal{U}$, it suffices to show that for each integer $r \geq 1$ we have $\mathcal{C}(\mathcal{T}^{r+1}, V) \subset \mathcal{C}(\mathcal{T}^r, V)$. Let $v \in \mathcal{C}(\mathcal{T}^{r+1}, V)$ and suppose that $v \notin \mathcal{C}(\mathcal{T}^r, V)$. Then there exists $t \in \mathcal{T}^r$ and $u \in V(t)$ such that $v_i < u_i \ \forall i \in \operatorname{supp} t$. Let us consider $t' = \frac{r}{r+1} t$. Clearly $\mathcal{A}^{t'}(\mathcal{E}) = \mathcal{A}^t(\mathcal{E})$, $U^t = U^{t'}$, $V(t) = V(t')$. Since $t' \in \mathcal{T}^{r+1}$ and $u \in V(t')$, we have got a contradiction. □

We are now ready to prove the main result of this section.

Theorem 3.1 *Under Assumptions* [A.1]–[A.3]*, the set of Edgeworth equilibria* $\mathcal{C}^E(\mathcal{E})$ *is nonempty.*

Proof. Let $v \in \bigcap_{r \geq 1} \mathcal{C}(\mathcal{T}^r, V)$ and $x \in \mathcal{A}(\mathcal{E})$ be such that $v_i \leq u_i(x_i) \ \forall i \in M$. We claim that $x \in \mathcal{C}^E(\mathcal{E})$. Indeed, if for some r, $x \notin \mathcal{C}^r(\mathcal{E})$, then there exist $t \in \mathcal{T}^r$ and $x' \in \mathcal{A}^t(\mathcal{E})$ such that $v_i \leq u_i(x_i) < u_i(x_i')$ for all $i \in \operatorname{supp} t$. Hence $v \notin \mathcal{C}(\mathcal{T}^r, V)$, a contradiction. □

Remark 3.1 Adding the assumption that the commodity space is finite dimensional, the consumption sets are closed and the utility functions are upper semicontinuous at every attainable consumption vector to the other assumptions of Theorem 3.1, it would be easy to deduce its conclusion from Proposition 3 in Florenzano [16]. Under analogous topological assumptions (relative to the weak*-topology on L), the same proof[3] can be given if the commodity space is an infinite dimensional Banach space which has a Banach predual. These two cases cover most of commodity spaces of economic interest. However, it should be noticed that the statement of Theorem 3.1 and its proof do not depend on the dimension of the commodity space or on its particular structure.

[3] Truncating the economy by an increasing sequence of closed balls of L, centered at 0 and containing all initial endowments, one obtains a sequence $(x_i^\nu)_{i=1}^m$ of Edgeworth equilibria of the truncated economies. The sequence $(u_i(x_i^\nu))_{i=1}^m$ belongs to $\mathcal{U}$ and has a converging subsequence. At the limit, from the definition of $\mathcal{U}$, one gets an allocation $(x_i)_{i=1}^m$. Using the upper semicontinuity of functions u_i, it is easily proved that this allocation is an Edgeworth equilibrium of $\mathcal{E}$.

Remark 3.2 As proved in Cheng [10], Dana–Le Van–Magnien [12], Allouch [4], in an infinite as well in a finite setting, Assumption [A.3] is strictly weaker than assuming that the set $\mathcal{A}$ of attainable and individually rational allocations is compact. It is even strictly weaker than assuming that every attainable and individually rational allocation can be Pareto dominated by an allocation belonging to some compact subset of $\mathcal{A}$ (for details, see Allouch [4]).

Remark 3.3 As easily seen and proved in Florenzano [16], if the commodity space L is a Hausdorff topological vector space and if the utility functions are lower semicontinuous at every attainable and individually rational consumption vector, an Edgeworth equilibrium whose existence is proved in Theorem 3.1 is actually an element of the fuzzy core, $\mathcal{C}^F(\mathcal{E})$, of the economy $\mathcal{E}$. As it is easily verified, the same holds true without continuity condition if Assumption [A.2] is replaced by:

[A$'$.2] For each i, $u_i : X_i \to R$ is concave.

