TWO REMARKS ON SOLUTIONS OF GROSS-PITAEVSKII EQUATIONS ON ZHIDKOV SPACES

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ABSTRACT. We consider the so-called Gross-Pitaevskii equations supplemented with non-standard boundary conditions. We prove two mathematical results concerned with the initial value problem for these equations in Zhidkov spaces.

1. INTRODUCTION

1.1. **Setting the problem.** This short article is concerned with the so-called Gross-Pitaevskii equations

$$iu_t + \Delta u + u(1 - |u|^2) = 0, \tag{1}$$

supplemented with non-standard boundary conditions that read $|u(t,x)| \to 1$ as $||x|| \to +\infty$. We also supplement this equation with initial condition u_0 that will be specified in the sequel. Here the unknown u maps $\mathbb{R}_t \times \mathbb{R}_x^D$ into $\mathbb{C}$. These equations with this non-standard boundary conditions occur in several physical contexts, as the Bose-Einstein condensation for suprafluids (see [3] and the references therein). The mathematical study of solitary waves for these equations was initiated in the pioneering work [1]. Here we are interested in two questions related to the Cauchy problem for (1). Throughout this article, we follow the framework developed by C. Gallo (see [3]), that extends to the multi-dimensional case the work of P. Zhidkov (see [8] and the references therein).

We now introduce the Zhidkov spaces as, for $k \geq 1$ integer

$$X^k(\mathbb{R}^D) = \{u \in L^\infty(\mathbb{R}^D) \cap UC(\mathbb{R}^D); \nabla u \in H^{k-1}(\mathbb{R}^D)\}. \tag{2}$$

Here $UC(\mathbb{R}^D)$ denotes the space of uniformly continuous functions and $H^{k-1}(\mathbb{R}^D)$ is the usual Sobolev space. This choice is suitable for the above boundary conditions. The norm on $X^k(\mathbb{R}^D)$ is

$$||u||_{X^k(\mathbb{R}^D)} = ||u||_{L^\infty(\mathbb{R}^D)} + ||\nabla u||_{H^{k-1}(\mathbb{R}^D)}. \tag{3}$$

C. Gallo has proved the following result

Date: 6 july 2005.

Theorem 1.1. *Assume $k > \frac{D}{2}$ and $u_0 \in X^k(\mathbb{R}^D)$. Then the initial value problem for Gross-Pitaevskii equation is locally well posed in $X^k(\mathbb{R}^D)$. If moreover $u_0 \in X^{k+2}(\mathbb{R}^D)$, then $u(t) \in X^{k+2}(\mathbb{R}^D)$.*

On the other word, due to the particular form of the boundary conditions, we define the energy associated to our problem as

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^D} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^D} (1-|u|^2)^2 dx. \tag{4}$$

C. Gallo also proved that for $E(u_0) < +\infty$ and for $D \leq 2$, then the energy $E(u(t))$ is conserved along the trajectories. As a consequence of this result, he has established that the one-dimensional initial value problem is globally well-posed in $X^1(\mathbb{R})$.
In this short article, our first result shows the persistence of the energy is valid on *any* dimension, i.e

Proposition 1.2. *For any $D \geq 2$, if u_0 in $X^k(\mathbb{R}^D)$ ($k > \frac{D}{2}$) is such that $E(u_0) < +\infty$, then the energy $E(u(t))$ is conserved along the trajectories.*

The proof relies on the study of the growth rate of the energy along annuli in $\mathbb{R}^D$.

Our second result states as follows

Proposition 1.3. *The initial value problem for Gross-Pitaevskii equations is globally well-posed in $X^2(\mathbb{R}^2)$ for initial data u_0 that is in this space and that has finite energy, i.e $E(u_0) < +\infty$.*

The proof relies on some Brezis-Gallouet inequality similar to those that appeared in [2].
After this work was completed, we learned that P. Gérard (see [5]) proved the global well-posedness for these equations, in the cases $D = 2$ or $D = 3$, in the energy space

$$E = \{u \in H^1_{loc}(\mathbb{R}^D); E(u) < +\infty\}. \tag{5}$$

See also [4] for where the author uses similar framework to handle more general nonlinearities and problem in exterior domains. For the sake of completeness, we would like to point out also the articles [6], [7] where the authors solve a parabolic regularization of the Gross-Pitaevskii equations, the so-called complex Ginzburg-Landau equations, in local Sobolev spaces. The next section is devoted to proving Proposition 1.2. We complete this article by the proof of Proposition 1.3 in a third section.

