SUPERCONNECTION AND FAMILY BERGMAN KERNELS

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Abstract. We establish an asymptotic expansion for families of Bergman kernels. The key idea is to use the superconnection as in the local family index theorem.

Superconnexion et noyaux de Bergman en famille

Résumé. Nous annonçons des résultats sur le développement asymptotique du noyau de Bergman en famille.

Let W, S be smooth compact complex manifolds. Let $\pi : W \to S$ be a holomorphic submersion with compact fiber X and $\dim_{\mathbb{C}} X = n$.

We will add a subscript $\mathbb{R}$ for the corresponding real objects. Thus TX is the holomorphic relative tangent bundle of π, and $T_{\mathbb{R}}X$ is the corresponding real vector bundle. Let $J^{T_{\mathbb{R}}X}$ be the complex structure on $T_{\mathbb{R}}X$.

Let E be a holomorphic vector bundle on W. Let L be a holomorphic line bundle on W. Let h^L, h^E be Hermitian metrics on L, E. Let ∇^L, ∇^E be the holomorphic Hermitian connections on (L, h^L), (E, h^E) with their curvatures R^L, R^E respectively. Set

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{0.1}$$

Then ω is a smooth real 2-form of complex type $(1,1)$ on W.

We suppose that ω defines a fiberwise Kähler form along the fiber X, i.e.

$$g^{T_{\mathbb{R}}X}(u,v) = \omega(u, J^{T_{\mathbb{R}}X} v) \tag{0.2}$$

defines a (fiberwise) Riemannian metric on $T_{\mathbb{R}}X$. We denote by h^{TX} the corresponding Hermitian metric on TX.

Let dv_X be the Riemannian volume form on $(X, g^{T_{\mathbb{R}}X})$.

By the Kodaira vanishing theorem, there exists $p_0 \in \mathbb{N}$ such that $H^0(X, (L^p \otimes E)|_X)$ forms a vector bundle, denoted by $H^0(X, L^p \otimes E)$, on S for $p > p_0$. From now on, we always assume $p > p_0$.

By the Grothendieck-Riemann-Roch Theorem, for $p > p_0$, we have

$$\operatorname{ch}(H^0(X, L^p \otimes E)) = \int_X \operatorname{Td}(TX) \operatorname{ch}(E) \operatorname{ch}(L^p) \quad \text{in } H^\bullet(S, \mathbb{R}). \tag{0.3}$$

Date: July 18, 2006.

The component in $H^0(S,\mathbb{R})$ of (0.3) is the Hirzebruch-Riemann-Roch Theorem, and as $p \to +\infty$,

$$\dim H^0(X, L^p\otimes E) = \mathrm{rk}(E)\int_X \frac{c_1(L)^n}{n!}p^n + \int_X \Big(c_1(E)+\frac{\mathrm{rk}(E)}{2}c_1(TX)\Big)\frac{c_1(L)^{n-1}}{(n-1)!}p^{n-1} + \mathscr{O}(p^{n-2}). \tag{0.4}$$

For $s\in S$, let $P_{p,s}$ be the orthogonal projection from $\mathscr{C}^\infty(X_s,(L^p\otimes E)_{X_s})$ onto $H^0(X_s,(L^p\otimes E)|_{X_s})$. Let $P_{p,s}(x,x')$ $(x,x'\in X_s, s\in S)$ be the smooth kernel of $P_{p,s}$ with respect to $dv_{X_s}(x')$. Then $P_{p,s}(x,x')$ is smooth on $s\in S$, and we denote it simply by $P_p(x,x')$, especially, $P_p(x,x)\in \mathrm{End}(E)_x$.

The results of [14], [16], [4] tell us that there exist $b_r\in\mathscr{C}^\infty(X_s,\mathrm{End}(E|_{X_s}))$ such that for any $k,l\in\mathbb{N}$, there exists $C>0$ such that

$$\Big|\frac{1}{p^n}P_{p,s}(x,x)-\sum_{r=0}^k b_r(x)p^{-r}\Big|_{\mathscr{C}^l(X_s)} \le C\,p^{-k-1}. \tag{0.5}$$

In [7], [15], Lu and Wang show that the first two coefficients b_0,b_1 coincide with the corresponding terms in the local Hirzebruch-Riemann-Roch Theorem, i.e. the leading terms in the Chern-Weil representative of $\mathrm{Td}(TX)\,\mathrm{ch}(E)\,\mathrm{ch}(L^p)$ with respect to the metrics h^{TX},h^L,h^E. We refer to [5], [9], [10] for alternate approaches as well as extensions to the symplectic case.