4 Walras equilibria of an arbitrage-free exchange economy

Recall that a couple (x, p) is said to be a *quasiequilibrium* of $\mathcal{E}$ if and only if $x \in \mathcal{A}(\mathcal{E})$, p is a linear functional on L, with $p \neq 0$, and for every $i \in M$,

$$p \cdot x_i = p \cdot e_i \quad \text{and} \quad u_i(x'_i) > u_i(x_i) \Rightarrow p \cdot x'_i \geq p \cdot x_i.$$

This quasiequilibrium is said to be *nontrivial* if for some $i \in M$,

$$\inf_{x'_i \in X_i} p \cdot x'_i < p \cdot e_i.$$

A quasiequilibrium (x, p) is a *Walras equilibrium* if $u_i(x'_i) > u_i(x_i)$ actually implies $p \cdot x'_i > p \cdot x_i$.

If for every $i \in M$, $\inf_{x'_i \in X_i} p \cdot x'_i < p \cdot e_i$ and if each $u_i : X_i \to \mathbb{R}$ either is a concave function or is lower semicontinuous at x_i, then a quasiequilibrium (x, p) is actually an equilibrium. Under some irreducubility assumption on the economy, the same holds true if (x, p) is a nontrivial quasiequilibrium. For this reason, in the following, we will be interested only in nontrivial quasiequilibria and will prove the existence of nontrivial quasiequilibria by decentralizing Edgeworth equilibria obtained via Theorem 3.1.

4.1 Decentralization under interiority assumptions

Let us first assume that the commodity space L is R^ℓ, the ℓ-dimensional Euclidean space. For each $x_i \in X_i$, we define the strictly preferred set to x_i by

$$P_i(x_i) = \{x'_i \in X_i \mid u_i(x_i) < u_i(x'_i)\}$$

and we set the two following assumptions:

[A.4] For each $i \in M$, u_i is lower semicontinuous at every attainable and individually rational consumption vector;

[A.5] If $x \in \mathcal{A}$ then for each $i \in M$, $x_i \in \overline{P_i(x_i)}$ (the closure of $P_i(x_i)$).[4]

Proposition 4.1 *Assume that the commodity space is finite dimensional. Then under* [A.1]–[A.5] (*resp.* [A.1], [A′.2], [A.3], [A.5]), *the economy* $\mathcal{E} = \big((X_i, u_i, e_i)_{i\in M}\big)$ *has a quasiequilibrium. This quasiequilibrium is nontrivial provided that* $e \in \operatorname{int} \sum_{i\in M} X_i$.

Proof. Let $\bar{x} \in \mathcal{C}^E(\mathcal{E})$. In view of Assumption [A.4] (resp. [A'.2]), by Remark 3.3 $\bar{x} \in \mathcal{C}^F(\mathcal{E})$. Let $G = \operatorname{co}(\bigcup_{i\in M}(P_i(\bar{x}_i) - \{e_i\}))$. The set G is nonempty since $\bar{x} \in \mathcal{A}$ and Assumption [A.5] imply that $P_i(\bar{x}_i) \neq \emptyset$. We first prove that $0 \notin G$. Indeed if not, there exists $\lambda = (\lambda_i)_{i\in M}$ such that $\lambda_i \geq 0$, for all i and $\sum_{i\in M} \lambda_i = 1$ and $(x_i) \in \prod_{i\in M} X_i$ such that

$$\sum_{i\in M} \lambda_i x_i = \sum_{i\in M} \lambda_i e_i$$

$$x_i \in P_i(\bar{x}_i),\ \forall i \text{ such that } \lambda_i > 0.$$

Thus the fuzzy coalition λ improves upon $\bar{x}$, which contradicts $\bar{x} \in \mathcal{C}^F(\mathcal{E})$.

Now, by the separation theorem for finite dimensional vector spaces, there exists $p \in R^\ell \setminus 0$ such that $p \cdot g \geq 0$, for all $g \in G$. From [A.5], one deduces that $p \cdot \bar{x}_i \geq p \cdot e_i$ for all $i \in M$. Since $\bar{x} \in \mathcal{C}^E(\mathcal{E})$, one has $\sum_{i\in M} \bar{x}_i = \sum_{i\in M} e_i$. Thus $p \cdot \bar{x}_i = p \cdot e_i$ for all $i \in M$ and $(\bar{x}, p)$ is a quasi-equilibrium of $\mathcal{E}$.