2. Persistence of the energy

2.1. **Splitting of the energy.** Consider a function u that has finite energy $E(u) < +\infty$. Set

$$C_j = \{x \in \mathbb{R}^D, j \leq |x| < j+1\}. \tag{6}$$

The kinetic energy of u expands as follows

$$\frac{1}{2}\int_{\mathbb{R}^D}|\nabla u|^2dx=\sum_{j=0}^{+\infty}k_j, \qquad k_j=\frac{1}{2}\int_{C_j}|\nabla u|^2dx. \tag{7}$$

The potential energy reads also

$$\frac{1}{4}\int_{\mathbb{R}^D}(1-|u|^2)^2dx=\sum_{j=0}^{+\infty}p_j, \qquad p_j=\frac{1}{4}\int_{C_j}(1-|u|^2)^2dx. \tag{8}$$

Consider now a function $u(t,x)$ that is solution to Gross-Pitaevskii equations and that starts from u_0 that has finite energy. Consider any T such that $u(t)$ belongs to Zhidkov space $X^2(\mathbb{R}^D)$ for $|t|\le T$. Obviously the kinetic part of the energy of $u(t)$ is bounded by the Zhidkov norm. We will prove below that the potential energy is also bounded, arguing by contradiction on the growth rate of $j\to p_j$.

2.2. **Growth rate of the potential energy.** In this section C will be a constant that depends on $\sup_{|t|\le T}||u(t)||_{X^2(\mathbb{R}^D)}$ and that may vary from one line to one another. On the one hand, we have

$$p_j\le\frac{1}{4}(1+||u||_{L^\infty}^2)^2\mathrm{vol}(C_j)\le Cj^{D-1}, \tag{9}$$

where $\mathrm{vol}(C_j)$ denotes the D dimensional volume of the annulus.

On the other hand, we introduce θ_j a radially symmetric function on $\mathbb{R}^D$ such that $\theta_j=1$ if $|x|\le j$, $\theta_j=0$ if $|x|\ge j+1$, and $|\nabla\theta_j|\le 2$ everywhere. Multiply (1) by $\bar u_t\theta_j$ and integrate the real part of the resulting equation over $\mathbb{R}^D$ to obtain

$$\frac{d}{dt}\left(\int_{\mathbb{R}^D}(\frac{1}{2}|\nabla u|^2+\frac{1}{4}(1-|u|^2)^2)\theta_j dx\right)=-\mathrm{Re}\left(\int_{\mathbb{R}^D}\bar u_t\nabla u.\nabla\theta_j dx\right). \tag{10}$$

We majorize the r.h.s of this equality as follows

$$|\mathrm{Re}\left(\int_{\mathbb{R}^D}\bar u_t\nabla u.\nabla\theta_j dx\right)|\le 2||u_t||_{L^2(C_j)}||\nabla u||_{L^2(C_j)}. \tag{11}$$

Introducing $K_j=\int_0^T k_jdt$ and $P_j=\int_0^T p_jdt$, we then infer from (10)-(11) (integrating in time and using Cauchy-Schwarz inequality)

$$\sum_{l=0}^{j-1}(k_l+p_l)\le E(u_0)+2\ (\int_0^T||u_t||_{L^2(C_j)}^2dt)^{1/2}(\int_0^T||\nabla u||_{L^2(C_j)}^2dt)^{1/2}. \tag{12}$$

Going back to the equation (1), we also have

$$(13)\quad ||u_t||_{L^2(C_j)} \leq ||\Delta u||_{L^2(C_j)} + ||u||_{L^\infty(C_j)}||1-|u|^2||_{L^2(C_j)} \leq C(1+\sqrt{p_j}).$$

Therefore

$$(14)\qquad \sum_{l=0}^{j-1}(k_l+p_l) \leq E(u_0) + C\sqrt{K_j}(1+\sqrt{P_j}).$$

Integrating once more in time, we then obtain

$$(15)\qquad \sum_{l=0}^{j-1}(K_l+P_l) \leq TE(u_0) + CT\sqrt{K_j}(1+\sqrt{P_j}).$$