By (0.3), in $H^2(S,\mathbb{R})$, as $p\to\infty$, we have

$$c_1(H^0(X,L^p\otimes E)) = \mathrm{rk}(E)\int_X \frac{c_1(L)^{n+1}}{(n+1)!}p^{n+1} + \int_X \Big(c_1(E)+\frac{\mathrm{rk}(E)}{2}c_1(TX)\Big)\frac{c_1(L)^{n}}{n!}p^{n} + \mathscr{O}(p^{n-1}). \tag{0.6}$$

Now, in view of the Bismut local family index theorem [2], it is nature to ask whether the analogue of (0.5) still holds on higher degree levels , which will involve the curvature of the vector bundle $H^0(X,L^p\otimes E)$ as in (0.6).

In this note, we announce some results on the existence of such an expansion, and compute the first two coefficients in the expansion.

To define a canonical connection on $H^0(X,L^p\otimes E)$, we need to introduce a horizontal sub-bundle T^HW of TW.

Let T^HW be a sub-bundle of TW such that

$$TW = T^HW\oplus TX. \tag{0.7}$$

Let P^{TX} be the projection from TW onto TX. For $U\in TS$, let $U^H\in T^HW$ be the lift of U.

Clearly, (0.7) induces canonically a decomposition $\Lambda(T^*_{\mathbb{R}}W)=\pi^*(\Lambda(T^*_{\mathbb{R}}S))\widehat{\otimes}\Lambda(T^*_{\mathbb{R}}X)$. For a differential form A on W, we will denote by $A^{(i)}$ its component in $\Lambda^i(T^*_{\mathbb{R}}S)\widehat{\otimes}\Lambda(T^*_{\mathbb{R}}X)$. Then $dv_X=(\omega^n)^{(0)}/n!$.

Let $T \in \Lambda^2(T^*_{\mathbb{R}}W)\otimes T_{\mathbb{R}}X$ be the tensor defined in the following way: for $U,V\in TS$, $X,Y\in TX$,

$$T(U^H,V^H) := -P^{TX}[U^H,V^H], \quad T(X,Y):=0, \tag{0.8}$$
$$T(U^H,X) := \frac{1}{2}(g^{TX})^{-1}(\mathcal{L}_{U^H}g^{TX})X.$$

Let $\nabla^{L^p\otimes E}$ be the connection on $L^p\otimes E$ induced by ∇^L,∇^E. For $U\in T_{\mathbb{R}}S$, $\sigma\in\mathscr{C}^\infty(S,H^0(X,L^p\otimes E))$, we define

$$\nabla_U^{H^0(X,L^p\otimes E)}\sigma = P_p\nabla_{U^H}^{L^p\otimes E}P_p\sigma. \tag{0.9}$$

Then $\nabla^{H^0(X,L^p\otimes E)}$ is a holomorphic connection on $H^0(X,L^p\otimes E)$ with curvature $R^{H^0(X,L^p\otimes E)}$, but it need not be a Hermitian connection with respect to the (usual) induced L^2 metric $h^{H^0(X,L^p\otimes E)}$ on $H^0(X,L^p\otimes E)$.

Let $\mathbf{k}\in T^*_{\mathbb{R}}W$ be such that for $U\in T_{\mathbb{R}}S$, $X\in T_{\mathbb{R}}X$,

$$\mathbf{k}(U^H) = \frac{1}{2}(L_{U^H}dv_X)/dv_X, \quad \mathbf{k}(X)=0. \tag{0.10}$$

Then

$$\nabla_U^{\mathrm{Ker}(D_p)} = P_p(\nabla_{U^H}^{L^p\otimes E}+\mathbf{k}(U^H))P_p, \tag{0.11}$$

is a canonical Hermitian connection on $(H^0(X,L^p\otimes E),h^{H^0(X,L^p\otimes E)})$ with curvature $R^{\mathrm{Ker}(D_p)}$, but it need not be holomorphic.

Let $R^{H^0(X,L^p\otimes E)}(x,x')$, $R^{\mathrm{Ker}(D_p)}(x,x')$ ($x,x'\in X_s, s\in S$) be the smooth kernel of the operator $R^{H^0(X,L^p\otimes E)}$, $R^{\mathrm{Ker}(D_p)}$ with respect to $dv_X(x')$. Then

$$R^{H^0(X,L^p\otimes E)}(x,x), \quad R^{\mathrm{Ker}(D_p)}(x,x)\in\Lambda^2(T^*S)\otimes\mathrm{End}(E_x). \tag{0.12}$$