The proof of the last assertion is obvious. □

Remark 4.1 In view of [A.4] (resp. [A′.2]), assuming either that each $e_i \in \operatorname{int} X_i$ or that $\mathcal{E}$ satisfies some irreducibility assumption, then the nontrivial quasiequilibrium $(\bar{x}, p)$ is a Walras equilibrium.

Proposition 4.1 extends Nielsen [23], Page and Wooders [26], Dana–Le Van–Magnien [12], in fact most of finite dimensional equilibrium existence results established under no-arbitrage conditions which imply the compactness of $\mathcal{U}$, as far as one forgets for some of them (Hart [18], Page [24], Page and Wooders [26]) the dependence on relative prices of consumption sets and preferences. A notable exception is Werner [33] who uses a different nonsatiation assumption. In [6], Allouch–Le Van–Page indicate how to modify the economy in order to apply Proposition 4.1 to the modified economy, getting then existence of quasiequilibrium in the original economy. Note that

[4] As remarked by several authors, a nonsatiation assumption in every component of an attainable allocation may be unreasonable (for example, if consumption sets coincide with the consumption space, it implies nonsatiation for each agent on the whole consumption space). We emphasize that we only assume here nonsatiation at every attainable and individually rational consumption vector.

Nielsen, Dana–Le Van–Magnien assume neither [A.4] nor [A$'$.2][5] but prove only the existence of a quasiequilibrium.

Let us now assume that the commodity space, L, is an infinite dimensional Hausdorff topological vector space and that for every $x \in \mathcal{A}$, $\text{int}(\bigcup_{i\in M}(P_i(x_i) - \{e_i\})) \neq \emptyset$. Using Hahn-Banach's theorem, we can mimic the proof of Proposition 4.1 in order to obtain:

Proposition 4.2 *If L is a Hausdorff topological vector space, assume* [A.1]–[A.5] (*resp.* [A.1], [A$'$.2], [A.3], [A.5]). *If* $\text{int}(\bigcup_{i\in M}(P_i(x_i) - \{e_i\})) \neq \emptyset$ *holds for every attainable and individually rational allocation x, then the economy $\mathcal{E} = \big((X_i, u_i, e_i)_{i\in M}\big)$ has a quasiequilibrium with continuous price. This quasiequilibrium is nontrivial if $e \in \text{int}\sum_{i\in M} X_i$. Under the same additional assumptions as in Remark 4.1, this nontrivial quasiequilibrium is an equilibrium.*

The previous result extends results of Cheng [10], Theorem 1 of Brown and Werner [9], Theorem 1 of Ishimoto [19], Theorem 1 of Dana–Le Van–Magnien [13]. As in the finite dimensional case, Dana–Le Van–Magnien assume neither [A.4] nor [A'.2] but prove only the existence of a quasiequilibrium. Assumptions [A.4] or [A'.2] would be of use for going from the obtained quasiequilibrium to equilibrium.

4.2 Decentralization under properness assumptions

To go further, from now on we make the following structural assumption on the commodity space:

[SA] L is a linear vector lattice (or Riesz space) endowed with a Hausdorff linear topology τ such that
- *(i)* L_+ is a closed cone in the τ-topology of L;
- *(ii)* L' is a sublattice of the order dual of L.

As well-known, [SA] holds true if L is a topological vector lattice but may be satisfied in other settings.

Under this assumption, different properness assumptions compensate the fact that consumption sets (hence preferred sets) may have empty interior or simply that $e \notin \text{int}\sum_{i\in M} X_i$. The first two definitions below correspond to the case where the consumption sets of the agents are equal to the positive cone ($\forall i \in I,\ X_i = L_+$). Recall that $e = \sum_{i\in M} e_i$ and that for each i, $P_i(x_i) = \{x'_i \in X_i \mid u_i(x_i) < u_i(x'_i)\}$ defines a preference correspondence (preference relation) $P_i : X_i \to X_i$.

[5] Due to their method of proof, they assume upper semicontinuity instead of lower semicontinuity of utility functions. The need for openess of preferred sets in consumption sets (or some substitute to this continuity property) is a drawback of the Edgeworth equilibrium approach to the Walras quasiequilibrium existence. Nevertheless such assumptions will be of use to go from the obtained quasiequilibrium to equilibrium.