We aim to prove that $\sum_{l=0}^{+\infty} P_l < +\infty$. Let us argue by contradiction; we now pretend that $\sum_{l=0}^{+\infty} P_l = +\infty$. Since $K_j \to 0$, we then infer from (15) that $P_j \to +\infty$. We now introduce $a = TE(u_0) + 2CT\sup_j(\sqrt{K_j})$ and $Q_j = \max(1, P_j)$. We then infer from (15), dropping some unnecessary terms, that $\frac{Q_{j-1}}{a^2} \leq \sqrt{\frac{Q_j}{a^2}}$.
There exists j_0 such that for $j \geq j_0$ then $Q_j \geq 2a^2$. Therefore $Q_j \geq a^2 2^{2^{j-j_0}}$. This contradicts (9). Therefore $\sum_{l=0}^{+\infty} P_l < +\infty$. Going back to (14) and letting $j \to +\infty$, we thus obtain

$$(16)\qquad E(u(t)) \leq E(u_0).$$

Since we can go backward in time in the equation, the reverse inequality is also valid. This completes the proof of Proposition 1.2.

3. Global existence result in the two-dimensional case

In this section c is a numerical constant that may vary from one line to one another.

3.1. **Brezis-Gallouët type inequality.** We first state and prove

Proposition 3.1. *Consider a function u that belongs to $X^2(\mathbb{R}^2)$ and such that $E(u) < +\infty$. There exists a constant c that is independent of u such that*

$$||u||_{L^\infty(\mathbb{R}^2)} \leq c(1+\sqrt{E(u)})\left(1+\log(1+||\Delta u||^2_{L^2(\mathbb{R}^2)})\right)^{1/2}.$$

Proof: Consider first a smooth function $\phi(x)$ in $H^2(\mathbb{R}^2)$. Consider $R > 0$. We have

(17)
$$||\hat{\phi}||_{L^1(\mathbb{R}^2_\xi)} \leq c\left(||\phi||_{H^1(\mathbb{R}^2)}(\int_{|\xi|\leq R}\frac{d\xi}{1+|\xi|^2})^{1/2} + ||\Delta\phi||_{L^2(\mathbb{R}^2)}(\int_{|\xi|\geq R}\frac{d\xi}{|\xi|^4})^{1/2}\right) \leq$$
$$c(||\phi||_{H^1(\mathbb{R}^2)}\log(1+R^2)^{1/2} + ||\Delta\phi||_{L^2(\mathbb{R}^2)}R^{-1}.$$

Consider u in $\cap_{k\geq 1}X^k(\mathbb{R}^2)$. Actually, by density results we just have to prove the Brezis-Gallouët inequality for smooth functions. We now chose $\phi(x) = 1 - |u(x)|^2$ in (17). On the other hand, we have, setting $E = E(u)$ for the sake of simplicity

$$||\phi||^2_{L^2(\mathbb{R}^2)} \leq 4E, \tag{18}$$

$$||\nabla\phi||_{L^2(\mathbb{R}^2)} \leq 2||u\nabla u||_{L^2(\mathbb{R}^2)} \leq 2||u||_{L^\infty(\mathbb{R}^2)}\sqrt{E}, \tag{19}$$

$$\begin{aligned} ||\Delta\phi||_{L^2(\mathbb{R}^2)} &\leq 2||\nabla u||^2_{L^4(\mathbb{R}^2)} + 2||u||_{L^\infty(\mathbb{R}^2)}||\Delta u||_{L^2(\mathbb{R}^2)} \\ &\leq c\left(||\nabla u||_{L^2(\mathbb{R}^2)}||\Delta u||_{L^2(\mathbb{R}^2)} + ||u||_{L^\infty(\mathbb{R}^2)}||\Delta u||_{L^2(\mathbb{R}^2)}\right) \\ &\leq c||\Delta u||_{L^2(\mathbb{R}^2)}(\sqrt{E} + ||u||_{L^\infty(\mathbb{R}^2)}), \end{aligned} \tag{20}$$

due to classical Gagliardo-Niremberg inequality.
We then infer from (17)-(20) that

$$\begin{aligned} -1 + ||u||^2_{L^\infty(\mathbb{R}^2)} \leq c\left(\sqrt{E}(1+||u||_{L^\infty(\mathbb{R}^2)})\right)(\log(1+R^2)^{1/2} + \\ c(||\Delta u||_{L^2}(||u||_{L^\infty(\mathbb{R}^2)} + \sqrt{E})R^{-1}) \end{aligned} \tag{21}$$

Then either $||u||_{L^\infty(\mathbb{R}^2)} \leq c(1+\sqrt{E})$ and Proposition 3.1 is valid. Or, the reverse inequality $\sqrt{E} \leq c||u||_{L^\infty(\mathbb{R}^2)}$ together with (21) implies