If

$$T^HW = \{u\in TW; \omega(u,\overline{X})=0 \text{ for any } X\in TX\}, \tag{0.13}$$

then the triple $(\pi, g^{T_{\mathbb{R}}X}, T^HW)$ defines a Kähler fibration in the sense of [3, Definition 1.4]. In this case, the connection $\nabla^{\mathrm{Ker}(D_p)}$ is the canonical holomorphic connection on $(H^0(X,L^p\otimes E),h^{H^0(X,L^p\otimes E)})$, and

$$\mathbf{k}=0, \quad \nabla^{\mathrm{Ker}(D_p)} = \nabla^{H^0(X,L^p\otimes E)}. \tag{0.14}$$

Let $\{w_i\}$ be an orthonormal frame of (TX,h^{TX}). Let $\{e_i\}$ be an orthonormal frame of $(T_{\mathbb{R}}X,g^{TX})$. Let $\{g_\alpha\}$ be an frame of TS and $\{g^\alpha\}$ its dual frame.

Theorem 0.1. *There exist smooth sections $b_{2,r}(x)\in\mathscr{C}^\infty(W,\Lambda^2(T^*S)\otimes\mathrm{End}(E_x))$ which are polynomials in R^{TX}, T, R^E (and R^L), their derivatives of order $\leqslant 2r-1$ (resp. $2r$) along the fiber X, with*

$$b_{2,0} = -2\pi\sqrt{-1}\frac{(\omega^{n+1})^{(2)}}{(\omega^n)^{(0)}}\,\mathrm{Id}_E, \tag{0.15}$$

such that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $x \in W$, $p \in \mathbb{N}$, $p > p_0$,

$$\Big|\frac{1}{p^{n+1}} R^{H^0(X,L^p\otimes E)}(x,x) - \sum_{r=0}^{k} b_{2,r}(x) p^{-r}\Big|_{\mathscr{C}^l(W)} \leqslant C_{k,l}\, p^{-k-1}. \tag{0.16}$$

For $R^{\mathrm{Ker}(D_p)}(x,x)$, we have the similar expansion as (0.16)*, with the same leading term $b_{2,0}$ in* (0.15)*, and the corresponding $b_{2,r}(x)$ depends also the derivative of $d\mathbf{k}$ of order $\leqslant 2r-1$ along the fiber X.*

If (0.13) *is verified, then*

$$b_{2,1} = \Big(\Big(\frac{1}{2}\langle R^{TX} w_i, \overline{w}_i\rangle + R^E + \frac{\sqrt{-1}}{4} g^\alpha \wedge \overline{g}^\beta \Delta_X(\omega(g^H_\alpha, \overline{g}^H_\beta))\Big)\omega^n\Big)^{(2)} /(\omega^n)^{(0)}, \tag{0.17}$$

here $\Delta_X = -\sum_i [(\nabla_{e_i})^2 - \nabla_{\nabla^{TX}_{e_i} e_i}]$ is the Bochner Laplacian along the fiber X.

If we take the trace on E and integrate along X in (0.16), from (0.15) and (0.17), we get a refinement of (0.6) on the level of differential forms, in the spirit of the Local Family Index Theorem.

From (0.15), we get

$$b_{2,0} = 2\pi g^\alpha \wedge \overline{g}^\beta \left[-\sqrt{-1}\omega(g^H_\alpha, \overline{g}^H_\beta) - \omega(g^H_\alpha, \overline{w}_j)\omega(\overline{g}^H_\beta, w_j)\right] \mathrm{Id}_E\,. \tag{0.18}$$

By (0.16), (0.18), the curvatures $R^{H^0(X,L^p\otimes E)}(x,x)$, $R^{\mathrm{Ker}(D_p)}(x,x)$ provide a natural approximation of the Monge-Ampère operator on the space of Kähler metrics. It must have relations with the existence problem of geodesics on the space of Kähler metrics (cf. [6], [8], [13], [12]).

The equation (0.18) gives also an exact local asymptotic behavior of the curvature estimates in [1, §6].

To prove Theorem 0.1, we will use the superconnection formation as in the local index theory. This is the main idea of our work. An important feature of superconnection is that its curvature is a second order differential operator along the fiber X, while the superconnection itself involves derivatives along the horizontal direction. This is also one of the points in the local index theory. Now, by combining the formal power series trick in [9], we get in fact a general and algorithmic way to compute the coefficients in the expansion. More details will appear in [11].

Remark. In this note, we have only formulated our results in the fiberwise positive holomorphic line bundle case. Actually, the results hold also in the fiberwise symplectic case, and we have the off-diagonal expansion results too.

Acknowledgements. The work of the second author was partially supported by MOEC and NNSFC. Part of work was done while the first author was visiting Centre de Recerca Matemàtica (CRM) in Barcelona during June and July, 2006. He would like to thank CRM for hospitality.

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