Definition 4.1 (Yannelis – Zame[34]) *The correspondence P_i is F-proper at $x_i \in L_+$ if there exists some $v_{x_i} \in L$, some τ-open 0-neighborhood V_{x_i} and for each $u \in V_{x_i}$, a real number $\overline{\lambda}_u > 0$ such that if $u \in V$ and if $0 < \lambda < \overline{\lambda}_u$, then $x_i + \lambda(v_{x_i} - u) \in P_i(x_i)$, provided $x_i + \lambda(v_{x_i} - u) \in L_+$.*

Definition 4.2 (Podczeck[28]) *Let K be a linear subspace of L. The correspondence P_i is E-proper relative to K at $x_i \in K_+$ if there exists some $v_{x_i} \in L_+$, some τ-open 0-neighborhood V_{x_i} and some subset $A_{x_i} \in L$ which is radial at x_i such that*

(a) $x_i + \alpha v_{x_i} \in P_i(x_i)$ for every sufficiently small real number $\alpha > 0$;
(b) $(P_i(x) + \Gamma_{x_i}) \cap K_+ \cap A_{x_i} \subset P_i(x_i)$, where Γ_{x_i} is the open cone

$$\Gamma_{x_i} = \{\lambda(v_{x_i} - u) \mid 0 < \lambda,\ u \in V_{x_i}\}.$$

As proved by Podczeck [28], for preferences defined by utility functions, uniform properness (as defined by Mas-Colell [22]) with a properness vector in K implies E-properness relative to K.

The two following definitions correspond to the case of general consumption sets.

Definition 4.3 (Tourky [31]) *A preference relation P_i is M-proper at $x_i \in \overline{P_i(x_i)}$ if there exist a convex lattice Z_{x_i} and a convex set $\widehat{P_i(x_i)}$ such that:*

1. *$\widehat{P_i(x_i)} \bigcap Z_{x_i} = P_i(x_i)$;*
2. *$x_i + e$ is an interior point of $\widehat{P_i(x_i)}$ and $P_i(x_i)$ is open in Z_{x_i};*
3. *0, $e_i \in Z_{x_i}$ and $Z_{x_i} + L_+ = Z_{x_i}$;*
4. *$(1 + \alpha)x_i \in Z_{x_i}$ for some $\alpha > 0$.*

Definition 4.4 (Florenzano – Marakulin [17]) *Let K be some order ideal of L containing e. The preference relation P_i is M-proper relative to K at $x_i \in K \cap \overline{P_i(x_i)}$ if there exist a convex sublattice Z_{x_i} of K and a convex set $\widehat{P_i(x_i)}$ such that:*

1. *$\widehat{P_i(x_i)} \bigcap (Z_{x_i} + L_+) = P_i(x_i)$;*
2. *$x_i + e$ is an interior point of $\widehat{P_i(x_i)}$;*
3. *0, $e_i \in Z_{x_i}$ and $Z_{x_i} + K_+ \subset Z_{x_i}$;*
4. *$x_i \in Z_{x_i}$.*

As proved in Podczeck [28], if L is locally convex and if for every i, $X_i = Z_{x_i} = L_+$, then M-properness at $x_i \in \operatorname{cl} P_i(x_i)$, as in Definition 4.3, implies E-properness relative to L with e as properness vector, while M-properness relative to K in Definition 4.4 implies E-properness relative to K with e as properness vector.

In case of consumption sets equal to the positive cone, using Proposition 4.2 of Deghdak and Florenzano [15], we have the following existence result where $L(e)$ denotes the principal order ideal generated by e.

Proposition 4.3 *Assume* [SA], [A.1] – [A.4] (*resp.* [SA], [A.1], [A′.2], [A.3]) *and that*

- *either $L(e)$ is τ-dense in L and each P_i is F-proper at every component x_i of an attainable and individually rational allocation, with a properness vector satisfying $x_i + v_{x_i} \in L(e)_+$ and a convex properness 0-neighborhood V_{x_i};*
- *or each P_i is E-proper relative to $L(e)$ at every component x_i of an attainable and individually rational allocation, with a properness vector satisfying $x_i + v_{x_i} \in L(e)_+$ and a convex properness 0-neighborhood V_{x_i}.*

Then the economy $\mathcal{E} = \big((X_i, u_i, e_i)_{i\in M}\big)$ has a nontrivial quasiequilibrium with continuous price.