(22)
$$(||u||_{L^\infty(\mathbb{R}^2)} + 1)(||u||_{L^\infty(\mathbb{R}^2)} - 1) \leq c\left(\sqrt{E}(1+||u||_{L^\infty(\mathbb{R}^2)})\right)(\log(1+R^2)^{1/2} +$$
$$c(||\Delta u||_{L^2}(||u||_{L^\infty(\mathbb{R}^2)} + 1)R^{-1}),$$

and then, dividing by $||u||_{L^\infty(\mathbb{R}^2)} + 1$,

$$||u||_{L^\infty(\mathbb{R}^2)} \leq 1 + c\left(\sqrt{E}\log(1+R^2)^{1/2} + ||\Delta u||_{L^2(\mathbb{R}^2)}R^{-1}\right). \tag{23}$$

Choosing $R = ||\Delta u||_{L^2} + 1$ completes the proof of the Proposition.

3.2. **The proof of Proposition 1.3.** Consider a solution of (1) that starts from u_0 in $X^2(\mathbb{R}^2)$. This solution is obtained by a fixed point argument on the Duhamel's form of the equation. Therefore it is classical to observe that either this solution is defined for all time, or that the X^2 norm of the solution blows up in finite time. We prove below that $||\nabla u||_{H^1(\mathbb{R}^2)}$ remains bounded. Since the L^2-norm of ∇u is bounded by the energy, it is standard to observe that we just have to bound $||\Delta u||_{L^2(\mathbb{R}^2)}$ along the trajectories. We first seek for an upper bound to $||u_t||_{L^2(\mathbb{R}^2)}$

Set $w(t) = u_t(t)$. Then $w(t)$ is solution in $L^2(\mathbb{R}^2)$

$$iw_t + \Delta w + w(1 - |u|^2) - 2\mathrm{Re}(w\bar{u})u = 0, \tag{24}$$

supplemented with initial data $w(0) = i\Delta u_0 + iu_0(1 - |u_0|^2)$ in $L^2(\mathbb{R}^2)$ (once again we use that u_0 has finite energy). Let us multiply (24) by $\bar{w}$ and let us integrate the imaginary part of the resulting equation. We obtain

$$\frac{1}{2}\frac{d}{dt}||w||^2_{L^2(\mathbb{R}^2)} = 2\int_{\mathbb{R}^2} \mathrm{Re}(w\bar{u})\mathrm{Im}(w\bar{u})dx \leq 2||u||^2_{L^\infty(\mathbb{R}^2)}||w||^2_{L^2(\mathbb{R}^2)}. \tag{25}$$

Using now Proposition 3.1, we infer from this inequality that

$$\frac{1}{2}\frac{d}{dt}||w||^2_{L^2(\mathbb{R}^2)} \leq c\,(1+E)\left(1 + \log(1 + ||\Delta u||^2_{L^2(\mathbb{R}^2)})\right)||w||^2_{L^2(\mathbb{R}^2)}. \tag{26}$$

On the other hand, going back once more to the equation (1), we also have

$$||\Delta u||_{L^2(\mathbb{R}^2)} \leq ||w||_{L^2(\mathbb{R}^2)} + 2||u||_{L^\infty(\mathbb{R}^2)}\sqrt{E} \leq ||w||_{L^2(\mathbb{R}^2)} + c(1+E)\log(1+||\Delta u||^2_{L^2(\mathbb{R}^2)})^{1/2}. \tag{27}$$

This last inequality implies that there exists a constant C_E that depends on E such that $||\Delta u||_{L^2(\mathbb{R}^2)} \leq C_E(1 + ||w||_{L^2(\mathbb{R}^2)})$. If we substitute this in (26), we infer that

$$\frac{1}{2}\frac{d}{dt}||w||^2_{L^2(\mathbb{R}^2)} \leq \tilde{C}_E(1 + \log(2 + ||w||^2_{L^2(\mathbb{R}^2)}))||w||^2_{L^2(\mathbb{R}^2)}. \tag{28}$$

Therefore the exist a, b depending on $||u_0||_{X^2(\mathbb{R}^2)}$ and on E such that $||w(t)||_{L^2(\mathbb{R}^2)} \leq e^{ae^{bt}}$. At this stage, we infer from this and from (27) that the L^2-norm of Δu cannot blow up in finite time. Then, due to Proposition 3.1, the L^∞ norm of u is also controlled. This completes the proof of Proposition 1.3.

ACKNOWLEDGEMENT

The author would like to thank Alberto Farina and Clément Gallo for valuable discussions about this work. Thanks also to the anonymous referees for helpful remarks and comments.

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