It is worth noticing that Assumption [A.5] is not needed in the previous proposition since local nonsatiation at each component of $x \in \mathcal{A}$ follows from the properness assumption.

Remark 4.2 In view of [A.4] (resp. [A'.2]), assuming either that for each $i \in M$, $\inf_{x_i' \in X_i} p \cdot x_i' < p \cdot e_i$ or that $\mathcal{E}$ satisfies some irreducibility assumption, then the nontrivial quasiequilibrium $(\bar{x}, p)$ is a Walras equilibrium.

With general consumption sets, we obtain the following existence results. Notice that in the second one we do not assume monotonicity of preferences.

Proposition 4.4 *Assume* [SA], [A.1]–[A.3] *and that, in addition, $e > 0$, each u_i is strictly increasing, each P_i is M-proper at every component of an attainable and individually rational allocation. Then $\mathcal{E}$ has a quasiequilibrium $(\bar{x}, p)$ with $p \in L'$ such that $p \cdot e > 0$. This quasiequilibrium is nontrivial if for each $i \in M$, $0 \in X_i$. Under* [A.4] *(resp.* [A′.2]*) and the same additional assumptions as in Remark 4.2, the nontrivial quasiequilibrium is an equilibrium.*

Proof. By Theorem 3.1, Edgeworth equilibria exist for the economy. Then the conclusion is immediate from Theorem 2.1 of Tourky [31]. □

Proposition 4.5 *For each attainable and individually rational allocation $x = (x_i)_{i\in I} \in \mathcal{A}$, let $L(u(x))$ be the principal order ideal generated by all x_i, e_i for $i \in M$. Assume* [SA], [A.1]-[A.4] (*resp.* [SA], [A.1], [A′.2], [A. 3]-[A.4]) *and that, in addition, $e > 0$, each P_i is M-proper relative to $L(u(x))$ at every component of an attainable and individually rational allocation. Then $\mathcal{E}$ has a quasiequilibrium $(\bar{x}, p)$ with $p \in L'$ such that $p \cdot e > 0$. This quasiequilibrium is nontrivial if for each $i \in M$, $0 \in X_i$. Under the same additional assumptions as in Remark 4.2, the nontrivial quasiequilibrium is an equilibrium.*

Proof. By Theorem 3.1 and Remark 3.3, the fuzzy core of this economy is nonempty. Then the conclusion is immediate from Corollary 4.1 of Florenzano–Marakulin [17]. □

Proposition 4.3 is a variant of Theorem 2 and Theorem 3 in Dana–Le Van–Magnien [13]. Actually, both results go to Mas-Colell [22], Araujo and

Monteiro [7] if one assumes monotonicity of preferences, and to Podczeck [28] without monotonicity. Propositions 4.4 and 4.5 have no antecedent in the literature on trade in financial assets.

5 Application to Finance

We now indicate how the results of the previous section can be used in some examples of financial markets found in the literature. Let us consider a finance model with m investors trading securities and having identical expectations on the security payoffs. In what follows, L denotes the *portfolio (vector) space*. The *space of contingent claims*, E, is a space of real-valued random variables on some underlying probability space (Ω, Σ, P), such as $L_p(\Omega, \Sigma, P)$ for $1 \leq p \leq \infty$. Let $R : L \to E$ be a one-to-one linear operator defining the payoff of a portfolio $x \in L$, and $M := R(L)$ be the *marketed contingent claim space*. Each investor i is given a *portfolio set* $X_i \subset L$ and an initial endowment $e_i \in X_i$. If agent i has a preference over E described by the utility function $v_i : E \to [-\infty, +\infty)$, we define the *portfolio utility function* of i, $u_i : X_i \to R$, as the indirect utility function given by $u_i(x_i) = v_i(R(x_i))$. The economy under consideration is either $(X_i, u_i, e_i)_{i=1}^m$ or $(R(X_i), v_i, R(e_i))_{i=1}^m$.

The finite dimensional case ($\dim L < \infty$, hence $\dim M < \infty$) has its own interest as an idealization of Hart's model. It is also useful in the equilibrium versions of the Arbitrage Pricing Theory and the Capital Asset Pricing Model, though their formulation involve an infinite dimensional portfolio space. Specifically, let us assume as Kim [20] that $E = L_2(\Omega, \Sigma, P)$, M is norm-closed, each individual portfolio set is equal to L, each investor i has a strictly concave von-Neumann-Morgenstern utility function v_i, and the economy has a finite dimensional factor subspace F, containing the payoff of the total initial endowment, the riskless payoff, and such that every marketed contingent claim in M is second-order stochastically dominated by an element of F. As observed by Kim [20], the equilibrium existence problem in the infinite dimensional economy $\big(M, v_i, R(e_i)\big)_{i=1}^m$ is then reduced to the equilibrium existence problem in an economy appropriately defined on the finite dimensional commodity space F. This reduction works in the equilibrium versions of the Arbitrage Pricing Theory as well as the Capital Asset Pricing Model, and provides an equilibrium existence result, as far as the assumptions made on the initial model (essentially the existence of a riskless marketed payoff and assumptions implying the compactness of $\mathcal{U}$) allow to apply Proposition 4.1 to the associated finite dimensional economy.

The CAPM can also be directly studied as an infinite dimensional model where the existence of a quasiequilibrium follows from Proposition 4.2. In [13], Dana–Le Van–Magnien work directly in the space M of marketed contingent claims, a norm-closed subspace of $E = L_2(\Omega, \Sigma, P)$. For each i, $X_i = M$ is the portfolio set of i, $e_i \in M$, $v_i : L_2(\Omega, \Sigma, P) \to R$ is "mean variance", in other words there exists a strictly concave function,

$U_i : \mathbb{R} \times [-\rho, +\infty[\to \mathbb{R}$, increasing in its first variable and strictly decreasing in the second one, such that $v_i(x_i) = U_i\big(E(x_i), \mathrm{var}(x_i)\big)$, $x_i \in X_i$, where $E(x_i)$ and $\mathrm{var}(x_i)$ denote the expectation and variance of x_i. One will find in [13], under two alternative sets of assumptions, a complete checking of the conditions of Proposition 4.2 for the model $\big(M, v_i, R(e_i)\big)_{i=1}^m$, especially of the interiority conditions. Note that the obtained equilibrium price is an element of $L_2(\Omega, \Sigma, P)$, proved to belong in fact to a 2-dimensional subspace of $L_2(\Omega, \Sigma, P)$.

Actually, most of equilibrium existence results for financial models with infinite dimensional portfolio spaces or marketed contingent claim spaces can be obtained using Proposition 4.2, that is under interiority conditions which are postulated or verified. These conditions are easier to satisfy if, as in Section 3 of [19], each X_i is equal to the whole space M, a norm-closed subspace of $E = L_p(\Omega, \Sigma, P)$ and each v_i is a state-separable concave utility function. An interesting example where individual portfolio sets differ from the whole portfolio space is given in Section 6 of Brown–Werner [9]. In this example, $E = L_\infty(\Omega, \Sigma, P)$ is the space of contingent claims. There is a countable collection of securities, one of them is the riskless asset $R_1(\omega) = 1$ for each $\omega \in \Omega$, and $L = \ell_1 = \{x \in R^\infty \mid \sum_{k=1}^\infty \mid x_k \mid < +\infty\}$ is the portfolio space. As returns $R_k,\ k = 1, 2, \ldots, +\infty$, are assumed to belong to a norm-bounded subset of $L_\infty^+(\Omega, \Sigma, P)$, then $R(x) = \sum_{k=1}^\infty x_k R_k \in L_\infty(\Omega, \Sigma, P)$ is defined for each $x \in L$. Assuming that investors are constrained to have non-negative end-of-period wealth, the portfolio set of each $i \in M$ is $X_i = \{x \in L \mid R(x) \geq 0\}$, $v_i : L_\infty^+(\Omega, \Sigma, P) \to R^+$ is a state-separable, concave, continuous utility function, $e_i \in X_i$. As the nonrisky asset belongs to the norm-interior of X_i, the interiority condition of Proposition 4.2 is satisfied. On the other hand, it follows from the definition of the portfolio sets that $\mathcal{U}$ is bounded. If one assumes in addition that $R(\ell_1)$ is a norm-closed subset of L_∞, that R is one-to-one and that v_i is continuous for the Mackey topology $\tau(L_\infty, L_1)$ associated to the duality $< L_\infty, L_1 >$, then it is easily seen that $\mathcal{U}$ is closed. The other assumptions are easy to check. The existence of a quasiequilibrium with a price in ℓ_∞, follows from Proposition 4.2.

Since consumption sets of agents are assumed to be equal to the positive cone, Proposition 4.3 may seem far from our main concern: the existence of equilibrium in models with short selling. However, a reordering of the portfolio space such that the portfolio sets of the agents become equal to the positive cone arises naturally in financial models. For example, in Aliprantis et al. [2], $E = L_p(\Omega, \Sigma, P), 1 \leq p \leq +\infty$. As in Brown–Werner [9], there is a countable collection of securities. The portfolio space is $L = \Phi$, the space of all eventually zero real sequences,

$$\Phi = \{x = (x_n) \in \mathbb{R}^\infty \mid x_n = 0 \text{ for all but a finite number of } n\}$$

endowed with the inductive limit topology. If investors are constrained to have non-negative end-of-period wealth, then $X_i = \{x \in L \mid R(x) \geq 0\}$ is the portfolio set of i, $v_i : L_p^+(\Omega, \Sigma, P) \to R^+$ is a state-separable, quasi-concave, continuous utility function such that the contingent claim $R(e)$ is desirable, $e_i \in X_i$. If R is one-to-one, L can be ordered by the partial order $x \geq_R x' \Leftrightarrow R(x) \geq R(x')$, so that each consumption set X_i is now equal to the positive cone of $(L, \geq_R)$. As in the previous paragraph, it follows from the definition of the portfolio sets that $\mathcal{U}$ is bounded. Aliprantis et al. [2] give a condition on the cone of positive payoff portfolios which implies that $\mathcal{U}$ is closed. In the absence of an interiority condition, the existence of equilibrium with a price in R^∞ follows from a properness assumption with e as properness vector. Properness of u_i in $(L, \geq_R)^+$ is inherited from properness of v_i in $L_p^+(\Omega, \Sigma, P)$. Conditions on the properties of v_i, which guarantee this properness, can be found in Le Van [21]. In order to apply Proposition 4.3, it remains to verify the structural assumption [SA], guaranteed in this model by the hypothesis made on the positive cone of $(L, \geq_R)$. Note that in other models, it may be hard to verify that, ordered by the the portfolio dominance, the portfolio space is a Riesz space. However, this verification is indispensable. Aliprantis–Monteiro–Tourky [3] give an example of an exchange economy with proper preferences on a positive cone defined on $\mathbb{R}^3$ which satisfies the assumptions of Proposition 4.3 but does not admit any nontrivial equilibrium.

When the investors are not constrained to have non negative end-of-period wealth, but have consumption sets which may differ from the whole portfolio set, an interiority condition may stay difficult to satisfy. In order to use Propositions 4.4 or 4.5 for obtaining an equilibrium existence result in an expected utility financial model, it remains to characterize M-properness in a framework à la Aliprantis et al. [2] (where after reordering of the portfolio space, the preferences of the agents are strictly monotone) or M-properness relative to $L(u(x))$ (as in Proposition 4.5) in a framework à la Brown-Werner [9] (where preferences cannot be assumed to be monotone). Such a characterization is left for future work.

References

1. Aliprantis, C. D., Brown, D. J., Burkinshaw, O.: Existence and optimality of competitive equilibria. Berlin Heidelberg New York: Springer 1989
2. Aliprantis, C. D., Brown, D. J., Polyrakis, I. A., Werner, J.: Portfolio dominance and optimality in infinite security markets. Journal of Mathematical Economics **30**, 347-366 (1998)
3. Aliprantis, C. D., Monteiro, P. K., Tourky, R.: Non-marketed options, non-existence of equilibria, and non-linear prices. Journal of Economic Theory forthcoming
4. Allouch, N.: An equilibrium existence result with short selling. Journal of Mathematical Economics **37**, 81-94, (2002)
5. Allouch, N., Le Van, C., Page, F. H. Jr.: The geometry of arbitrage and the existence of competitive equilibrium. Journal of Mathematical Economics **38**, 373-392 (2002)
6. Allouch, N., Le Van, C., Page, F. H. Jr.: Arbitrage, equilibrium and nonsatiation. Warwick Economic Research Papers No 623, (2001)

7. Araujo, A., Monteiro, P. K.: Equilibrium without uniform conditions. Journal of Economic Theory **48**, 416-427 (1989)
8. Aubin, J.P.: Mathematical Methods of Games and Economic Theory. Amsterdam New York Oxford: North Holland 1979
9. Brown, D. J., Werner, J.: Arbitrage and existence of equilibrium in infinite asset markets. Review of Economic Studies **62**, 101-114 (1995)
10. Cheng, H. C.: Asset market equilibrium in infinite dimensional complete markets. Journal of Mathematical Economics **20**, 137-152 (1991)
11. Chichilnisky, G. Heal, G. M.: Competitive equilibrium in Sobolev spaces without bounds on short sales. Journal of Economic Theory **59**, 364-384 (1993)
12. Dana, R.-A., Le Van, C., Magnien, F.: On the dif and only if erent notions of arbitrage and existence of equilibrium. Journal of Economic Theory **86**, 169-193 (1999)
13. Dana, R.-A., Le Van, C., Magnien, F.: General equilibrium in asset markets with or without short-selling. Journal of Mathematical Analysis and Applications **206**, 567-588 (1997)
14. Debreu, G., Scarf, H.: A limit theorem on the core of an economy. International Economic Review **4**, 235-246 (1963)
15. Deghdak, M., Florenzano, M.: Decentralizing Edgeworth equilibria in economies with many commodities. Economic Theory **14**, 297-310 (1999)
16. Florenzano, M.: Edgeworth equilibria, fuzzy core and equilibria of a production economy without ordered preferences, *Journal of Mathematical Analysis and Applications* **153**, 18-36 (1990)
17. Florenzano, M., Marakulin, V. M.: Production equilibria in vector lattices. Economic Theory **17**, 577-598 (2001)
18. Hart, O.: On the existence of an equilibrium in a securities model. Journal of Economic Theory **9**, 293-311 (1974)
19. Ishimito, O.: Equilibrium in infinite asset markets with short selling. University of Windsor Research Papers, (1995)
20. Kim, C.: Stochastic dominance, Pareto optimality, and equilibrium asset pricing. Review of Economic Studies **65**, 341-356 (1998)
21. Le Van, C.: Complete characterization of Yannelis–Zame and Chichilnisky–Kalman–Mas-Colell conditions on preferences for separable concave functions defined in L^p_+ and L^p. Economic Theory **8**, 155-166 (1996)
22. Mas-Colell, A.: The price equilibrium existence problem in topological vector lattices. Econometrica **54**, 1039-1054 (1986)
23. Nielsen, L. T.: Asset market equilibrium with short selling. Review of Economic Studies **56**, 467-474 (1989)
24. Page, F. H. Jr.: On equilibrium in Hart's securities exchange model. Journal of Economic Theory **41**, 392-404 (1987),
25. Page, F. H. Jr., Wooders, M. H.: A necessary and sufficient condition for compactness of individually rational and feasible outcomes and the existence of an equilibrium. Economics Letters **52**, 153-162 (1996)
26. Page, F. H. Jr., Wooders, M. H.: Arbitrage with price-dependent preferences: Equilibrium and market stability. Fields Institute Communications **23**, 189-211 (1999)
27. Page, F. H. Jr., Wooders, M. H., Monteiro P. K.: Inconsequential arbitrage. Journal of Mathematical Economics **34**, 439-469 (2000)
28. Podczeck, K.: Equilibria in vector lattices without ordered preferences or uniform properness, Journal of Mathematical Economics **25**, 465-485 (1996)

29. Scarf, H.: The core of a n person game. Econometrica **38**, 50-69 (1967)
30. Shapley, L., Vohra, R.: On Kakutani's fixed point theorem, the K-K-M theorem and the core of a balanced game. Economic Theory **1**, 108-116 (1991)
31. Tourky, R.: A new approach to the limit theorem on the core of an economy in vector lattices. Journal of Economic Theory **78**, 321-328 (1998)
32. Vohra, R.: On Scarf's theorem on the non-emptiness of the core: A direct proof through Kakutani's fixed point theorem. Brown University Working Paper, (1987)
33. Werner, J.: Arbitrage and the existence of competitive equilibrium, Econometrica **55**, 1403-1418 (1987)
34. Yannelis, N. C., Zame, W. R.: Equilibria in Banach lattices without ordered preferences, Journal of Mathematical Economics **15**, 85-110 (1986)